Problem Set 5

Due April 16

Model Theory

Math 506, Spring 2004.

1. Let K be a field and let T be the theory of infinite K-vector spaces as in Problem 5 of the last Problem Set. Let V be an infinite K-vector space and A a subset of V. Show that $\mathrm{acl}_V(A)$ is the K-subspace of V spanned by A.

2. Use properties of model-theoretic algebraic closure in algebraically closed fields to prove the following facts. Here p is a prime number or 0, and F the prime field of characteristic p (that is, $F = \mathbb{F}_p$ if p is a prime and $F = \mathbb{Q}$ otherwise).

   a) Let k be a field of characteristic p and let K_1 and K_2 be algebraic closures of k, that is, algebraically closed extension fields of k which are algebraic over k (in the sense of fields). Show that there is an isomorphism $K_1 \to K_2$ which is the identity on k.

   b) Let K be an algebraically closed field of characteristic p. We call a subset B of K **algebraically independent** if $P(X_1, ..., X_n) \in F[X_1, ..., X_n]$ is a non-zero polynomial and $b_1, ...,$ $b_n \in B$ are distinct elements of B, then $P(b_1, ..., b_n) \neq 0$. We call B a **transcendence basis** for K if B is algebraically independent and K is algebraic over the subfield $F(B)$ of K generated by B.

      i. Show that there exists a transcendence basis for K.

      ii. Show that B is a transcendence basis for K if and only if B is a minimal subset of K with the property that K is algebraic over $F(B)$.

      iii. Show that K is determined, up to isomorphism, by the cardinality of a transcendence basis for K.

3. Let $\mathcal{M}$ be an $\mathcal{L}$-structure and $A \subseteq M$. We say that $b \in M$ is **definable over A** in $\mathcal{M}$ if there is a formula $\varphi(x, y_1, ..., y_n)$ and $a \in A^n$ such that

$$\mathcal{M} \models \varphi(b, a) \wedge \forall y(\varphi(y, a) \rightarrow y = b),$$

   that is, the set $\{b\}$ is A-definable. Let $\mathrm{dcl}_{\mathcal{M}}(A) := \mathrm{dcl}(A) := \{b \in M\colon b \text{ is definable over } A\}$, the **definable closure of A** in $\mathcal{M}$.

   a) Show that $\mathrm{dcl}(A)$ is the universe of a substructure of $\mathcal{M}$, and $A \subseteq \mathrm{dcl}(A) \subseteq \mathrm{acl}(A)$.

   b) Show that b is definable over A if and only if for some $n \geqslant 1$ there is an $\emptyset$-definable function $f\colon M^n \to M$ and $a \in A^n$ such that $f(a) = b$.

   c) Suppose that b is definable over A and σ is an automorphism of $\mathcal{M}$ such that $\sigma(a) = a$ for all $a \in A$. Show that $\sigma(b) = b$.

   d) Show that $\mathrm{dcl}(\mathrm{dcl}(A)) = \mathrm{dcl}(A)$.

4. Let $\mathcal{L}$ be a language which contains a binary relation symbol $<$. Suppose that $\mathcal{M}$ is an $\mathcal{L}$-structure in which $<^{\mathcal{M}}$ is a linear ordering. Show that $\mathrm{acl}_{\mathcal{M}}(A) = \mathrm{dcl}_{\mathcal{M}}(A)$ for all $A \subseteq M$.

5. [Optional.] Give an example of a structure $\mathcal{M}$ (in some language $\mathcal{L}$) such that $\mathrm{acl}_{\mathcal{M}}(A) \neq \mathrm{dcl}_{\mathcal{M}}(A)$ for some subset A of M.