Problem Set 6

Solutions

Model Theory

Math 506, Spring 2004.

1. Let $\mathcal{L} = \{0, 1, +, -, \cdot\}$ be the language of rings and consider the field $\mathbb{R}$ of real numbers as an $\mathcal{L}$-structure as usual. Is the $\mathcal{L}$-theory $\mathrm{Th}(\mathbb{R})$ model-complete? Does $\mathrm{Th}(\mathbb{R})$ admit quantifier elimination?

   Solution. A field K is real closed if and only if $\{a^2 : a \in K\}$ is a positive cone of K, and every polynomial in $K[X]$ of degree $\geqslant 3$ is reducible. These statements can be formulated as an $\mathcal{L}$-theory. Therefore every model of $\mathrm{Th}(\mathbb{R})$ is a real closed field.

   Let $K \subseteq L$ be an extension of models of $\mathrm{Th}(\mathbb{R})$. Then K and L can be equipped with unique orderings making them into ordered fields; with these orderings $K \subseteq L$ is an extension of ordered fields (in other words, the ordering of K is the restriction of the ordering of L). The ordered fields K and L are then models of the $\mathcal{L} \cup \{<\}$-theory RCF. Therefore (by q.e. qua model-completeness of RCF) we have $K \preccurlyeq L$ in the language $\mathcal{L} \cup \{<\}$ and hence also in the language $\mathcal{L}$. This shows that $\mathrm{Th}(\mathbb{R})$ is model-complete.

   We claim that $\mathrm{Th}(\mathbb{R})$ does *not* admit q.e.: a field whose theory admits quantifier elimination in the language $\mathcal{L}$ is strongly minimal (same proof as for ACF), but $\mathbb{R}$ is not strongly minimal: the set $\{x \in \mathbb{R} : x \geqslant 0\}$ is definable (by the formula $\varphi(x) = \exists y (y^2 = x)$) but neither finite nor cofinite.

2. Let $\mathcal{L}_E = \{L, B, C, A, D\}$ be the following language: L and B are ternary relation symbols; C and A are 6-ary relation symbols, and G is a 4-ary relation symbol. We let $\mathcal{E} = (E, L^{\mathcal{E}}, B^{\mathcal{E}}, C^{\mathcal{E}}, A^{\mathcal{E}}, D^{\mathcal{E}})$ be the $\mathcal{L}_E$-structure with universe $E = \mathbb{R}^2$ where

   i. $\mathcal{E} \vDash L^{\mathcal{E}}(a, b, c)$ if and only if a, b, c are collinear, and $\mathcal{E} \vDash B^{\mathcal{E}}(a, b, c)$ if and only if a, b, c are collinear and c lies between a and b;

   ii. $\mathcal{E} \vDash C^{\mathcal{E}}(a, b, c, a', b', c')$ if and only if the triangles with vertices a, b, c and a', b', c', respectively, are congruent; $\mathcal{E} \vDash A^{\mathcal{E}}(a, b, c, a', b', c')$ if and only if the angle between the line segments $a\,b$ and $b\,c$ is the same as the angle between $a'\,b'$ and $b'\,c'$;

   iii. $\mathcal{E} \vDash D^{\mathcal{E}}(a, b, a', b')$ if and only if the distance between a and b is the same as the distance between a' and b'.

   Show that $\mathrm{Th}(\mathcal{E})$ is decidable. (Decidability of plane euclidean geometry; Tarski 1948.)

   Solution. We know from class that the theory of $\mathbb{R}$, construed as a structure in the language $\mathcal{L} = \{0, 1, +, -, \cdot, <\}$, is decidable. (A consequence of the recursive axiomatization of $\mathrm{Th}(\mathbb{R})$.) Therefore it is enough to describe an algorithm which, given as input an $\mathcal{L}_E$-sentence φ, produces an $\mathcal{L}$-sentence φ^* such that $\mathcal{E} \vDash \varphi$ if and only if $\mathbb{R} \vDash \varphi^*$. This is easy, by interpreting the relation symbols in $\mathcal{L}_E$ by the corresponding (definable) relations on cartesian powers of $\mathbb{R}^2$— for example, the $\mathcal{L}_E$-sentence

$$\varphi = \forall a \forall b \forall c (L(a, b, c) \rightarrow a = b \vee B(a, b, c) \vee B(a, c, b) \vee B(c, b, a))$$

(which holds in $\mathcal{E}$) will be replaced by the $\mathcal{L}$-sentence

$\varphi^* = \forall a_1 \forall a_2 \forall b_1 \forall b_2 \forall c_1 \forall c_2 (L^*(a_1, a_2, b_1, b_2, c_1, c_2) \rightarrow (a_1 = b_1 \wedge a_2 = b_2) \vee B^*(a_1, a_2, b_1, b_2, c_1, c_2) \vee B^*(a_1, a_2, c_1, c_2, b_1, b_2) \vee B^*(c_1, c_2, b_1, b_2, a_1, a_2))$,

where

$$L^*(a_1, a_2, b_1, b_2, c_1, c_2) := (a_1 = b_1 \wedge a_2 = b_2) \vee \exists \lambda (\lambda (b_1 - a_1) = c_1 - a_1 \wedge \lambda (b_2 - a_2) = c_2 - a_2)$$

and

$B^*(a_1, a_2, b_1, b_2, c_1, c_2) := (a_1 \neq b_1 \vee a_2 \neq b_2) \wedge \exists \lambda (0 < \lambda < 1 \wedge \lambda (b_1 - a_1) = c_1 - a_1 \wedge \lambda (b_2 - a_2) = c_2 - a_2)$.

3. Let F be an ordered field.

   a) Show that F is real closed if and only if for every $P(X) \in F[X]$ and $a < b$ in F such that $P(a)P(b) < 0$ there exists $c \in F$ with $P(c) = 0$ ("P has the intermediate value property").

   b) Construe F as an $\mathcal{L}$-structure as usual, where $\mathcal{L} = \{0, 1, +, -, \cdot, <\}$. Use (a) to show that if $\mathrm{Th}(F)$ is o-minimal then F is real closed.

Solution.

a) Suppose first that every $P \in F[X]$ has the intermediate value property. Let $a > 0$ and put $P(X) := X^2 - a$. Then $P(0) < 0$ and $P(1 + a) > 0$, hence there is $c \in F$ with $c^2 = a$. Now let $P(X) = X^n + \sum_{i=0}^{n-1} a_i X^i$ where $a_i \in F$ and n is odd. Then for $M := 1 + |a_{n-1}| + \cdots + |a_0|$ we have $P(M) > 0$ and $P(-M) < 0$. This follows from

$$P(\pm M)/(\pm M)^n = 1 + a_{n-1} (\pm M)^{-1} + \cdots + a_0 (\pm M)^{-n} > 0,$$

since

$$|a_{n-1} (\pm M)^{-1} + \cdots + a_0 (\pm M)^{-n}| \leqslant (|a_{n-1}| + \cdots + |a_0|) \cdot |M|^{-1} < 1$$

by the triangle inequality, which holds for $|\cdot|$. (Why?) Hence $P(c) = 0$ for some $c \in F$ with $-M < c < M$. Therefore F is real closed. Conversely, suppose that F is real closed, and let $P(X) \in F[X]$, $a < b$ in F with $P(a)P(b) < 0$. We may assume that $P(a) < 0 < P(b)$ and P irreducible (why?). If P is not linear, then $P(X) = (X - d)^2 + e^2$ with $d, e \in F$, $e \neq 0$; but then $P(x) > 0$ for all $x \in F$, which contradicts $P(a) < 0$. So $P(X)$ is linear, and hence has a zero in the interval (a, b).

b) Suppose that F is o-minimal. We show that every $P(X) \in F[X]$ has the intermediate value property. Let $a < b$ with $P(a)P(b) < 0$. Then both $A := \{x \in F: P(x) > 0\}$ and $B := \{x \in F: P(x) < 0\}$ are non-empty. As remarked in class, P gives rise to a continuous function $x \mapsto P(x)$. Hence both A and B are open. By o-minimality it follows that A and B are disjoint unions of open intervals in F, and their union is F. This is impossible.

4. Recall that a function $f: \mathbb{R} \rightarrow \mathbb{R}$ is called **semialgebraic** if its graph

$$\Gamma(f) = \{(x, f(x)): x \in \mathbb{R}\} \subseteq \mathbb{R}^2$$

is semialgebraic.

a) We say that a function $f: \mathbb{R} \to \mathbb{R}$ is **algebraic** if there is a non-zero polynomial $P(X, Y) \in \mathbb{R}[X, Y]$ such that $P(x, f(x)) = 0$ for all $x \in \mathbb{R}$. Show that every semialgebraic function $\mathbb{R} \to \mathbb{R}$ is algebraic.

b) Use (a) to show that the exponential function $x \mapsto e^x: \mathbb{R} \to \mathbb{R}$ is not semialgebraic.

Solution.

a) The graph of a semialgebraic function $f: \mathbb{R} \to \mathbb{R}$ is a finite union of semialgebraic sets of the form

$$\{(x, y) \in \mathbb{R} \times \mathbb{R}: P_1(x, y) = \cdots = P_m(x, y) = 0, Q_1(x, y) > 0, \ldots, Q_n(x, y) > 0\}$$

where P_i, $Q_j \in \mathbb{R}[X, Y]$ and at least one among the P_i non-zero, since otherwise, the graph of f would contain a non-empty open subset of $\mathbb{R}$. If we take P to be the product of these non-zero polynomials, then $P(x, f(x)) = 0$ for all $x \in \mathbb{R}$.

b) By (a) it is enough to show the following: if $p_0, \ldots, p_n \in \mathbb{R}[X]$ are such that

$$p_n(x)\, e^{nx} + p_{n-1}(x)\, e^{(n-1)x} + \cdots + p_0(x) = 0 \qquad \text{for all } x \in \mathbb{R},$$

then $p_0 = \cdots = p_n = 0$. Suppose that $p_n \neq 0$. Then for all sufficiently large $x > 0$ we have, $p_n(x) \neq 0$, hence by the triangle inequality (see the argument in 3.(a)):

$$e^x \leqslant 1 + |p_{n-1}(x)/p_n(x)| + \cdots + |p_0(x)/p_n(x)|$$

The right-hand side can be majorized by x^d for some $d > 0$. But it is well-known that $e^x \geqslant x^d$ for all sufficiently large x, a contradiction.

c) Clear from (a) and (b).