Valued Differential Fields

Matthias Aschenbrenner

UCLA

I'll report on joint work with Lou van den Dries and Joris van der Hoeven.

About three years ago we achieved decisive results on the model theory of the valued differential field $\mathbb{T}$ of *transseries,* akin to what Tarski did for the field of reals in the 1940s.

Some of our work also applies to rather general

valued differential fields.

In this talk I will introduce the basic concepts, mention some examples, and then focus on the existence and uniqueness of *spherically complete immediate extensions* in this setting.

(Lou will talk about the notion of *differential-henselianity.*)

A **valued field** is a field K equipped with a valuation

$$v\colon K^{\times} \to \Gamma, \qquad \text{extended by } v(0) := \infty > \Gamma.$$

Here $\Gamma = v(K^{\times})$ is the **value group** (an ordered abelian group).

$\mathcal{O} := \{f : vf \geqslant 0\}$ (a local ring, the **valuation ring** of K)

$\mathcal{o} := \{f : vf > 0\}$ (maximal ideal of $\mathcal{O}$)

$\boldsymbol{k} := \mathcal{O}/\mathcal{o}$ (the **residue field** of K).

We can also encode v by its associated **dominance relation**:

$$f \preccurlyeq g \quad :\Longleftrightarrow \quad vf \geqslant vg \qquad \text{“}g \text{ dominates } f\text{”}.$$

Example (Hahn fields)

$$\boldsymbol{k}(\!(t^{\Gamma})\!) = \left\{ \begin{array}{l} \text{all formal sums } f = \sum_{\gamma\in\Gamma} f_{\gamma} t^{\gamma} \text{ with } f_{\gamma} \in \boldsymbol{k} \text{ such} \\ \text{that } \operatorname{supp} f := \{\gamma \in \Gamma : f_{\gamma} \neq 0\} \text{ is well-ordered.} \end{array} \right.$$

The valuation topology

Let K be a valued field. For $a \in K$, $\gamma \in \Gamma$,

$$B_a(\gamma) := \{ f \in K : v(a - f) > \gamma \}$$

(open ball with center a and radius γ)

$$\overline{B_a}(\gamma) := \{ f \in K : v(a - f) \geqslant \gamma \}$$

(closed ball with center a and radius γ)

The open balls form the basis for the **valuation topology** on K.

A **nest of balls** is a collection of balls in K any two of which meet. Each nest of balls in K is totally ordered by inclusion.

One says that K is **spherically complete** if every nonempty nest of closed balls in K has a point in its intersection.

Archetypical example: $\boldsymbol{k}((t^\Gamma))$ is spherically complete.

Extensions of valued fields

Let K, L be valued fields. Then L is an **extension** of K if L contains K as a subfield and for $f, g \in K$:

$$f \preccurlyeq g \qquad \Longleftrightarrow \qquad f \preccurlyeq_L g.$$

In this case we can identify

- Γ with an ordered subgroup of Γ_L, and
- $\boldsymbol{k} = \mathcal{O}/\scriptstyle{\mathcal{O}}$ with a subfield of $\boldsymbol{k}_L = \mathcal{O}_L/\scriptstyle{\mathcal{O}}_L$.

Such an extension L of K is **immediate** if $\Gamma = \Gamma_L$ and $\boldsymbol{k} = \boldsymbol{k}_L$.

Example

The valued field $\boldsymbol{k}(\!(t^\Gamma)\!)$ is an immediate extension of its valued subfield $\boldsymbol{k}(t^\Gamma)$.

Fact

spherically complete $\Longleftrightarrow$ no proper immediate extensions.

Classical results on valued fields

Theorem (Krull, 1932)

Each valued field has a spherically complete immediate extension.

Theorem (Kaplansky, 1942)

Let K be a valued field with residue field of characteristic zero. Then any two spherically complete immediate extensions of K are isomorphic over K.

Thus any valued field with value group Γ and residue field $\boldsymbol{k}$ of characteristic zero embeds into the Hahn field $\boldsymbol{k}((t^\Gamma))$.

Let K be a differential field (always of characteristic 0), with derivation ∂. As usual

$$f' = \partial(f),\ f'' = \partial^2(f),\ \ldots,\ f^{(n)} = \partial^n(f),\ \ldots$$

The **constant field** of K is $C = C_K = \{f \in K : f' = 0\}$.

The ring of **differential polynomials** in the indeterminate Y with coefficients in K is denoted by $K\{Y\}$.

For $\phi \neq 0$, we denote by K^ϕ the **compositional conjugate** of K by ϕ: the field K equipped with the derivation $\phi^{-1}\partial$.

For $P \in K\{Y\}$ there is $P^\phi \in K^\phi\{Y\}$ with $P(y) = P^\phi(y)$ for all y:

$$Y^\phi = Y, \qquad (Y')^\phi = \phi Y', \qquad (Y'')^\phi = \phi^2 Y'' + \phi' Y', \qquad \ldots$$

(This will play an important role later.)

Small derivation and continuity

Let now K be a differential field equipped with a valuation.

We are interested in the case where the derivation ∂ of K is *continuous* (for the valuation topology).

This will be the case if the derivation is **small** in the sense that $\partial\smallo \subseteq \smallo$. Partial converse: if ∂ is continuous, then some multiple $\phi^{-1}\partial$ with $\phi \in K^\times$ is small.

If ∂ is small then $\partial\mathcal{O} \subseteq \mathcal{O}$, and so ∂ induces a derivation $\partial_{\boldsymbol{k}}$ on $\boldsymbol{k}$.

If ∂ is small and $\partial_{\boldsymbol{k}} \neq 0$ then for $\phi \in \smallo$ the derivation $\phi^{-1}\partial$ is no longer small. We therefore collect the "good" multipliers:

$$\Gamma(\partial) := \left\{ v\phi : \phi \neq 0, \phi^{-1}\partial \text{ is small} \right\}$$

(a nonempty downward closed subset of Γ).

Also important:

$$S(\partial) := \left\{ \gamma \in \Gamma : \gamma + \Gamma(\partial) = \Gamma(\partial) \right\} \quad \text{(a convex subgroup of } \Gamma\text{)}.$$

Valued differential fields

Definition (for this talk)

A **valued differential field** is a differential field equipped with a valuation whose residue field has characteristic zero and whose derivation is continuous.

Examples (with small derivation)

1. $\boldsymbol{k}(t)$ and $\boldsymbol{k}(\!(t)\!)$ with the t-adic valuation and $\partial = t\frac{d}{dt}$, with $\boldsymbol{k}$ any field of characteristic zero;
2. Hardy fields, with $\mathcal{O} = \{\text{germs of bounded functions}\}$;
3. $\mathbb{T}$, the differential field of transseries;
4. $\boldsymbol{k}(\!(t^\Gamma)\!)$, where $\boldsymbol{k}$ is any differential field, Γ any ordered abelian group, and $\partial\left(\sum_\gamma f_\gamma t^\gamma\right) = \sum_\gamma f'_\gamma t^\gamma$.

Spherically complete valued differential fields

Let K be a valued differential field. By an **extension** of K we mean a valued differential field extension of K. An extension L of K is **strict** if for all $\phi \in K^\times$,

$$\partial \smallo \subseteq \phi \smallo \quad \Rightarrow \quad \partial_L \smallo_L \subseteq \phi \smallo_L, \qquad \partial \mathcal{O} \subseteq \phi \smallo \quad \Rightarrow \quad \partial_L \mathcal{O}_L \subseteq \phi \smallo_L.$$

Theorem (differential analogue of KRULL's theorem)

The valued differential field K has an immediate strict extension that is spherically complete.

For proving this theorem we can assume that $\Gamma \neq \{0\}$ and that $\partial \neq 0$ is small, whenever convenient.

In our book on $\mathbb{T}$ we proved this theorem when ∂ is small and the induced derivation $\partial_{\boldsymbol{k}} \neq 0$.

Spherically complete valued differential fields

This case does not cover the valued differential field $\mathbb{T}$, which has small derivation with $\boldsymbol{k} \cong \mathbb{R} = C_{\mathbb{T}}$.

In the proof of the theorem we first work under the following assumptions:

Γ *has no smallest positive element and* $S(\partial) = \{0\}$*.*

Key technique: given $P \in K\{Y\}$, the set of monomials in $P^{\phi} \in K^{\phi}\{Y\}$ with coefficient of minimal valuation stabilizes as $v\phi$ increases in $\Gamma(\partial)$, allowing us to identify the "eventual dominant degree" (*Newton degree*) of P.

Under the assumptions above this turns into a useful tool for constructing zeros of P in immediate extensions of K.

Spherically complete valued differential fields

To prove the theorem in general, we use coarsening and specialization by $\Delta = S(\partial)$ to reduce to these special cases:

Definition

Let Δ be a convex subgroup of Γ, with ordered quotient group $\dot{\Gamma} := \Gamma/\Delta$. Then K with its valuation replaced by

$$K^\times \xrightarrow{v} \Gamma \xrightarrow{\gamma \mapsto \gamma+\Delta} \dot{\Gamma}$$

is a valued differential field, called the **coarsening** of K by Δ. Its residue field $\dot{K}$ also carries a valuation

$$\dot{K}^\times \to \Delta$$

and a derivation making it a valued differential field, called a **specialization** of K.

Uniqueness of spherically complete extensions

Say that K **has the uniqueness property** if it has up to isomorphism over K a unique spherically complete immediate strict extension (the analogue of KAPLANSKY's theorem holds).

Examples

- $\Gamma = \{0\}$ or $\partial = 0$
- $\Gamma \cong \mathbb{Z}$ (since then "spherically complete" = "complete").

We also have an example of a valued differential field K with spherically complete immediate strict extensions which are non-isomorphic over K.

This has to do with a certain equation $y' + ay = 1$ ($a \in K$) having no solution in any immediate strict extension of K.

Uniqueness of spherically complete extensions

We say that a differential field F is **linearly surjective** if for all $a_1, \ldots, a_n, b \in F$ there is some $y \in F$ such that

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y = b.$$

If K is spherically complete with small derivation, then

$$\boldsymbol{k} \text{ linearly surjective} \quad \Longrightarrow \quad K \text{ linearly surjective.}$$

Conjecture

If K has small derivation and $\boldsymbol{k}$ is linearly surjective then K has the uniqueness property.

This has been shown in various cases ($\rightarrow$ LOU's talk).

M. A., L. van den Dries, J. van der Hoeven, *Asymptotic Differential Algebra and Model Theory of Transseries,* Ann. of Math. Stud., vol. 195, Princeton University Press, 2017.

______, *Maximal immediate extensions of valued differential fields,* `arXiv:1701.06691`