Homework 3

Metamathematics II

Math 503, Spring 2006

Due April 28.

1. Let $\mathcal{L}$ be a numerical language and Σ be a set of $\mathcal{L}$-sentences. Suppose $\Sigma \vdash S^m0 \neq S^n0$ whenever $m \neq n$. (This condition is satisfied for $\Sigma = \mathrm{N}$.) Show that if the function $F\colon \mathbb{N}^m \to \mathbb{N}$ is Σ-represented by the $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_m, y)$, then the graph of F (as a subset of $\mathbb{N}^{m+1}$) is Σ-represented by $\varphi(x_1, \ldots, x_m, y)$.
2. Suppose $\mathcal{A} \models \mathrm{N}$. Show that there is a unique $\mathcal{L}(\mathrm{N})$-embedding $\iota\colon \mathcal{N} \to \mathcal{A}$, and show that ι satisfies, for all $a \in A$ and $n \in \mathbb{N}$:
   (a) if $a <^{\mathcal{A}} \iota(n)$, then $a = \iota(m)$ for some $m \in \mathbb{N}$ with $m < n$;
   (b) if $a \notin \iota(\mathbb{N})$, then $\iota(n) <^{\mathcal{A}} a$.
3. Suppose Σ is a computable and consistent set of sentences in the finite numerical language $\mathcal{L}$. Show that every Σ-representable set $R \subseteq \mathbb{N}^n$ is computable. (You may appeal to the Church-Turing thesis.)
4. Suppose $\mathcal{L}$ is a finite numerical language and $\Sigma \supseteq \mathrm{N}$ is consistent.
   (a) Show that $\ulcorner \mathrm{Th}(\Sigma) \urcorner$ is not Σ-representable.
   (b) A **truth definition for** Σ is an $\mathcal{L}$-formula $t(y)$ such that for all $\mathcal{L}$-sentences σ,
   $$\Sigma \vdash \sigma \longleftrightarrow t(S^n0), \qquad \text{where } n = \ulcorner \sigma \urcorner.$$
   Show that there is no truth definition for Σ.
   (Hint: use the Fixed Point Lemma.)