Problem Set 5

Solutions

Model Theory

Math 506, Spring 2004.

1. Let K be a field and let T be the theory of infinite K-vector spaces as in Problem 5 of the last Problem Set. Let V be an infinite K-vector space and A a subset of V. Show that $\operatorname{acl}_V(A)$ is the K-subspace of V spanned by A.

   Solution. If $b \in V$ is in the K-subspace of V spanned by A, then $a = \lambda_1 a_1 + \cdots + \lambda_n a_n$ for some $\lambda_i \in K$ and $a_i \in A$, and $x = b$ is the only solution to the formula $\varphi(x, a_1, \ldots, a_n)$, where φ is the $\mathcal{L}$-formula $x = \mu_{\lambda_1}(y_1) + \cdots + \mu_{\lambda_n}(y_n)$. Hence $b \in \operatorname{acl}(A)$. Conversely, let $\varphi(x, y_1, \ldots, y_n)$ be an $\mathcal{L}$-formula and $a = (a_1, \ldots, a_n) \in A^n$ such that $\mathcal{M} \vDash \varphi(b, a)$ and there are only finitely many $b' \in M$ such that $\mathcal{M} \vDash \varphi(b', a)$. By q.e. (see last Problem Set) we may assume that φ is quantifier-free, in fact, that φ is a disjunction of formulas of the form

$$\bigwedge_{i=1}^{r} \left(\sum_j \lambda_{ij} y_j\right) + \lambda_i x = 0 \wedge \bigwedge_{i=1}^{s} \left(\sum_j \lambda'_{ij} y_j\right) + \lambda'_i x \neq 0$$

   where $\lambda_{ij}, \lambda_i, \lambda'_{ij}, \lambda'_i \in K$. Replacing φ with such a disjunct which is satisfied by $(x, y) = (b, a)$, we may assume that φ is of this form. Since $\varphi(x, a)$ only has finitely many solutions we must have $\lambda_i \neq 0$ for some i. Then $b = \frac{1}{\lambda_i}\left(-\sum_j \lambda_{ij} a_j\right)$, showing that b is in the subspace of V spanned by A.

2. Use properties of model-theoretic algebraic closure in algebraically closed fields to prove the following facts. Here p is a prime number or 0, and F the prime field of characteristic p (that is, $F = \mathbb{F}_p$ if p is a prime and $F = \mathbb{Q}$ otherwise).

   a) Let k be a field of characteristic p and let K_1 and K_2 be algebraic closures of k, that is, algebraically closed extension fields of k which are algebraic over k (in the sense of fields). Show that there is an isomorphism $K_1 \to K_2$ which is the identity on k.

   b) Let K be an algebraically closed field of characteristic p. We call a subset B of K **algebraically independent** if $P(X_1, \ldots, X_n) \in F[X_1, \ldots., X_n]$ is a non-zero polynomial and $b_1, \ldots, b_n \in B$ are distinct elements of B, then $P(b_1, \ldots, b_n) \neq 0$. We call B a **transcendence basis** for K if B is algebraically independent and K is algebraic over the subfield $F(B)$ of K generated by B.

      i. Show that there exists a transcendence basis for K.

      ii. Show that B is a transcendence basis for K if and only if B is a minimal subset of K with the property that K is algebraic over $F(B)$.

      iii. Show that K is determined, up to isomorphism, by the cardinality of a transcendence basis for K.

Solution.

a) The theory ACF of algebraically closed fields eliminates quantifiers, hence the identity map on k is elementary with respect to K_1 and K_2. By Proposition (2.3.6) from class it can be extended to a bijective map $\mathrm{acl}_{K_1}(k) \to \mathrm{acl}_{K_2}(k)$, which is also elementary with respect to K_1, K_2. But since K_i is algebraic over k, we have $K_i = \mathrm{acl}_{K_i}(k)$ for $i = 1, 2$. This proves the existence of the desired isomorphism $K_1 \to K_2$ which is the identity on k.

b)

i. Let B be a basis for K in the pre-geometry acl_K. We claim that B is a transcendence basis for K. Let $P(X_1, ..., X_n) \in F[X_1, ..., X_n]$ be non-zero, and let $b_1, ..., b_n \in B$ be distinct elements of B with $P(b_1, ..., b_n) = 0$. Then we can write

$$P(X_1, ..., X_n) = \sum_{i=0}^{d} Q_i(X_1, ..., X_{n-1})\, X_n^i$$

for certain $Q_i \in F[X_1, ..., X_{n-1}]$, and we can assume $Q_d(b_1, ..., b_{n-1}) \neq 0$. This yields that we have

$$b_n \in \mathrm{acl}_K(\{b_1, ..., b_{n-1}\}) \subseteq \mathrm{acl}_K(B \setminus \{b_n\}),$$

which contradicts the fact that B is acl-independent. Furthermore, we have $K = \mathrm{acl}_K(B)$, so K is algebraic over the subfield $F(B)$ generated by B, by (2.3.2) in class. Hence B is a transcendence basis for K.

ii. By part (4) of Theorem (2.3.12) it suffices to show that $B \subseteq K$ spans K in the sense of acl_K if and only if K is algebraic over $F(B)$, and that $B \subseteq K$ is a transcendence basis if and only if it is a basis for K with respect to acl_K. The first statement is clear by the fact (2.3.2) that $\mathrm{acl}_K(A) = F(A)^{\mathrm{alg}}$ for all $A \subseteq K$, and the "$\Longleftarrow$"-direction of the second one was proved in (i). The reverse implication "$\Longrightarrow$" follows from the first statement and (2.3.2).

iii. Follows immediately from the fact that $B \subseteq K$ is a transcendence basis for K if and only if B is a basis for K with respect to acl_K, and part (3) of Theorem (2.3.16) from class.

3. Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $A \subseteq M$. We say that $b \in M$ is **definable over A** in $\mathcal{M}$ if there is a formula $\varphi(x, y_1, ..., y_n)$ and $a \in A^n$ such that

$$\mathcal{M} \vDash \varphi(b, a) \wedge \forall y(\varphi(y, a) \to y = b),$$

that is, $\{b\}$ is A-definable. Let $\mathrm{dcl}_{\mathcal{M}}(A) := \mathrm{dcl}(A) := \{b \in M : b \text{ is definable over } A\}$, the **definable closure of A** in $\mathcal{M}$.

a) Show that $\mathrm{dcl}(A)$ is the universe of a substructure of $\mathcal{M}$, and $A \subseteq \mathrm{dcl}(A) \subseteq \mathrm{acl}(A)$.

b) Show that b is definable over A if and only if for some $n \geqslant 1$ there is an $\emptyset$-definable function $f: M^n \to M$ and $a \in A^n$ such that $f(a) = b$.

c) Suppose that b is definable over A and σ is an automorphism of $\mathcal{M}$ such that $\sigma(a) = a$ for all $a \in A$. Show that $\sigma(b) = b$.

d) Show that $\mathrm{dcl}(\mathrm{dcl}(A)) = \mathrm{dcl}(A)$.

Solution.

a) If f is an n-ary function symbol from $\mathcal{L}$ and $b_1, \ldots, b_n \in \mathrm{dcl}(A)$, defined by formulas $\varphi_1(x, y), \ldots, \varphi_n(x, y)$, where $y = (y_1, \ldots, y_m)$, using parameters $a = (a_1, \ldots, a_m)$ from A, then $f^{\mathcal{M}}(b_1, \ldots, b_n)$ is defined by the formula

$$\psi(z, y) := \exists x_1 \cdots \exists x_n \left(\bigwedge_{i=1}^{n} \varphi(x_i, y) \wedge z = f(x_1, \ldots, x_n) \right)$$

using the same parameters a. Therefore $f^{\mathcal{M}}(b_1, \ldots, b_n) \in \mathrm{dcl}(A)$. If c is a constant symbol from $\mathcal{L}$ then $c^{\mathcal{M}} \in \mathrm{dcl}(A)$ is witness by the formula $x = c$. Hence $\mathrm{dcl}(A)$ is the universe of a substructure of $\mathcal{M}$. The formula $x = y$ can be used to show that $A \subseteq \mathrm{dcl}(A)$, and $\mathrm{dcl}(A) \subseteq \mathrm{acl}(A)$ follows from the definitions.

b) Suppose that for some $n \geqslant 1$ there is an $\emptyset$-definable function $f: M^n \to M$ and $a \in A^n$ such that $f(a) = b$. Let $\varphi(x, y_1, \ldots, y_n)$ be an $\mathcal{L}$-formula such that $(d, e) \in \Gamma(f) \Longleftrightarrow \mathcal{M} \vDash \varphi(e, d)$ for all $d \in M^n$, $e \in M$. Then $\mathcal{M} \vDash \varphi(b, a)$, and b is the only solution to $\varphi(x, a)$, since $\Gamma(f)$ is the graph of a function. Hence $b \in \mathrm{dcl}(A)$. Conversely, suppose that $b \in \mathrm{dcl}(A)$. Take a formula $\varphi(x, y_1, \ldots, y_n)$ and $a \in A^n$ witnessing this. Suppose first that $n \geqslant 1$. Then the formula $\delta(y_1, \ldots, y_n) = \exists x(\varphi(x, y_1, \ldots, y_n) \wedge \forall z(z \neq x \to \neg\varphi(z, y_1, \ldots, y_n)))$ defines a subset D of M^n containing a, and $\gamma(y_1, \ldots, y_n, x) := \varphi(x, y_1, \ldots, y_n) \wedge \delta(y_1, \ldots, y_n)$ defines the graph of a function $D \to M$ with $a \mapsto b$. Now $\psi(y_1, \ldots, y_n, x) := \gamma \vee (\neg\delta \wedge x = y_1)$ defines a function $M^n \to M$ with $a \mapsto b$, as required. Now suppose $n = 0$. Then b is the unique solution to $\varphi(x)$, hence $\psi(y, x) := \varphi(x)$ defines the constant function $M \to M$ with value b.

c) Clear since automorphisms preserve the truth of formulas.

d) The inclusion $\supseteq$ follows from (a). For $\subseteq$ let $b \in \mathrm{dcl}(\mathrm{dcl}(A))$. By (b) there exists a definable function $f: M^n \to M$ and $a = (a_1, \ldots, a_n) \in \mathrm{dcl}(A)^n$ such that $f(a) = b$, for some $n \geqslant 1$. By (b) again there exists for each i a definable function g_i: $M^{m_i} \to M$ and $c_i \in A^{m_i}$ such that $g_i(c_i) = a_i$, for some $m_i \geqslant 1$. Put $m := m_1 + \cdots + m_n$ and define g: $M^m \to M^n$ by $g(z_1, \ldots, z_m) := (g_1(z_1), \ldots, g_n(z_m))$ for all $z_i \in M^{m_i}$. Then g is a definable function, hence so is $h := f \circ g: M^m \to M$, with $h(c_1, \ldots, c_m) = a$. This shows that $b \in \mathrm{dcl}(A)$, by (b).

4. Let $\mathcal{L}$ be a language which contains a binary relation symbol $<$. Suppose that $\mathcal{M}$ is an $\mathcal{L}$-structure in which $<^{\mathcal{M}}$ is a linear ordering. Show that $\mathrm{acl}_{\mathcal{M}}(A) = \mathrm{dcl}_{\mathcal{M}}(A)$ for all $A \subseteq M$.

   Solution. Let $b \in \mathrm{acl}(A)$ and $\psi(x, y_1, \ldots, y_n)$ be an $\mathcal{L}$-formula, $a = (a_1, \ldots, a_n) \in A^n$ such that $\mathcal{M} \vDash \psi(b, a)$, and $\psi(x, a)$ has only finitely many solutions in $\mathcal{M}$. Let $b_1 <^{\mathcal{M}} \cdots <^{\mathcal{M}} b_m$ be all the different solutions to this formula, so $b = b_j$ for some $j \in \{1, \ldots, m\}$. Now define an $\mathcal{L}$-formula φ by $\varphi(x, y_1, \ldots, y_n) := \psi(x, y_1, \ldots, y_n) \wedge$"there are exactly $j - 1$ solutions of $\psi(x, y_1, \ldots, y_n)$ smaller than x." It is clear that $\mathcal{M} \vDash \varphi(b, a)$ and that b is the only solution to $\varphi(x, a)$. Hence $b \in \mathrm{dcl}(A)$, showing that $\mathrm{acl}(A) \subseteq \mathrm{dcl}(A)$. The reverse inclusion holds by 3. (a).

5. [Optional.] Give an example of a structure $\mathcal{M}$ (in some language $\mathcal{L}$) such that $\mathrm{acl}_{\mathcal{M}}(A) \neq \mathrm{dcl}_{\mathcal{M}}(A)$ for some subset A of M.

   Solution. Let $\mathcal{L} = \{0, 1, +, -, \cdot\}$ and let $K := \mathbb{Q}^{\mathrm{alg}}$ be the algebraic closure of the field $A := \mathbb{Q}$, considered as an $\mathcal{L}$-structure as usual. Obvisously $\mathrm{acl}(A) = K$. We claim that $\mathrm{dcl}(A) = A$ ($\neq K$). For this, let $a \in \mathrm{dcl}(\mathbb{Q})$, and let $P(X) \in \mathbb{Q}[X]$ be the minimal polynolmial of a over $\mathbb{Q}$. For every zero b of $P(X)$ in K there exists an automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{alg}}|\mathbb{Q})$ with $\sigma(a) = b$. By 4. (c) we get $a = b$. Hence a is the only zero of $P(X)$ in $\mathbb{Q}^{\mathrm{alg}}$. Since P is separable, this implies $P(X) = X - c$ for some $c \in \mathbb{Q}$, therefore $a = c \in \mathbb{Q}$.