Homework 1

Metamathematics II

Math 503, Spring 2006

Due February 13.

1. Show that the following subsets of $\mathbb{N}$ are diophantine:
   (a) the set of all composite numbers;
   (b) the set of numbers which are not powers of 2;
   (c) the set of numbers which are not perfect squares.
2. Show that the function $F\colon \mathbb{N} \to \mathbb{N}$ given by
$$F(n) = 1 + 2 + \cdots + n \qquad \text{for each } n \in \mathbb{N}$$
is diophantine.
3. Suppose $F\colon S \to \mathbb{N}$, where $S \subseteq \mathbb{N}^m$, is a diophantine function. Let
$$G_1\colon T \to N, \ldots, G_m\colon T \to \mathbb{N},$$
where $T \subseteq \mathbb{N}^n$, be diophantine functions such that $\big(G_1(y), \ldots, G_m(y)\big) \in S$ for all $y \in T$. Show that the function
$$y \mapsto F\big(G_1(y), \ldots, G_m(y)\big)\colon T \to \mathbb{N}$$
is diophantine.
4. Let $G_1, G_2\colon T \to \mathbb{N}$, where $T \subseteq \mathbb{N}^n$, be diophantine functions. Use the previous problem to show that
$$y \mapsto G_1(y) + G_2(y), y \mapsto G_1(y) \cdot G_2(y)$$
are diophantine functions $T \to \mathbb{N}$.
5. Let $S \subseteq \mathbb{N}^n$ be diophantine. Show that there are polynomials $P_1, \ldots, P_n \in \mathbb{Z}[X_1, \ldots, X_m]$ (for some positive $m \in \mathbb{N}$) with $S = P(\mathbb{N}^m) \cap \mathbb{N}^n$, where $P(x) := \big(P_1(x), \ldots, P_n(x)\big)$ for $x \in \mathbb{N}^m$.
6. Let $a, b \in \mathbb{N}$, $b \geqslant 2$. Show that if there exists $n \in \mathbb{N}$ such that $a = \alpha_b(n)$ then
$$z^2 - \big((b/2)^2 - 1\big)a^2 = 1 \qquad \text{for some } z \in \mathbb{Q}.$$
7. Let $b, m, n \in \mathbb{N}$, $b \geqslant 2$. Show that
$$n < m \quad \Longrightarrow \quad \alpha_b(n) < \alpha_b(m).$$
8. Let $b_1, b_2, q \in \mathbb{N}$, $b_1, b_2 \geqslant 2$, $q > 0$. Show that
$$b_1 \equiv b_2 \mod q \quad \Longrightarrow \quad \alpha_{b_1}(n) \equiv \alpha_{b_2}(n) \mod q \quad \text{for all } n \in \mathbb{N}.$$