Problem Set 2
Due Friday, September 29.

Foundations of Number Theory

Math 435, Fall 2006

1. Compute the following gcd's using the Euclidean Algorithm and write $\gcd(x, y)$ in the form $sx + ty$ $(s, t \in \mathbb{Z})$:

$$\gcd(14, 35); \qquad \gcd(11, 15); \qquad \gcd(4081, 2585).$$

2. Let $a, b, c \in \mathbb{Z}$.
   (a) Suppose we can write $1 = sa + tb$ for some $s, t \in \mathbb{Z}$. Show that a and b are coprime.
   (b) Without using facts about prime numbers, show: if a and c are coprime, and b and c are coprime, then ab and c are coprime. (Hint: if $1 = sa + tc = s'b + t'c$ with $s, t, s', t' \in \mathbb{Z}$, consider $(sa + tc)(s'b + t'c)$ and use part (a).)
   (c) Recall that the Fibonacci sequence $F_1, F_2, \ldots$ is defined by $F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ for all $n > 2$. Show that F_n and F_{n+1} are coprime, for all $n > 0$. (Use (a).)
3. Show that the following statements are equivalent, for $a, b \in \mathbb{N}$:
   (a) $\gcd(b, x) = a$ for some $x \in \mathbb{N}$;
   (b) $\operatorname{lcm}(a, y) = b$ for some $y \in \mathbb{N}$;
   (c) $a | b$.
4. Show that

$$(2n + 1, 2n^2 + 2n, 2n^2 + 2n + 1)$$

   is a Pythagorean triple, for every $n > 0$. Is every Pythagorean triple of this form?
5. Let $s, t \in \mathbb{N}$, $s > t > 0$, $\gcd(s, t) = 1$, of distinct parity. Show that then (x, y, z) defined by

$$x = 2st, \quad y = s^2 - t^2, \quad z = s^2 + t^2$$

   is a primitive Pythagorean triple. Show that a different pair (s, t) gives rise to a different triple (x, y, z).
6. (Extra credit.) Show that if p is an odd prime number, then $24 | p^3 - p$.