WHITNEY'S EXTENSION PROBLEM IN O-MINIMAL STRUCTURES

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Abstract. In 1934, H. Whitney asked how one can determine whether a real-valued function on a closed subset of $\mathbb{R}^n$ is the restriction of a C^m-function on $\mathbb{R}^n$. A complete answer to this question was found much later by C. Fefferman in the early 2000s. Here, we work in an o-minimal expansion of a real closed field and solve the C^1-case of Whitney's Extension Problem in this context. Our main tool is a definable version of Michael's Selection Theorem, and we include other another applications of this theorem, to solving linear equations in the ring of definable continuous functions.

Introduction

The long history of Whitney's Extension Problem began in 1934, when H. Whitney presented a series of papers [46, 47, 48]. In the first paper, Whitney's Extension Theorem, which can be regarded as a partial converse of Taylor's Theorem, was proved (see Section 1 below); it later became an important tool in differential topology (see [31]). In the the latter two papers, Whitney answered special cases of the following question:

Question (Whitney's Extension Problem; $\mathrm{WEP}_{n,m}$). Let $f\colon X \to \mathbb{R}$ be a continuous function, where X is a closed subset of $\mathbb{R}^n$. How can we determine whether f is the restriction of a C^m-function on $\mathbb{R}^n$?

An answer to this question in the case $n = 1$ was given in [47], and, judging from the title, Whitney planned to solve the general case also; however, the continuation of this paper never appeared. In 1958 G. Glaeser [20] introduced the notion of an "iterated paratangent bundle" and used it to give an answer to the above question when n is arbitrary and $m = 1$. The concept of paratangent bundles had significant influence on later work in this area. After gradual progress on Whitney's original question by Fefferman, Brudnyi, Shvartsman, Zobin, and others (see, e.g., [8, 9, 10, 40, 41, 42, 51, 52]), in 2004 C. Fefferman [18] gave a complete answer to Whitney's Extension Problem, i.e., provided a necessary and sufficient condition for the existence of a C^m-extension of functions defined on closed subsets of $\mathbb{R}^n$.

In 1997, K. Kurdyka and W. Pawłucki [28] showed a subanalytic version of Whitney's Extension Theorem. Later Pawłucki together with E. Bierstone and P. Milman [4] introduced an analogue of iterated paratangent bundles (which became an inspiration for Fefferman's proof of $\mathrm{WEP}_{n,m}$; see [17]) and showed that if $f\colon X \to \mathbb{R}$ is a subanalytic function on a compact subanalytic subset X of $\mathbb{R}^n$ which is the restriction to X of a C^m-function $\mathbb{R}^n \to \mathbb{R}$, then there is a constant $r = r(X, m, n) \in \{0, \dots, m\}$ (depending only on X, m, and n) and a subanalytic C^{m-r}-extension $\mathbb{R}^n \to \mathbb{R}$ of f. Therefore, this raises the interesting question whether we can find an extension which preserves both subanalyticity and the order of differentiability.

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The category of subanalytic sets possesses many good topological and geometric properties, most of which are shared by the category of semialgebraic sets. (See [5, 6].) In model theory, a source of these good properties has been isolated, and is known as *o-minimality*: an *o-minimal expansion* $\boldsymbol{R}$ *of the ordered field of real numbers* is defined to be a class of subsets of $\mathbb{R}^n$ (for varying n) which

(1) is closed under (finite) intersections and unions, complements, finite cartesian products, and linear projections;
(2) contains all algebraic subsets of $\mathbb{R}^n$; and
(3) contains only those subsets of $\mathbb{R}$ which have finitely many connected components (the *o-minimality* axiom).

The archetypical example of such an o-minimal expansion of the ordered field of real numbers is the class of semialgebraic sets (i.e., sets defined by finite boolean combinations of polynomial inequalities); another example is the class of *finitely subanalytic* sets, i.e., the subsets of $\mathbb{R}^n$ which are subanalytic when viewed as subsets of real projective n-space [15]. In recent years, many new examples of o-minimal structures have been constructed, often by sophisticated uses of elimination theory and desingularization (see, e.g., [25, 38, 43, 49]).

Let $\boldsymbol{R}$ be an o-minimal expansion of the ordered field of real numbers. Following the usual terminology of logicians, a set $S \subseteq \mathbb{R}^n$ which belongs to $\boldsymbol{R}$ is said to be *definable* (in $\boldsymbol{R}$). A map $f\colon S \to \mathbb{R}^n$, where $S \subseteq \mathbb{R}^m$, is said to be definable (in $\boldsymbol{R}$) if its graph $\Gamma(f) \subseteq \mathbb{R}^{m+n}$ is. It is often routine to verify that topological and geometric constructions of a "finitary" nature preserve definability; for example, if $X \subseteq \mathbb{R}^m$ is definable, then so are the closure and the interior of X. This leads to the development of a "tame topology" [14] as envisaged by Grothendieck's *esquisse d'un programme* [21]. This is witnessed by the remarkable fact that although the o-minimality axiom only refers to subsets of the line, the classical finiteness theorems for semialgebraic and subanalytic sets and maps (cell decompositions, Whitney stratifications, triangulation, trivialization, etc.) continue to hold for definable objects in $\boldsymbol{R}$ living in higher-dimensional euclidean spaces; see, e.g., [16].

In o-minimal expansions of the ordered field $\mathbb{R}$, definable versions of Whitney's Extension Theorem and $\mathrm{WEP}_{n,m}$ can be considered. In Section 2 below we do show that $\mathrm{WEP}_{1,m}$ has a simple solution in the case of functions definable in $\boldsymbol{R}$. In [45], the second-named author proved a definable version of Whitney's Extension Theorem in $\boldsymbol{R}$; see Section 1 below. (An alternative proof was given by Kurdyka and Pawłucki [29].) In the present paper we use this result to treat the C^1-case $\mathrm{WEP}_{n,1}$ of the Whitney Extension Problem for definable functions; our main result is the following theorem. Given $A \subseteq \mathbb{R}^N$, a family $(f_a)_{a\in A}$ of functions $f_a\colon X_a \to \mathbb{R}$ ($X_a \subseteq \mathbb{R}^n$) is said to be definable if the map $(a,x) \mapsto f_a(x)\colon X \to \mathbb{R}$ is definable, where $X = \big\{(a,x) \in \mathbb{R}^N \times \mathbb{R}^n : x \in X_a\big\}$.

Theorem. *Let $\boldsymbol{R}$ be an o-minimal expansion of the ordered field of real numbers, and let $(f_a)_{a\in\mathbb{R}^N}$ be a definable family of functions $f_a\colon X_a \to \mathbb{R}$, where $X_a \subseteq \mathbb{R}^n$ is closed. Then the set A consisting of all $a \in \mathbb{R}^N$ such that f_a has an extension to a C^1-function $\mathbb{R}^n \to \mathbb{R}$ is definable. Moreover, if $a \in A$, then f_a extends to a C^1-function $\mathbb{R}^n \to \mathbb{R}$ which is definable in $\boldsymbol{R}$; in fact, there exists a definable family $(\widetilde{f}_a)_{a\in A}$ of C^1-functions on $\mathbb{R}^n$ such that $\widetilde{f}_a \restriction X_a = f_a$ for each $a \in A$.*

Thus, for example, if $f\colon X \to \mathbb{R}$ is semialgebraic, where $X \subseteq \mathbb{R}^n$ is closed, and f extends to a C^1-function F on $\mathbb{R}^n$, then F can be taken to be semialgebraic. This can be seen as providing an answer to the C^1-case of a question posed by Bierstone and Milman (see [50]). Our proof follows the argument for $\mathrm{WEP}_{n,1}$ given by Klartag and Zobin [26], which in turn rests on a use of Michael's Selection Theorem from general topology. Therefore, a study of properties of definable set-valued maps and a definable version of this selection theorem occupy most of this paper. (Sections 3 and 4.) In a companion paper [2] we investigate the Michael Selection Theorem for the class of semilinear sets and maps.

It may be worthwhile to explain why we couldn't simply mimic Glaeser's original argument for the C^1-case of Whitney's Extension Problem in order to prove the theorem above. First, Glaeser [20, Proposition IV of Section 5, p. 43] defines his "linearized paratangent space ptgl^1 of order 1" in a completely abstract way, simply taking the intersection of all set-valued mappings satisfying certain properties. But there is no way to guarantee that ptgl^1, so introduced, is definable (since only intersections of *definable* families of sets result in definable objects, and not an abstract set-theoretic intersection of this kind). Now, using ptgl^1, Glaeser then goes on to give a criterion for the existence of a C^1-extension [20, p. 44]. This criterion and the proof that it works are indeed constructive (modulo an argument in the proof of the "Lemme" on p. 44 employing sequences, which can probably be replaced by appeals to the Curve Selection Lemma of o-minimality [14, Chapter 6, §1]). In the following Section 6, he then shows how to obtain ptgl^1 in a constructive manner. However, justifying this construction involves, among other things, a theorem of Baire on semicontinuous functions on Baire spaces, for which Glaeser refers to Bourbaki's book on General Topology (p. 47). He then goes on, in the proof, to make other non-constructive twists (extracting a subsequence from a certain convergent sequence), and another application of Baire Category appears on p. 49. We could not see how to make Glaesers arguments constructive in the way necessary for a truly "o-minimal proof" leading to our main result.

Let us also briefly discuss why we believe that such a proof is desirable. One says that $\boldsymbol{R}$ is *polynomially bounded* if for each definable function $f\colon \mathbb{R} \to \mathbb{R}$ there is an integer $N \geq 0$ such that $|f(x)| \leq x^N$ for all sufficiently large x. In some ways, functions definable in polynomially bounded o-minimal expansions of the ordered field of reals resemble real analytic functions [34]. This may suggest the potential adaptability of classical techniques for Whitney Extension Problems to the o-minimal context. However, such tools are not available in absence of polynomial boundedness, yet there are plenty of examples of o-minimal expansions of the real field which are not polynomially bounded: by Wilkie [49] there is an o-minimal expansion of the ordered field of reals which contains the real exponential function $x \mapsto e^x\colon \mathbb{R} \to \mathbb{R}$, and indeed, every o-minimal expansion $\boldsymbol{R}$ of the ordered field of reals is contained in one in which the exponential function is definable [43]. (As an aside, we note here a remarkable dichotomy discovered by Miller [33]: if $\boldsymbol{R}$ is not polynomially bounded, then this is so because the exponential function is definable in $\boldsymbol{R}$.)

The virtue of the argument of Klartag and Zobin is that it neatly isolates the non-constructive input in the form of the Michael Selection Theorem; hence proving a definable version of this theorem, which we believe to be of independent interest, became the centerpiece of our paper. To illustrate the usefulness of this definable version of Michael Selection, we include another application:

Corollary. *With $\boldsymbol{R}$ as above, let $f, g_1, \dots, g_m\colon \mathbb{R}^n \to \mathbb{R}$ be definable functions. If there are continuous functions $y_1, \dots, y_m\colon \mathbb{R}^n \to \mathbb{R}$ such that*

$$f = g_1 y_1 + \cdots + g_m y_m, \tag{$*$}$$

then there are also definable *continuous functions $y_1, \dots, y_m$ with this property.*

See Section 6 below for a more precise statement. For the case where $f, g_1, \dots, g_m$ are given by real polynomials and $\boldsymbol{R}$ is the ordered field of real numbers (hence "definable = semialgebraic"), this was shown by Fefferman and Kollár [19, Corollary 29, (1)] using algebraic-geometric techniques specific to polynomials. (Kollár and Nowak [27] showed that in this situation, in general one cannot choose $y_1, \dots, y_m$ to be continuous *rational* functions.) Our approach follows the method to construct continuous solutions y_i to $(*)$ from [19, Section 1] using affine bundles, Glaeser refinements, and Michael's Theorem.

In the rest of this paper, we more generally work in an o-minimal expansion $\boldsymbol{R}$ of a real closed ordered field R (not necessarily the reals). This allows for applications of model-theoretic compactness; see, e.g., Section 6 below or [45, Section 6]. We assume that readers have a working knowledge of o-minimality. (See [14] or [16] for the necessary background.) "Definable" always means "definable in $\boldsymbol{R}$, possibly with parameters."

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Conventions and notations. Throughout this paper, d, k, l, m, and n will range over the set $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ of natural numbers. For a set $S \subseteq R^n$ we denote by $\operatorname{cl} S = \operatorname{cl}(S)$ the closure, by $\partial S = \partial(S) := \operatorname{cl}(S) \setminus S$ the frontier, and by $\operatorname{int} S = \operatorname{int}(S)$ the interior of S. We denote the euclidean norm on R^n by $\|\cdot\|$ and the associated metric by $(x, y) \mapsto d(x, y) := \|x - y\|$. For $r \in R^{>0}$ and $x \in R^n$ we let

$$B_r(x) := \big\{y \in R^n : d(x, y) < r\big\}$$

be the open ball of radius r around x and

$$\overline{B}_r(x) := \big\{y \in R^n : d(x, y) \leq r\big\}$$

be the closed ball of radius r around x. Given $x \in R^n$, for a non-empty definable set $S \subseteq R^n$ let $d(x, S) := \inf_{y \in S} d(x, y) \in R^{\geq 0}$ be the distance between x and S, and $d(x, \emptyset) := +\infty$. For $E \subseteq R^n \times R^m$ and $x \in R^n$, let

$$E_x = \big\{y \in R^m : (x, y) \in E\big\}.$$

1. Definable Whitney Extension Theorem

In this section, a definable version of Whitney's Extension Theorem and related terminology needed will be introduced (see [45]). We let X be a definable subset of R^n, and fix some m. We say that $f\colon X \to R$ is C^m if there exists a definable C^m-function $F\colon U \to R$ on an open neighborhood U of X with $F \restriction X = f$. We let $\alpha = (\alpha_1, \dots, \alpha_n)$ range over $\mathbb{N}^n$, and let $D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$ and $|\alpha| := \alpha_1 + \cdots + \alpha_n$.

Definition 1.1. A **jet of order** m on X is a family $F = (F^\alpha)_{|\alpha| \leq m}$ of definable continuous functions $F^\alpha\colon X \to R$. If $f\colon U \to R$ is a definable C^m-function on an open neighborhood U of X, then we obtain a jet $J^m_X(f) := \big((D^\alpha f) \restriction X\big)_{|\alpha| \leq m}$ of order m on X. If F is a jet of order m on X and $X' \subseteq X$ is definable, then $F \restriction X' := (F^\alpha \restriction X')_{|\alpha| \leq m}$ is a jet of order m on X'. Let F be a jet of order m on X. For every $a \in R^n$, $x \in X$, we define

$$T^m_a F(x) = \sum_{|\alpha| \leq m} F^\alpha(a) \frac{(x - a)^\alpha}{\alpha!}$$

and

$$R_a^m F(x) = F - J_X^m(T_a^m F(x)).$$

We say that F is a **definable C^m-Whitney field** ($F \in \mathscr{E}^m(X)$) if for all $x_0 \in X$ and $|\alpha| \leq m$,

$$(R_x^m F)^\alpha(x') = o(\|x - x'\|^{m-|\alpha|}) \quad \text{as } X \ni x, x' \to x_0, \tag{1}$$

equivalently, if

$$|T_x^m F(z) - T_{x'}^m F(z)| = o(\|x - z\|^m + \|x' - z\|^m) \quad \text{for } z \in R^n \text{ and } X \ni x, x' \to x_0.$$

(See [30, 35].) Note that if $F \in \mathscr{E}^m(X)$ and $X' \subseteq X$ is definable, then $F \restriction X' \in \mathscr{E}^m(X')$. Given a jet F of order m on X, we say that a C^m-function $f\colon R^n \to R$ is an **extension** of F if $J_X^m(f) = F$.

The following is shown in [45]. (See also [29].)

Theorem 1.2 (Definable Whitney Extension Theorem)**.** *Suppose X is closed, and let $F \in \mathscr{E}^m(X)$ and $q \in \mathbb{N}$. Then F has a definable C^m-extension which is C^q on $R^n \setminus X$.*

The classical Whitney Extension Theorem is the same statement in the case where $R = \mathbb{R}$, of course without the definability requirements in both the hypothesis and conclusion; here the C^m-extension of F can even be chosen to be analytic on $\mathbb{R}^n \setminus X$, by [47].

2. The One-dimensional Case

In his paper [47], H. Whitney introduced the concept of difference quotients and used it to answer $\mathrm{WEP}_{1,m}$. Even though this concept is very natural, it is quite complicated to verify the resulting conditions in practice. In this section, we show that if we work in an o-minimal context, the answer to the definable $\mathrm{WEP}_{1,m}$ becomes a lot simpler. We first establish an estimate related to Taylor's formula. For $x, y \in R$ we denote by

$$[x, y] := \{rx + (1-r)y : r \in [0,1]\}$$

the line segment between x and y.

Lemma 2.1. *Let $f\colon (a,b) \to R$ be a definable C^m-function where $a, b \in R$, $a < b$. Assume, for $l \leq m$, that the lth derivative $f^{(l)}$ of f has an extension to a continuous function $f_l\colon [a,b] \to R$, and consider the m-jet $F = (f_l)_{l \leq m}$ on $[a,b]$. Let $x, y \in [a,b]$ and $\epsilon > 0$, and suppose $|f_m(s) - f_m(t)| \leq \epsilon$ for all $s, t \in [x,y]$. Then*

$$\left|(R_x^m F)^{(l)}(y)\right| \leq \epsilon \frac{|x-y|^{m-l}}{(m-l)!} \qquad \textit{for all } l \leq m.$$

Proof. Let $l \leq m$. Suppose $x < y$, and for $0 < \epsilon_0 < y - x$ let $x_{\epsilon_0} := x + \frac{1}{2}\epsilon_0$ and $y_{\epsilon_0} := y - \frac{1}{2}\epsilon_0$. By Taylor's Theorem (see [14, Chapter 7, Exercise (2.12)]), there exists $z_{\epsilon_0} \in (x_{\epsilon_0}, y_{\epsilon_0})$ such that

$$f^{(l)}(y_{\epsilon_0}) = f^{(m)}(z_{\epsilon_0}) \frac{(y_{\epsilon_0} - x_{\epsilon_0})^{m-l}}{(m-l)!} + \sum_{k \leq m-l-1} f^{(l+k)}(x_{\epsilon_0}) \frac{(y_{\epsilon_0} - x_{\epsilon_0})^k}{k!}.$$

By definable Skolem functions, we may assume that $\epsilon_0 \mapsto z_{\epsilon_0}\colon (0, y-x) \to [x,y]$ is definable, and so by the Monotonicity Theorem, $z := \lim_{\epsilon_0 \to 0^+} z_{\epsilon_0} \in [x,y]$ exists. By continuity

$$f_l(y) = f_m(z) \frac{(y-x)^{m-l}}{(m-l)!} + \sum_{k \leq m-l-1} f_{l+k}(x) \frac{(y-x)^k}{k!}.$$

Therefore

$$\begin{aligned}
\left|(R_x^m F)^{(l)}(y)\right| &= \left| f_l(y) - \sum_{k\leq m-l} f_{l+k}(x)\frac{(y-x)^k}{k!}\right| \\
&= \left| (f_m(z) - f_m(x))\frac{(y-x)^{m-l}}{(m-l)!}\right| \\
&\leq |f_m(z) - f_m(x)| \cdot \frac{|x-y|^{m-l}}{(m-l)!} \\
&\leq \epsilon\frac{|x-y|^{m-l}}{(m-l)!}
\end{aligned}$$

as claimed. Similarly one deals with the case $y < x$. $\square$

Theorem 2.2. *Let $f\colon X \to R$ be definable and continuous where $X \subseteq R$ is closed. Suppose $f \restriction \operatorname{int} X$ is C^m and $f^{(l)} \restriction \big([-r,r] \cap \operatorname{int} X\big)$ is uniformly continuous for all $l \leq m$ and $r \in R^{>0}$. Then f is the restriction of a definable C^m-function $R \to R$.*

Proof. If $\dim X = 0$, then this is obvious (since then X is finite). Suppose $\dim X = 1$. By the Definable Whitney's Extension Theorem (Theorem 1.2), it is enough to find a C^m-Whitney field $F = (f_l)_{l\leq m}$ with $f_0 = f$ on X. It is enough to construct F on each definably connected component of X, and hence we may assume that X is definably connected. (For isolated points $x \in X$, we can simply let $f_l(x) = 0$ for all $l = 1,\dots,m$.) We also assume $X = [a,b]$ where $a,b \in R$, $a < b$. (The case where X is unbounded is similar.) Let $f_l\colon [a,b] \to R$ be the continuous extension of the lth derivative $f^{(l)}$ of $f \restriction (a,b)$. Define $F := (f_l)_{l\leq m}$. To prove that $F \in \mathscr{E}^m([a,b])$, by Taylor's Theorem, it is enough to only check (1) in Definition 1.1 for $x_0 \in \{a,b\}$. Let $\epsilon > 0$ be given. Since f_m is continuous on $[a,b]$, there exists $\delta \in (0, b-a)$ such that $|f_m(s) - f_m(t)| \leq \epsilon$ for all $s,t \in [a, a+\delta]$. By Lemma 2.1, for $x,y \in [a,a+\delta]$ and $l \leq m$, we have

$$\frac{\left|(R_x^m F)^{(l)}(y)\right|}{|x-y|^{m-l}} \leq \frac{\epsilon}{(m-l)!}.$$

Hence (1) holds for $x_0 = a$; similarly, (1) also holds for $x_0 = b$. So $F \in \mathscr{E}^m([a,b])$. $\square$

3. Continuity of Definable Set-valued Maps

The discussion of more advanced topics about $\mathrm{WEP}_{n,m}$ requires an investigation of set-valued maps; therefore, we devote this section to topological properties of definable set-valued maps. (See [3, 24] for classical studies.)

Notation. Let X, Y be sets. We use the notation $T\colon X \rightrightarrows Y$ to denote a map $T\colon X \to 2^Y$, and call such T a **set-valued map.** Let $T\colon X \rightrightarrows Y$ be a set-valued map. The **domain** of T is the set of $x \in X$ with $T(x) \neq \emptyset$. The **graph** of T is the subset

$$\Gamma(T) := \big\{(x,y) \in X \times Y : y \in T(x)\big\}$$

of $X \times Y$. Note that every map $f\colon X \to Y$ gives rise to a set-valued map $X \rightrightarrows Y$, whose graph is the graph of the map f.

Let $\mathcal{T} = (T_x)_{x\in X}$ be a family of subsets of R^m, where $X \subseteq R^n$. As usual we say that $\mathcal{T}$ is definable if the set $\bigcup_{x\in X} \{x\} \times T_x$ is definable. Then $\mathcal{T}$ gives rise to a set-valued map $T\colon X \rightrightarrows R^m$ given by $T(x) := T_x$ for $x \in X$. A set-valued map $X \rightrightarrows R^m$ ($X \subseteq R^n$) which arises in this way from a definable family of subsets of R^m is said to be **definable.**

In the rest of this section, we fix a definable set-valued map $T\colon X \rightrightarrows R^m$ ($X \subseteq R^n$).

Definition 3.1. We say that

(1) T is **lower semicontinuous (l.s.c.)** if, for every $x \in X$, $y \in T(x)$, and neighborhood V of y, there is a neighborhood U of x such that $T(x') \cap V \neq \emptyset$ for all $x' \in U \cap X$;
(2) T is **upper semicontinuous (u.s.c.)** or **closed** if $\Gamma(T)$ is closed in $X \times R^m$, that is: for every $x \in X$, $y \in R^m \setminus T(x)$, there are neighborhoods U of x and V of y such that $T(x') \cap V = \emptyset$ for all $x' \in U \cap X$;
(3) T is **continuous** if T is both l.s.c. and u.s.c.

What we call lower semicontinuous (upper semicontinuous) is sometimes called "inner semicontinuous" (respectively "outer semicontinuous"), for example in [13, 37]. In [3], "upper semicontinuous" is reserved for a slightly more restrictive concept (which can be shown to agree with ours if $\bigcup_{x\in X} T(x)$ is bounded; cf. [3, Proposition 1.4.8]); we prefer the terminology "closed."

Remarks.

(1) A definable map $X \to R^m$ is continuous in the usual sense iff it is continuous in the sense of the previous definition, when viewed as a set-valued map $X \rightrightarrows R^m$.
(2) If T is closed, then $T(x)$ is closed for every $x \in X$ (but of course, the converse of this implication fails).
(3) If $f\colon X' \to X$ is a definable continuous map, where $X' \subseteq R^{n'}$, and T is l.s.c. (closed), then $T \circ f$ is l.s.c. (closed, respectively). In particular, if X' is a definable subset of X, and T is l.s.c. (closed), then $T \restriction X'$ is l.s.c. (closed, respectively).
(4) Suppose $X = X_1 \cup X_2$ where X_1, X_2 are definable subsets of X with $\mathrm{cl}(X_1) \cap X_2 = X_1 \cap \mathrm{cl}(X_2) = \emptyset$. If $T \restriction X_1$ and $T \restriction X_2$ are l.s.c. (u.s.c., respectively), then so is T.

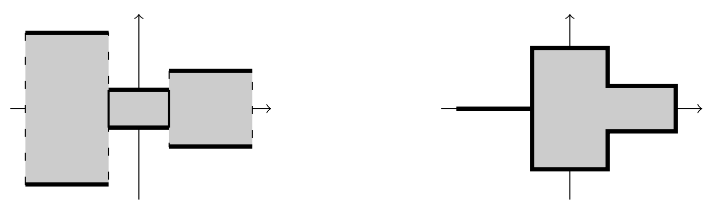

FIGURE 1. A lower semicontinuous set-valued map (left); a closed set-valued map (right).

One powerful consequence of the o-minimality axiom is the Cell Decomposition Theorem, which implies that every definable continuous map is piecewise continuous. The main goal in this section is to show an analogue for set-valued maps. We let $\pi\colon X \times R^n \to X$ denote the natural projection onto X. First, we show that T is piecewise l.s.c.:

Lemma 3.2. *There is a finite partition $\mathscr{C}$ of X into definable sets such that $T \restriction C$ is l.s.c., for every $C \in \mathscr{C}$.*

Proof. We prove this lemma by induction on $d = \dim X$. If $d = 0$, then X is a finite set; so this case is trivial. Assume the lemma holds for all definable set-valued maps whose domain has dimension $< d$. By the Cell Decomposition Theorem, take a cell decomposition $\mathscr{D}$ of R^n compatible with X. The induction hypothesis applies to $T \restriction D$ for each $D \in \mathscr{D}$ with

$\dim D < d$; hence we may assume that X is a cell. Moreover, we may also assume that X is an open cell in R^d, since every cell of dimension d is definably homeomorphic to an open cell in R^d. Let

$$K = \Big\{(x,y) \in X \times R^n : \ y \in T(x) \text{ and } \\ (\exists \epsilon > 0)\,(\forall \delta > 0)\,(\exists x' \in B_\delta(x) \cap X)\,(\forall y' \in T(x'))\ \|y - y'\| > \epsilon\Big\}$$

be the set of witnesses of lower semi-discontinuity of T. Obviously, $T \restriction \big(X \setminus \pi(K)\big)$ is l.s.c. Thus it remains to show the following claim:

Claim. $\dim(\pi(K)) < d$.

Suppose not. Then $\pi(K)$ has non-empty interior. By definable Skolem functions and the Cell Decomposition Theorem, we may assume that there is a definable continuous map $f\colon U \to R^m$, where $U \subseteq R^d$ is open, such that $\Gamma(f) \subseteq K$. Let $\big(x, f(x)\big) \in K$. Then there exists $\epsilon > 0$ such that, for every $\delta > 0$, there is $x' \in B_\delta(x) \cap U$ with $\|f(x) - f(x')\| > \epsilon$. This contradicts the continuity of f at x. □

Next we show that if T has closed values, then T is also piecewise closed:

Lemma 3.3. *Suppose that $T(x)$ is closed, for every $x \in X$. There is a finite partition $\mathscr{C}$ of X into definable sets such that $T \restriction C$ is closed, for every $C \in \mathscr{C}$.*

Proof. Similarly to the proof of Lemma 3.2, we show this by induction on $d = \dim X$. The case $d = 0$ is obvious. Suppose the statement holds true for definable set-valued maps with closed values whose domain has dimension $< d$. The induction hypothesis, the Cell Decomposition Theorem, and a similar argument as in the beginning of the proof of Lemma 3.2, allow us to reduce to the case that X is an open cell. Let $S := \Gamma(T)$ and $K := \pi(\partial S)$. It is sufficient to show that $\dim K < d$. Assume $\dim K = d$. By definable Skolem functions and the Cell Decomposition Theorem, there exist a non-empty definable bounded open set $V \subseteq \operatorname{cl} V \subseteq K$ and a definable continuous function $f\colon \operatorname{cl} V \to R^m$ such that $\Gamma(f) \subseteq \partial S$. For each $x \in \operatorname{cl} V$ we have $d\big(f(x), T(x)\big) > 0$, because $T(x)$ is closed. After shrinking V suitably, we may assume that the function $x \mapsto d\big(f(x), T(x)\big)\colon \operatorname{cl} V \to R^{>0}$ is continuous. Since $\operatorname{cl} V$ is closed and bounded, there is $\Delta > 0$ such that

$$d\big(f(x), T(x)\big) > \Delta \qquad \text{for all } x \in \operatorname{cl} V.$$

Pick $x_0 \in V$ and $\delta_0 > 0$ such that $B_{\delta_0}(x_0) \subseteq V$. By continuity of f, take $0 < \delta < \delta_0$ such that

$$\|f(x_0) - f(x)\| < \frac{\Delta}{3} \qquad \text{for every } x \in B_{\delta_0}(x_0).$$

So, $\|f(x) - f(x')\| < \frac{2\Delta}{3}$ for all $x, x' \in B_\delta(x_0)$. Hence,

$$\begin{aligned} d\big((x, f(x)), \{x'\} \times T(x')\big) &\geq d\big(f(x), T(x')\big) \\ &\geq d\big(f(x'), T(x')\big) - \big\|\big(f(x), f(x')\big)\big\| \\ &> \Delta - \frac{2\Delta}{3} = \frac{\Delta}{3} \end{aligned}$$

for $x, x' \in B_\delta(x_0)$. Thus,

$$d\big(\Gamma(f \restriction B_\delta(x_0)), \Gamma(T \restriction B_\delta(x_0))\big) > \frac{\Delta}{3},$$

and so,

$$(x_0, f(x_0)) \in (B_\delta(x_0) \times R^m) \cap B_{\frac{\Delta}{3}}(x_0, f(x_0)) \subseteq (X \times R^m) \setminus S,$$

which contradicts $(x_0, f(x_0)) \in \partial S$. □

Lemmas 3.2 and 3.3 in combination with the Cell Decomposition Theorem immediately yield the following theorem:

Theorem 3.4. *Suppose $T(x)$ is closed, for every $x \in X$. Then there is a cell decomposition $\mathscr{C}$ of R^n compatible with X such that $T \restriction C$ is continuous, for every $C \in \mathscr{C}$.*

Remark. A version of Theorem 3.4 in the case where R is the ordered field of real numbers was shown in [13], with a longer proof. (See the "main result," Theorem 32, and its Corollary 33, in [13].)

4. Definable Michael's Selection Theorem

In this section we treat a definable version of the well-known Michael Selection Theorem [32] for set-valued maps. The classical version of this theorem plays a crucial role in the approach to solving $\mathrm{WEP}_{n,1}$ by Klartag and Zobin [26]. Classically, this theorem is shown by a non-constructive iterative procedure; see [3, Section 9.1] or [23, 24] for expositions. By definable Skolem functions, every definable set-valued map $T\colon X \rightrightarrows R^m$ with domain X has a definable selection, i.e., a definable map $f\colon X \to R^m$ such that $\Gamma(f) \subseteq \Gamma(T)$; here, we prove a strengthening of this fact under suitable additional hypotheses on T:

Theorem 4.1 (Definable Michael's Selection Theorem)**.** *Let X be a closed subset of R^n and $T\colon X \rightrightarrows R^m$ be a definable l.s.c. set-valued map such that $T(x)$ is non-empty, closed, and convex for every $x \in X$. Then T has a continuous definable selection.*

In the proof, we use:

Lemma 4.2. *Let $T\colon R \rightrightarrows R^m$ be a definable set-valued map with domain $(0,1)$. Let $(0,y) \in \mathrm{cl}\big(\Gamma(T)\big)$. Then there is a definable continuous $f\colon (0,\epsilon) \to R^m$, for some $\epsilon > 0$, such that $f(t) \in T(t)$ for all $t \in (0,\epsilon)$ and $\lim\limits_{t\to 0^+} f(t) = y$.*

Proof. By Definable Curve Selection [14, Chapter 6, §1], there is a definable continuous injective path $\gamma\colon (0,\epsilon_0) \to \Gamma(T)$, where $\epsilon_0 \in R^{>0}$, such that $\lim\limits_{s\to 0^+} \gamma(s) = (0,y)$. We may assume that γ^{-1} is also continuous. Let $P = \gamma\big((0,\epsilon_0)\big) \subseteq R \times R^m$; clearly, $\dim P = 1$. Let $\pi\colon R \times R^m \to R$ be the projection onto the first coordinate; then there are only finitely many $t \in \pi(P)$ with $\dim(P_t) = 1$. After making ϵ_0 smaller, we may assume that $\dim(P_t) = 0$ for every $t \in \pi(P)$. It is sufficient to show that there is an $\epsilon > 0$ such that if $0 < t < \epsilon$, then $|P_t| = 1$. Suppose not. By definable Skolem functions, there exist $\epsilon_1 > 0$ and definable continuous maps $g_1, g_2\colon (0,\epsilon_1] \to R^m$ such that $\Gamma(g_i) \subseteq P$ for $i = 1,2$ and $g_1(t) \neq g_2(t)$ for every $t \in (0,\epsilon_1]$. Since $\lim\limits_{s\to 0^+} \gamma(s) = (0,y)$, the functions g_1, g_2 are bounded; and thus $\lim\limits_{t\to 0^+} g_i(t)$ exist for $i = 1,2$. Since γ^{-1} is a continuous injective definable map and $\Gamma(g_1)$, $\Gamma(g_2)$ are definably connected, $I_1 := \gamma^{-1}\big(\Gamma(g_1)\big)$ and $I_2 := \gamma^{-1}\big(\Gamma(g_2)\big)$ are disjoint definably connected subsets of $(0,\epsilon_0)$. Pick an $i \in \{1,2\}$ such that $0 \notin \mathrm{cl}(I_i)$. Then $I_i = [a,\epsilon_0)$ where $0 < a < \epsilon_0$ or $I_i = (a,b]$ where $0 < a < b < \epsilon_0$. By continuity of γ^{-1}, in the first case we have $\gamma(a) = \left(0, \lim\limits_{t\to 0^+} g_i(t)\right)$, and in the second case $\gamma(b) = \left(0, \lim\limits_{t\to 0^+} g_i(t)\right)$. Obviously, $\left(0, \lim\limits_{t\to 0^+} g_i(t)\right) \notin \Gamma(T)$ (since the domain of T is $(0,1)$), but both $\gamma(a)$ and $\gamma(b)$ are in $\Gamma(T)$, a contradiction. □

Proof of Theorem 4.1. We proceed by induction on $d = \dim X$. If $d = 0$, then X is a finite set and the statement is obvious. Suppose the theorem holds for all set-valued maps satisfying the hypotheses, on a domain of dimension $< d$. Let $X' := \mathrm{cl}\{x \in X : T \text{ is not continuous at } x\}$. By Theorem 3.4, X' is a definable closed subset of X, $\dim X' < \dim X$ and $T \restriction (X \setminus X')$ is continuous. Therefore, by induction hypothesis, we can take a definable continuous selection $f\colon X' \to R^m$ of $T \restriction X'$. Since X' is closed, by the definable Tietze Extension Theorem (see, e.g., [1, Section 6.2]), we can further take a definable continuous map $g\colon R^n \to R^m$ such that $g \restriction X' = f$. Since $T(x)$ is closed and convex, for each $y' \in R^n$ there is a unique $y \in T(x)$ with $d(y', y) = d\big(y', T(x)\big)$. (See, e.g., [1, Lemma 2.12].) Define $F\colon X \to R^m$ by

$$F(x) = \text{the unique } y \in T(x) \text{ such that } d\big(g(x), y\big) = d\big(g(x), T(x)\big).$$

To finish the proof, it remains to show that F is continuous. Let $x_0 \in X$ and $\gamma\colon (0,1) \to X$ such that $\lim_{t\to 0^+} \gamma(t) = x_0$; we need to show that $\lim_{t\to 0^+} F\big(\gamma(t)\big) = F(x_0)$.

Claim. Let $\epsilon > 0$. Then

$$\|g(x_0) - F(\gamma(t))\| \leq \|g(x_0) - F(x_0)\| + \epsilon \qquad \text{as } t \to 0^+.$$

Proof of claim. Since T is l.s.c., by Lemma 4.2, after replacing γ by a suitable reparametrization of $\gamma \restriction (0, \epsilon_0)$, for some $\epsilon_0 \in (0,1)$, we obtain a definable continuous function $h\colon \gamma\big((0,1)\big) \to R^m$ such that $h\big(\gamma(t)\big) \in T\big(\gamma(t)\big)$ for $t \in (0,1)$ and $\lim_{t\to 0^+} h\big(\gamma(t)\big) = F(x_0)$. By continuity of g at x_0, take $\delta > 0$ such that for all $x_1 \in R^n$ with $\|x_1 - x_0\| < \delta$, we have $\|g(x_1) - g(x_0)\| < \frac{1}{3}\epsilon$. Let then $t_0 \in (0,1)$ be such that for $0 < t \leq t_0$ we have $\|\gamma(t) - x_0\| < \delta$ and $\big\|h\big(\gamma(t)\big) - F(x_0)\big\| < \frac{1}{3}\epsilon$. By the definition of F,

$$\big\|F\big(\gamma(t)\big) - g\big(\gamma(t)\big)\big\| \leq \big\|h\big(\gamma(t)\big) - g\big(\gamma(t)\big)\big\| \qquad \text{for all } t \in (0,1).$$

Moreover, for $0 < t \leq t_0$ we have

$$\begin{aligned}
\big\|h\big(\gamma(t)\big) - g\big(\gamma(t)\big)\big\| &\leq \big\|h\big(\gamma(t)\big) - F(x_0)\big\| + \|F(x_0) - g(x_0)\| + \big\|g(x_0) - g\big(\gamma(t)\big)\big\| \\
&\leq \tfrac{1}{3}\epsilon + \|F(x_0) - g(x_0)\| + \tfrac{1}{3}\epsilon \\
&= \|g(x_0) - F(x_0)\| + \tfrac{2}{3}\epsilon
\end{aligned}$$

and hence

$$\begin{aligned}
\big\|g(x_0) - F\big(\gamma(t)\big)\big\| &\leq \big\|g(x_0) - g\big(\gamma(t)\big)\big\| + \big\|g\big(\gamma(t)\big) - F\big(\gamma(t)\big)\big\| \\
&\leq \tfrac{1}{3}\epsilon + \big\|h\big(\gamma(t)\big) - g\big(\gamma(t)\big)\big\| \\
&\leq \tfrac{1}{3}\epsilon + \|g(x_0) - F(x_0)\| + \tfrac{2}{3}\epsilon \\
&= \|g(x_0) - F(x_0)\| + \epsilon
\end{aligned}$$

as required. $\square$

Hence $y_0 = \lim_{t\to 0^+} F\big(\gamma(t)\big)$ exists in R^m, and $\|g(x_0) - y_0\| \leq \|g(x_0) - F(x_0)\| + \epsilon$ for every $\epsilon > 0$; that is, $\|g(x_0) - y_0\| \leq \|g(x_0) - F(x_0)\|$. Thus if $x_0 \in X'$, then $F(x_0) = g(x_0)$ and hence $y_0 = g(x_0) = F(x_0)$. Now suppose $x_0 \in X \setminus X'$. Then by closedness of $T \restriction (X \setminus X')$, we have $y_0 \in T(x_0)$, so by definition of F we obtain $y_0 = F(x_0)$. Therefore F is continuous at x_0. $\square$

We do not know whether Theorem 4.1 continues to hold if $\boldsymbol{R}$ is merely assumed to be definably complete (i.e., every non-empty bounded definable subset of R has a supremum in R).

Corollary 4.3. *Let T be as in the previous theorem, and let $X_0 \subseteq X$ be definable and closed. Then every continuous definable selection of $T \upharpoonright X_0$ extends to a continuous definable selection of T. In particular, given distinct $x_1, \dots, x_N \in X$ and $y_i \in T(x_i)$ for $i = 1, \dots, N$, there exists a continuous definable selection f of T with $f(x_i) = y_i$ for $i = 1, \dots, N$.*

Proof. Let $f_0 \colon X_0 \to R^m$ be a continuous definable selection of $T \upharpoonright X_0$. Let $T_0 \colon X \rightrightarrows R^m$ be a set-valued map given by

$$T_0(x) = \begin{cases} T(x) & \text{if } x \in X \setminus X_0, \\ \big\{f_0(x)\big\} & \text{if } x \in X_0. \end{cases}$$

It is easy to verify that T_0 is l.s.c. Now apply Theorem 4.1 to T_0. □

Remark. The closedness of X_0 in the above corollary is necessary. Consider $X_0 = (0, +\infty)$ and $T \colon \mathbb{R} \rightrightarrows \mathbb{R}$ where $T(x) = \mathbb{R}$ for every $x \in \mathbb{R}$. Then $x \mapsto \frac{1}{x} \colon (0, +\infty) \to \mathbb{R}$ is a continuous selection of $T \upharpoonright X_0$ without continuous extension.

Let H_m denote the set of closed bounded non-empty convex definable subsets of R^m. Equip H_m with the Hausdorff metric d_H: for $A, B \in H_m$ set

$$d_H(A, B) := \sup\left(\big\{d(y, A) : y \in B\big\} \cup \big\{d(y, B) : y \in A\big\}\right).$$

Every map $X \to H_m$ is a set-valued map $X \rightrightarrows R^m$, and so it makes sense to talk about definable maps $X \to H_m$. From Theorem 4.1 we immediately obtain:

Corollary 4.4. *Every continuous definable map $X \to H_m$, where $X \subseteq R^n$ is closed, has a continuous definable selection.*

It is well-known that for $R = \mathbb{R}$, every Lipschitz map $X \to H_m$, where $X \subseteq \mathbb{R}^n$, has a Lipschitz selection. See, e.g., [3, Theorem 9.4]; the construction given there uses Steiner points (and hence integration). We do not know whether a definable Lipschitz map $X \to H_m$ always has a definable Lipschitz selection.

We finish this section with another standard application of Michael's Selection Theorem, about approximating possibly discontinuous maps by continuous ones (cf. [12, Section 7.2]). For a definable map $f \colon X \to R^m$, where $X \subseteq R^n$, we set

$$||f|| := \sup\big\{||f(x)|| : x \in X\big\} \in R^{\geq 0} \cup \{+\infty\}.$$

Corollary 4.5. *Let $f, g \colon X \to R^m$ be definable, where $X \subseteq R^n$ is closed, and suppose f is continuous and g is bounded. Then for each $\epsilon > 0$ there exists a continuous definable and bounded $\overline{g} \colon X \to R^m$ with $||f - \overline{g}|| \leq ||f - g|| + \epsilon$.*

Proof. Note that if $||f - g|| = \infty$, then we can take $\overline{g} = 0$. Therefore from now on we assume that $||f - g|| < \infty$ (so f is also bounded). Take $r > 0$ such that $g(X) \subseteq \overline{B}_r(0)$, and let $\lambda := ||f - g|| + \epsilon$. For $x \in X$ define

$$T(x) := \big\{y \in \overline{B}_r(0) : ||f(x) - y|| \leq \lambda\big\}.$$

Then $T(x) \neq \emptyset$ since $g(x) \in T(x)$, and $T(x)$ is closed and convex. By Theorem 4.1, it remains to show that T is l.s.c. Let $x \in X$, $y \in T(x)$, and $V \subseteq R^m$ be an open neighborhood of y. Thus $||f(x) - y|| \leq \lambda$. Since $||f(x) - g(x)|| < \lambda$, by considering the line segment between y and $g(x)$, we see that we may take some $y' \in V$ with $||f(x) - y'|| < \lambda$. Since f is continuous at x, we now let U be an open neighborhood of x such that $||f(x') - f(x)|| < \lambda - ||f(x) - y'||$ for all $x' \in U \cap X$. Thus $||f(x') - y'|| < \lambda$ for all $x' \in U \cap X$, i.e., $y' \in T(x') \cap V$ for all $x' \in U \cap X$. □

In addition, by [44, Corollary 1.2], one can achieve differentiability of the approximation up to some fixed finite order.

Corollary 4.6. *Let $f, g \colon X \to R^m$ be definable, where $X \subseteq R^n$ is closed, and suppose f is continuous and g is bounded. Then for each $m > 0$ and $\epsilon > 0$ there exists a C^m, definable and bounded $\overline{g} \colon X \to R^m$ with $||f - \overline{g}|| \leq ||f - g|| + \epsilon$.*

5. Affine Bundles

In this section, following [18], we introduce affine bundles and Glaeser refinements and establish basic facts about them used later. The proofs are routine adaptations of those given in [26] and are only included for the convenience of the reader. Throughout this section we let X denote a subset of R^n. For notational simplicity, from now on we often denote a set-valued map and its graph by the same letter. A set-valued map $H \colon X \rightrightarrows R^m$ such that for each $x \in X$, $H(x)$ is empty or an affine subspace of R^m, is called an **affine bundle** on X. In the rest of this section, we let $H \colon X \rightrightarrows R^m$ be an affine bundle on X. Define the **Glaeser refinement** H' of H by

$$H'(x_0) := \big\{ y_0 \in H(x_0) : d\big(y_0, H(x)\big) \to 0 \text{ as } x \to x_0 \text{ in } X \big\} \quad \text{for } x_0 \in X.$$

That is, $(x_0, y_0) \in H$ is in H' iff for every $\epsilon > 0$ there is some $\delta > 0$ such that for every $x \in X \cap B_\delta(x_0)$, there is $y \in H(x)$ such that $||y_0 - y|| < \epsilon$. Clearly if H is definable, then so is H'. Since $H' \subseteq H$, every selection (continuous or not) of H' is a selection of H; conversely, each continuous selection of H is also a selection of H'. Moreover:

Proposition 5.1. *The Glaeser refinement H' of H is an affine bundle on X.*

Proof. Let $p_0 \in H'(x_0)$. To prove that $H'(x_0)$ is an affine subspace of R^m, let $q_0, r_0 \in H'(x_0)$, $a, b \in R$. It suffices to show that $y_0 := a(q_0 - p_0) + b(r_0 - p_0) + p_0 \in H'(x_0)$. Let $\epsilon > 0$, and take $\delta > 0$ such that for all $x_1 \in X \cap B_\delta(x_0)$, there exist $p_1, q_1, r_1 \in H(x_1)$ such that

$$||p_0 - p_1||, ||q_0 - q_1||, ||r_0 - r_1|| < \frac{\epsilon}{3(|a| + |b| + 1)}.$$

For such $x_1 \in X \cap B_\delta(x_0)$ and p_1, q_1, r_1, we have $y_1 := a(q_1 - p_1) + b(r_1 - p_1) + p_1 \in H(x_1)$ and $||y_0 - y_1|| < \epsilon$. Thus $y_0 \in H'(x_0)$. □

We say that H is **stable under Glaeser refinement** if $H' = H$. Clearly if H is stable under Glaeser refinement, then H is l.s.c.

Lemma 5.2. *Let $x_0 \in X$. Then*

$$\dim H'(x_0) \leq \liminf_{X \ni x \to x_0} \dim H(x).$$

Proof. We may assume $H'(x_0) \neq \emptyset$. Let $d = \dim H'(x_0)$ and $p_0, \dots, p_d \in H'(x_0)$ be such that $p_1 - p_0, \dots, p_d - p_0$ are R-linearly independent. Let $\epsilon > 0$. By definition of H', there exists some $\delta > 0$ such that for all $x \in B_\delta(x_0)$ we obtain $q_0, \dots, q_d \in H(x)$ with $\|p_i - q_i\| < \epsilon$; for sufficiently small ϵ, $q_1 - q_0, \dots, q_d - q_0$ are R-linearly independent, so $\dim H(x) \geq d$. □

Starting with an affine bundle $H_0 = H \colon X \rightrightarrows R^m$ on X, we inductively define a sequence $(H^{(l)})_{l \in \mathbb{N}}$ of affine bundles on X by setting $H^{(l+1)} := (H^{(l)})'$ for each $l \in \mathbb{N}$. The previous lemma has a remarkable consequence:

Lemma 5.3. *Let $x_0 \in X$ be such that $H(x_0) \neq \emptyset$. Then*

$$\dim H^{(2k+1)}(x_0) \geq m - k \quad \Longrightarrow \quad H^{(l)}(x_0) = H^{(2k+1)}(x_0) \text{ for all } l \geq 2k + 1.$$

Proof. We proceed by induction on k. If $k = 0$, then the implication asserts that if $H'(x_0) = H(x_0)$, then $H^{(l)}(x_0) = H'(x_0)$ for all $l \geq 1$. Now if $H'(x_0) = H(x_0)$, then $H(x) = H(x_0)$ for all $x \in X$ in a neighborhood of x_0, by the previous lemma, and hence $H^{(l)}(x) = H(x_0)$ for all $x \in X$ in a neighborhood of x_0 and all $l \geq 1$. For the induction step, assume the implication holds for a certain value of k. Suppose $\dim H^{(2k+3)}(x_0) \geq m - k - 1$. If $\dim H^{(2k+1)}(x_0) \geq m - k$, then by inductive hypothesis we obtain $H^{(l)}(x_0) = H^{(2k+1)}(x_0) = H^{(2k+3)}(x_0)$ for every $l \geq 2k + 1$. Assume $\dim H^{(2k+1)}(x_0) \leq m - k - 1$. Then

$$\dim H^{(2k+1)}(x_0) \leq m - k - 1 \leq \dim H^{(2k+3)}(x_0) \leq \dim H^{(2k+1)}(x_0)$$

and so

$$\dim H^{(2k+1)}(x_0) = \dim H^{(2k+2)}(x_0) = \dim H^{(2k+3)}(x_0) = m - k - 1.$$

By Lemma 5.2, $\dim H^{(2k+1)}(x) \geq m - k - 1$ for all $x \in X$ in a neighborhood of x_0.

Claim. $H^{(2k+1)}(x) = H^{(2k+2)}(x)$ for all $x \in X$ in a neighborhood of x_0.

Proof of claim. Suppose not. Then, for every $\delta > 0$, there is $x \in B_\delta(x_0) \cap X$ with

$$\dim H^{(2k+1)}(x) > \dim H^{(2k+2)}(x), \text{ i.e., } \dim H^{(2k+2)}(x) \leq m - k - 2.$$

By Lemma 5.2 again,

$$m - k - 1 = \dim H^{(2k+3)}(x_0) \leq \liminf_{x \to x_0} \dim H^{(2k+2)}(x) \leq m - k - 2,$$

a contradiction. □

By the above claim, for $l \geq 2k + 3$, there exists $\delta > 0$ (depending on l) such that $H^{(l)}(x) = H^{(2k+3)}(x)$ for all $x \in B_\delta(x_0) \cap X$. □

If $H(x) \neq \emptyset$ for some $x \in X$, we put $l_* := 2m + 1$; otherwise put $l_* := 0$. With these notations, by the preceding lemma we have:

Corollary 5.4. *$H^{(l)} = H^{(l_*)}$ for $l \geq l_*$.*

Hence the sequence $\left(H^{(l)}\right)_{l \in \mathbb{N}}$ of affine bundles on X constructed from $H^{(0)} = H$ as above by iterated Glaeser refinements eventually stabilizes: $H^{(2m+1)} = H^{(2m+2)} = \cdots$. We let $H^{(*)} := H^{(2m+1)}$ be the eventual value of this sequence. Thus $H^{(*)}$ is an affine bundle on X which is stable under Glaeser refinement, and hence l.s.c. The definable Michael Selection Theorem (Theorem 4.1) now yields:

Corollary 5.5. *Suppose H is definable. Then H has a continuous selection iff $H^{(*)}(x) \neq \emptyset$ for all $x \in X$, and in this case, H even has a definable continuous selection.*

6. Linear Equations in Continuous Functions

In this section we give an application of Corollary 5.5; the material in this section is not used later in the paper. Let $f, g_1, \dots, g_m$ be definable maps $X \to R^k$, where $X \subseteq R^n$. We consider the linear equation

$$(*) \qquad f = g_1 y_1 + \cdots + g_m y_m$$

in unknown continuous functions $y_1, \dots, y_m \colon X \to R$.

Theorem 6.1. *If there are continuous functions $y_1, \dots, y_m \colon X \to R$ solving $(*)$, then there are also* definable *continuous functions $y_1, \dots, y_m$ with this property.*

This follows immediately from Corollary 5.5 above, applied to the definable affine bundle $H\colon X \rightrightarrows R^m$ on X given by

$$H(x) := \big\{(y_1,\dots,y_m)\in R^m : f(x) = g_1(x)y_1+\cdots+g_m(x)y_m\big\} \quad \text{for } x\in X.$$

In fact we have a version of this theorem for definable families:

Corollary 6.2. *Let $(f_a)_{a\in R^N}$ and $(g_{i,a})_{a\in R^N}$ $(i=1,\dots,m)$ be definable families of maps $f_a, g_{1,a},\dots,g_{k,a}\colon X_a\to R^k$ $(X_a\subseteq R^n)$. Then the set $A\subseteq R^N$ given by*

$$a\in A \quad\iff\quad \begin{cases} \text{there are continuous maps } y_1,\dots,y_m\colon X_a\to R^k \\ \text{such that } f_a = g_{1,a}y_1+\cdots+g_{m,a}y_m \end{cases}$$

is definable. Moreover, there are definable families $(y_{i,a})_{a\in A}$ of continuous maps

$$y_{i,a}\colon X_a\to R^k \qquad (i=1,\dots,m)$$

such that for all $a\in A$, we have $f_a = g_{1,a}y_{1,a}+\cdots+g_{m,a}y_{m,a}$.

The definability of A follows by noting that $H^{(*)}$ is defined uniformly in $f, g_1,\dots,g_m$. The rest is a routine application of the Compactness Theorem of first-order logic (see, e.g., [45, Section 6] for a similar argument). Indeed, let $\mathcal{L}$ be the language of $\boldsymbol{R}$, assumed to include a name for each element of R, so that every definable set in R is definable by an $\mathcal{L}$-formula. Let $\alpha(x)$ be an $\mathcal{L}$-formula defining A in $\boldsymbol{R}$. For each m-tuple $\psi=(\psi_1,\dots,\psi_m)$ of formulas $\psi_i(x,y,z)$, where the lengths of x, y, and z are n, k, and N, respectively, let $\chi_\psi(z)$ be a formula such that, for each $a\in R^N$, $\chi_\psi(a)$ holds in $\boldsymbol{R}$ precisely when $\psi_i(x,y,a)$ defines the graph of a continuous map $y_i\colon X_a\to R^k$ $(i=1,\dots,m)$ such that $f_a=g_{1,a}y_1+\cdots+g_{m,a}y_m$. Next, add N fresh constants $c_1,\dots,c_N$ to $\mathcal{L}$ and call the resulting language $\mathcal{L}'$. For notational convenience, write $c=(c_1,\dots,c_N)$. By our main theorem, the $\mathcal{L}'$-theory

$$\operatorname{Th}(\boldsymbol{R})\cup\big\{\alpha(c)\big\}\cup\big\{\neg\chi_\psi(c) : \psi=(\psi_1,\dots,\psi_m) \text{ is a tuple of } \mathcal{L}\text{-formulas } \psi_i=\psi_i(x,y,z)\big\}$$

is inconsistent. Therefore, by the Compactness Theorem, there are m-tuples

$$\psi_1(x,y,z),\dots,\psi_M(x,y,z)$$

of formulas such that, for each $a\in A$, one of the tuples $\psi_i(x,y,a)$ defines the graphs of continuous maps $y_1,\dots,y_m\colon X_a\to R^k$ such that $f_a=g_{1,a}y_1+\cdots+g_{m,a}y_m$. We can now easily construct a single m-tuple of formulas $\psi(x,y,z)$ which works for every $a\in A$, i.e., for each $a\in A$, the components of $\psi(x,y,a)$ define the graphs of continuous maps $y_1,\dots,y_m\colon X_a\to R^k$ with $f_a=g_{1,a}y_1+\cdots+g_{m,a}y_m$.

We finish this section with a special case of the corollary above.

Corollary 6.3. *Let $f(C,X), g_1(C,X),\dots,g_m(C,X)\in\mathbb{R}[C,X]$ where $C=(C_1,\dots,C_N)$, $X=(X_1,\dots,X_n)$ are disjoint tuples of distinct indeterminates. Then the set A consisting of all $c\in\mathbb{R}^N$ such that there are continuous functions $y_1,\dots,y_m\colon\mathbb{R}^n\to\mathbb{R}$ such that*

$$f(c,x)=g_1(c,x)y_1(x)+\cdots+g_m(c,x)y_m(x) \qquad \textit{for all } x\in\mathbb{R}^n$$

is semialgebraic. Moreover, there are semialgebraic functions $y_1,\dots,y_m\colon\mathbb{R}^N\times\mathbb{R}^n\to\mathbb{R}$ such that for all $c\in A$ and $i=1,\dots,m$, the function $y_i(c,-)\colon\mathbb{R}^n\to\mathbb{R}$ is continuous and

$$f(c,x)=g_1(c,x)y_1(c,x)+\cdots+g_m(c,x)y_m(c,x) \qquad \textit{for all } x\in\mathbb{R}^n.$$

Theorem 6.1 remains true for finite systems of linear equations instead of a single equation $(*)$. It is natural to wonder whether Theorem 6.1 also has an analogue for *homogeneous* equations. Let $X \subseteq R^n$ be definable, let $C(X)$ denote the R-algebra of continuous functions $X \to R$, and let $C(X)_{\text{def}}$ be its subalgebra consisting of the definable continuous functions $X \to R$. The m-tuples $(y_1, \dots, y_m) \in C(X)^m$ satisfying

$$g_1 y_1 + \cdots + g_m y_m = 0$$

form a $C(X)$-submodule M of $C(X)^m$. *Is the $C(X)$-module M generated by $M \cap C(X)^m_{\text{def}}$?*

In algebraic terms, Theorem 6.1 says that $C(X)_{\text{def}}$ is *pure* in $C(X)$ (viewed as a $C(X)_{\text{def}}$-module in the natural way). A positive answer to the question above would complement this by saying that $C(X)$ is a *flat* (and hence, by *faithfully flat*) $C(X)_{\text{def}}$-module. However, this question has a negative answer, as was pointed out to us by C. Fefferman: For this, let $\boldsymbol{R}$ be the ordered field of real numbers (so "definable" = "semialgebraic") and $X = \mathbb{R}^2$, and consider the homogeneous equation

$$g_1 y_1 + g_2 y_2 = 0 \quad \text{where } g_1(x,y) = x^2 + y^2,\ g_2(x,y) = xy \text{ for } x, y \in \mathbb{R}.$$

Then M is isomorphic to the maximal ideal

$$\mathfrak{m} := \{ f \in C(\mathbb{R}^2) : f(0) = 0 \}$$

of $C(\mathbb{R}^2)$, the isomorphism being given by $(y_1, y_2) \mapsto y_2$. However, $\mathfrak{m}$ is not generated by $\mathfrak{m}_{\text{def}} := \mathfrak{m} \cap C(\mathbb{R}^2)_{\text{def}}$, due to the asymptotics of semialgebraic functions. (For example, the function $h \in C(\mathbb{R}^2)$ given by $h(x,y) = 1/\log x$ if $x > 0$ and $h(x,y) = 0$ if $x \leq 0$ is easily seen to not to belong to the ideal of $C(\mathbb{R}^2)$ generated by $\mathfrak{m}_{\text{def}}$.)

Another attempt to generalize Theorem 6.1 involves the model theory of modules, and for the rest of this section we assume that the reader is familiar with the basics of this theory; see, for example, [22, Appendix A.1]. We view $C(X)$ and $C(X)_{\text{def}}$ as structures in the language of $C(X)_{\text{def}}$-modules in the natural way.

Question. Are $C(X)$ and $C(X)_{\text{def}}$ elementarily equivalent?

By [39] and Theorem 6.1, a positive answer to this question would imply that $C(X)_{\text{def}}$ is an elementary substructure of $C(X)$. Answering this question amounts to showing that all Baur-Monk invariants $\operatorname{Inv}(\phi, \psi, -)$ of $C(X)$ and $C(X)_{\text{def}}$ agree. Since $C(X)_{\text{def}}$ contains an infinite subfield, each of these invariants is either 1 or ∞. Moreover, $\operatorname{Inv}(\phi, \psi, C(X)_{\text{def}}) \leq \operatorname{Inv}(\phi, \psi, C(X))$ since $C(X)_{\text{def}}$ is pure in $C(X)$. Hence the question above has a positive answer if $\operatorname{Inv}(\phi, \psi, C(X)) > 1 \Rightarrow \operatorname{Inv}(\phi, \psi, C(X)_{\text{def}}) > 1$, for all positive primitive formulas ϕ, ψ in a single free variable y. Here is the simplest non-trivial instance of this question:

Question. Let $a_1, \dots, a_k, b_1, \dots, b_l, f, g \in C(X)_{\text{def}}$. Suppose some $y \in C(X)$ satisfies

$$\begin{aligned} fy &= a_1 y_1 + \cdots + a_k y_k && \text{for some } y_1, \dots, y_k \in C(X), \text{ and} \\ gy &\neq b_1 z_1 + \cdots + b_l z_l && \text{for all } z_1, \dots, z_l \in C(X). \end{aligned}$$

Is there some $y \in C(X)_{\text{def}}$ satisfying the same conditions?

We do not know the answer to this question. If $l = 0$ (so the second condition reads "$gy \neq 0$"), then the answer is yes, since given a continuous solution y we can simply pick a point $x_0 \in X$ where $g(x_0)y(x_0)$ is nonzero, consider the affine bundle

$$H(x) = \{ (y_0, \dots, y_k) : f(x)y_0 = a_1(x)y_1 + \cdots + a_k(x)y_k, \text{ and if } x = x_0, \text{ then } y_0 = y(x_0) \}$$

and proceed as in the proof of Theorem 6.1. See [36] for a description of the Baur-Monk invariants $\operatorname{Inv}(\phi, \psi, C(X)_{\text{def}})$ in the case when X has dimension 1.

7. C^1-Whitney's Extension Problem

In this section, we follow the idea given in [26] to solve $\mathrm{WEP}_{n,1}$. Throughout this section, we fix a definable closed subset X of R^n and a definable continuous function $f\colon X \to R$.

Definition 7.1. We say that a definable affine bundle $H\colon X \rightrightarrows R \times R^n$ on X is a **holding space for** f if whenever $F \in C^1(R^n)$ is definable with $F = f$ on X, then

$$\left\{ \left(x, F(x), \frac{\partial F}{\partial x_1}(x), \dots, \frac{\partial F}{\partial x_n}(x) \right) : x \in X \right\} \subseteq H.$$

We can think of a holding space for f as a collection of potential Taylor polynomials of extensions of f to a C^1-function $U \to R$ on a neighborhood U of X: Let $\mathscr{P}_n$ be the R-vector space of linear polynomials in n indeterminates with coefficients from R. For a fixed $x_0 \in X$, there is a one-to-one correspondence between $R \times R^n$ and $\mathscr{P}_n$ given by

$$(a, u) \longleftrightarrow p(a, u)(x) = a + \langle u, x - x_0 \rangle.$$

Therefore, we may also think of $H \subseteq X \times (R \times R^n)$ as a subset of $X \times \mathscr{P}_n$.

Obviously,

$$H_0 := \big\{ (x, f(x), u) \in X \times R \times R^n : x \in X,\ u \in R^n \big\}$$

is a holding space for f. We call H_0 the **trivial** holding space for f. Clearly H_0 contains every holding space for f, and $\dim H_0(x) = n$ for each $x \in X$.

Holding spaces usually contain too much information; for example, consider the trivial holding space for f. In order to cut down on the insignificant information, C^1-Glaeser refinements are introduced:

Definition 7.2. Let $H \subseteq X \times \mathscr{P}_n$. The **$C^1$-Glaeser refinement** $\widetilde{H}$ of H is defined as follows: we let $(x_0, p_0) \in \widetilde{H}$ if and only if $(x_0, p_0) \in H$, and for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x_1, x_2 \in X \cap B_\delta(x_0)$, there exist $p_1 \in H(x_1)$ and $p_2 \in H(x_2)$ satisfying the following inequalities (with the convention $0^0 = 0$):

$$|D^\alpha (p_i - p_j)(x_i)| \leq \epsilon \left\| x_i - x_j \right\|^{1-|\alpha|} \quad \text{for } i, j = 0, 1, 2 \text{ and } \alpha \text{ with } |\alpha| \leq 1. \tag{2}$$

That is, $(x_0, p_0) \in H$ with $p_0 = (a_0, u_0)$ is in $\widetilde{H}$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x_1, x_2 \in X \cap B_\delta(x_0)$, there are $p_i = (a_i, u_i) \in H(x_i)$, $i = 1, 2$, such that

$$\begin{cases} |a_i + \langle u_i, x_j - x_i \rangle - a_j| \leq \epsilon \|x_i - x_j\|, \\ ||u_i - u_j|| \leq \epsilon, \end{cases}$$

for $i, j = 0, 1, 2$.

Note that $\widetilde{H} \subseteq H$, and if H is definable, then so is $\widetilde{H}$. Here is how $\widetilde{H}$ relates to the notion of Glaeser refinement H' of H introduced in the previous section:

Proposition 7.3. *Suppose $H \subseteq H_0$. Then $\widetilde{H} \subseteq H'$.*

Proof. Let $\big(x_0, f(x_0), u_0\big) \in \widetilde{H}$ and $\epsilon > 0$ be given. By continuity of f and the definition of $\widetilde{H}$, there is $\delta > 0$ such that for every $x_1 \in B_\delta(x_0) \cap X$ there exists $\big(f(x_1), u_1\big) \in H(x_1)$ where $|f(x_1) - f(x_0)| < \epsilon/\sqrt{2}$ and $\|u_1 - u_0\| < \epsilon/\sqrt{2}$. Therefore, $\big\| \big(f(x_1), u_1\big) - \big(f(x_0), u_0\big) \big\| < \epsilon$. Thus $\big(x_0, f(x_0), u_0\big) \in H'$. $\square$

We say that $H \subseteq X \times \mathscr{P}_n$ is **stable under C^1-Glaeser refinement** if $\widetilde{H} = H$. By Proposition 7.3, if H is a holding space for f which is stable under C^1-Glaeser refinement, then $H = H' = \widetilde{H}$ is l.s.c.

Proposition 7.4. *Let $H\colon X \rightrightarrows \mathscr{P}_n$ be an affine bundle on X. Then $\widetilde{H}\colon X \rightrightarrows \mathscr{P}_n$ is an affine bundle on X.*

Proof. Let $p_0 \in \widetilde{H}(x_0)$. To prove that $\widetilde{H}(x_0)$ is affine, let $q_0, r_0 \in \widetilde{H}(x_0)$, $a, b \in R$. It is enough to show that $a(q_0 - p_0) + b(r_0 - p_0) \in \widetilde{H}(x_0) - p_0$. Let $\epsilon > 0$, and take $\delta > 0$ such that for all $x_1, x_2 \in X \cap B_\delta(x_0)$, there exist $p_1, q_1, r_1 \in H(x_1)$ and $p_2, q_2, r_2 \in H(x_2)$ with

$$\begin{aligned}
|D^\alpha(p_i - p_j)(x_i)| &\le \frac{\epsilon}{2(|a| + |b| + 1)} \|x_i - x_j\|^{1-|\alpha|}, \\
|D^\alpha(q_i - q_j)(x_i)| &\le \frac{\epsilon}{2(|a| + |b| + 1)} \|x_i - x_j\|^{1-|\alpha|}, \\
|D^\alpha(r_i - r_j)(x_i)| &\le \frac{\epsilon}{2(|a| + |b| + 1)} \|x_i - x_j\|^{1-|\alpha|}
\end{aligned}$$

for $i, j = 0, 1, 2$ and α with $|\alpha| \le 1$. Let $x_1, x_2 \in X \cap B_\delta(x_0)$, and fix such witnesses p_1, q_1, r_1 and p_2, q_2, r_2. Then $a(q_i - p_i) + b(r_i - p_i) \in H(x_i) - p_i$ for $i = 1, 2$. Hence

$$\begin{aligned}
&\big|D^\alpha\big[(a(q_i - p_i) + b(r_i - p_i) + p_i) - (a(q_j - p_j) + b(r_j - p_j) + p_j)\big](x_i)\big| \\
&\quad \le \big|D^\alpha\big[(a + b + 1)(p_i - p_j)\big](x_i)\big| + |D^\alpha(a(q_i - q_j))(x_i)| + |D^\alpha(b(r_i - r_j))(x_i)| \\
&\quad \le 2(|a| + |b| + 1)\frac{\epsilon}{2(|a| + |b| + 1)} \|x_i - x_j\|^{1-|\alpha|} \\
&\quad = \epsilon \|x_i - x_j\|^{1-|\alpha|}
\end{aligned}$$

for $i, j = 0, 1, 2$ and α with $|\alpha| \le 1$. Thus $a(q_0 - p_0) + b(r_0 - p_0) + p_0 \in \widetilde{H}(x_0)$ as desired. $\square$

The above proposition and the definition of continuous differentiability imply that the class of holding spaces for C^1-functions is closed under C^1-Glaeser refinement:

Corollary 7.5. *Suppose f is C^1. If H is a holding space for f, then so is $\widetilde{H}$.*

By iterating the process of taking C^1-Glaeser refinements, we obtain a decreasing sequence $(H_l)_{l\in\mathbb{N}}$ of subsets of $X \times \mathscr{P}_n$ as follows:

$$\begin{aligned}
H_0 &:= \text{ the trivial holding space for } f, \\
H_{l+1} &:= \widetilde{H_l}.
\end{aligned}$$

We call $(H_l)_{l\in\mathbb{N}}$ the **sequence of holding spaces for** f. By induction on l using Proposition 7.3 we obtain $H_l \subseteq H^{(l)}$.

By Proposition 7.4, if $l \in \mathbb{N}$ and $x \in X$ such that $H_l(x) \ne \emptyset$, then $H_l(x)$ is an affine subspace of $\mathscr{P}_n$. This together with the definition of C^1-Glaeser refinement and Taylor's Theorem implies the following corollary:

Corollary 7.6. *If $H_{l+1}(x)$ is non-empty for every $x \in X$, and H_l is a holding space for f, then H_{l+1} is a holding space for f. In particular, if $H_l(x)$ is non-empty for every $l \in \mathbb{N}$ and $x \in X$, then H_l is a holding space for f for all $l \in \mathbb{N}$.*

If there exists some $l_* \in \mathbb{N}$ such that H_{l_*} is stable under C^1-Glaeser refinement, then we call $H_* := H_{l_*}$ the **stable holding space** for f. In [18], by an argument originating in [20] and adapted in [4], it was shown in the classical real context that every continuous function has a stable holding space. This remains true for definable continuous functions in our setting:

Lemma 7.7. *Let H be a holding space for f, and $x_0 \in X$. Then*

$$\dim \widetilde{H}(x_0) \le \liminf_{X \ni x \to x_0} \dim H(x).$$

This follows from Lemma 5.2 and Proposition 7.3. As in the proof of Lemma 5.3, the previous lemma implies:

Lemma 7.8. *Let $x_0 \in X$ be such that $H(x_0) \neq \emptyset$, and set $m := \mathscr{P}_n$. Then*

$$\dim H^{(2k+1)}(x_0) \geq m - k \quad \Longrightarrow \quad H^{(l)}(x_0) = H^{(2k+1)}(x_0) \text{ for all } l \geq 2k+1.$$

Corollary 7.9. *Let $l_* := 2 \dim \mathscr{P}_n + 1 = 2n + 3$. Then $H_l = H_{l_*}$ for $l \geq l_*$, so f has stable holding space $H_* = H_{l_*}$.*

The following lemma exhibits a certain uniformity of the C^1-Glaeser refinement:

Lemma 7.10. *Let H be a holding space for f, and $x_0 \in X$. If $\big(f(x_0), u_0\big) \in \widetilde{H}(x_0)$, then for every $\epsilon > 0$, there is $\delta > 0$ such that*

$$|f(x) + \langle u_0, x' - x\rangle - f(x')| \leq \epsilon \,\|x - x'\| \qquad \text{for all } x, x' \in X \cap B_\delta(x_0).$$

Proof. Suppose that there is $\epsilon > 0$ such that for every $\delta > 0$ there are $x, x' \in X \cap B_\delta(x_0)$ with

$$|f(x) + \langle u_0, x' - x\rangle - f(x')| > \epsilon \,\|x - x'\|\,.$$

Let $\delta > 0$. Let $x, x' \in X \cap B_\delta(x_0)$ be witnesses of the above statement and $\big(f(x), u\big) \in H(x)$ with $\|u - u_0\| \leq \frac{\epsilon}{2}$. Then

$$\begin{aligned} |f(x) + \langle u, x' - x\rangle - f(x')| &\geq |f(x) + \langle u_0, x' - x\rangle - f(x')| - |\langle u - u_0, x' - x\rangle| \\ &\geq |f(x) + \langle u_0, x' - x\rangle - f(x')| - \|u - u_0\| \cdot \|x' - x\| \\ &> \epsilon \,\|x' - x\| - \frac{\epsilon}{2}\,\|x' - x\| = \frac{\epsilon}{2}\,\|x' - x\|\,. \end{aligned}$$

Thus, $\big(f(x_0), u_0\big) \notin \widetilde{H}(x_0)$. □

Lemma 7.11. *Let H_* be the stable holding space for f. Then f is the restriction of a definable C^1-function $R^n \to R$ iff H_* admits a continuous definable selection.*

Proof. The forward direction being trivial, we let $g = (g_1, \dots, g_n) \colon X \to R^n$ be a definable continuous map such that $\Gamma\big((f, g)\big) \subseteq H_*$ where (f, g) is the map

$$x \mapsto \big(f(x), g(x)\big) \colon X \to R \times R^n.$$

Let $F = (F_\alpha)_{|\alpha| \leq 1}$ where $F_{\bar{0}} := f$ and $F_{e_i} := g_i$ for $i = 1, \dots, n$. (Here, $e_1, \dots, e_n \in \mathbb{N}^n$ are the standard basis vectors of R^n.) By the Definable Whitney Extension Theorem, it is sufficient to prove that F is a C^1-Whitney field. Since g is continuous, it is enough to show the following:

$$|f(x) + \langle g(x), x' - x\rangle - f(x')| = o(\|x' - x\|) \qquad \text{for } x, x', x_0 \in X \text{ with } x, x' \to x_0.$$

Let $\epsilon > 0$ and $x_0 \in X$ be given. By continuity of g, we can take $\delta_1 > 0$ such that

$$\|g(x) - g(x_0)\| < \frac{\epsilon}{2} \quad \text{for all } x \in B_{\delta_1}(x_0) \cap X.$$

By Lemma 7.10 and since H_* is stable under C^1-Glaeser refinement, there is $\delta_2 > 0$ such that, for all $x, x' \in B_{\delta_2}(x_0) \cap X$,

$$|f(x) + \langle g(x_0), x' - x\rangle - f(x')| < \frac{\epsilon}{2}\,\|x' - x\|\,.$$

Set $\delta = \min\{\delta_1, \delta_2\}$. Thus,

$$\begin{aligned} |f(x) + \langle g(x), x' - x\rangle - f(x')| &\leq |f(x) + \langle g(x_0), x' - x\rangle - f(x')| + \\ &\quad |\langle g(x) - g(x_0), x' - x\rangle| \\ &< \frac{\epsilon}{2}\|x' - x\| + \|g(x) - g(x_0)\| \cdot \|x' - x\| \\ &< \epsilon\|x' - x\| \end{aligned}$$

for any $x, x' \in B_\delta(x_0) \cap X$. This yields the claim. □

Combining the Definable Michael Selection Theorem (Theorem 4.1) with Proposition 7.3 and the previous lemma, we obtain our main result:

Theorem 7.12. *Let $f\colon X \to R$ be a definable continuous function where $X \subseteq R^n$ is closed, and let H_* be its stable holding space. Then f is the restriction of a definable C^1-function $R^n \to R$ iff $H_*(x) \neq \emptyset$ for every $x \in X$.*

From this theorem, the theorem stated in the introduction follows by an application of the Compactness Theorem in a similar way as at the end of Section 5.

We finish with answering a special case of the following question of van den Dries, posed in lectures at Urbana in 1997. Let $f\colon X \to R$ be a definable function where $X \subseteq R^n$ is closed. Recall that we say that f is C^m if it extends to a definable C^m-function on an open neighborhood of X.

Question. Suppose that for each $x \in X$ there is some $\delta > 0$ such that $f \restriction B_\delta(x) \cap X$ is C^m. Is f then C^m?

The local nature of the C^1-Glaeser refinement and Theorem 7.12 allows us to give a positive answer in the case $m = 1$. Given $H \subseteq X \times \mathscr{P}_n$ and $Y \subseteq X$, let $H \restriction Y := H \cap (Y \times \mathscr{P}_n)$.

Lemma 7.13. *Let (H_l) be the sequence of holding spaces for f. Let $x \in X$ and $\delta > 0$, and let (H'_l) be the sequence of holding spaces for $f \restriction \overline{B}_\delta(x) \cap X$. Then for all $l \in \mathbb{N}$:*

$$H'_l \restriction \overline{B}_{\delta/2^l}(x) \cap X \subseteq H_l \restriction \overline{B}_{\delta/2^l}(x) \cap X \tag{3}$$

Proof. Clearly $H_0 \restriction Y$ is the trivial holding space of $f \restriction Y$, for each definable closed $Y \subseteq X$. Suppose we have already shown (3) for some value of l. Let $(x_0, p_0) \in \big(\overline{B}_{\delta/2^{l+1}}(x) \cap X\big) \times \mathscr{P}_n$ be given. Then

$$(x_0, p_0) \in H'_{l+1} \iff \begin{cases} (x_0,p_0) \in H'_l, \text{ and for all } \epsilon > 0 \text{ there is some } \delta_0 > 0 \\ \text{such that for all } x_1, x_2 \in B_{\delta_0}(x_0) \cap \overline{B}_\delta(x) \cap X \text{ there} \\ \text{are } p_i \in H'_l(x_i)\ (i = 1, 2) \text{ such that the inequali-} \\ \text{ties (2) in Definition 7.2 hold.} \end{cases} \tag{4}$$

On the other hand,

$$(x_0, p_0) \in H_{l+1} \restriction \overline{B}_\delta(x) \cap X \iff \begin{cases} (x_0,p_0) \in H_l, \text{ and for all } \epsilon > 0 \text{ there is} \\ \text{some } \delta_0 > 0 \text{ such that for all } x_1, x_2 \in \\ B_{\delta_0}(x_0) \cap X \text{ there are } p_i \in H_l(x_i)\ (i = \\ 1, 2) \text{ such that (2) holds.} \end{cases}$$

Suppose now that $(x_0, p_0) \in H'_{l+1}$. So $(x_0, p_0) \in H'_l$ and $x_0 \in \overline{B}_{\delta/2^{l+1}}(x) \cap X \subseteq \overline{B}_{\delta/2^l}(x) \cap X$, hence $(x_0, p_0) \in H_l$ by inductive hypothesis. Given $\epsilon > 0$ we may choose $\delta_0 > 0$ as in (4) to additionally satisfy $\delta_0 \leq \delta/2^{l+1}$, and then $B_{\delta_0}(x_0) \subseteq B_{\delta/2^l}(x) \subseteq \overline{B}_\delta(x)$. Together with the inductive hypothesis, this yields $(x_0, p_0) \in H_{l+1}$. □

By the previous lemma and Theorem 7.12, we obtain:

Corollary 7.14. *Let $X \subseteq R^n$ be closed and $f\colon X \to R$ be definable, and suppose that for each $x \in X$ there is some $\delta > 0$ such that $f \restriction B_\delta(x) \cap X$ is C^1. Then f is C^1.*

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