**Problem Set 6**

Due April 30.

*Model Theory*

Math 506, Spring 2004.

1. Let $\mathcal{L} = \{0, 1, +, -, \cdot\}$ be the language of rings and consider the field $\mathbb{R}$ of real numbers as an $\mathcal{L}$-structure as usual. Is the $\mathcal{L}$-theory $\mathrm{Th}(\mathbb{R})$ model-complete? Does $\mathrm{Th}(\mathbb{R})$ admit quantifier elimination?

2. Let $\mathcal{L}_E = \{L, B, C, A, D\}$ be the following language: $L$ and $B$ are ternary relation symbols; $C$ and $A$ are 6-ary relation symbols, and $D$ is a 4-ary relation symbol. We let $\mathcal{E} = (E, L^{\mathcal{E}}, B^{\mathcal{E}}, C^{\mathcal{E}}, A^{\mathcal{E}}, D^{\mathcal{E}})$ be the $\mathcal{L}_E$-structure with universe $E = \mathbb{R}^2$ where for all $a, b, c, a', b', c' \in E$:

   i. $\mathcal{E} \models L^{\mathcal{E}}(a, b, c)$ if and only if $a, b, c$ are collinear, and $\mathcal{E} \models B^{\mathcal{E}}(a, b, c)$ if and only if $a, b, c$ are collinear and $c$ lies between $a$ and $b$;

   ii. $\mathcal{E} \models C^{\mathcal{E}}(a, b, c, a', b', c')$ if and only if the triangles with vertices $a, b, c$ and $a', b', c'$, respectively, are congruent; $\mathcal{E} \models A^{\mathcal{E}}(a, b, c, a', b', c')$ if and only if the angle between the line segments $a\,b$ and $b\,c$ is the same as the angle between $a'\,b'$ and $b'\,c'$;

   iii. $\mathcal{E} \models D^{\mathcal{E}}(a, b, a', b')$ if and only if the distance between $a$ and $b$ is the same as the distance between $a'$ and $b'$.

   Show that $\mathrm{Th}(\mathcal{E})$ is decidable. (Decidability of plane euclidean geometry; Tarski 1948.)

3. Let $F$ be an ordered field.

   a) Show that $F$ is real closed if and only if for every $P(X) \in F[X]$ and $a < b$ in $F$ such that $P(a)P(b) < 0$ there exists $c \in F$ with $P(c) = 0$ ("$P$ has the intermediate value property").

   b) Construe $F$ as an $\mathcal{L}$-structure as usual, where $\mathcal{L} = \{0, 1, +, -, \cdot, <\}$. Use (a) to show that if $\mathrm{Th}(F)$ is o-minimal then $F$ is real closed.

4. Recall that a function $f\colon \mathbb{R} \to \mathbb{R}$ is called **semialgebraic** if its graph

   $$\Gamma(f) = \{(x, f(x)) \colon x \in \mathbb{R}\} \subseteq \mathbb{R}^2$$

   is semialgebraic.

   a) We say that a function $f\colon \mathbb{R} \to \mathbb{R}$ is **algebraic** if there is a non-zero polynomial $P(X, Y) \in \mathbb{R}[X, Y]$ such that $P(x, f(x)) = 0$ for all $x \in \mathbb{R}$. Show that every semialgebraic function $\mathbb{R} \to \mathbb{R}$ is algebraic.

   b) Use (a) to show that the exponential function $x \mapsto e^x \colon \mathbb{R} \to \mathbb{R}$ is not semialgebraic.