Hardy’s Dream

Matthias Aschenbrenner

universität
wien

Kurt Gödel Research Center for Mathematical Logic
Faculty of Mathematics

Introduction

I will give a—mainly historical—overview of the development of a fascinating subject on the borderline of algebra and analysis.

At the end of my talk I will be reporting on very recent joint work with *Lou van den Dries* and *Joris van der Hoeven*.

This, in some sense, brings to a conclusion investigations started more than a hundred years ago by G. H. Hardy (1877–1947), who dreamt of

> *an all-inclusive, maximally stable algebra of "totally formalizable functions".* (Jean Écalle, 1993)

I will try to explain how one can interpret this statement, how our theorems (at least partially) realize this dream, and the role of mathematical logic in the endeavor.

du Bois-Reymond

Our story, however, begins with

Paul du Bois-Reymond (1831–1889)

- born in Berlin into a family from Neuchâtel (then Prussian);
- studies in Zürich, Königsberg, Berlin;
- initially pursued medicine like his famous brother Émile (1818–1896);
- PhD with E. Kummer in Berlin: *De aequilibrio fluidorum* (1853);
- taught *Gymnasium* in Berlin, then professor in Heidelberg (1865);
- held professorships in Freiburg, Tübingen, and Technische Hochschule Charlottenburg, Berlin.

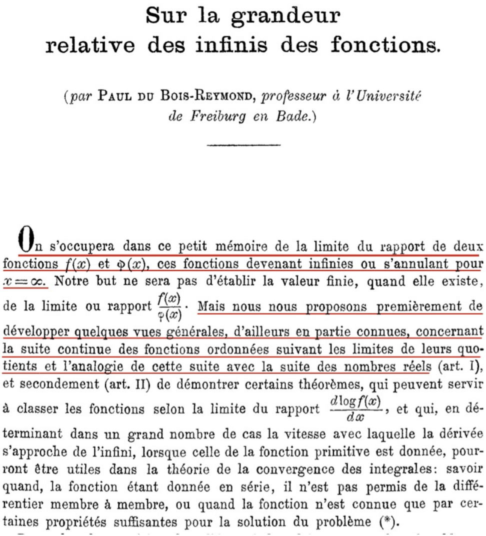
du Bois-Reymond

Nowadays, he is mainly remembered for his work in the calculus of variations, and for giving the first example of a continuous function whose Fourier series diverges at a point.

He also had an abiding interest in the philosophy of mathematics, in particular concerning the continuum and the concept of function.

Around 1870 he begun studying the possible growth behaviors of—real-valued, one-variable, continuous—functions.

The “Infinitärcalcül”



Sur la grandeur relative des infinis des fonctions.

(*par* Paul du Bois-Reymond, *professeur à l'Université de Freiburg en Bade.*)

On s'occupera dans ce petit mémoire de la limite du rapport de deux fonctions $f(x)$ et $\varphi(x)$, ces fonctions devenant infinies ou s'annulant pour $x = \infty$. Notre but ne sera pas d'établir la valeur finie, quand elle existe, de la limite ou rapport $\frac{f(x)}{\varphi(x)}$. Mais nous nous proposons premièrement de développer quelques vues générales, d'ailleurs en partie connues, concernant la suite continue des fonctions ordonnées suivant les limites de leurs quotients et l'analogie de cette suite avec la suite des nombres réels (art. I), et secondement (art. II) de démontrer certains théorèmes, qui peuvent servir à classer les fonctions selon la limite du rapport $\frac{d \log f(x)}{dx}$, et qui, en déterminant dans un grand nombre de cas la vitesse avec laquelle la dérivée s'approche de l'infini, lorsque celle de la fonction primitive est donnée, pourront être utiles dans la théorie de la convergence des integrales: savoir quand, la fonction étant donnée en série, il n'est pas permis de la différentier membre à membre, ou quand la fonction n'est connue que par certaines propriétés suffisantes pour la solution du problème (*).

We will occupy ourselves in this little memoir with the limit of the ratio of two functions $f(x)$ *and* $\varphi(x)$*, these functions becoming infinite or vanishing for* $x = \infty$ […] *We first propose to develop some general viewpoints, some already known, about the continuous sequence of functions ordered according to the limits of their quotients and the analogy of that sequence with the sequence of real numbers,* […]

The “Infinitärcalcül”

He introduced the following useful notations, for (eventually non-vanishing) functions $f, \varphi \colon (a, +\infty) \to \mathbb{R}$:

$$f \prec \varphi \quad :\Longleftrightarrow \quad \lim_{t \to +\infty} \frac{f(t)}{\varphi(t)} = 0,$$

$$f \asymp \varphi \quad :\Longleftrightarrow \quad \lim_{t \to +\infty} \frac{f(t)}{\varphi(t)} \in \mathbb{R} \setminus \{0\}.$$

So for example,

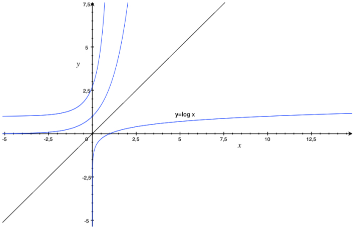
$$\log x \prec x \prec \mathrm{e}^x \prec \mathrm{e}^{\mathrm{e}^x}$$

$$x \prec x^p \ (p > 1)$$

$$cx^r \asymp x^r \ (c, r \in \mathbb{R},\ c \neq 0)$$

but $\quad f \not\prec \varphi, \quad f \not\asymp \varphi, \quad \varphi \not\prec f$

for $f = x(2 + \sin x)$, $\varphi = x$.



The “Infinitärcalcül”

Main motivation: to construct an ideal series that can serve as a boundary between convergent and divergent series. (Similarly for integrals.)

Comparison with Bertrand’s series

Can the convergence/divergence of all series with positive terms be settled by comparison with a real multiple of a series of the form

$$\sum_n \frac{1}{n \log n \log\log n \cdots \log_{m-1} n (\log_m n)^p} \qquad (m \in \mathbb{N},\ p \in \mathbb{R})$$

where $\log_m = \log\log \cdots \log$ (m times)?

(Converges for $p > 1$, diverges for $p \leqslant 1$.)

He later shows (*Crelle’s Journal*, 1873) that the answer is “no” (in the process inventing the “diagonal argument” a bit earlier than Cantor).

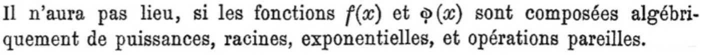
The "Infinitärcalcül"

The iterates of the logarithm form a "scale" of infinitely growing functions which are linearly ordered under $\prec$:

$$\cdots \prec \log_m x \prec \log_{m-1} x \prec \cdots \prec \log x \prec x \prec \mathrm{e}^x \prec \mathrm{e}^{\mathrm{e}^x} \prec \cdots$$

- Such scales resemble the real number line somewhat: given $f \prec g$ there is always some h with $f \prec h \prec g$.
- But they are inherently non-archimedean: $(\log x)^n \prec x$ for all n.
- And of course, we already saw that in general none of the relations $f \prec \varphi$, $f \asymp \varphi$, or $\varphi \prec f$ might hold.

Du Bois-Reymond states (without proof):

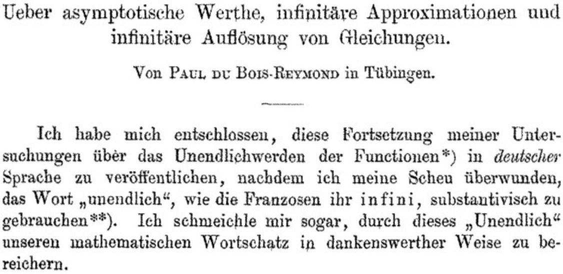


Il n'aura pas lieu, si les fonctions $f(x)$ et $\varphi(x)$ sont composées algébriquement de puissances, racines, exponentielles, et opérations pareilles.

This does not happen if the functions $f(x)$ and $\varphi(x)$ are composed of powers, roots, exponentials, and similar operations.

The “Infinitärcalcül”

Throughout his life he further developed these ideas to compute with functions belonging to a common scale like the one above, in particular to determine their “infinity”: the equivalence class w. r. t. $\asymp$.



Ueber asymptotische Werthe, infinitäre Approximationen und infinitäre Auflösung von Gleichungen.

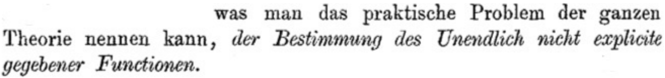
Von Paul du Bois-Reymond in Tübingen.

Ich habe mich entschlossen, diese Fortsetzung meiner Untersuchungen über das Unendlichwerden der Functionen*) in *deutscher* Sprache zu veröffentlichen, nachdem ich meine Scheu überwunden, das Wort „unendlich“, wie die Franzosen ihr infini, substantivisch zu gebrauchen**). Ich schmeichle mir sogar, durch dieses „Unendlich“ unseren mathematischen Wortschatz in dankenswerther Weise zu bereichern.

> *I decided to publish this continuation of my research on functions becoming infinite in German after I overcame my aversion to using the word ‘infinite’ as a substantive, like the French their ‘infini’. I even flatter myself that, by this ‘infinite’, I have enriched our mathematical vocabulary in a noteworthy way.*

The “Infinitärcalcül”

In this long paper in *Mathematische Annalen* (1875), he took first steps towards treating



was man das praktische Problem der ganzen Theorie nennen kann, *der Bestimmung des Unendlich nicht explicite gegebener Functionen.*

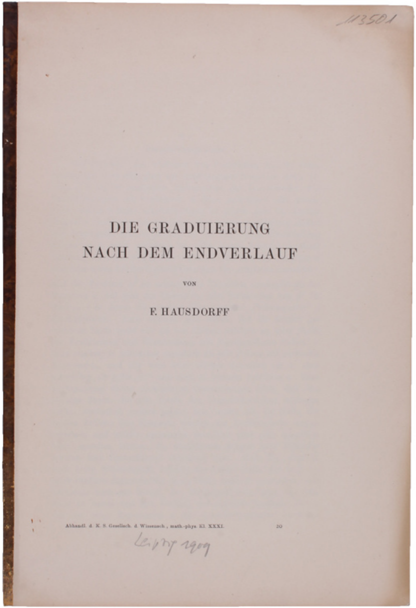
> *what one can call the practical problem of the whole theory, the determination of the infinity of a non-explicitly given function.*

Unfortunately, du Bois-Reymond faced vociferous opposition by Georg Cantor, who accused him of trying to

> *infect mathematics with the cholera bacillus of infinitesimals.* (Letter to Vivanti, 1893)

Perhaps for this reason, his work was mainly forgotten until it was revived by F. Hausdorff and G. H. Hardy early in the 20th century.

Hardy’s work



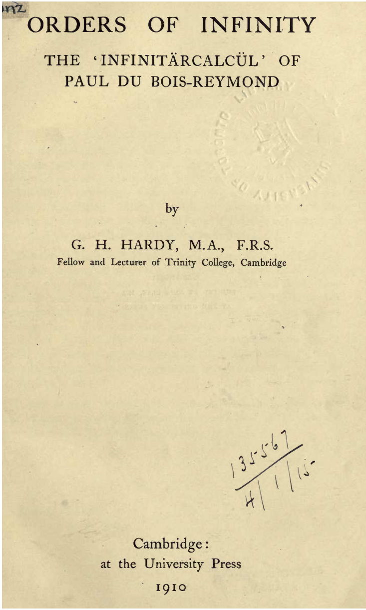
DIE GRADUIERUNG NACH DEM ENDVERLAUF

VON

F. HAUSDORFF

Abhandl. d. K. S. Gesellsch. d. Wissensch., math.-phys. Kl. XXXI. 20

Leipzig 1909



ORDERS OF INFINITY

THE ‘INFINITÄRCALCÜL’ OF PAUL DU BOIS-REYMOND

by

G. H. HARDY, M.A., F.R.S.

Fellow and Lecturer of Trinity College, Cambridge

Cambridge:
at the University Press
1910

Hardy’s work

Hardy put du Bois-Reymond’s ideas on a firm footing.

In particular, he constructed the field of **logarithmico-exponential** (LE) **functions**:

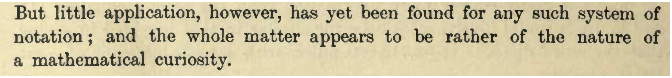
real-valued functions built up from constants and the variable x *using* $+$, $\times$, $\div$, *exponentiation, and logarithms.*

$$x^{\sqrt{2}}, \quad e^{e^x + x^2}, \quad \sinh x = \tfrac{1}{2}(e^x - e^{-x}), \quad \log\left(\frac{x+1}{x-1}\right), \quad \ldots$$

He showed that such a function, when defined on an interval $(a, +\infty)$, has eventually constant sign, and its derivative is also an LE-function.

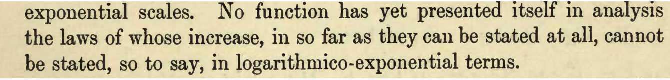
Hardy’s work

Like Cantor, Hardy remained critical of du Bois Reymond’s attempts to compute with “infinities”:



> But little application, however, has yet been found for any such system of notation; and the whole matter appears to be rather of the nature of a mathematical curiosity.

But he did recognize the significance of these ideas for analysis:

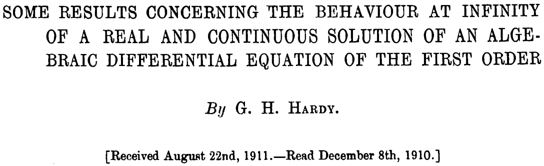


> exponential scales. No function has yet presented itself in analysis the laws of whose increase, in so far as they can be stated at all, cannot be stated, so to say, in logarithmico-exponential terms.

Nevertheless, the collection of LE-functions lacks some closure properties that make it useful for a comprehensive theory.

For example, $\int e^{x^2}\ dx$ is not an LE-function. (Liouville, 1830s)

Hardy’s work



SOME RESULTS CONCERNING THE BEHAVIOUR AT INFINITY OF A REAL AND CONTINUOUS SOLUTION OF AN ALGEBRAIC DIFFERENTIAL EQUATION OF THE FIRST ORDER

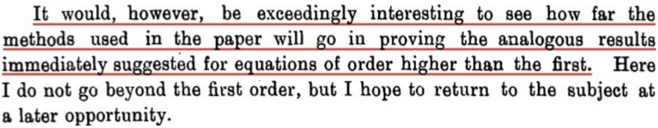
By G. H. Hardy.

[Received August 22nd, 1911.—Read December 8th, 1910.]

Perhaps motivated by this, Hardy undertook first studies on the asymptotic behavior of solutions $y = y(t)$ to order 1 differential equations

$P(y, y') = 0$ where P is a polynomial with coefficients in $\mathbb{R}(x)$.

$$5x^2y^9(y')^3 - (2x^3+1)y' + 2y^2 - 4 = 0$$



It would, however, be exceedingly interesting to see how far the methods used in the paper will go in proving the analogous results immediately suggested for equations of order higher than the first. Here I do not go beyond the first order, but I hope to return to the subject at a later opportunity.

The modern setting: Hardy fields

This is due to Bourbaki (1951).

Focussing at behavior near infinity

We say that continuous functions f, g **have the same germ** at $+\infty$ if they agree on some half-line $(a, +\infty)$.

The **germ** of f is its equivalence class with respect to the equivalence relation of "having the same germ."

One can add and multiply germs in the natural way: they form a commutative ring $\mathcal{C}$.

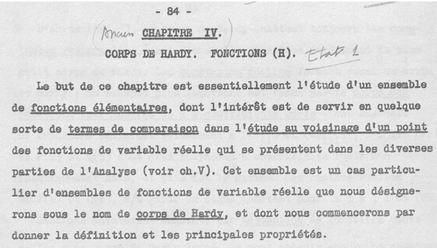
Call a germ f **differentiable** if it has a representative which is a C^1 function, and then denote by f' the germ of the derivative.

The differentiable germs form a subring $\mathcal{C}^1$ of $\mathcal{C}$.

The modern setting: Hardy fields

Definition (Bourbaki)

A **Hardy field** is a subfield of $\mathcal{C}^1$ which is closed under taking derivatives.



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(Ancien CHAPITRE IV.)

CORPS DE HARDY. FONCTIONS (H). Etat 1

Le but de ce chapitre est essentiellement l'étude d'un ensemble de fonctions élémentaires, dont l'intérêt est de servir en quelque sorte de termes de comparaison dans l'étude au voisinage d'un point des fonctions de variable réelle qui se présentent dans les diverses parties de l'Analyse (voir ch.V). Cet ensemble est un cas particulier d'ensembles de fonctions de variable réelle que nous désignerons sous le nom de corps de Hardy, et dont nous commencerons par donner la définition et les principales propriétés.

Easy examples

$$\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{R}(x) \subseteq \mathbb{R}(x, \mathrm{e}^x) \subseteq \mathbb{R}(x, \mathrm{e}^x, \log x)$$

This innocuous-looking definition has a wealth of consequences.

The modern setting: Hardy fields

Let H be a Hardy field and $f \in H$. Then

$$f \neq 0 \quad \Longrightarrow \quad \frac{1}{f} \in H \quad \Longrightarrow \quad \begin{cases} f(t) > 0 \text{ eventually, or} \\ f(t) < 0 \text{ eventually.} \end{cases}$$

Consequently:

- H carries an ordering making H an ordered field:

$$f > 0 \qquad :\Longleftrightarrow \qquad f(t) > 0 \text{ eventually;}$$

- f is eventually monotonic, and

$$\lim_{t \to +\infty} f(t) \in \mathbb{R} \cup \{\pm\infty\} \qquad \text{exists.}$$

So a germ like $\sin x$ cannot be in a Hardy field.

The modern setting: Hardy fields

Recall du Bois-Reymond's notations:

$$f \prec g \iff \lim_{t \to +\infty} \frac{f(t)}{g(t)} = 0, \qquad f \asymp g \iff \lim_{t \to +\infty} \frac{f(t)}{g(t)} \in \mathbb{R}\setminus\{0\}.$$

In a Hardy field, for all $f, g \neq 0$ exactly one of the relations

$$f \prec g, \quad f \asymp g, \quad g \prec f$$

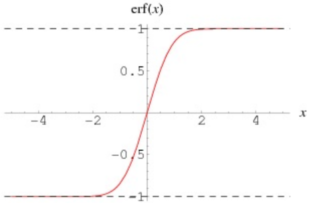
holds. The map

$$f \mapsto \begin{pmatrix}\text{equivalence class of } f \\ \text{with respect to } \asymp\end{pmatrix} \qquad \text{(for } f \in H,\, f \neq 0\text{)}$$

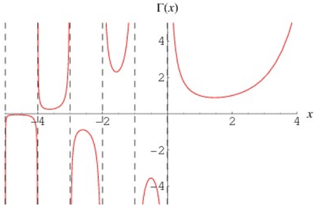
is an example of a **valuation** on H.

This allows us to harness the tools of *valuation theory*, a well-developed chapter of algebra, for the study of Hardy fields.

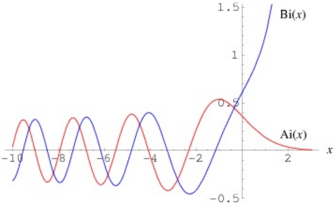
Examples of functions in Hardy fields



$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-t^2}\, dt$$



$$\Gamma(x) = \int_0^\infty t^{x-1}\, \mathrm{e}^{-t}\, dt$$



Ai, Bi are $\mathbb{R}$-linearly independent solutions to $y'' - xy = 0$

Algebraic differential equations over Hardy fields

Let $P \in H[Y_0, Y_1, \ldots, Y_r]$ be a polynomial of positive degree.

> *When is there some y in a Hardy field extension of H solving the equation $P(y) = P(y, y', \ldots, y^{(r)}) = 0$?*

Some answers in basic cases were given over the decades by Hausdorff, Hardy, Bourbaki, Rosenlicht, Boshernitzan … For example:

Every solution y (in $\mathcal{C}^1$) of an equation

$$y' + fy = g \qquad (f, g \in H)$$

is contained in some Hardy field extension of H.

Hence $H(\mathbb{R})$ and $H(x)$ are Hardy fields, and if $h \in H$, then so are

$$H(\textstyle\int h), \quad H(\mathrm{e}^h), \quad H(\log h) \text{ when } h > 0.$$

($\Longrightarrow$ Hardy's field of LE-functions is indeed a Hardy field!)

The ultimate extension result

We can now state our recent theorem, an intermediate value property for algebraic differential equations:

Theorem (A.–van den Dries–van der Hoeven)

Let $f < g$ in H be such that

$$P(f) < 0 < P(g).$$

Then there is some y in a Hardy field extending H satisfying

$$P(y) = 0 \quad \textit{and} \quad f < y < g.$$

In a sense, this theorem justifies du Bois-Reymond's intuition that his "orders of infinity" do share many properties with the real continuum—but probably not in a way that he envisaged.

The ultimate extension result

The analogy with $\mathbb{R}$ and $\mathbb{C} = \mathbb{R} + \mathbb{R}i$ (where $i = \sqrt{-1}$) goes further:

Corollary

❶ *There are* y, z *in a Hardy field extension of* H *such that* $P(y + z\mathrm{i}) = 0$.

❷ *If* P *has odd degree, then there is some* y *in a Hardy field extension of* H *with* $P(y) = 0$.

Thus for example, there is some y satisfying

$$(y'')^5 + \sqrt{2}\,\mathrm{e}^x\,(y'')^4 y''' - x\log x\, y^2 y'' + yy' - \Gamma = 0$$

in a Hardy field containing $\mathbb{R}$, e^x, $\log x$, Γ!

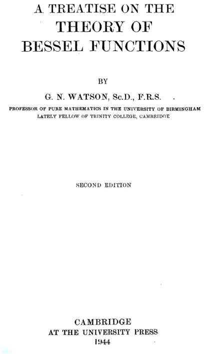
The ultimate extension result

The main applications of our theorem are to systems of algebraic differential equations (including asymptotic side conditions).

But as a byproduct, even for *linear* differential equations of order 2 like the Bessel equation

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0 \qquad (\alpha \in \mathbb{R})$$

it can give useful new information:



A TREATISE ON THE
THEORY OF
BESSEL FUNCTIONS

BY

G. N. WATSON, Sc.D., F.R.S.
PROFESSOR OF PURE MATHEMATICS IN THE UNIVERSITY OF BIRMINGHAM
LATELY FELLOW OF TRINITY COLLEGE, CAMBRIDGE

SECOND EDITION

CAMBRIDGE
AT THE UNIVERSITY PRESS
1944

Corollary

There is a unique germ ϕ in a Hardy field with $\phi - x \preccurlyeq 1/x$ such that the solutions of the Bessel equation are exactly the germs of the form

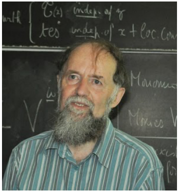
$$y = \frac{c}{\sqrt{x\phi'}}\cos(\phi + d) \qquad (c, d \in \mathbb{R}).$$

Under the hood of the proof

It is conceptually easier to focus on *maximal* Hardy fields—those that cannot be extended further.

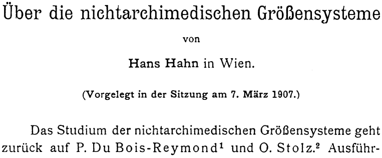
The key is to show is that these all share the same logical properties as the differential field $\mathbb{T}$ of *transseries*.

Transseries are formal objects which can be used to model the complete asymptotic behavior of germs in Hardy fields: not just their "infinities", in du Bois-Reymond's parlance.



They were invented by the analyst Jean Écalle (and independently, by the logicians Dahn and Göring) in the 1980s, and are based on Hans Hahn's "generalized power series" (1907).

Under the hood of the proof



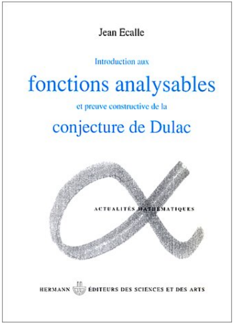
Über die nichtarchimedischen Größensysteme

von

Hans Hahn in Wien.

(Vorgelegt in der Sitzung am 7. März 1907.)

Das Studium der nichtarchimedischen Größensysteme geht zurück auf P. Du Bois-Reymond[1] und O. Stolz.[2] Ausführ-



Jean Ecalle

Introduction aux

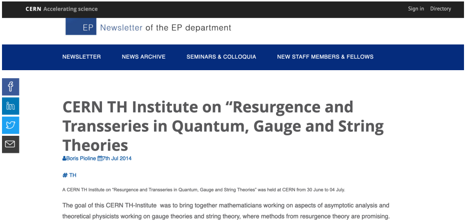
fonctions analysables

et preuve constructive de la

conjecture de Dulac

ACTUALITÉS MATHÉMATIQUES

HERMANN ÉDITEURS DES SCIENCES ET DES ARTS



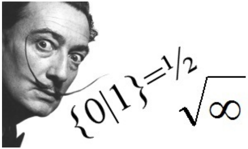
CERN TH Institute on “Resurgence and Transseries in Quantum, Gauge and String Theories”

Boris Pioline 7th Jul 2014

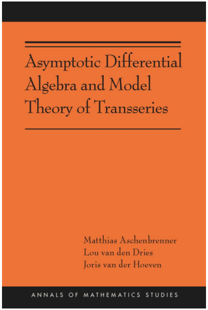
TH

A CERN TH Institute on “Resurgence and Transseries in Quantum, Gauge and String Theories” was held at CERN from 30 June to 04 July.

The goal of this CERN TH-Institute was to bring together mathematicians working on aspects of asymptotic analysis and theoretical physicists working on gauge theories and string theory, where methods from resurgence theory are promising.



Under the hood of the proof



Asymptotic Differential Algebra and Model Theory of Transseries

Matthias Aschenbrenner
Lou van den Dries
Joris van der Hoeven

ANNALS OF MATHEMATICS STUDIES

About six years ago, we finished the proof of a theorem complete describing the elementary theory of $\mathbb{T}$.

This essentially amounts to establishing an elimination theory for systems of algebraic differential equations over $\mathbb{T}$.

Abraham Robinson taught us how to approach such “logical” results algebraically.

The proof of our theorem, besides refinements of the tools from our book, requires analytic arguments in an essential way:

- fixed point theorems;
- uniform distribution mod 1

The details are lengthy. …

Coda

Corollary (obtained using Gödel’s Completeness Theorem)

<u>There is</u> an algorithm which takes as input a polynomial

$$P \in \mathbb{R}(x)[Y_0, \dots, Y_r]$$

and decides whether there is some y in a Hardy field such that

$$P(y) = P(y, y', \dots, y^{(r)}) = 0.$$

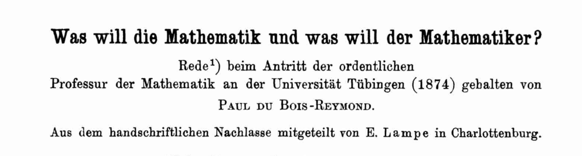
Works much more generally, e.g., for systems, also involving $<$, $\prec$, like:

$$x^2 y_1'' - (y_3')^7 \prec y_2, \qquad y_2^2 = 4y_3 + (3x^5 + 1)y_1 - 9, \qquad y_3 < x.$$

Challenge

Devise such a “practical” algorithm!

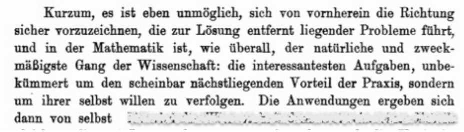
Coda



Was will die Mathematik und was will der Mathematiker?

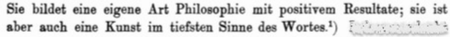
Rede1) beim Antritt der ordentlichen
Professur der Mathematik an der Universität Tübingen (1874) gebalten von
Paul du Bois-Reymond.

Aus dem handschriftlichen Nachlasse mitgeteilt von E. Lampe in Charlottenburg.



Kurzum, es ist eben unmöglich, sich von vornherein die Richtung sicher vorzuzeichnen, die zur Lösung entfernt liegender Probleme führt, und in der Mathematik ist, wie überall, der natürliche und zweckmäßigste Gang der Wissenschaft: die interessantesten Aufgaben, unbekümmert um den scheinbar nächstliegenden Vorteil der Praxis, sondern um ihrer selbst willen zu verfolgen. Die Anwendungen ergeben sich dann von selbst

In short, it is simply impossible to determine from the outset with certainty the direction that will lead to the solution of remote problems, and in mathematics, as everywhere else, the natural and most expedient course of science is this: to pursue the most interesting problems for their own sake, unconcerned about the apparent regard of practical applicability. The applications will then appear by themselves.



Sie bildet eine eigene Art Philosophie mit positivem Resultate; sie ist aber auch eine Kunst im tiefsten Sinne des Wortes.1)

It is a kind of philosophy with positive results; but it is also an art in the deepest sense of the word.