THE SURREAL NUMBERS AS A UNIVERSAL H-FIELD

MATTHIAS ASCHENBRENNER, LOU VAN DEN DRIES, AND JORIS VAN DER HOEVEN

ABSTRACT. We show that the natural embedding of the differential field of transseries into Conway's field of surreal numbers with the Berarducci-Mantova derivation is an elementary embedding. We also prove that any Hardy field embeds into the field of surreals with the Berarducci-Mantova derivation.

INTRODUCTION

Berarducci and Mantova [3, Theorem B] have recently constructed a derivation (denoted by ∂_{BM} below) on Conway's ordered field $\mathbf{No}$ of surreal numbers that makes the latter a Liouville closed H-field with constant field $\mathbb{R}$. The standard example of such an object is the ordered differential field $\mathbb{T}$ of transseries, and the question arises whether $\mathbf{No}$ with ∂_{BM} is elementarily equivalent to $\mathbb{T}$. Below we give a positive answer in a stronger form: Theorem 1. Throughout this paper we consider $\mathbf{No}$ as a differential field with derivation ∂_{BM}.

Both $\mathbf{No}$ and $\mathbb{T}$ are also exponential fields; the exponential function exp on $\mathbf{No}$ is defined in Gonshor [9]. We refer to [2, Appendix A] for the precise construction of $\mathbb{T}$, but the "generating element" x of $\mathbb{T}$ there will be denoted by $x_{\mathbb{T}}$ here, since we prefer to have x range here over arbitrary surreal numbers. It is folklore (but see Section 5 for a proof) that there is a unique embedding $\iota\colon \mathbb{T} \to \mathbf{No}$ of ordered exponential fields with $\iota(x_{\mathbb{T}}) = \omega$ that is the identity on $\mathbb{R}$ and respects infinite sums. It follows easily from Wilkie's theorem [13] and other known facts that ι is an elementary embedding of ordered exponential fields; see Section 5 for details. The analogue for the derivation instead of the exponentiation requires more effort:

Theorem 1. *The mapping $\iota\colon \mathbb{T} \to \mathbf{No}$ is an elementary embedding of ordered differential fields.*

This answers a question posed in [3]. The main tools for proving this result come from [2, Theorems 15.0.1 and 16.0.1]. These tools enable us to reduce the proof of Theorem 1 to exhibiting $\mathbf{No}$ as a directed union of subfields $\mathbb{R}[[\omega^{\Gamma}]]$ that are closed under ∂_{BM} and where Γ is an ordered additive subgroup of $\mathbf{No}$ having a smallest nontrivial archimedean class; exhibiting $\mathbf{No}$ as such a directed union makes up an important part of our paper. (As a byproduct we get a new proof that $\partial_{\mathrm{BM}}(\mathbf{No}) = \mathbf{No}$.) We use the same kind of reduction to obtain:

Theorem 2. *The surreals of countable length form a subfield of $\mathbf{No}$ closed under ∂_{BM}. As a differential subfield of $\mathbf{No}$ it is an elementary submodel of $\mathbf{No}$.*

This also uses a result of Esterle [8] and its consequence that for any countable ordinal α, any well-ordered set of surreals of length $< \alpha$ is countable: Lemma 4.3.

Finally, we establish an embedding result for H-fields:

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Theorem 3. *Every H-field with small derivation and constant field* $\mathbb{R}$ *can be embedded over* $\mathbb{R}$ *as an ordered differential field into* $\mathbf{No}$.

Thus every Hardy field extending $\mathbb{R}$ embeds over $\mathbb{R}$ as an ordered differential field into $\mathbf{No}$. Despite these excellent properties of ∂_{BM}, Schmeling's thesis [12] gives us reason to believe that ∂_{BM} is not yet the "best" derivation on $\mathbf{No}$. We expect to address this issue in later papers.

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1. Preliminaries

Here we fix notation and terminology and summarize the results from [2, 3, 9] that we need as background material and as tools in our proofs.

Notations and terminology. Below, m, n range over $\mathbb{N} = \{0, 1, 2, \dots\}$, and α, β and μ, ν range over ordinals. (The letter λ will serve another purpose, as in [3].)

As in [9], a *surreal number* is by definition a function $a\colon \mu \to \{-, +\}$ on an ordinal $\mu = \{\alpha : \ \alpha < \mu\}$. For such a we let $l(a) := \mu$ be the *length* of a. From now on we let a, b, x, y be surreal numbers. The class $\mathbf{No}$ of surreal numbers carries a canonical linear ordering $<$: $a < b$ iff a is lexicographically less than b, where by convention we set $a(\mu) := 0$ for $\mu \geqslant l(a)$ and linearly order $\{-, 0, +\}$ by $- < 0 < +$. We also have the canonical partial ordering $<_s$ on $\mathbf{No}$ given by: $a <_s b$ ("a is simpler than b") iff a is a proper initial segment of b, that is, $l(a) < l(b)$, and $a|_\mu = b|_\mu$ for $\mu := l(a)$. For sets $A, B \subseteq \mathbf{No}$ with $A < B$ (that is, $a < b$ for all $a \in A$ and $b \in B$) we let $x = A|B$ mean that x is the simplest surreal with $A < x < B$, as in [9] and [3]. We also use the terms "canonical representation" and "monomial representation" (of a surreal number) as in [3].

The ordinal α is identified with the surreal $a\colon \alpha \to \{-, +\}$ with $a(\beta) = +$ for all $\beta < \alpha$. A useful fact is the equivalence $\alpha < x \Longleftrightarrow \alpha \dot{+} 1 \leqslant_s x$, where $\alpha \dot{+} 1$ is the successor ordinal to α. The subclass of $\mathbf{No}$ consisting of the ordinals is denoted by $\mathbf{On}$. A set $S \subseteq \mathbf{No}$ is said to be *initial* if $x \in S$ whenever $x <_s y \in S$. As in [5] we set $\mathbf{No}(\alpha) = \big\{x : \ l(x) < \alpha\big\}$, an initial subset of $\mathbf{No}$.

We refer to [9] or [3] for the inductive definitions of the binary operations of addition and multiplication on $\mathbf{No}$ that make $\mathbf{No}$ into a real closed field, with the ordinal 0 as its zero element and the ordinal 1 as its multiplicative identity. The field ordering of this real closed field is the above lexicographic linear ordering $<$. This field $\mathbf{No}$ contains $\mathbb{R}$ as an initial subfield in the way specified in [9]. The field sum $\alpha + n$ equals the ordinal sum $\alpha \dot{+} n$. Each initial set $\mathbf{No}(\omega^\alpha)$ underlies an additive subgroup of $\mathbf{No}$; see [5].

Let Γ be an (additively written) ordered abelian group. Then we set

$$\Gamma^{>} \ := \ \{\gamma \in \Gamma : \ \gamma > 0\}.$$

We use this notation also for the underlying additive groups of $\mathbf{No}$ and $\mathbb{R}$, so $\mathbf{No}^{>} = \{a : \ a > 0\}$, and $\mathbb{R}^{>} := \{r \in \mathbb{R} : \ r > 0\}$. For $\gamma \in \Gamma$ we define

$$[\gamma] \ := \ \big\{\delta \in \Gamma : \ |\delta| \leqslant n|\gamma| \text{ and } |\gamma| \leqslant n|\delta| \text{ for some } n \geqslant 1\big\},$$

the *archimedean class of* γ (in Γ). The archimedean classes of elements of Γ partition the set Γ, and we totally order this set of archimedean classes by

$$[\gamma_1] \ < \ [\gamma_2] \quad :\Longleftrightarrow \quad n|\gamma_1| \ < \ |\gamma_2| \text{ for all } n \geqslant 1.$$

Thus the least archimedean class is $[0] = \{0\}$, the *trivial* archimedean class.

The convex hull of $\mathbb{R}$ in $\mathbf{No}$ is a valuation ring V of the field $\mathbf{No}$. We consider $\mathbf{No}$ accordingly as a *valued* field whose (Krull) valuation v has V as its valuation ring. For any (Krull) valued field K with valuation v and elements $f, g \in K$ we let $f \preccurlyeq g$, $f \prec g$, $f \asymp g$, $f \sim g$ abbreviate $v(f) \geqslant v(g)$, $v(f) > v(g)$, $v(f) = v(g)$, and $v(f-g) > vf$. (See [2, Section 3.1].) We shall use these notations in particular for the valued field $\mathbf{No}$.

The omega map, the Conway normal form, and summability. We assume familiarity with Conway's omega map $x \mapsto \omega^x \colon \mathbf{No} \to \mathbf{No}^{>}$. Recall that ω^x is the simplest positive element in its archimedean class; so $\omega^x \prec \omega^y$ whenever $x < y$. See [9] for details, including the proof that each a has a unique representation

$$a \ = \ \sum_x a_x \omega^x \qquad \text{(the Conway normal form of } a\text{)}$$

with real coefficients a_x such that $E(a) := \{x : \ a_x \neq 0\}$ is a subset of $\mathbf{No}$ (not just a subclass) and is reverse well-ordered. This will be the meaning of $E(a)$ and a_x throughout. The *leading monomial of* a is ω^x with $x = \max E(a)$, for $a \neq 0$. The *terms* of a are the $a_x\omega^x$ with $a_x \neq 0$. The omega map extends the usual ordinal exponentiation $\alpha \mapsto \omega^\alpha$. Given any set $S \subseteq \mathbf{No}$ we let $\mathbb{R}[[\omega^S]]$ denote the additive subgroup of $\mathbf{No}$ consisting of the surreals a with $E(a) \subseteq S$.

Let $(a_i)_{i \in I}$ be a family of surreals; this includes I being a set. We say that (a_i) is *summable* (or that $\sum_i a_i$ exists) if $\bigcup_i E(a_i)$ is reverse well-ordered, and for each x there are only finitely many $i \in I$ with $x \in E(a_i)$; in that case we set $\sum_i a_i := \sum_x \left(\sum_i a_{i,x}\right) \omega^x$. If S is a subset of $\mathbf{No}$, then for any summable family (a_i) in $\mathbb{R}[[\omega^S]]$ we have $\sum_i a_i \in \mathbb{R}[[\omega^S]]$.

As in [3], we let $\mathfrak{M}$ denote the class of *monomials* ω^x; so $\mathfrak{M}$ is a multiplicative subgroup of $\mathbf{No}^\times$. The Conway normal form allows us to consider any surreal number a as a *generalized series*

$$a \ = \ \sum_{\mathfrak{m} \in \mathfrak{M}} a_{\mathfrak{m}} \mathfrak{m}$$

with coefficients $a_{\mathfrak{m}} \in \mathbb{R}$, monomials $\mathfrak{m} \in \mathfrak{M}$, and reverse well-ordered *support* $\operatorname{supp} a := \{\mathfrak{m} \in \mathfrak{M} : a_{\mathfrak{m}} \neq 0\} = \omega^{E(a)}$. This makes the above notion of summability for surreal numbers coincide with the corresponding notion for generalized series from [12, Section 1.5].

Next, $\mathbb{J} := \left\{a : \ E(a) \subseteq \mathbf{No}^{>}\right\}$ is the class of *purely infinite* surreals, an additive subgroup of $\mathbf{No}$ that is moreover closed under multiplication. Thus $\mathfrak{M} \cap \mathbb{J} = \mathfrak{M}^{\succ 1}$, and $\mathbf{No} = \mathbb{J} \oplus \mathbb{R} \oplus \mathbf{No}^{\prec 1}$.

Exponentiation, and the functions g and h. Gonshor [9] gave an inductive definition of the exponential function $\exp\colon \mathbf{No} \to \mathbf{No}^{>}$, and established its basic properties. These include exp being an order-preserving isomorphism from the additive group of $\mathbf{No}$ onto its multiplicative group of positive elements. The inverse of exp is of course denoted by $\log\colon \mathbf{No}^{>} \to \mathbf{No}$. The nth iterate of the map $\exp\colon \mathbf{No} \to \mathbf{No}$ is denoted by $\exp_n$, so $\exp_0$ is the identity map on $\mathbf{No}$, and

$\exp_1(x) = \exp(x)$. Also $e^x := \exp(x)$. The logarithmic map log maps $\mathbf{No}^{>\mathbb{N}}$ into itself; the nth iterate of the restriction of log to a map $\mathbf{No}^{>\mathbb{N}} \to \mathbf{No}^{>\mathbb{N}}$ is denoted by $\log_n$, so $\log_0$ is the identity map on $\mathbf{No}^{>\mathbb{N}}$ and $\log_1(x) = \log(x)$ for $x > \mathbb{N}$.

The exponential map exp and the omega-map $x \mapsto \omega^x$ are related by the order preserving bijection $g\colon \mathbf{No}^{>} \to \mathbf{No}$, which satisfies

$$\exp(\omega^x) \;=\; \omega^{\omega^{g(x)}} \qquad \text{for all } x > 0.$$

We have $g(n) = n$ for all n. More generally, Theorem 10.14 in [9] says that $g(\alpha) = \alpha$ unless $\varepsilon \leqslant \alpha < \varepsilon + \omega$ for some ε-number, in which case $g(\alpha) = \alpha + 1$. (An *ε-number* is an ordinal ε such that $\omega^\varepsilon = \varepsilon$.) We shall need $g(x)$ mainly in the other extreme case where x has the form $\omega^{-\alpha}$. Here Theorem 10.15 in [9] gives $g(\omega^{-\alpha}) = -\alpha + 1$.

We also use the inverse $h\colon \mathbf{No} \to \mathbf{No}^{>}$ of g. Note that

$$\omega^{\omega^y} \;=\; \exp(\omega^{h(y)}) \qquad \text{for all } y.$$

The result above for $g(\omega^{-\alpha})$ yields $h(-\alpha+1) = \omega^{-\alpha}$, from which we get

$$\log \omega^{\omega^{-\alpha+1}} \;=\; \omega^{\omega^{-\alpha}}.$$

Applying this to the ordinal $\alpha + 1$ instead of α we get

$$\log \omega^{\omega^{-\alpha}} \;=\; \omega^{\omega^{-(\alpha+1)}}.$$

From [9] we have $\exp(\mathbb{J}) = \mathfrak{M}$. Thus besides the Conway normal form and the series representation, any surreal number a also has a unique representation

$$a \;=\; \sum_{j \in \mathbb{J}} a_j\, e^j \qquad \text{(exponential normal form of } a\text{)}$$

with real coefficients a_j and reverse well-ordered $\{j \in \mathbb{J} : a_j \neq 0\}$; this is also called the *Ressayre form of a*. For nonzero a with leading monomial e^b, $b \in \mathbb{J}$, we set $\ell(a) := b$. Then $-\ell\colon \mathbf{No}^{\times} \to \mathbb{J}$ is a (Krull) valuation on the field $\mathbf{No}$, and

$$\big\{a : -\ell(a) \geqslant 0\big\} \;=\; \big\{a : \ |a| \leqslant r \text{ for some } r \in \mathbb{R}^{\geqslant 0}\big\} \;=\; V,$$

so we may consider $-\ell$ as the valuation of our valued field $\mathbf{No}$. Important in [3] is also the class $\mathfrak{A}$ of *log-atomic* surreals, consisting of the $a > \mathbb{N}$ all whose iterated logarithms $\log_n a$ lie in $\mathfrak{M}$. We have $\mathfrak{A} \subseteq \mathfrak{M}^{\succ 1}$ and $\exp(\mathfrak{A}) = \log(\mathfrak{A}) = \mathfrak{A}$. It follows from $\mathfrak{A} \subseteq \mathfrak{M}$ that if $x, y \in \mathfrak{A}$ and $x < y$, then $x \prec y$. (In [3] the class of log-atomic surreals is denoted by $\mathbb{L}$, but this notation conflicts with ours in other papers.)

Surreal derivations. We summarize here some results from [3] as needed, and add a few remarks. A *surreal derivation* is a derivation ∂ on the field $\mathbf{No}$ such that

- (SD1) $\big\{a : \partial(a) = 0\big\} = \mathbb{R}$;
- (SD2) $\partial(a) > 0$ for all $a > \mathbb{R}$;
- (SD3) $\partial\big(\exp(a)\big) = \partial(a)\exp(a)$ for all a;
- (SD4) for any summable family (a_i) of surreals, the family $\big(\partial(a_i)\big)$ is also summable, and $\partial\left(\sum_i a_i\right) = \sum_i \partial(a_i)$.

The ordered field $\mathbf{No}$ equipped with any surreal derivation is an H-field; this doesn't need (SD3) or (SD4). The particular derivation ∂_{BM} is surreal, maps $\mathfrak{A}$ into $\mathfrak{M}$, and is obtained in [3] as a special case of a rather general construction. Before we get to that, we mention Proposition 6.5 and Theorem 6.32 from that paper:

(BM1) If ∂ is a surreal derivation, then for all $x, y > \mathbb{N}$ with $x - y > \mathbb{N}$ we have

$$\log \partial(x) - \log \partial(y) \ \prec\ x - y.$$

(BM2) Any map $D\colon \mathfrak{A} \to \mathbb{R}^{>}\mathfrak{M}$ such that for all $x, y \in \mathfrak{A}$,

$$D(\exp x) \ =\ D(x) \exp x, \quad \log D(x) - \log D(y) \ \prec\ \max(x, y),$$

extends to a surreal derivation.

Thus (BM2) is a partial converse to (BM1), although the condition in (BM2) that D takes only values in $\mathbb{R}^{>}\mathfrak{M}$ seems a rather severe restriction. We define a *pre-derivation* to be a map $D\colon \mathfrak{A} \to \mathbb{R}^{>}\mathfrak{M}$ as in (BM2). Note that if D is a pre-derivation, then

$$D(a) = \left(\prod_{m<n} \log_m a\right) \cdot D(\log_n a) \qquad \text{for all } a \in \mathfrak{A} \text{ and all } n. \tag{$*$}$$

A pre-derivation D actually extends canonically to a surreal derivation ∂_D. To define ∂_D in terms of D we rely on the notion of *path derivatives*, introduced in [10], further developed in [12], and adapted to the surreal setting in [3]. A *path* is a function $P\colon \mathbb{N} \to \mathbb{R}^{\times}\mathfrak{M}$ such that $P(n+1)$ is a term of $\ell(P(n))$, for all n. Given x, the paths P such that $P(0)$ is a term of x are the elements of a set $\mathcal{P}(x)$. For $x \in \mathfrak{A}$ there is a unique path $P \in \mathcal{P}(x)$; it is given by $P(n) = \log_n x$. Thus if P is a path and $P(m) \in \mathfrak{A}$, then $P(n) = \log_{n-m} P(m)$ for all $n \geqslant m$, so $P(n) \in \mathfrak{A}$ for all $n \geqslant m$.

Let D be a pre-derivation. The *path derivative* $\partial_D(P) \in \mathbb{R}\mathfrak{M}$ for a path P is defined as follows, with $(*)$ guaranteeing independence of n in (1):

(1) if $P(n) \in \mathfrak{A}$, then $\partial_D(P) := \big(\prod_{m<n} P(m)\big) \cdot D(P(n))$;
(2) if $P(n) \notin \mathfrak{A}$ for all n, then $\partial_D(P) := 0$.

The rationale behind path derivatives is the following proposition:

(BM3) For each a the family $\big(\partial_D(P)\big)_{P \in \mathcal{P}(a)}$ is summable.

This result is stated in [3, Proposition 6.20] only for one particular pre-derivation, but, as the authors mention, the proof extends to any pre-derivation. In view of (BM3) we can now define $\partial_D\colon \mathbf{No} \to \mathbf{No}$ by

$$\partial_D(a) \ :=\ \sum_{P \in \mathcal{P}(a)} \partial_D(P).$$

It follows from $(*)$ that ∂_D extends D, and the arguments in [3, Section 6] show that ∂_D is a surreal derivation.

Results from [2]**.** To state the relevant facts, we recall from [1] or [2] that an H-*field* is by definition an ordered differential field K with derivation ∂ and constant field $C = \big\{f \in K :\ \partial(f) = 0\big\}$ such that:

(H1) $\partial(f) > 0$ for all $f \in K$ with $f > C$;
(H2) $\mathcal{O} = C + \mathcal{o}$, where $\mathcal{O}$ is the convex hull of C in K, and $\mathcal{o}$ is the maximal ideal of the valuation ring $\mathcal{O}$.

Let K be an H-field, and let $\mathcal{O}$ and $\mathcal{o}$ be as in (H2). Thus K is a valued field with valuation ring $\mathcal{O}$. We consider K in the natural way as an $\mathcal{L}$-structure, where

$$\mathcal{L} := \{\, 0,\ 1,\ +,\ -,\ \times,\ \partial,\ \leqslant,\ \preccurlyeq \,\}$$

is the language of ordered valued differential fields; in particular,

$$f \preccurlyeq g \iff f \in \mathcal{O}g \iff |f| \leqslant c|g| \text{ for some } c \geqslant 0 \text{ in } C.$$

Given $f \in K$ we also write f' instead of $\partial(f)$, and we set $f^\dagger := f'/f$ for $f \neq 0$, so $(fg)^\dagger = f^\dagger + g^\dagger$ and $(1/f)^\dagger = -f^\dagger$ for $f, g \in K^\times$. A useful subset of the value group $\Gamma := v(K^\times)$ of the valued field K is

$$\Psi \;:=\; \Psi_K \;:=\; \big\{v(f^\dagger) :\; f \in K^\times,\; f \not\asymp 1\big\} \;=\; \big\{v(f^\dagger) :\; f \in K,\; f > C\big\}.$$

As in [2] we call K *grounded* if Ψ has a largest element. For the convenience of the reader we include a proof of the following wellknown fact.

Lemma 1.1. *Assume K has constant field $C = \mathbb{R}$. Then K is grounded iff Γ has a smallest nontrivial archimedean class.*

Proof. Let $f, g \in K$, $f, g > C$. Suppose the archimedean class $\big[v(f)\big] = \big[v(1/f)\big]$ of $v(f)$ is greater than $\big[v(g)\big]$. This means: $v(f) < nv(g) = v(g^n) < 0$ for all $n \geqslant 1$. Hence $f^\dagger > (g^n)^\dagger = ng^\dagger > 0$ for all $n \geqslant 1$, by [1, Lemma 1.4], so $v(f^\dagger) < v(g^\dagger)$. A similar argument (which doesn't need $C = \mathbb{R}$) shows that if $\big[v(f)\big] = \big[v(g)\big]$, then $v(f^\dagger) = v(g^\dagger)$. Thus we have an order-reversing bijection $\big[v(f)\big] \mapsto v(f^\dagger)$ ($f \in K$, $f > C$) from the set of nontrivial archimedean classes of Γ onto Ψ. □

An *H-subfield of K* is by definition an ordered differential subfield of K that is an H-field. In [2] we axiomatized the elementary (= first-order) theory of the H-field $\mathbb{T}$ of transseries. This (complete) theory is called $T^{\text{nl}}_{\text{small}}$ there and its models are exactly the H-fields K satisfying the following (first-order) conditions:

(1) the derivation of K is small, that is, $\partial\smallo \subseteq \smallo$;
(2) K is Liouville closed;
(3) K is ω-free;
(4) K is newtonian.

(An H-field K is said to be *Liouville closed* if it is real closed and for all $f \in K$ there exists $g \in K$ with $g' = f$ and an $h \in K^\times$ such that $h^\dagger = f$; for the definition of "ω-free" and "newtonian" we refer to the Introduction of [2].) Dropping the smallness axiom (1), we get the incomplete but model complete theory T^{nl}; see [2, Chapter 16]. The H-field $\mathbb{T}$ satisfies (3) and (4) by [2, Corollary 11.7.15 and Theorem 15.0.1], which for an arbitrary H-field K amount to the following:

If $\partial K = K$ and K is a directed union of spherically complete grounded H-subfields, then K is ω-free and newtonian.

The condition $\partial K = K$ is automatically satisfied if K is a directed union of spherically complete grounded H-subfields E such that for some $\phi \in E$ we have $v(\phi) = \max \Psi_E$ and $\phi \in \partial K$, by [2, Corollary 15.2.4].

2. Infinite Products and Log-atomic Surreals

The pre-derivation D in [3] with $\partial_D = \partial_{\text{BM}}$ is defined by a certain identity. Towards the end of this section we give this identity a more suggestive form, which we found useful. But we begin with some remarks on ε-numbers, which play an important role in the next sections.

Remarks on ε-numbers. Throughout this paper ε will denote an ε-number, that is, ε is an ordinal such that $\omega^\varepsilon = \varepsilon$.

Lemma 2.1. *For any α there is a least ε-number $\varepsilon(\alpha) \geqslant \alpha$. Moreover, if α is infinite, then* $\operatorname{card}(\varepsilon(\alpha)) = \operatorname{card}(\alpha)$.

Proof. The recursion defining ω^α as a function of α easily yields that this function is strictly increasing, with $\omega^\alpha \geqslant \alpha$, $\operatorname{card}(\omega^\alpha) = \max\big(\aleph_0, \operatorname{card}(\alpha)\big)$, and thus $\operatorname{card}(\omega^\alpha) = \operatorname{card}(\alpha)$ if α is infinite. Now define α_n as a function of n by the recursion $\alpha_0 = \alpha$ and $\alpha_{n+1} = \omega^{\alpha_n}$. Then $\sup_n \alpha_n$ is clearly the least ε-number $\geqslant \alpha$, and it has the same cardinality as α if the latter is infinite. □

If κ is an uncountable cardinal, then by the remarks in the proof above we have $\omega^\alpha < \kappa$ for all $\alpha < \kappa$. Thus uncountable cardinals are ε-numbers. The least ε-number is denoted by ε_0, as usual, so $\varepsilon_0 = \sup_n \omega_n$ where the ω_n are defined by the recursion $\omega_0 = \omega$ and $\omega_{n+1} = \omega^{\omega_n}$.

Infinite products of monomials. Recall that $\mathfrak{M}$ is the multiplicative group of monomials ω^a. For a family $(\mathfrak{m}_i)$ in $\mathfrak{M}$ we say that $\prod_i \mathfrak{m}_i$ exists if $\sum_i a_i$ exists, with $\mathfrak{m}_i = \omega^{a_i}$ for all i, and in that case, we set

$$\prod_i \mathfrak{m}_i \ :=\ \omega^{\sum_i a_i} \in \mathfrak{M}.$$

The rules for manipulating these infinite products are easy consequences of those for infinite sums, and we shall freely use them below. Note in particular that if $(\mathfrak{m}_i)$ is a family in $\mathfrak{M}$ and $\prod_i \mathfrak{m}_i$ exists, then $\prod_i \mathfrak{m}_i^{-1}$ exists and equals $(\prod_i \mathfrak{m}_i)^{-1}$.

In our definition of infinite products we could have represented monomials as exponentials of elements in $\mathbb{J}$ instead of as powers of ω. Indeed, the equivalence between these options follows from the next two lemmas:

Lemma 2.2. *Let (a_i) be a summable family in $\mathbb{J}$. Then $\prod_i \exp(a_i)$ exists, and*

$$\exp\left(\sum_i a_i\right) \ =\ \prod_i \exp(a_i).$$

Proof. We have $a_i = \sum_{x>0} a_{i,x}\omega^x$, so by [9, Theorem 10.13],

$$\exp(a_i) \ =\ \omega^{b_i}, \quad b_i \ :=\ \sum_{x>0} a_{i,x}\omega^{g(x)},$$

so $E(b_i) = g(E(a_i))$. Since $\sum_i a_i$ exists, so does $\sum_i b_i$, and hence $\prod_i \exp(a_i) = \prod_i \omega^{b_i}$ exists, and $\prod_i \exp(a_i) = \omega^{\sum_i b_i}$. Moreover, with $\sum_i a_i = \sum_{x>0} a_x\omega^x$, we have $\sum_i b_i = \sum_{x>0} a_x\omega^{g(x)}$. Hence again by [9, Theorem 10.13],

$$\prod_i \exp(a_i) \ =\ \omega^{\sum_{x>0} a_x\omega^{g(x)}} \ =\ \exp\left(\sum_{x>0} a_x\omega^x\right) \ =\ \exp\left(\sum_i a_i\right),$$

as claimed. □

Lemma 2.3. *Let $(\mathfrak{m}_i)$ be a family in $\mathfrak{M}$ such that $\prod_i \mathfrak{m}_i$ exists. Then $\sum_i \log \mathfrak{m}_i$ exists, and $\log \prod_i \mathfrak{m}_i = \sum_i \log \mathfrak{m}_i$.*

Proof. We have $\mathfrak{m}_i = \exp(a_i)$ with $a_i \in \mathbb{J}$, so $a_i = \sum_{x>0} a_{i,x}\omega^x$, hence

$$\mathfrak{m}_i \;=\; \omega^{b_i}, \qquad b_i \;:=\; \sum_{x>0} a_{i,x}\omega^{g(x)}$$

by [9, Theorem 10.13]. Since the product $\prod_i \mathfrak{m}_i$ exists, so does $\sum_i b_i$, and therefore $\sum_i a_i = \sum_i \log \mathfrak{m}_i$ exists. Moreover, and again by [9, Theorem 10.13],

$$\prod_i \mathfrak{m}_i \;=\; \omega^{\sum_i b_i} \;=\; \omega^{\sum_{x>0} a_x \omega^{g(x)}} \;=\; \exp\left(\sum_{x>0} a_x\omega^x\right), \quad a_x := \sum_i a_{i,x},$$

and so $\log \prod_i \mathfrak{m}_i = \sum_{x>0} a_x\omega^x = \sum_i a_i$. □

Log-atomic surreals. Recall that $\mathfrak{A} \subseteq \mathfrak{M}^{\succ 1}$ is the class of log-atomic surreals. See [3, Sections 1, 5] for the order-preserving bijection $x \mapsto \lambda_x \colon \mathbf{No} \to \mathfrak{A}$ and for the fact that $\lambda_x \leqslant_s \lambda_y$ iff $x \leqslant_s y$. It follows from $\exp(\omega^x) = \omega^{\omega^{g(x)}}$ that $\mathfrak{A} \subseteq \omega^{\mathfrak{M}}$. Thus for any well-ordered index set I and strictly decreasing map $i \mapsto \lambda_i \colon I \to \mathfrak{A}$ the product $\prod_i \lambda_i$ exists. We shall use Proposition 2.6 and Corollary 2.9 below to define the pre-derivation $\partial_{\mathrm{BM}}|_{\mathfrak{A}}$.

Lemma 2.4. *Let $\mathfrak{m} = A|B$ be a monomial representation with $\mathfrak{m} \succ 1$. Then*

$$\exp(\mathfrak{m}) \;=\; \big(\mathfrak{m}^{\mathbb{N}} \cup \exp(A)\big) \,\big|\, \exp(B).$$

Proof. For $\mathfrak{m}' < \mathfrak{m}$ with $\mathfrak{m}' <_s \mathfrak{m}$ we have $\mathfrak{m}' \leqslant a$ for some $a \in A$ (since $A < \mathfrak{m}' < \mathfrak{m} < B$ gives $\mathfrak{m} \leqslant_s \mathfrak{m}'$). Likewise, for $\mathfrak{m} < \mathfrak{m}'' <_s \mathfrak{m}$, we have $b \leqslant \mathfrak{m}''$ for some $b \in B$. It follows that for $\mathfrak{m}'$ as above and $k \in \mathbb{N}^{\geqslant 1}$ we have $\exp(\mathfrak{m}')^k \leqslant \exp(a)$ for some $a \in A$, and that for $\mathfrak{m}''$ as above and $k \in \mathbb{N}^{\geqslant 1}$ we have $\exp(b) \leqslant \exp(\mathfrak{m}'')^{1/k}$ for some $b \in B$. This yields the desired result in view of [3, Theorem 3.8 (1)]. □

The monomial representation $\omega = \mathbb{N}|\emptyset$ shows that in the conclusion of Lemma 2.4 we cannot drop $\mathfrak{m}^{\mathbb{N}}$. Below we use the binary relations $\asymp^L$ and $\prec^L$ from [3]. Let $x = \{x'\}|\{x''\}$ be the canonical representation of x, and let j, k range over $\mathbb{N}^{\geqslant 1}$. Then by [3, Definition 5.12], the defining representation of λ_x is given by

$$\lambda_x \;=\; \big\{k, \exp_j\big(k \log_j(\lambda_{x'})\big)\big\} \,\big|\, \big\{\exp_j\big(\tfrac{1}{k}\log_j(\lambda_{x''})\big)\big\}\,.$$

Proposition 2.5. *We have $\lambda_{x+1} = \exp(\lambda_x)$, and thus $\lambda_{x-1} = \log(\lambda_x)$.*

Proof. Let $x = \{x'\}|\{x''\}$ be the canonical representation of x. Then $1 = 0|\emptyset$ gives $x+1 = \{x, x'+1\}|\{x''+1\}$. Assume inductively that $\lambda_{x'+1} = \exp(\lambda_{x'})$ and $\lambda_{x''+1} = \exp(\lambda_{x''})$ for all x' and x''. With j, k ranging over $\mathbb{N}^{\geqslant 1}$, [3, 5.15] gives

$$\begin{aligned}\lambda_{x+1} \;&=\; \big\{k, \exp_j\big(k\log_j(\lambda_x)\big), \exp_j\big(k\log_j(\lambda_{x'+1})\big)\big\} \,\big|\, \big\{\exp_j\big(\tfrac{1}{k}\log_j(\lambda_{x''+1})\big)\big\} \\ &=\; \big\{k, \exp_j\big(k\log_j(\lambda_x)\big), \exp_j\big(k\log_{j-1}(\lambda_{x'})\big)\big\} \,\big|\, \big\{\exp_j\big(\tfrac{1}{k}\log_{j-1}(\lambda_{x''})\big)\big\}\,.\end{aligned}$$

The defining representation $\lambda_x = A|B$ is monomial, and the above gives $\lambda_{x+1} = \mathbb{N} \cup S \cup \exp(A)|\exp(B)$ where S includes $\lambda_x^{\mathbb{N}}$ and all elements of S are $\asymp^L \lambda_x$. Since $\lambda_x \prec^L \exp(\lambda_x)$, it follows easily from Lemma 2.4 that $\lambda_{x+1} = \exp(\lambda_x)$. □

Lemma 2.6. *We have $\lambda_{-\alpha} = \omega^{\omega^{-\alpha}}$.*

Proof. By induction on α. The case $\alpha = 0$ holds since $\lambda_0 = \omega$. Assuming it holds for a certain α, we have

$$\lambda_{-(\alpha+1)} \;=\; \log \lambda_{-\alpha} \;=\; \log \omega^{\omega^{-\alpha}} \;=\; \omega^{\omega^{-(\alpha+1)}}.$$

Next, let μ be an infinite limit ordinal. Then $-\mu = \emptyset|\{-\alpha : \alpha < \mu\}$, and so by [3, 5.15] and with j, k ranging over $\mathbb{N}^{\geqslant 1}$ we have

$$\lambda_{-\mu} \;=\; \mathbb{N} \,\big|\, \big\{\exp_j\!\big(\tfrac{1}{k}\log_j \lambda_{-\alpha}\big)\big\}\,.$$

Now $\exp_j\!\big(\frac{1}{k}\log_j \lambda_{-\alpha}\big) \asymp^L \lambda_{-\alpha} \succ^L \lambda_{-\beta}$ when $\alpha < \beta$, so by cofinality and the inductive assumption we have

$$\lambda_{-\mu} \;=\; \mathbb{N}\,\big|\big\{\omega^{\omega^{-\alpha}} :\ \alpha < \mu\big\}.$$

From $\mathbb{N} < \omega^{\omega^{-\mu}} < \omega^{\omega^{-\alpha}}$ for all $\alpha < \mu$, we get $\lambda_{-\mu} \leqslant_s \omega^{\omega^{-\mu}}$. Take a such that $\lambda_{-\mu} = \omega^{\omega^{-a}}$. Then $\lambda_{-\mu} < \omega^{\omega^{-\alpha}}$ for $\alpha < \mu$ gives $\omega^{-a} < \omega^{-\alpha}$ for all $\alpha < \mu$, and thus $a > \alpha$ for all $\alpha < \mu$. This yields $\mu \leqslant_s a$, and thus $\omega^{\omega^{-\mu}} \leqslant_s \lambda_{-\mu}$, hence $a = \mu$. □

Lemma 2.7. *For $\lambda \in \mathfrak{A}$ we have: $\lambda < \lambda_{-\alpha} \iff \lambda_{-(\alpha+1)} \leqslant_s \lambda$.*

Proof. For $\lambda = \lambda_x$ we have the equivalences

$$\begin{aligned}\lambda_x < \lambda_{-\alpha} &\iff x < -\alpha \iff \alpha < -x \iff \alpha+1 \leqslant_s -x \\ &\iff -(\alpha+1) \leqslant_s x \iff \lambda_{-(\alpha+1)} \leqslant_s \lambda_x.\end{aligned}$$
□

Transfinitely iterating the logarithm function. In view of $\lambda_{-n} = \log_n \omega$ and the proof of Lemma 2.6 it is suggestive to think of $\lambda_{-\alpha}$ as the α times iterated function log evaluated at ω. Accordingly we set $\log_\alpha \omega := \lambda_{-\alpha}$. We note that for $\beta < \alpha$ we have $-\beta <_s -\alpha$, so $\omega^{-\beta} <_s \omega^{-\alpha}$, and thus $\log_\beta \omega <_s \log_\alpha \omega$.

Lemma 2.8. *Suppose α is an infinite limit ordinal. Then $\log_\alpha \omega$ is the simplest surreal $x > \mathbb{N}$ such that $x < \log_\beta \omega$ for all $\beta < \alpha$.*

Proof. First, $\mathbb{N} < \log_\alpha \omega < \log_\beta \omega$ for all $\beta < \alpha$. Let x be the simplest surreal $> \mathbb{N}$ such that $x < \log_\beta \omega$ for all $\beta < \alpha$. Then x is the simplest positive element in its archimedean class, so $x = \omega^y$ with $y > 0$. Then $x = \omega^y < \omega^{\omega^{-\beta}}$ for $\beta < \alpha$ gives $y < \omega^{-\beta}$ for all $\beta < \alpha$. Then y is the simplest positive element in its archimedean class: if $0 < y_0 \leqslant_s y$ and $y_0 \leqslant ny$, then $\omega^{y_0} \leqslant_s \omega^y = x$ and $\mathbb{N} < \omega^{y_0} \leqslant x^n < \log_\beta \omega$ for all $\beta < \alpha$, so $\omega^{y_0} = \omega^y$, and thus $y_0 = y$. Hence $y = \omega^z$ with $z < -\beta$ for all $\beta < \alpha$, and thus $z \leqslant -\alpha \leqslant_s z$. Therefore, $\omega^{-\alpha} \leqslant_s \omega^z = y$, so

$$\log_\alpha \omega \;=\; \omega^{\omega^{-\alpha}} \;\leqslant_s\; \omega^y \;=\; x,$$

and thus $\log_\alpha \omega = x$. □

The surreals $\log_\alpha \omega$ occur in the definition of ∂_{BM} later in this section.

The κ-numbers. The definition of ∂_{BM} in [3] also involves the surreals $\kappa_x \in \mathfrak{A}$ defined by Kuhlmann and Matusinski [11]. This is only needed for $x = -\alpha$, and it follows from the results in [11] that $\kappa_{-\alpha} = \omega^{\omega^{-\omega\alpha}}$, where $\omega\alpha$ is the usual ordinal product. Thus in view of Lemma 2.6:

Corollary 2.9. *We have $\kappa_{-\alpha} \;=\; \lambda_{-\omega\alpha} \;=\; \omega^{\omega^{-\omega\alpha}} \;=\; \log_{\omega\alpha}\omega$.*

We also use the binary relations $\preccurlyeq^K$, $\succ^K$, and $\asymp^K$ on $\mathbf{No}^{>\mathbb{N}}$ defined by

$$\begin{aligned} x \preccurlyeq^K y &\iff x \leqslant \exp_n(y) \text{ for some } n,\\ x \succ^K y &\iff x > \exp_n(y) \text{ for all } n,\\ x \asymp^K y &\iff x \preccurlyeq^K y \text{ and } y \preccurlyeq^K x.\end{aligned}$$

We refer to [3, 5.3] for proofs of some basic facts about these relations and the κ_x such as: $\asymp^K$ is an equivalence relation on $\mathbf{No}^{>\mathbb{N}}$ with convex equivalence classes, every $\asymp^K$-equivalence class has a unique element κ_x in it, and this element is the simplest element of this equivalence class. Also, $\kappa_x \leqslant_s \kappa_y$ iff $x \leqslant_s y$.

Defining the pre-derivation for ∂_{BM}. The pre-derivation D with $\partial_D = \partial_{\mathrm{BM}}$ is denoted by $\partial_{\mathbb{L}}$ in [3, Definition 6.7], and by $\partial_{\mathfrak{A}}$ in this paper. It is given by

$$\partial_{\mathfrak{A}}(\lambda) \ := \ \prod_n \log_n \lambda \Big/ \prod_\alpha \log_\alpha \omega$$

with α in the denominator ranging over the ordinals such that $\log_\alpha \omega \geqslant \log_n \lambda$ for some n; to facilitate comparison with [3] we note that this condition on α is equivalent to $\lambda \preccurlyeq^K \log_\alpha \omega$. (The products on the right exist, since $\log_n \lambda$ and $\log_\alpha \omega$ are strictly decreasing as functions of n and α, respectively.) The above defining identity for $\partial_{\mathfrak{A}}$ simplifies the expression in [3] by our use of infinite products (instead of exponentials of infinite sums), and of Lemma 2.6 and Corollary 2.9 (to get rid of κ-numbers). As [3, Section 9] shows, $\partial_{\mathfrak{A}}$ is in a certain technical sense the *simplest* pre-derivation.

If $\lambda > \exp_n \omega$ for all n, then $\partial_{\mathfrak{A}}(\lambda) = \prod_n \log_n \lambda$. Another special case is $\partial_{\mathfrak{A}}(\log_\alpha \omega) = 1 \big/ \prod_{\beta<\alpha} \log_\beta \omega$, in particular, $\partial_{\mathfrak{A}}(\omega) = 1$. For ε-numbers we get the following (not needed later, but included as an example):

Lemma 2.10. *We have* $\log_n \varepsilon = \omega^{\omega^{\varepsilon-n}}$. *Hence* $\varepsilon \in \mathfrak{A}$ *and*

$$\partial_{\mathfrak{A}}(\varepsilon) \ = \ \omega^{\omega^{\varepsilon}+\omega^{\varepsilon-1}+\omega^{\varepsilon-2}+\cdots} \ = \ \omega^{\varepsilon/(1-\omega^{-1})}.$$

Proof. From [9, pp. 179, 180] we get that if b, as a sequence of pluses and minuses, equals ε followed by $\varepsilon\omega n$ minuses, with $n \geqslant 1$ and $\varepsilon\omega n$ being the ordinal product, then $b = \omega^{\varepsilon-n}$, and $g(b) = \varepsilon - (n-1)$. In other words,

$$g\big(\omega^{\varepsilon-n}\big) \ = \ \varepsilon - (n-1) \qquad (n \geqslant 1).$$

Using this we prove the lemma by induction on n. The case $n = 0$ is clear. Assume inductively that $\log_n \varepsilon = \omega^{\omega^{\varepsilon-n}}$. Since $g\big(\omega^{\varepsilon-(n+1)}\big) = \varepsilon - n$, this gives

$$\exp\left(\omega^{\omega^{\varepsilon-(n+1)}}\right) \ = \ \omega^{\omega^{\varepsilon-n}},$$

from which we get $\log_{n+1} \varepsilon = \omega^{\omega^{\varepsilon-(n+1)}}$, as desired. □

3. Exhibiting **No** as a Suitable Directed Union

At the end of Section 1 we explained how proving $\mathbb{T} \equiv \mathbf{No}$ (as differential fields) reduces to representing **No** as a directed union of spherically complete grounded H-subfields. In this section we obtain such a representation. The reader should beware of considering **No** itself as spherically complete, even though the Conway normal form is sometimes summarized as "$\mathbf{No} = \mathbb{R}(\!(\omega^{\mathbf{No}})\!)$". This is misleading, however, since it suggests that a series like $\sum_\alpha \omega^{-\alpha}$, where the sum is over all ordinals α, is a surreal number. It might perhaps be viewed as a surreal number in a strictly larger set-theoretic universe, but not in the one we are (tacitly) working in. A better way of understanding **No** as a valued field is as the directed union $\bigcup_\Gamma \mathbb{R}[[\omega^\Gamma]]$ with Γ ranging over the subsets of **No** that underly an additive subgroup of **No**; for example, any α gives $\mathbf{No}(\omega^\alpha)$ as such a Γ. For any such Γ the corresponding $\mathbb{R}[[\omega^\Gamma]]$

is indeed a spherically complete valued subfield of **No**, but in general $\mathbb{R}[[\omega^\Gamma]]$ is not closed under ∂_{BM}, and even if it is, it might not be grounded.

In this section we show that for $S = \mathbf{No}(\varepsilon) \cup \{-\varepsilon\}$, with ε any ε-number, the Hahn subgroup $\Gamma = \mathbb{R}[[\omega^S]]$ of **No** gives rise to a spherically complete valued subfield $\mathbb{R}[[\omega^\Gamma]]$ that is closed under ∂_{BM} and grounded as an H-subfield of **No**.

A length bound for h. This very useful bound is as follows:

Lemma 3.1. $l\big(h(y)\big) \ \leqslant\ \omega^{l(y)+1}$.

Proof. By [9, p. 172] the canonical representation $y = \{y'\}|\{y''\}$ yields

$$h(y) \ = \ \big\{0, h(y')\big\}\big|\big\{h(y''), \omega^y/2^n\big\}.$$

We can assume inductively that the lemma holds for the y' and y'' instead of y, and thus $l\big(h(y')\big) \leqslant\ \omega^{l(y')+1} < \omega^{l(y)+1}$ for all y', and likewise with y'' instead of y'. Also, $l(\omega^y/2^n) \leqslant l(\omega^y)l(1/2^n) < \omega^{l(y)}\omega = \omega^{l(y)+1}$, using [5, Lemmas 3.6 and 4.1]. Now appeal to [9, Theorem 2.3]. □

Recall from Section 1 that $h(-\alpha) = \omega^{-(\alpha+1)}$, and so $h(0) = \omega^{-1}$ shows that for $y = 0$ the upper bound in Lemma 3.1 is attained.

Some spherically complete initial subfields of No. In this subsection we fix an initial subset I of **No**. Then $\Gamma := \mathbb{R}[[\omega^I]]$ is an initial additive subgroup of **No** by the proof of Theorem 18 in [7]. (That theorem considers Hahn fields rather than the Hahn group Γ, but the same ideas work; we stress that it is the proof of that theorem rather than its statement that matters here.) Moreover, as Philip Ehrlich mentioned to one of us:

Lemma 3.2. *Suppose I has a least element a. Then $a = -\alpha$ for some α, and Γ has a least nontrivial archimedean class represented by ω^a.*

Proof. Taking the longest initial segment of a consisting of minus signs we get the largest ordinal α with $-\alpha \leqslant_s a$. Then $-\alpha \in I$ and $-\alpha \leqslant a$, so $-\alpha = a$. □

Since Γ is initial and an ordered additive group it leads to the initial subfield $K := \mathbb{R}[[\omega^\Gamma]]$ of **No**. Note that K is spherically complete, and if (a_i) is a family in K for which $\sum_i a_i$ exists, then $\sum_i a_i \in K$. Now $\Gamma = \mathbb{R}[[\omega^I]]$ is also closed under infinite sums, so if $(\mathfrak{m}_i)$ is a family in $\mathfrak{M} \cap K$ such that $\prod_i \mathfrak{m}_i$ exists, then $\prod_i \mathfrak{m}_i \in K$. Thus K is closed under infinite sums, and also under infinite products of monomials. This is very useful in showing that for suitable choices of I the field K is closed under certain surreal derivations. Note however, that if I has a least element, then $K^{>\mathbb{N}}$ is not closed under log: if $-\alpha$ is the least element of I, then $\log_\alpha \omega = \omega^{\omega^{-\alpha}} \in K$, but $\log_{\alpha+1}\omega \notin K$, as $-(\alpha+1) \notin I$.

In order to discuss examples we set $a^r := \exp(r \log a)$ for $a > 0$ and $r \in \mathbb{R}$, and note agreement with the previously defined ω^r when $a = \omega$. Moreover,

$$(\log_\alpha \omega)^r \ = \ \omega^{r\omega^{-\alpha}} \qquad (r \in \mathbb{R}),$$

by the definition of a^r, using also $g\big(\omega^{-(\alpha+1)}\big) = -\alpha$ and [9, Theorem 10.13].

Examples. For $I = \{0\}$ we get $\Gamma = \mathbb{R}$ and $K = \mathbb{R}[[\omega^{\mathbb{R}}]]$; note that K is closed under ∂_{BM}, but $\omega \in K$ and $\log \omega = \omega^{1/\omega} \notin K$.

For $I = \{0, -1\}$ we have $\Gamma = \mathbb{R} + \mathbb{R}\omega^{-1}$, so $\omega^\Gamma = \omega^{\mathbb{R}}(\log \omega)^{\mathbb{R}}$, and thus $K = \mathbb{R}[[\omega^{\mathbb{R}}(\log \omega)^{\mathbb{R}}]]$, which is again closed under ∂_{BM}.

Let $I = \{\alpha : \alpha \leqslant \varepsilon\}$. Then $\varepsilon = \omega^{\omega^{\varepsilon}} \in K$, but Lemma 2.10 gives $\log \varepsilon \notin K$, since $\varepsilon - 1 \notin I$ and so $\omega^{\varepsilon-1} \notin \Gamma$. Likewise we get $\partial_{\mathrm{BM}}(\varepsilon) \notin K$.

Lemma 3.3. *If $I = \{a : l(a) < \alpha\}$ or $I = \{a : l(a) \leqslant \alpha\}$, then $I \subseteq \Gamma \subseteq K$.*

Proof. Suppose $I = \{a : l(a) < \alpha\}$. (The case $I = \{a : l(a) \leqslant \alpha\}$ is handled in the same way.) Let $a \in I$. Then $a = \sum_x a_x \omega^x$, and if $x \in E(a)$, then $l(x) \leqslant l(\omega^x) \leqslant l(a) < \alpha$ by [5, Lemmas 3.4, 4.1, and 4.2], so $x \in I$. Thus $a \in \Gamma$. This proves $I \subseteq \Gamma$. Next, if $b \in \Gamma$, then $b = \sum_{x \in I} b_x \omega^x$, and so $b \in K$ in view of $I \subseteq \Gamma$. □

The next lemma will also be crucial:

Lemma 3.4. *Suppose $h(I) \subseteq \Gamma$. Then $\log K^{>} \subseteq K$ and for each $a \in K$ and term t of a we have: t and all terms of $\ell(t)$ lie in K.*

Proof. Let $a \in K^{>}$ have leading monomial $\mathfrak{m} = \omega^b$ with $b = \sum_{y \in I} b_y \omega^y$; to get $\log a \in K$, it is enough that $\log \mathfrak{m} \in K$; the latter holds because $\log \mathfrak{m} = \sum_y b_y \omega^{h(y)}$. This proves $\log K^{>} \subseteq K$.

Next, let $a \in K$ and let t be a term of a; we have to show that t and all terms of $\ell(t)$ lie in K. As $K \supseteq \mathbb{R}$ is initial, it does contain the term t of its element a. We have $t = r\omega^b$ with $r \in \mathbb{R}^{\times}$ and $b \in \Gamma$, so $b = \sum_{y \in I} b_y \omega^y$, and thus $\omega^b = \exp\big(\sum_{y \in I} b_y \omega^{h(y)}\big)$. Hence $\ell(t) = \ell(r\omega^b) = \sum_{y \in I} b_y \omega^{h(y)}$ and each of its terms $b_y \omega^{h(y)}$ lies obviously in K. □

Corollary 3.5. *If $h(I) \subseteq \Gamma$ and D is a pre-derivation with $D(K \cap \mathfrak{A}) \subseteq K$, then $\partial_D(K) \subseteq K$.*

Proof. Use the definition of ∂_D from Section 1, the fact that K is closed under infinite sums, and Lemma 3.4. □

Corollary 3.6. *Suppose $h(I) \subseteq \Gamma$. Then $\partial_{\mathrm{BM}}(K) \subseteq K$.*

Proof. Let $\lambda \in K \cap \mathfrak{A}$; by Corollary 3.5 we just need to get $\partial_{\mathfrak{A}}(\lambda) \in K$. Since K is closed under infinite products, it is enough for this to get $\log_n \lambda \in K$ for all n (which is the case by Lemma 3.4), and $\lambda_{-\alpha} \in K$ for all α such that $\lambda \preccurlyeq^K \lambda_{-\alpha}$. Given such α, take n with $\log_n \lambda < \lambda_{-\alpha}$. Then $\lambda_{-\alpha} \leqslant_s \lambda_{-(\alpha+1)} \leqslant_s \log_n \lambda \in K$ by Lemma 2.7, and so $\lambda_{-\alpha} \in K$ because K is initial. □

It can happen that $h(I) \not\subseteq \Gamma$ and that K is nevertheless closed under ∂_{BM}. The next lemma gives a useful criterion for that. To see why that lemma holds, consider a surreal derivation ∂, and note that from $\omega^{\omega^y} = \exp(\omega^{h(y)})$ we get

$$\partial\big(\omega^{\omega^y}\big) \;=\; \omega^{\omega^y} \cdot \partial(\omega^{h(y)}),$$

so for any monomial $\mathfrak{m} = \omega^b \in K$ we have $b = \sum_{y \in I} b_y \omega^y$, and thus

$$\mathfrak{m} \;=\; \exp\left(\sum_{y \in I} b_y \omega^{h(y)}\right), \qquad \partial(\mathfrak{m}) \;=\; \mathfrak{m} \cdot \sum_{y \in I} b_y \partial\big(\omega^{h(y)}\big).$$

This leads to:

Lemma 3.7. *Given a surreal derivation ∂, the following are equivalent:*

(1) *K is closed under ∂;*
(2) *$\partial(\omega^{\omega^y}) \in K$ for all $y \in I$;*
(3) *$\partial(\omega^{h(y)}) \in K$ for all $y \in I$.*

The surreal fields K_ε. Given the ε-number ε, we have the initial set $I := \mathbf{No}(\varepsilon)$, with the corresponding $\Gamma := \mathbb{R}[[\omega^I]]$ and $K := \mathbb{R}[[\omega^\Gamma]]$. In view of Lemmas 3.1 and 3.3 we have $h(I) \subseteq I \subseteq \Gamma$, so $\partial_{\mathrm{BM}}(K) \subseteq K$ by Corollary 3.6. Thus K is a spherically complete initial H-subfield of $\mathbf{No}$. However, I has no least element, so K is not grounded. We repair this by just augmenting I by $-\varepsilon$: set $I_\varepsilon := I \cup \{-\varepsilon\}$. Then I_ε is still initial, with least element $-\varepsilon$, and so we have the corresponding $\Gamma_\varepsilon := \mathbb{R}[[\omega^{I_\varepsilon}]]$ and $K_\varepsilon := \mathbb{R}[[\omega^{\Gamma_\varepsilon}]]$. To get $\partial_{\mathrm{BM}}(K_\varepsilon) \subseteq K_\varepsilon$ we note that $K \subseteq K_\varepsilon$, and so it suffices by Lemma 3.7 that $\partial_{\mathfrak{A}}(\omega^{\omega^{-\varepsilon}}) \in K_\varepsilon$. But $\omega^{\omega^{-\varepsilon}} = \log_\varepsilon \omega$, and

$$\partial_{\mathfrak{A}}(\log_\varepsilon \omega) \;=\; 1 \Big/ \prod_{\alpha<\varepsilon} \log_\alpha \omega,$$

which lies in K, and hence in K_ε. Thus K_ε is a grounded H-subfield of $\mathbf{No}$, and

$$\mathbf{No} \;=\; \bigcup_\varepsilon K_\varepsilon.$$

Note that Corollary 3.6 does not apply to I_ε, since $h(-\varepsilon) = \omega^{-(\varepsilon+1)} \notin \Gamma$; this is why we did the less direct construction via $I = \mathbf{No}(\varepsilon)$.

Since $\omega^{-\varepsilon}$ represents the smallest archimedean class of Γ_ε, we have

$$\max \Psi_{K_\varepsilon} \;=\; v\big((\omega^{\omega^{-\varepsilon}})^\dagger\big) \;=\; v\big((\log_\varepsilon \omega)^\dagger\big)$$

by the proof of Lemma 1.1. In view of $(\log_\varepsilon \omega)^\dagger = (\log_{\varepsilon+1} \omega)'$ and the remarks at the end of Section 1, the representation of $\mathbf{No}$ as an increasing union $\bigcup_\varepsilon K_\varepsilon$ of spherically complete grounded H-subfields now gives $\partial_{\mathrm{BM}}(\mathbf{No}) = \mathbf{No}$. (The proof of $\partial_{\mathrm{BM}}(\mathbf{No}) = \mathbf{No}$ in [3, Section 7] is different.) Thus by the results stated at the end of Section 1 we conclude that $\mathbf{No} \equiv \mathbb{T}$, as differential fields.

4. The Case of Restricted Length

A set $S \subseteq \mathbf{No}$ is said to be of *countable type* if $l(a)$ is countable for all $a \in S$, and all well-ordered subsets of S as well as all reverse well-ordered subsets of S are countable. (Note that $l(a)$ is countable for every $a \in \mathbf{No}(\omega_1)$, but that $\mathbf{No}(\omega_1)$ is not of countable type, since it has the set of countable ordinals as an uncountable well-ordered subset.)

Proposition 4.1. *Suppose the subset S of $\mathbf{No}$ is of countable type. Then the additive subgroup $\mathbb{R}[[\omega^S]]$ of $\mathbf{No}$ is also of countable type.*

Proof. The case $\alpha = 1$ of Esterle [8, Lemme 2.2] and the remarks following it yield that every well-ordered subset of $\mathbb{R}[[\omega^S]]$ is countable. Hence every reverse well-ordered subset of $\mathbb{R}[[\omega^S]]$ is countable as well. Let $a \in \mathbb{R}[[\omega^S]]$. Then $a = \sum_{s\in E(a)} a_s\omega^s$. Now $E(a) \subseteq S$ is countable, so the well-ordered set $-E(a)$ has order type $\mu < \omega_1$. Since ω_1 is regular, we have a countable ordinal ν such that $l(s) \leqslant \nu$ for all $s \in E(a)$. Then $l(\omega^s) \leqslant \omega^\nu$ for all $s \in E(a)$ by [5, Lemma 4.1], hence $l(a_s\omega^s) \leqslant \omega^{\nu+1}$ for all $s \in E(a)$ by [5, Proposition 3.6]. Thus

$$l(a) \;\leqslant\; \mu \cdot \omega^{\nu+1} \;<\; \omega_1,$$

by [9, Theorem 5.12], or [5, Lemma 4.2, (3)]. □

As an example, consider $S := \mathbf{No}(\omega)$, the set of of dyadic numbers. Then S is of countable type, and so $\mathbb{R}[[\omega^S]]$ is of countable type. Nevertheless, $l\big(\mathbb{R}[[\omega^S]]\big)$ is cofinal in ω_1: given any countable ordinal μ, take an order reversing injective map $\alpha \mapsto s_\alpha \colon \mu \to S$; then $a := \sum_\alpha \omega^{s_\alpha} \in \mathbb{R}[[\omega^S]]$ has $l(a) \geqslant \mu$, by [9, p. 63].

Let κ be any infinite cardinal. Esterle [8, Lemme 2.2] actually tells us for any set $S \subseteq \mathbf{No}$: if all well-ordered subsets and all reverse well-ordered subsets of S have size $\leqslant \kappa$, then this remains true for the set $\mathbb{R}[[\omega^S]] \subseteq \mathbf{No}$. The next cardinal κ^+ is regular, so the arguments in the proof of Proposition 4.1 go through to give the following, where we call $S \subseteq \mathbf{No}$ *of type* κ if $l(a) \leqslant \kappa$ for all $a \in S$ and all well-ordered subsets of S and all reverse well-ordered subsets of S have size $\leqslant \kappa$.

Corollary 4.2. *If $S \subseteq \mathbf{No}$ is of type κ, then so is $\mathbb{R}[[\omega^S]]$.*

Next we show that for countable μ the set $\mathbf{No}(\mu)$ is of countable type. Every element of $\mathbf{No}(\mu)$ has clearly countable length, for countable μ, and $\mathbf{No}(\mu)$ is closed under $x \mapsto -x$, so the assertion above reduces to:

Lemma 4.3. *Suppose the ordinal μ is countable. Then every well-ordered subset of $\mathbf{No}(\mu)$ is countable.*

This may remind the reader of the well-known property of the ordered set $\mathbb{R}$ that every well-ordered subset of $\mathbb{R}$ is countable. Here is a quick proof using that $\mathbb{R}$ has a countable dense subset $\mathbb{Q}$: given any embedding $\alpha \mapsto r_\alpha$ of an infinite cardinal κ into $\mathbb{R}$, pick for each $\alpha < \kappa$ a rational q_α such that $r_\alpha < q_\alpha < r_{\alpha+1}$; it follows that $\kappa = \aleph_0$. However, such a countable density argument cannot be used for ordered sets $\mathbf{No}(\mu)$ when μ is a countable limit ordinal $> \omega$:

Lemma 4.4. *Let μ be an infinite limit ordinal. Then the ordered set $\mathbf{No}(\mu)$ is dense without endpoints. If $\mu > \omega$, then there exists a collection of $2^{\aleph_0}$ pairwise disjoint open intervals in $\mathbf{No}(\mu)$, which has therefore no countable dense subset.*

Proof. The ordinals $\alpha < \mu$ are cofinal in this ordered set, and there is no largest such α. For $a < b$ in this ordered set, take $\alpha \leqslant l(a), l(b)$ such that $a|_\alpha = b|_\alpha$ and $a(\alpha) < b(\alpha)$. If $l(b) > \alpha$, then $b(\alpha) = +$, so $a < b- < b$. If $l(a) > \alpha$, then $a(\alpha) = -$, so $a < a+ < b$. Note that $b-, a+ \in \mathbf{No}(\mu)$, as μ is a limit ordinal,

Next, assume $\mu > \omega$. For each nondyadic $r \in \mathbb{R} \subseteq \mathbf{No}$, we have the surreals $r-$ and $r+$ of length $\omega+1$, and so we obtain the pairwise disjoint open intervals $(r-, r+)$ in $\mathbf{No}(\mu)$. □

Proof of Lemma 4.3. For $a \in \mathbf{No}(\mu)$ we define $\widehat{a} \colon \mu \to \mathbb{R}$ by

$$\widehat{a}(\alpha) \;=\; \begin{cases} -1 & \text{if } a(\alpha) = -, \\ 0 & \text{if } a(\alpha) = 0, \\ 1 & \text{if } a(\alpha) = +, \end{cases}$$

For $S = \{\alpha : \ \alpha < \mu\}$ this yields an order-preserving injective map

$$a \mapsto \sum_{\alpha<\mu} \widehat{a}(\alpha)\omega^{-\alpha} \ : \ \mathbf{No}(\mu) \to \mathbb{R}[[\omega^S]].$$

It remains to appeal to Proposition 4.1. □

Essentially the same argument yields the following generalization:

Corollary 4.5. *If κ is an infinite cardinal and μ is an ordinal of cardinality $\leqslant \kappa$, then each well-ordered subset of $\mathbf{No}(\mu)$ has cardinality $\leqslant \kappa$.*

Note that for a countable ε-number ε the initial set $I_\varepsilon = \mathbf{No}(\varepsilon) \cup \{-\varepsilon\}$ is of countable type by Lemma 4.3, and hence Γ_ε and K_ε are as well by Proposition 4.1. Taking the union over all such countable ε we obtain the set $\mathbf{No}(\omega_1)$ of all surreals of countable length as an increasing union of spherically complete grounded H-subfields K_ε of $\mathbf{No}$. As in Section 3 and using also the model completeness of $T^{\mathrm{nl}}_{\mathrm{small}} = \mathrm{Th}(\mathbb{T})$ this yields Theorem 2. The results above lead moreover to the following generalization:

Corollary 4.6. *Let κ be any uncountable cardinal. Then the subfield $\mathbf{No}(\kappa)$ of $\mathbf{No}$ is closed under ∂_{BM}, and $\mathbf{No}(\kappa) \prec \mathbf{No}$, as ordered differential fields.*

Proof. If κ is regular we can argue as for ω_1, using Corollaries 4.2 and 4.5 instead of Proposition 4.1 and Lemma 4.3. If κ is singular, use that it is the supremum of the uncountable regular cardinals below it. $\square$

5. Constructing Embeddings

So far we have just worked inside $\mathbf{No}$ and established Theorem 2. In this section we turn to $\mathbb{T}$ and prove the embedding results: Theorems 1 and 3.

Embedding $\mathbb{T}$ into No. Given a Hahn field $\mathbb{R}[[G]]$ over $\mathbb{R}$ we define a map $F\colon \mathbb{R}[[G]] \to \mathbf{No}$ to be *strongly additive* if for every summable family (f_i) in $\mathbb{R}[[G]]$ the family $\big(F(f_i)\big)$ is summable in $\mathbf{No}$ and $F\left(\sum_i f_i\right) = \sum_i F(f_i)$. We refer to [2, Appendix A] for the construction of $\mathbb{T}$ as an exponential ordered field. In this construction $\mathbb{T}$ is a subfield of a Hahn field $\mathbb{R}[[G^{\mathrm{LE}}]]$: in fact, G^{LE} is a certain directed union of ordered subgroups $G_m{\downarrow_n}$, and $\mathbb{T}$ is the corresponding directed union of the Hahn fields $\mathbb{R}[[G_m{\downarrow_n}]]$. A map $F\colon \mathbb{T} \to \mathbf{No}$ is said to be *strongly additive* if its restriction to each $\mathbb{R}[[G_m{\downarrow_n}]]$ is strongly additive.

Proposition 5.1. *There is a unique strongly additive embedding $\iota\colon \mathbb{T} \to \mathbf{No}$ of exponential ordered fields that is the identity on $\mathbb{R}$ and such that $\iota(x_{\mathbb{T}}) = \omega$.*

Proof. We use the notations from [2, Appendix A] except that the x there is $x_{\mathbb{T}}$ here. The construction of $\mathbb{T}$ there begins with the Hahn field $E_0 = \mathbb{R}[[x_{\mathbb{T}}^{\mathbb{R}}]]$, and there is clearly a (unique) strongly additive ordered field embedding $i_0\colon E_0 \to \mathbf{No}$ such that $i_0(r) = r$ and $i_0(x_{\mathbb{T}}^r) = \omega^r$ for all $r \in \mathbb{R}$. Moreover, $i_0(\mathrm{e}^b) = \exp(i_0(b))$ for all $b \in B_0$, and $\exp(i_0(a)) > i_0(E_0)$ for all $a \in A_0^>$. Assume inductively that we have an extension of i_0 to a strongly additive ordered field embedding $i_m\colon E_m = \mathbb{R}[[G_m]] \to \mathbf{No}$ such that $i_m(\mathrm{e}^b) = \exp(i_m(b))$ for all $b \in B_m$, and $\exp(i_m(a)) > i_m(E_m)$ for all $a \in A_m^>$. Then one checks easily that i_m extends (uniquely) to a strongly additive ordered field embedding $i_{m+1}\colon E_{m+1} \to \mathbf{No}$ such that $i_{m+1}(\mathrm{e}^b) = \exp(i_{m+1}(b))$ for all $b \in B_{m+1}$, and $\exp(i_{m+1}(a)) > i_{m+1}(E_{m+1})$ for all $a \in A_{m+1}^>$. Taking a union over all m we obtain an embedding

$$\iota_0 \;:=\; \bigcup_m i_m \;:\; \mathbb{R}[[x_{\mathbb{T}}^{\mathbb{R}}]]^{\mathrm{E}} \;=\; \bigcup_m \mathbb{R}[[G_m]] \to \mathbf{No}$$

of ordered exponential fields. Replacing in the above $\ell_0 = x_{\mathbb{T}}$, G_m, ω, by $\ell_n = \log_n x_{\mathbb{T}}$, $G_m{\downarrow_n}$, $\log_n \omega$, respectively, we obtain likewise an embedding

$$\iota_n \;:\; \mathbb{R}[[\ell_n^{\mathbb{R}}]]^{\mathrm{E}} \;=\; \bigcup_m \mathbb{R}[[G_m{\downarrow_n}]] \to \mathbf{No}$$

of ordered exponential fields with $\iota_n(\ell_n) = \log_n \omega$. Each ι_{n+1} extends ι_n, so we can take the union over all n to get an embedding $\iota\colon \mathbb{T} \to \mathbf{No}$ as claimed. The uniqueness holds because the smallest subfield of $\mathbb{T}$ that contains $\mathbb{R}(x_{\mathbb{T}})$ and is closed under exponentiation, taking logarithms of positive elements, and summation of summable families is $\mathbb{T}$ itself. □

Next we apply the model completeness of the theory of the exponential ordered field of real numbers (Wilkie [13]). By [6] and [5], respectively, the ordered exponential fields $\mathbb{T}$ and $\mathbf{No}$ are models of this theory, and so $\iota\colon \mathbb{T} \to \mathbf{No}$ is an elementary embedding of ordered exponential fields.

It is easy to check that $\iota\colon \mathbb{T} \to \mathbf{No}$ is also an embedding of ordered differential fields. In view of $\mathbb{T} \equiv \mathbf{No}$ (as differential fields), and the model completeness of $T^{\text{nl}}_{\text{small}}$ mentioned at the end of Section 1 we conclude that ι is an elementary embedding of ordered differential fields: Theorem 1.

Is ι an elementary embedding of *ordered differential exponential fields*? We don't know; this is related to the open problem from [2] to extend the model-theoretic results there about $\mathbb{T}$ as a differential field to $\mathbb{T}$ as a differential exponential field.

It follows easily from the construction of $\mathbb{T}$ and ι that all surreal derivations ∂ with $\partial(\omega) = 1$ agree on $\iota(\mathbb{T})$.

Proposition 5.2. *Here are some further properties of the map* ι*:*

(1) $\iota(G^{\text{LE}}) = \mathfrak{M} \cap \iota(\mathbb{T})$*;*
(2) $\iota(\mathbb{T})$ *is truncation closed;*
(3) $\iota(\mathbb{T})$ *is of countable type; in particular,* $\iota(\mathbb{T}) \subseteq \mathbf{No}(\omega_1)$.

Proof. Induction on m gives $\iota(G_m) \subseteq \mathfrak{M}$, where we use at the inductive step that $G_{m+1} = \exp(A_m)G_m$ and $\iota(A_m) \subseteq \mathbb{J}$, the latter being a consequence of $\iota(G_m) \subseteq \mathfrak{M}$. Likewise, $\iota(G_m{\downarrow_n}) \subseteq \mathfrak{M}$ for all m, n, and thus $\iota(G^{\text{LE}}) \subseteq \mathfrak{M}$. Since ι respects infinite sums of monomials, this yields (1), and (2) is then an immediate consequence using also that $\mathbb{T}$ is truncation closed in $\mathbb{R}[[G^{\text{LE}}]]$. As to (3), using the results in Section 4 one shows by induction on m that $\iota(G_m)$, and likewise each $\iota(G_m{\downarrow_n})$, has countable type. Hence $\iota(G^{\text{LE}})$ has countable type, and so does $\iota(\mathbb{T})$. □

Question (Elliot Kaplan): can (2) be improved to $\iota(\mathbb{T})$ being initial?

Embedding H-fields into No. Let ε be an ε-number; for example, ε could be any uncountable cardinal. We recall from [5] that $\mathbf{No}(\varepsilon)$ is a real closed subfield of $\mathbf{No}$ containing $\mathbb{R}$. We consider $\mathbf{No}(\varepsilon)$ as a valued subfield of $\mathbf{No}$ with (divisible) ordered value group $v\big(\mathbf{No}(\varepsilon)^{\times}\big)$. We shall need an easy auxiliary result:

Lemma 5.3. *Let κ be a regular uncountable cardinal. Then the underlying ordered sets of $\mathbf{No}(\kappa)$ and $v\big(\mathbf{No}(\kappa)^{\times}\big)$ are κ-saturated.*

Proof. Let $A, B \subseteq \mathbf{No}(\kappa)$ have cardinality $< \kappa$, with $A < B$. The regularity of κ yields an ordinal $\alpha < \kappa$ such that $l(A \cup B) < \alpha$. By [9, Theorem 2.3] this gives a surreal a with $l(a) \leqslant \alpha$ such that $A < a < B$, and then $a \in \mathbf{No}(\kappa)$. Thus $\mathbf{No}(\kappa)$ is κ-saturated as an ordered set. Next, let $P, Q \subseteq \mathbf{No}(\kappa)^{>}$ have cardinality $< \kappa$, with $v(P) > v(Q)$. Set $A := \{np :\ n \geqslant 1,\ p \in P\}$ and $B := \{q/n :\ n \geqslant 1,\ q \in Q\}$. Then $A < B$, and so the above gives $a \in \mathbf{No}(\kappa)$ with $A < a < B$. Then $v(P) > v(a) > v(Q)$, showing that $v\big(\mathbf{No}(\kappa)^{\times}\big)$ is κ-saturated as an ordered set. □

For Theorem 3 we need a sharpening of the model completeness of the theory T^{nl} of $\upomega$-free newtonian Liouville closed H-fields, namely, the quantifier elimination (QE) explained in [2, Introduction to Chapter 16]. The relevant first-order language for

QE has in addition to $\mathcal{L}$ extra unary predicate symbols $\mathrm{I}, \boldsymbol{\Lambda}, \boldsymbol{\Omega}$, to be interpreted in a model L of T^{nl} as sets $\mathrm{I}(L), \boldsymbol{\Lambda}(L), \boldsymbol{\Omega}(L) \subseteq L$ according to their defining axioms:

$$\begin{aligned} \mathrm{I}(a) &\iff a = y' \text{ for some } y \prec 1 \text{ in } L, \\ \boldsymbol{\Lambda}(a) &\iff a = -y^{\dagger\dagger} \text{ for some } y \succ 1 \text{ in } L, \\ \boldsymbol{\Omega}(a) &\iff 4y'' + ay = 0 \text{ for some } y \in L^{\times}. \end{aligned}$$

The sets $\mathrm{I}(L), \boldsymbol{\Lambda}(L), \boldsymbol{\Omega}(L) \subseteq L$ are convex; their role with respect to QE is like that of the set of squares in a real closed field. For more on this, see [2, Introduction]. A $\boldsymbol{\Lambda\Omega}$*-field* is a substructure $\boldsymbol{K} = (K, I, \Lambda, \Omega)$ of such an expanded model $(L, \dots)$ of T^{nl} for which K is an H-subfield of L. This notion of a $\boldsymbol{\Lambda\Omega}$-field is studied in detail in [2, Section 16.3], from which we take in particular the fact that any $\boldsymbol{\omega}$-free H-field K has a unique expansion to a $\boldsymbol{\Lambda\Omega}$-field $\boldsymbol{K} = (K, I, \Lambda, \Omega)$. The proof below assumes familiarity with several other results from [2, Section 16.3].

Proof of Theorem 3. Let $\mathbf{No}_{\Lambda\Omega}$ be the expansion of $\mathbf{No}$ to a $\boldsymbol{\Lambda\Omega}$-field, and let K be any H-field with small derivation and constant field $\mathbb{R}$. In order to embed K over $\mathbb{R}$ into $\mathbf{No}$, we first expand K to a $\boldsymbol{\Lambda\Omega}$-field $\boldsymbol{K} = (K, I, \Lambda, \Omega)$ with $1 \notin I$; this can be done in at least one way, and at most two ways, and $1 \notin I$ guarantees that all $\boldsymbol{\Lambda\Omega}$-field extensions of $\boldsymbol{K}$ have small derivation. We claim that $\boldsymbol{K}$ can be embedded into $\mathbf{No}_{\Lambda\Omega}$. The ordered field $\mathbb{R}$ with the trivial derivation is an H-field and expands to the $\boldsymbol{\Lambda\Omega}$-field $\boldsymbol{R} := \big(\mathbb{R}, \{0\}, (-\infty, 0], (-\infty, 0]\big)$. The inclusion of $\mathbb{R}$ into K and into $\mathbf{No}$ are embeddings of $\boldsymbol{R}$ into $\boldsymbol{K}$ and $\mathbf{No}_{\Lambda\Omega}$, respectively. By taking $\boldsymbol{E} := \boldsymbol{R}$, our claim reduces therefore to proving the following more general statement:

Claim. Let $\boldsymbol{E} \subseteq \boldsymbol{K}$ be an extension of $\boldsymbol{\Lambda\Omega}$-fields with $\mathbb{R}$ as their common constant field, and let $i \colon \boldsymbol{E} \to \mathbf{No}_{\Lambda\Omega}$ be an embedding of $\boldsymbol{\Lambda\Omega}$-fields that is the identity on $\mathbb{R}$. Then i extends to an embedding $\boldsymbol{K} \to \mathbf{No}_{\Lambda\Omega}$ of $\boldsymbol{\Lambda\Omega}$-fields.

To prove this we first extend $\boldsymbol{K}$ to make it $\boldsymbol{\omega}$-free, newtonian, and Liouville closed; by [2, 16.4.1 and 14.5.10] this can be done without changing its constant field. Next we apply [2, 16.4.1] again, but this time to $\boldsymbol{E}$, to arrange that $\boldsymbol{E}$ is $\boldsymbol{\omega}$-free. Take a regular uncountable cardinal $\kappa > \operatorname{card}(K)$ such that $i(E) \subseteq \mathbf{No}(\kappa)$, where E is the underlying set of $\boldsymbol{E}$. By Corollary 4.6 we have $\mathbf{No}(\kappa) \prec \mathbf{No}$. In view of Lemma 5.3 and [2, 16.2.3] we can then extend i to an embedding $K \to \mathbf{No}(\kappa)$. $\square$

Final remarks. Suppose the H-field K has small derivation and constant field $\mathbb{R}$. Then Theorem 3 yields an embedding $i \colon K \to \mathbf{No}$ over $\mathbb{R}$. Under some reasonable further conditions, like K being $\boldsymbol{\omega}$-free and newtonian, can we take i such that $i(K)$ is truncation closed, or even initial? The interest of such a result would depend on how canonical the derivation ∂_{BM} is deemed to be. As already mentioned at the end of the introduction, we doubt that ∂_{BM} is optimal: the condition on pre-derivations to take values in $\mathbb{R}^{>}\mathfrak{M}$ seems too narrow. But even with this restriction one can construct pre-derivations $D \neq \partial_{\mathfrak{L}}$ such that Theorems 1 and 3 go through for $\mathbf{No}$ equipped with ∂_D instead of with ∂_{BM}, with only minor changes in the proofs.

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Department of Mathematics, University of California, Los Angeles, Los Angeles, CA 90095, U.S.A.
E-mail address: matthias@math.ucla.edu

Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, U.S.A.
E-mail address: vddries@math.uiuc.edu

École Polytechnique, 91128 Palaiseau Cedex, France
E-mail address: vdhoeven@lix.polytechnique.fr