Problem Set 5
Due Friday, December 1.

Foundations of Number Theory

Math 435, Fall 2006

1. (10+10 pts.)
   (a) Find the smallest integer $n > 3$ such that
   $$3|n, \qquad 5|(n+2), \quad \text{and} \quad 7|(n+4).$$
   (b) Find the smallest integer $n > 2$ such that
   $$2|n, \qquad 3|(n+1), \qquad 4|(n+2), \qquad 5|(n+3), \quad \text{and} \quad 6|(n+4).$$
2. (20 pts.) Find all solutions $x \in \mathbb{Z}$ to the following system of congruences:
   $$\begin{aligned} 5x &\equiv 2 \mod 3 \\ 4x &\equiv 7 \mod 9 \\ 2x &\equiv 4 \mod 10. \end{aligned}$$
3. (10 pts.) Show that if $d \in \mathbb{Z}$ with $d \equiv 3 \bmod 4$, then the diophantine equation $x^2 - dy^2 = -1$ has no solution.
4. (10 pts.) Determine whether the congruence $x^2 \equiv 2 \bmod 77$ has a solution.
5. (10 pts.) Let p be a prime with $p \equiv 1 \bmod 12$. Show that 3 is a quadratic residue mod p.
6. (10 pts.) Let $a \in \mathbb{Z}$, and let $p > 3$ be a prime divisor of $a^2 + 3$. Show that $p \equiv 1 \bmod 3$.
7. (10 pts.) For $p \in \{3, 5, 7\}$ find a Carmichael number of the form $q_1q_2q_3$, where each q_i is a prime number with $q_i \equiv 1 \bmod p-1$ for $i = 1, 2, 3$.
8. (10 pts.) Find an integer a such that $x^2 \equiv a \bmod 385$ has exactly 8 solutions up to congruence mod 385.
9. (20 pts. extra credit.) Show that
   $$1!\,3!\,5! \cdots (p-2)! \equiv \pm 1 \mod p$$
   for every odd prime p.