Problem Set 1
Solutions

Foundations of Number Theory

Math 435, Fall 2006

1. (20 pts.) Put $a_n := n^3 + 2n$ and $b_n := 5^{2n} - 1$ for every $n \in \mathbb{N}$, $n \geq 1$. We show $3|a_n$ and $24|b_n$ for every $n \geq 1$, by induction on n. *Base step:* We have $a_1 = 3$ and $b_1 = 24$, so the claims hold trivially for $n = 1$. *Inductive step:* Suppose we have shown $3|a_n$ and $24|b_n$ for a certain $n \geq 1$. We compute

$$a_{n+1} = (n+1)^3 + 2(n+1) = n^3 + 3n^2 + 5n + 3$$

and thus

$$a_{n+1} = a_n + 3(n^2 + n + 1).$$

Since $3|a_n$, this yields $3|a_{n+1}$. Similarly

$$b_{n+1} = 5^{2(n+1)} - 1 = 5^{2n} \cdot 25 - 1 = (5^{2n} - 1) \cdot 25 + 25 - 1 = b_n + 24$$

and $24|b_n$ yields $24|b_{n+1}$.

2. (10 pts.) We proceed by induction on $n = 1, 2, \ldots$. *Base step:* If $n = 1$, then $(1+x)^1 \geq 1 + 1 \cdot x$ for all $x \in \mathbb{R}$. *Inductive step:* Suppose the claim holds for n, and we want to show it for $n+1$ in place of n. That is, we want to show $(1+x)^{n+1} \geq 1 + (n+1)x$ if $1 + x > 0$. Now by inductive hypothesis and since $1 + x > 0$, we have

$$(1+x)^{n+1} = (1+x)^n(1+x) \geq (1+nx)(1+x).$$

But

$$(1+nx)(1+x) = 1 + nx + x + nx^2 = 1 + (n+1)x + nx^2 \geq 1 + (n+1)x,$$

since $nx^2 \geq 0$.

3. (20 pts.) For $n \in \mathbb{N}$ we have $(2n+1)^2 = 4n(n+1)+1$ and $2|n(n+1)$; hence $(2n+1)^2$ has remainder 1 upon division by 8. Now suppose $m, n \in \mathbb{Z}$ are odd; then $m^2 = 8a + 1$ and $n^2 = 8b + 1$ for some $a, b \in \mathbb{Z}$, hence $(m+n)(m-n) = m^2 - n^2 = 8(a-b)$ is divisible by 8.

4. (10 pts.) The mistake is simply that in the inductive step, after taking out the two cats, there might not be any cats left: if $n = 1$, then we are left with *no* cats at all, so it is meaningless to say that this "rest of the set has $n - 1$ cats of color x." So we cannot conclude that the first cat must have color x as well. The moral of the story is: in proving the inductive step $n \to n+1$, we have to be careful and make sure that the proof applies to *all* $n \geq 1$. (Or $n \geq k$, if we start the induction at k, say.)

5. (20 pts.) We proceed by induction on $n = 1, 2, \ldots$. *Base step:* If $n = 1$, then

$$1^2 = 1 = \frac{1}{6}1(1+1)(2 \cdot 1 + 1).$$

Inductive step: Suppose the statement holds for n:

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1),$$

and we we want to show that it holds for $n+1$. That is, we want to show:

$$1^2 + 2^2 + \cdots + (n+1)^2 = \frac{1}{6}(n+1)(n+2)(2(n+1)+1).$$

We compute that

$$\frac{1}{6}n(n+1)(2n+1) = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

and

$$\frac{1}{6}(n+1)(n+2)(2(n+1)+1) = \frac{1}{3}n^3 + \frac{3}{2}n^2 + \frac{13}{6}n + 1.$$

By inductive hypothesis and using these equalities, we get:

$$\begin{aligned}
1^2 + 2^2 + \cdots + (n+1)^2 &= (1^2 + 2^2 + \cdots + n^2) + (n+1)^2 \\
&= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n + (n+1)^2 \\
&= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n + n^2 + 2n + 1 \\
&= \frac{1}{3}n^3 + \frac{3}{2}n^2 + \frac{13}{6}n + 1 \\
&= \frac{1}{6}(n+1)(n+2)(2(n+1)+1).
\end{aligned}$$

6. (20 pts.) There are $\frac{1}{2}n(n+1)+1$ many regions. We prove this by induction on n. For $n = 1$ lines, there are $2 = \frac{1}{2}1(1+1)+1$ regions. If we have n lines, and we add another one, then we obtain $n+1$ new regions. (Draw a picture for $n = 1, 2, 3, 4$!) So we have $\frac{1}{2}n(n+1)+1+(n+1) = \frac{1}{2}(n+1)(n+2)+1$ regions.

7. (20 pts. extra credit.) Let $n \geq 1$ be a natural number. For every $k \geq 1$ let f_k be the remainder of F_k upon division by n, so $0 \leq f_k < n$. Among the n^2+1 pairs $(f_1, f_2), (f_2, f_3), \ldots, (f_m, f_{m+1})$, where $m = n^2+1$, there are two identical pairs (since only n^2 distinct pairs both of whose components come from $\{0, \ldots, n-1\}$ exist). Suppose $(f_k, f_{k+1}) = (f_l, f_{l+1})$ with $1 \leq k < l \leq m$; choose k minimal. One shows easily (check!) that $k > 1$, since otherwise $(f_{k-1}, f_k) = (f_{l-1}, f_l)$, contradicting minimality of k. Hence $(f_l, f_{l+1}) = (f_1, f_2) = (1, 1)$, so $F_{l-1} = F_{l+1} - F_l$ is divisible by n, and $1 \leq l - 1 \leq n^2$.

Total: 100 pts. + 20 pts. extra credit