DEGREE BOUNDS FOR GRÖBNER BASES IN ALGEBRAS OF SOLVABLE TYPE

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ABSTRACT. We establish doubly-exponential degree bounds for Gröbner bases in certain algebras of solvable type over a field (as introduced by Kandri-Rody and Weispfenning). The class of algebras considered here includes commutative polynomial rings, Weyl algebras, and universal enveloping algebras of finite-dimensional Lie algebras. For the computation of these bounds, we adapt a method due to Dubé based on a generalization of Stanley decompositions. Our bounds yield doubly-exponential degree bounds for ideal membership and syzygies, generalizing the classical results of Hermann and Seidenberg (in the commutative case) and Grigoriev (in the case of Weyl algebras).

INTRODUCTION

The algorithmic aspects of Weyl algebras were first explored by Galligo [11], Takayama [37] and others in the mid-1980s. They laid out the theory of Gröbner bases in this slightly non-commutative setting. Since then, Gröbner bases in Weyl algebras have been widely used for practical computations in algorithmic D-module theory as promoted in [32]. In the early 1990s, Kandri-Rody and Weispfenning [17], by isolating the features of Weyl algebras which permit Gröbner basis theory to work, extended this theory to a larger class of non-comutative algebras, which they termed *algebras of solvable type* over a given coefficient field K. This class of algebras includes the universal enveloping algebras of finite-dimensional Lie algebras over K, by a theorem attributed to Poincaré, Birkhoff and Witt. (For this reason, algebras of solvable type are sometimes called *PBW-algebras*; see, e.g., [5, 31]. Another designation in use is *polynomial rings of solvable type.*) Working implementations of these algorithms exist and are in widespread use; see [12, Section 2.6] and [21]. Similar extensions of Gröbner basis theory to non-commutative algebras were studied by Apel [2] and Mora [28]. See Sections 2 and 3 below for a recapitulation of the basic definitions, and [5] for a comprehensive introduction to this circle of ideas.

In this paper we are interested in degree bounds for left Gröbner bases in algebras of solvable type. It follows trivially from the case of commutative polynomials (as treated in [26]) and Section 5.2 below that the degrees of the elements of the reduced Gröbner basis of a left ideal I in an algebra of solvable type may depend doubly-exponentially on the maximum of the degrees of given generating elements of I. In view of the popularity of this kind of non-commutative Gröbner basis theory, it is surprising that little seems to be known about *upper degree bounds* for Gröbner bases (and, by extension, about the worst-case complexity of Buchberger's algorithm) in this setting. Perhaps it was believed that the upper degree bound for one-sided Gröbner bases, at least in the context of Weyl algebras, also follows from the commutative polynomial case by passing to the associated graded algebra for a certain filtration (which turns out to be nothing but a commutative polynomial

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ring over the given coefficient field). If true, the problem would have boiled down to the doubly-exponential degree bounds for Gröbner bases in commutative polynomial rings over fields found in the 1980s (see, e.g., [27]). However, we would like to emphasize that we *could not find* and we *do not believe there exists* a simple way to establish such a degree bound by reducing the question to commutative algebra. (See Section 3.6 for further discussion.)

A general uniform degree bound for left Gröbner bases in algebras of solvable type was established by Kredel and Weispfenning [18] (using parametric Gröbner bases). They showed that, given an admissible ordering $\leqslant$ on $\mathbb{N}^N$, there exists a computable function $(d,m) \mapsto B(d,m)$ with the following property: for every solvable algebra R over some field, generated by N generators whose commutator relations have degree at most d, every left ideal of R generated by m elements of R of degree at most d has a Gröbner basis (with respect to $\leqslant$) whose elements have degree at most $B(d,m)$.

In contrast to this, here we are mainly interested in finding explicit, doubly-exponential degree bounds. We follow a road to establish such bounds paved by Dubé [9], who gave a self-contained and constructive combinatorial argument for the existence of a doubly-exponential degree bound for Gröbner bases in commutative polynomial rings over a field of arbitrary characteristic. Earlier proofs of results of this type (as in [27]) proceed by first homogenizing and then placing the ideal under consideration into *generic coordinates.* The drawback of this method is that it seems difficult to adapt it to situations as general as the ones considered here; for example, it only works smoothly in characteristic zero. (See also [13] for the delicacies involved in using automorphisms of the Weyl algebra.) The main new technical tool in [9] are decompositions, called *cone decompositions,* of commutative polynomial rings over a field K into a direct sum of finitely many K-linear subspaces of a certain type. These decompositions generalize the *Stanley decompositions* of a given finitely generated commutative graded K-algebra R studied in [36]. A Stanley decomposition of R encodes a lot of information about R; for example, the Hilbert function of R can be easily read off from it. It has been noted in several other places in the literature that Stanley decompositions are ideally suited to avoid the assumption of general position, and, for example, can also be used to circumvent the use of generic hyperplane sections in the proof of Gotzmann's Regularity Theorem [24].

The present paper grew out of an attempt by the authors to better understand Dubé's article [9]. We modified the notions of cone decompositions and the argument of [9] to work for a subclass of the class of algebras of solvable type over an arbitrary coefficient field K, namely the ones whose commutation relations are given by *quadric* polynomials. (This restriction was necessary in order to be able to freely homogenize the algebras and ideals under consideration.) We refer to Section 2 below for precise definitions, and only note here that this class of algebras includes commutative polynomial rings, as well as Weyl algebras and the universal enveloping algebra of a finite-dimensional Lie algebra. Many more examples of quadric algebras of solvable type can be found in [22, Section I.5]. (E.g., Clifford algebras, in particular Grassmann algebras, as well as q-Heisenberg algebras and the Manin algebra of 2×2-quantum matrices.)

Let now K be a field, and let $R = K\langle x\rangle$ be a quadric K-algebra of solvable type with respect to $x = (x_1, \dots, x_N)$ and an admissible ordering $\leqslant$ of $\mathbb{N}^N$. Our main theorem is:

Theorem 0.1. *Every left ideal of R generated by elements of degree at most d has a Gröbner basis consisting of elements of degree at most*

$$D(N,d) := 2\left(\frac{d^2}{2}+d\right)^{2^{N-1}}.$$

Theorem 0.1 is deduced from the homogeneous case: we'll first show that if R is homogeneous and $f_1,\dots,f_n$ are homogeneous, then the elements of a reduced Gröbner basis of a left ideal of R generated by elements of degree at most d have degree at most $D(N-1,d)$, and the obtain the bound in Theorem 0.1 by dehomogenizing. Our theorem also yields uniform bounds for reduced Gröbner bases in the inhomogeneous case. (See [19, 38] for non-explicit uniform degree bounds for reduced Gröbner bases in commutative polynomial rings over fields.) For example, if the admissible ordering $\leqslant$ is degree-compatible, then the reduced Gröbner basis of every left ideal of R generated by elements of degree at most d consists of elements of degree at most $D(N,d)$. (Corollary 5.9.) In the case where the admissible ordering is not degree-compatible, the issues are somewhat more subtle. Therefore, we restrict ourselves to admissible orderings which can be represented by rational weights; this encompasses most admissible orderings used in practice, such as the lexicographic ordering. (See Section 1 for the definition.)

Corollary 0.2. *Suppose that the admissible ordering $\leqslant$ can be represented by rational weights. Then there exists a constant C, which only depends on $\leqslant$, with the following property: the elements of the reduced Gröbner basis with respect to $\leqslant$ of every left ideal of R generated by elements of degree at most d have degree at most $\big(C\cdot D(N,d)\big)^{N+1}$.*

It is routine to deduce from Theorem 0.1:

Corollary 0.3. *Suppose the admissible ordering $\leqslant$ is degree-compatible. Let $f_1,\dots,f_n\in R$ be of degree at most d, and let $f\in R$. If there are $y_1,\dots,y_n\in R$ such that*

$$y_1f_1+\cdots+y_nf_n=f,$$

then there are such y_i of degree at most $\deg(f)+D(N,d)$. Moreover, the left module of solutions to the linear homogeneous equation

$$y_1f_1+\cdots+y_nf_n=0$$

is generated by solutions all of whose components have degree at most $3D(N,d)$.

For $R=K[x_1,\dots,x_N]$, this corollary is essentially a classical result due to Hermann [16] (corrected and extended by Seidenberg [33]). In the case where R is a Weyl algebra, the first statement in this corollary also partly generalizes a result of Grigoriev [13] who showed that if a system of linear equations

$$y_1a_{1j}+\cdots+y_na_{nj}=b_j \qquad (j=1,\dots,m) \tag{$*$}$$

with coefficients $a_{ij},b_j\in R$ of degree at most d has a solution $(y_1,\dots,y_n)$ in R, then this system admits such a solution with $\deg(y_i)\leqslant (md)^{2^{O(N)}}$ for $i=1,\dots,n$. The methods of [13] are quite different from ours, and follow the lead of Hermann and Seidenberg. By arguments as in [3, Corollary 3.4 and Lemma 4.2] one may obtain uniform degree bounds on solutions to systems of linear equations such as $(*)$ by reduction to Corollary 0.3 (the case $m=1$); however, this yields bounds of the form $d^{2^{O(mN)}}$, worse than those obtained by Grigoriev. (Similarly if one tries to use Nagata's "idealization" technique as in [1].) Probably, Corollary 0.3 could be extended from a single linear equation to systems of linear equations with our techniques, by considering Gröbner bases of submodules of finitely generated free modules over R; we shall leave this to another occasion.

By virtue of an observation from [5], our main theorem and its Corollary 0.3, although ostensibly only about one-sided ideals, also have consequences for their two-sided counterparts:

Corollary 0.4. *Let $f_1, \dots, f_n \in R$ be of degree at most d, and let $f \in R$. The two-sided ideal of R generated by $f_1, \dots, f_n$ has a Gröbner basis whose elements have degree at most $D(2N, d)$. If $\leqslant$ is degree-compatible, and the equation*

$$f = y_1 f_1 z_1 + \cdots + y_n f_n z_n$$

has a solution $(y_1, \dots, y_n, z_1, \dots, z_n) \in R^{2n}$, then this equation also has such a solution where

$$\deg(y_i), \deg(y_i') \leqslant \deg(f) + D(2N, d) \qquad \textit{for } i = 1, \dots, n.$$

Weyl algebras are simple (i.e., their only two-sided ideals are the trivial ones). Hence in this case, the previous corollary is vacuous; however, there do exist many non-commutative non-simple algebras satisfying the hypotheses stated before Theorem 0.1, for example, among the universal enveloping algebras of finite-dimensional Lie algebras.

As shown in [29], Gröbner basis theory also extends in a straightforward way to certain K-algebras closely related to Weyl algebras, namely the rings $R_n(K)$ of partial differential operators with rational functions in $K(x) = K(x_1, \dots, x_n)$ as coefficients. Here $R_n(K)$ is the K-algebra generated by $K(x)$ and pairwise distinct symbols $\partial_1, \dots, \partial_n$ subject to the commutation relations

$$\partial_i \partial_j = \partial_j \partial_i, \quad \partial_i c(x) = c(x)\partial_i + \frac{\partial c(x)}{\partial x_i} \qquad (1 \leqslant i \leqslant j \leqslant n,\ c(x) \in K(x)).$$

By [32, Proposition 1.4.13], our main theorem implies the existence of a doubly-exponential degree bound for Gröbner bases for left ideals in $R_n(K)$: every left ideal of $R_n(K)$ generated by elements of degree at most d has a Gröbner basis with respect to a given admissible ordering $\leqslant$ of $\mathbb{N}^n$ consisting of elements of degree at most $D(2n, d)$. As above, this result can then be used to prove an analogue of Corollary 0.3 for $R_n(K)$ (also partially generalizing [13]); we omit the details.

Assume now that K has characteristic zero, and let $R = A_n(K)$ be the n-th Weyl algebra. A proper left ideal I of R is called *holonomic* if the Gelfand-Kirillov dimension of R/I equals n, exactly half of the dimension of R. The *Bernstein inequality*, versions of which are also known as the *Fundamental Theorems of Algebraic Analysis* (see Theorems 1.4.5 and 1.4.6 of [32]), states that $n \leqslant \dim R/I < 2n$. Therefore, holonomic ideals are proper ideals of the minimal possible dimension, which brings up an analogy with zero-dimensional ideals in the commutative polynomial setting. Now, there is a bound on the degrees of the elements of a reduced Gröbner basis of a zero-dimensional ideal in a commutative polynomial ring over a field generated in degree at most d that is (*single*) exponential. Namely, this is the Bézout bound: d^n, where n is the number of indeterminates. (See, e.g., [20].) Holonomic ideals of R are closely related to zero-dimensional left ideals of the algebra $R_n(K) = K(x) \otimes_{K[x]} R$ of differential operators with coefficients in rational functions: if I is a holonomic ideal of R, then the left ideal of $R_n(K)$ generated by I is zero-dimensional, and if conversely J is a zero-dimensional left ideal of $R_n(K)$ then $J \cap R$ is a holonomic ideal of R; see [32, Corollary 1.4.14 and Theorem 1.4.15]. It turns out that only a *weak Bézout* bound can be drawn (cf. [14]) for zero-dimensional ideals of $R_n(K)$, which is doubly-exponential.

So far, to our knowledge, a (single) exponential bound for the degrees of elements in Gröbner bases has been produced only for one very special class of holonomic ideals used in a particular application. These are the *GKZ-hypergeometric ideals,* with a homogeneity assumption (cf. [32, Corollary 4.1.2]). It would be interesting to see if holonomicity (zero-dimensionality) implies a general exponential bound in the algebras $A_n(K)$ ($R_n(K)$, respectively), as well as whether there is a better bound for ideals of minimal possible dimension in solvable algebras in general.

Finally, we'd like to mention that although our study is limited to the most frequently used type of bases, Gröbner bases, there are other kinds of "standard bases" for ideals that may be introduced for algebras of solvable type. For example, [15] explores involutive bases in the Weyl algebra; for one type of bases, Janet bases, there is a recent complexity result established in [8].

0.1. **Organization of the paper.** Sections 1 and 2 mainly have preliminary character, and deal with monomials and generalities on K-algebras, respectively. In Section 3 we review the fundamentals of Gröbner basis theory for algebras of solvable type. In Section 4 we adapt Dubé's method to the non-commutative situation, and in Section 5 we prove the main theorem and its corollaries.

1. Monomials and Monomial Ideals

In this section we collect a few notations and conventions concerning multi-indices, monomials and monomial ideals.

1.1. **Multi-indices.** Throughout this note, we let d, m, N and n range over the set $\mathbb{N} = \{0, 1, 2, \dots\}$ of natural numbers, and α, β, γ and λ range over $\mathbb{N}^N$. We let $\mathbb{N}^0 = \{0\}$ by convention, and identify $\mathbb{N}^N$ with the subset $\mathbb{N}^N \times \{0\}$ of $\mathbb{N}^{N+1}$ in the natural way. We think of the elements of $\mathbb{N}^N$ as *multi-indices.* A **semigroup ordering** of $\mathbb{N}^N$ is a total ordering $\leqslant$ of $\mathbb{N}^N$ such that $\alpha \leqslant \beta \Rightarrow \alpha + \gamma \leqslant \beta + \gamma$ for all α, β, γ. An **admissible ordering** (of $\mathbb{N}^N$) is a semigroup ordering of $\mathbb{N}^N$ having $(0, \dots, 0)$ as its smallest element. It is well-known that any admissible ordering is a well-ordering. For $\alpha = (\alpha_1, \dots, \alpha_N)$ we put $|\alpha| := \alpha_1 + \cdots + \alpha_N$. An ordering $\leqslant$ of $\mathbb{N}^N$ is said to be **degree-compatible** if $|\alpha| < |\beta| \Rightarrow \alpha \leqslant \beta$ for all α, β. Given total orderings $\leqslant_1$ of $\mathbb{N}^{N_1}$ and $\leqslant_2$ of $\mathbb{N}^{N_2}$ (where $N_1, N_2 \in \mathbb{N}$), the **lexicographic product** of $\leqslant_1$ and $\leqslant_2$ is the total ordering $\leqslant$ of $\mathbb{N}^{N_1+N_2} = \mathbb{N}^{N_1} \times \mathbb{N}^{N_2}$ defined by

$$(\alpha_1, \beta_1) \leqslant (\alpha_2, \beta_2) \qquad :\Longleftrightarrow \qquad \alpha_1 < \alpha_2, \text{ or } \alpha_1 = \alpha_2 \text{ and } \beta_1 \leqslant \beta_2,$$

for $\alpha_1, \alpha_2 \in \mathbb{N}^{N_1}$ and $\beta_1, \beta_2 \in \mathbb{N}^{N_2}$. Note that the lexicographic product of $\leqslant_1$ and $\leqslant_2$ extends the ordering $\leqslant_1$ of $\mathbb{N}^{N_1}$. It is easy to see that if $\leqslant_1$, $\leqslant_2$ are semigroup orderings, then so is the lexicographic product of $\leqslant_1$ and $\leqslant_2$, and similarly with "admissible" in place of "semigroup." The N-fold lexicographic product of the usual ordering of $\mathbb{N}$ is an admissible ordering of $\mathbb{N}^N$ called the **lexicographic ordering** of $\mathbb{N}^N$, denoted by $\leqslant_{\text{lex}}$. An example of a degree-compatible admissible ordering of $\mathbb{N}^N$ is the **degree-lexicographic ordering** $\leqslant_{\text{dlex}}$, defined by

$$\alpha \leqslant_{\text{dlex}} \beta \qquad :\Longleftrightarrow \qquad |\alpha| < |\beta|, \text{ or } |\alpha| = |\beta| \text{ and } \alpha \leqslant_{\text{lex}} \beta.$$

In the rest of this subsection we fix an admissible ordering $\leqslant$ of $\mathbb{N}^N$.

Proposition 1.1. *There exists a non-singular $N \times N$-matrix A with real entries such that for all α, β:*

$$\alpha \leqslant \beta \qquad \Longleftrightarrow \qquad A\alpha \leqslant_{\text{lex}} A\beta. \tag{1.1}$$

Here, on the right-hand side of the equivalence, we view the multi-indices α and β as column vectors, and $\leqslant_{\text{lex}}$ denotes the lexicographic ordering of $\mathbb{R}^N$.

For a proof see, e.g., [30]. We shall also need the following refinement. (A variant was stated in [10], with an incorrect proof.)

Lemma 1.2. *One can choose A such that, in addition to the property stated in the previous proposition, all entries of A are non-negative.*

Proof. In this proof we let i, j, k, l range over $\{1, \dots, N\}$. Let $A = \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix}$ be as in the proposition, where $a_i = (a_{i1}, \dots, a_{iN}) \in \mathbb{R}^N$ for every i. We first note that if i, l are such that $a_{il} \neq 0$ and $a_{ij} = 0$ for every j with $j < l$, then $a_{il} > 0$ (since our admissible ordering of $\mathbb{N}^N$ is a well-ordering). We now inductively define $b_1, \dots, b_N \in \mathbb{R}^N$ such that

(1) all entries b_{ij} of b_i are non-negative;
(2) if $b_{kj} > 0$ for some $k < i$ then $b_{ij} > 0$; and
(3) if $b_{ij} = 0$ then $a_{ij} = 0$.

Put $b_1 := a_1$; then clearly (1), (2) and (3) hold for $i = 1$. Suppose we have already defined $b_1, \dots, b_{i-1} \in \mathbb{R}^N$, for some $i > 1$, such that (1), (2) and (3) hold for $1, \dots, i-1$ in place of i, and for every j. Then set

$$b_i := a_i + \left(1 - \min_{b_{i-1,l} > 0} \frac{a_{il}}{b_{i-1,l}}\right) b_{i-1}.$$

We check that (1), (2) and (3) continue to hold for the index i and every j. If $b_{i-1,j} > 0$, then

$$b_{ij} = a_{ij} + \left(1 - \min_{b_{i-1,l} > 0} \frac{a_{il}}{b_{i-1,l}}\right) b_{i-1,j} > a_{ij} - \min_{b_{i-1,l} > 0} \frac{a_{il}}{b_{i-1,l}} b_{i-1,j} \geqslant 0,$$

hence (1)–(3) clearly hold. Now suppose $b_{i-1,j} = 0$. Then $a_{kj} = 0$ for every $k < i$, by (2) and (3), hence $b_{ij} = a_{ij} \geqslant 0$, so (1) holds; (2) and (3) hold trivially.

Clearly (1.1) is satisfied with $B = \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix}$ in place of A, and B is non-singular with non-negative entries. $\square$

We say that an $N \times N$-matrix A with real entries **represents** $\leqslant$ if (1.1) holds for all α, β. We also say that $\leqslant$ can be **represented by rational weights** if there is a non-singular $N \times N$-matrix A with rational entries representing $\leqslant$. Note that if $\leqslant$ can be represented by rational weights, then there is an $N \times N$-matrix with non-negative integer entries representing $\leqslant$ (by the proof of the lemma above, and after multiplying A by a suitable positive integer). Many common admissible orderings (for example, the lexicographic and degree-lexicographic ones) can be represented by rational weights. Although not every admissible ordering is so representable, every admissible ordering can be "finitely approximated" by one that is: given a finite set of multi-indices from $\mathbb{N}^N$, there exists an $N \times N$-matrix A with non-negative rational entries (but not necessarily non-singular) such that (1.1) holds for all $\alpha, \beta \in S$. (For logicians, this is immediate from the fact that the theory of divisible ordered abelian groups is model-complete, see [25].)

Let now A be a non-singular $N \times N$-matrix with non-negative integer entries a_{ij} representing $\leqslant$. For a multi-index α let $\mathrm{wt}_1(\alpha), \dots, \mathrm{wt}_N(\alpha)$ denote the entries of the column vector $A\alpha \in \mathbb{N}^N$. Note that $\mathrm{wt}_i(\alpha) \leqslant ||A||\,|\alpha|$ for $i = 1, \dots, N$, where $||A||$ is the largest among the entries of A. Given an integer $D > 1$ we define a weight function $\mathrm{wt} = \mathrm{wt}_{D,A}$ (taking non-negative integer values) on the set $\mathbb{N}^N$ by

$$\mathrm{wt}(\alpha) := \mathrm{wt}_1(\alpha)D^{N-1} + \mathrm{wt}_2(\alpha)D^{N-2} + \cdots + \mathrm{wt}_{N-1}(\alpha)D + \mathrm{wt}_N(\alpha).$$

We have $\mathrm{wt}(\alpha) = 0$ if and only if $\alpha = 0$. Moreover

$$|\alpha| \leqslant \mathrm{wt}(\alpha) \leqslant ||A||\,|\alpha|\,\frac{D^N - 1}{D - 1}. \tag{1.2}$$

(For the inequality on the right use that $\sum_i a_{ij} > 0$ for every j, since A is non-singular and $a_{ij} \geqslant 0$.) The weight function wt represents $\leqslant$ for multi-indices with small degree:

$$\alpha \leqslant \beta \quad\Longleftrightarrow\quad \mathrm{wt}(\alpha) \leqslant \mathrm{wt}(\beta), \qquad \text{if } |\alpha|, |\beta| < \frac{D}{||A||}.$$

We also have

$$\mathrm{wt}(\alpha + \beta) = \mathrm{wt}(\alpha) + \mathrm{wt}(\beta) \qquad \text{for all } \alpha, \beta.$$

1.2. **Monomials and K-linear spaces.** In the rest of this section we fix a positive N, we let K denote a field, and we let R be a K-linear space. A **monomial basis** of R is family $\{x^\alpha\}_\alpha$ of elements of R, indexed by the multi-indices in $\mathbb{N}^N$, which forms a basis of R. Of course, every K-linear space of countably infinite dimension has a monomial basis, for every positive N, but in the applications in the next sections, a specific monomial basis will always be given to us beforehand. Thus, in the following we assume that a monomial basis $\{x^\alpha\}_\alpha$ of R is fixed. We call a basis element x^α of R a **monomial** (of R), and we denote by $x^\diamond$ the set of monomials of R. Every element f of R can be uniquely written in the form

$$f = \sum_\alpha f_\alpha x^\alpha \qquad \text{where } f_\alpha \in K, \text{ with } f_\alpha = 0 \text{ for all but finitely many } \alpha,$$

and we define the **support** of such an f as

$$\operatorname{supp} f := \{x^\alpha : f_\alpha \neq 0\}.$$

We have $x^\alpha \neq x^\beta$ whenever $\alpha \neq \beta$, so every ordering $\leqslant$ of $\mathbb{N}^N$ yields an ordering (also denoted by $\leqslant$) of $x^\diamond$ in a natural way:

$$x^\alpha \leqslant x^\beta \quad :\Longleftrightarrow \quad \alpha \leqslant \beta \qquad \text{for multi-indices } \alpha, \beta.$$

We make $x^\diamond$ into a commutative monoid by defining

$$x^\alpha * x^\beta := x^{\alpha+\beta} \qquad \text{for multi-indices } \alpha, \beta.$$

Then the map

$$\alpha \mapsto x^\alpha \colon \mathbb{N}^N \to x^\diamond$$

is an isomorphism of monoids. A tuple of generators of the monoid $x^\diamond$ is given by $x = (x_1, \dots, x_N)$ where $x_i = x^{\varepsilon_i}$, with $\varepsilon_i =$ the i-th unit vector in $\mathbb{N}^N$.

There is a unique binary operation on R extending the operation $*$ on $x^\diamond$ and making the K-linear space R into a K-algebra. With this multiplication operation, of course, R is nothing but the ring $K[x]$ of polynomials in indeterminates $x = (x_1, \dots, x_N)$ with coefficients from K: the unique K-linear bijection $K[x] \to R$ which for each multi-index α sends the monomial $x_1^{\alpha_1} \cdots x_N^{\alpha_N}$ of $K[x]$ to the basis element x^α of R, is an isomorphism of K-algebras. However, in our applications below, the K-linear space R

will already come equipped with a binary operation making it into a K-algebra, and this operation will usually not agree with $*$ on $x^\diamond$ (in fact, not even restrict to an operation on $x^\diamond$). In order to clearly separate the combinatorial objects arising in the study of the (generally, non-commutative) K-algebras later on, we chose to introduce the extra bit of terminology concerning monomial bases.

A monomial x^α divides a monomial x^β (or x^β is divisible by x^α) if $x^\beta = x^\alpha * x^\gamma$ for some multi-index γ; in symbols: $x^\alpha | x^\beta$. If I is an ideal of $x^\diamond$, that is, if $x^\alpha \in I \Rightarrow x^\alpha * x^\beta \in I$ for all α, β, then there exist $x^{\alpha(1)}, \dots, x^{\alpha(k)} \in I$ such that each monomial in I is divisible by some $x^{\alpha(i)}$. (By Dickson's Lemma, [17, Lemma 1.1].) Given monomials x^α and x^β, the **least common multiple** of x^α and x^β is the monomial

$$\operatorname{lcm}(x^\alpha, x^\beta) = x^\gamma \qquad \text{where } \gamma_i = \max\{\alpha_i, \beta_i\} \text{ for } i = 1, \dots, N.$$

Let now $\leqslant$ be a total ordering of $\mathbb{N}^N$. Given a non-zero element f of R, there is a unique λ such that

$$f = f_\lambda x^\lambda + \sum_{\alpha < \lambda} f_\alpha x^\alpha, \qquad f_\lambda \neq 0.$$

We call

$$\operatorname{lc}(f) = f_\lambda, \quad \operatorname{lm}(f) = x^\lambda, \quad \operatorname{lt}(f) = f_\lambda x^\lambda$$

the **leading coefficient**, **leading monomial**, and **leading term**, respectively, of f with respect to $\leqslant$. It is convenient to define $\operatorname{lm}(0) := 0$ and extend $\leqslant$ to a total ordering on the set $x^\diamond \cup \{0\}$ by declaring $0 < x^\alpha$ for all α. We also declare $\operatorname{lc}(0) := \operatorname{lt}(0) := 0$. We extend the notation lm to subsets of R by a slight abuse: for $S \subseteq R$ put

$$\operatorname{lm}(S) := \big\{ \operatorname{lm}(f) : 0 \neq f \in S \big\} \subseteq x^\diamond.$$

1.3. **Monomial cones and monomial ideals.** By abuse of notation, we write $y \subseteq x$ to indicate that y is a subset of $\{x_1, \dots, x_N\}$, and for $y \subseteq x$ we let $y^\diamond$ be the submonoid of $(x^\diamond, *)$ generated by y. (So $\varnothing^\diamond = \{1\}$.)

A **monomial cone** defined by a pair (w, y), where $w \in x^\diamond$ and $y \subseteq x$, is the K-linear subspace $C(w, y)$ of R generated by $w * y^\diamond$. Note that $C(w, \varnothing) = \{0\}$ for every $w \in x^\diamond$, and $C(1, x) = R$. Also, if $y \subseteq y' \subseteq x$ then $C(w, y) \subseteq C(w, y')$. We refer to [9, Section 3] for how to represent monomial cones graphically in the (slightly misleading) case $N = 2$. If we identify R with the commutative polynomial ring $R = K[x]$ as explained above, then $C(w, y)$ is nothing but the K-linear subspace $wK[y]$ of $K[x]$.

We say that a K-linear subspace I of R is a **monomial ideal** if I is spanned by monomials, and $C(w, x) \subseteq I$ for all monomials $w \in I$. (Hence, if $R = K[x]$, then I is a monomial ideal of $K[x]$ in the usual sense of the word.) A set of **generators** for a monomial ideal I of R is defined to be a set of monomials F such that $I = \sum_{w \in F} C(w, x)$ (so the set $F * x^\diamond$ generates I as a K-linear space). A K-linear subspace of R is a monomial ideal if and only if the set of monomials in I is an ideal of $(x^\diamond, *)$. Every monomial subspace of R has a unique minimal set of generators, which is finite.

Given a monomial ideal I of R and a monomial w we put

$$(I : w) := \text{the } K\text{-linear subspace of } R \text{ generated by } \{v \in x^\diamond : w * v \in I\},$$

a monomial ideal of R containing I.

Let now M be a K-linear subspace of R generated by monomials, and let I be a monomial ideal of R. Then the K-linear subspace $M \cap I$ of M has a natural complement: we have

$$M = (M \cap I) \oplus \operatorname{nf}_I(M),$$

where $\operatorname{nf}_I(M)$ denotes the K-linear subspace of R generated by the monomials in $M \setminus I$.

2. Preliminaries on Algebras over Fields

In this section we let K be a field (of arbitrary characteristic). All K-algebras will be assumed to be associative with unit element 1. Given a subset G of a K-algebra R we denote by (G) the *left* ideal of R generated by G. We also let $\leqslant$ be an admissible ordering of $\mathbb{N}^N$.

2.1. **Multi-filtered K-algebras and modules.** A **multi-filtration** on R (indexed by $\mathbb{N}^N$) is a family $\{R_{(\leqslant\alpha)}\}_\alpha$ of K-linear subspaces of R such that:

(1) $1 \in R_{(\leqslant 0)}$;
(2) $\alpha \leqslant \beta \Rightarrow R_{(\leqslant\alpha)} \subseteq R_{(\leqslant\beta)}$;
(3) $R_{(\leqslant\alpha)} \cdot R_{(\leqslant\beta)} \subseteq R_{(\leqslant\alpha+\beta)}$;
(4) $\bigcup_\alpha R_{(\leqslant\alpha)} = R$.

A **multi-filtered K-algebra** is a K-algebra equipped with a multi-filtration. Suppose R is a multi-filtered K-algebra. A **multi-filtration** on a left R-module M (indexed by $\mathbb{N}^N$) is a family $\{M_{(\leqslant\alpha)}\}_\alpha$ of K-linear subspaces of M such that:

(1) $\alpha \leqslant \beta \Rightarrow M_{(\leqslant\alpha)} \subseteq M_{(\leqslant\beta)}$;
(2) $R_{(\leqslant\alpha)} \cdot M_{(\leqslant\beta)} \subseteq M_{(\leqslant\alpha+\beta)}$;
(3) $\bigcup_\alpha M_{(\leqslant\alpha)} = M$.

A **multi-filtered left R-module** is a left R-module equipped with a multi-filtration. Suppose now that in addition M is a multi-filtered left R-module. For every α the set $M_{(<\alpha)} := \bigcup_{\beta<\alpha} M_{(\leqslant\alpha)}$ is a K-linear subspace of M. Here $M_{(<0)} := \{0\}$ by convention. For every non-zero $f \in M$ there exists a unique α with $f \in M_{(\leqslant\alpha)} \setminus M_{(<\alpha)}$, and we call $\alpha = \deg(f)$ the **degree** of f. Given a left R-submodule M' of M, we always construe M' as a multi-filtered left R-module by means of the multi-filtration $\{M'_{(\leqslant\alpha)}\}_\alpha$ given by $M'_{(\leqslant\alpha)} := M' \cap M_{(\leqslant\alpha)}$ for every α, and we make the quotient M/M' into a multi-filtered left R-module by the multi-filtration induced on M/M' from M by the natural surjection $M \to M/M'$:

$$(M/M')_{(\leqslant\alpha)} := (M_{(\leqslant\alpha)} + M')/M' \quad \text{for every } \alpha.$$

For a two-sided ideal I of R, the induced filtration makes R/I a multi-filtered K-algebra.

2.2. **Multi-graded K-algebras and modules.** A **multi-grading** on R (indexed by $\mathbb{N}^N$) is a family $\{R_{(\alpha)}\}_\alpha$ of K-linear subspaces of R such that

(1) $R = \bigoplus_\alpha R_{(\alpha)}$ (internal direct sum of K-linear subspaces of R);
(2) $R_{(\alpha)} \cdot R_{(\beta)} \subseteq R_{(\alpha+\beta)}$ for all multi-indices α, β.

A K-algebra equipped with a multi-grading is called a **multi-graded K-algebra**. Suppose R is multi-graded. A **multi-grading** on a left R-module M (indexed by $\mathbb{N}^N$) is a family $\{M_{(\alpha)}\}_\alpha$ of K-linear subspaces of M such that

(1) $M = \bigoplus_\alpha M_{(\alpha)}$;
(2) $R_{(\alpha)} \cdot M_{(\beta)} \subseteq M_{(\alpha+\beta)}$ for all α, β.

A left R-module equipped with a multi-grading is called a **multi-graded left R-module**. Let M be a multi-graded left R-module. We call the K-linear subspace $M_{(\alpha)}$ of M the **homogeneous component** of degree α of M. We always view R as a multi-filtered K-algebra, and M as a multi-filtered left R-module by means of the natural multi-filtrations $\{R_{(\leqslant\alpha)}\}_\alpha$ and $\{M_{(\leqslant\alpha)}\}_\alpha$ given by

$$R_{(\leqslant\alpha)} := \bigoplus_{\beta\leqslant\alpha} R_{(\beta)}, \quad M_{(\leqslant\alpha)} := \bigoplus_{\beta\leqslant\alpha} M_{(\beta)} \qquad \text{for every } \alpha.$$

Every $f \in M$ has a unique representation in the form $f = \sum_\alpha f_{(\alpha)}$ where $f_{(\alpha)} \in M_{(\alpha)}$ for all α, and $f_{(\alpha)} = 0$ for all but finitely many α. We call $f_{(\alpha)}$ the homogeneous component of degree α of f. Similarly, given a K-linear subspace V of M which is homogeneous (i.e., for $f \in M$ we have $f \in V$ if and only if $f_{(\alpha)} \in V$ for each α), the homogeneous component of degree α of V is denoted by $V_{(\alpha)} := V \cap M_{(\alpha)}$, so

$$V = \bigoplus_\alpha V_{(\alpha)} \qquad \text{(internal direct sum of } K\text{-linear subspaces of } M).$$

If M' is a homogeneous left R-submodule of M, then the $M'_{(\alpha)}$ furnish M' with a multi-grading making M' a multi-graded left R-module, and we make M/M' into a multi-graded left R-module by the multi-grading induced from M:

$$(M/M')_{(\alpha)} := (M_{(\alpha)} + M')/M' \quad \text{for every } \alpha.$$

The multi-filtration of M/M' associated to this multi-grading agrees with the multi-filtration of M/M' induced from the multi-filtered left R-module M. If I is a two-sided ideal of R, then R/I a multi-graded K-algebra by means of the induced multi-grading.

2.3. **The associated multi-graded algebra.** Suppose R is multi-filtered, and let M be a multi-filtered left R-module M. Consider the left R-module

$$\operatorname{gr} M = \bigoplus_\alpha (\operatorname{gr} M)_{(\alpha)} \qquad \text{with } (\operatorname{gr} M)_{(\alpha)} = M_{(\leqslant\alpha)}/M_{(<\alpha)}$$

with

$$(f + R_{(<\alpha)}) \cdot (g + M_{(<\beta)}) = f \cdot g + M_{(<\alpha+\beta)}$$

for all α, β, and $f \in R_{(\leqslant\alpha)}$, $g \in M_{(\leqslant\beta)}$. For $M = R$ we obtain a multi-graded K-algebra $\operatorname{gr} R$, called the **multi-graded K-algebra associated to R.** In general, $\operatorname{gr} M$ is a multi-graded left $\operatorname{gr} R$-module, the **multi-graded left $\operatorname{gr} R$-module associated to M.** For non-zero $f \in M$ of degree α we denote by

$$\operatorname{gr} f := f + M_{(<\alpha)} \in (\operatorname{gr} M)_{(\alpha)}$$

the **initial form** (or **symbol**) of f, and we put $\operatorname{gr} 0 := 0 \in \operatorname{gr} M$. Given a left R-submodule M' of M, the inclusion $M' \to M$ induces an embedding $\operatorname{gr} M' \to \operatorname{gr} M$ of multi-graded left R-modules, and we identify $\operatorname{gr} M'$ with its image under this embedding.

2.4. **The Rees algebra.** Suppose R is multi-filtered. The **Rees algebra** of R is the multi-graded K-algebra

$$R^* = \bigoplus_\alpha (R^*)_{(\alpha)} \qquad \text{with } (R^*)_{(\alpha)} = R_{(\leqslant\alpha)}.$$

For a non-zero element f of R of degree α we let $f^* := f \in (R^*)_{(\alpha)}$ be the **homogenization** of f; by convention $0^* := 0$. Let I be a two-sided ideal of R. We let I^* be the two-sided ideal of R^* generated by all f^* with $f \in I$; the ideal I^* is homogeneous, and is called the **homogenization** of I. The natural surjection $R \to R/I$ is a morphism of multi-filtered K-algebras which induces a surjective morphism $R^* \to (R/I)^*$ of multi-graded K-algebras whose kernel is I^*; the induced homomorphism $R^*/I^* \to (R/I)^*$ is an isomorphism of multi-graded K-algebras. The natural inclusions $(R^*)_{(\alpha)} = R_{(\leqslant\alpha)} \subseteq R$ combine to a K-linear map $h \mapsto h_*\colon R^* \to R$ which is a surjective homomorphism of multi-graded K-algebras satisfying $(f^*)_* = f$ for all $f \in R$. For $h \in R^*$ the element h_* of R is called the **dehomogenization** of h. We extend this notation to subsets of R^*: $H_* := \{h_* : h \in H\}$ for $H \subseteq R^*$. If J is a left ideal of R^*, then J_* is a left ideal of R. Hence if $H \subseteq R^*$ then $(H)_* = (H_*)$.

2.5. **Filtered and graded algebras.** By a **filtered K-algebra** we will mean an multi-filtered algebra whose filtration is indexed by $\mathbb{N}$, and similarly a multi-graded K-algebra whose grading is indexed by $\mathbb{N}$ is just called a **graded K-algebra.** Analogous terminology will be used in the case of left R-modules. (Most of our multi-filtered or multi-graded objects will actually be filtered, respectively graded; we introduced the more general concepts in order to be able to speak about the associated multi-graded algebra of an algebra of solvable type with respect to the "fine filtration"; see Corollary 2.5.)

Suppose $R = \bigcup_d R_{(\leqslant d)}$ is a filtered K-algebra. We denote by t the **canonical element** of R^*, that is, the unit 1 of R, considered as an element of $(R^*)_{(1)} = R_{(\leqslant 1)}$. In this case the graded K-algebra associated to R and the Rees algebra of R are related as follows: the natural surjections

$$(R^*)_{(d)} = R_{(\leqslant d)} \to R_{(\leqslant d)}/R_{(<d)} = (\operatorname{gr} R)_{(d)}$$

combine to a surjective K-algebra morphism $R^* \to \operatorname{gr} R$ which has kernel R^*t and hence induces an isomorphism

$$R^*/R^*t \xrightarrow{\cong} \operatorname{gr} R \tag{2.1}$$

of graded K-algebras.

2.6. **Homogenization of graded algebras.** Suppose now that $R = \bigoplus_d R_{(d)}$ is a graded K-algebra. We make the ring $R[T]$ of polynomials in one commuting indeterminate T over R into a graded K-algebra using the grading

$$R[T] = \bigoplus_d R[T]_d \qquad \text{with } R[T]_{(d)} := \bigoplus_{i+j=d} R_{(i)}T^j.$$

The K-linear map $R[T] \to R^*$ with $fT^j \mapsto ft^j$ for all $f \in R_{(i)}$ and $i, j \in \mathbb{N}$ is an isomorphism of graded K-algebras. In the following we always identify the Rees algebra of a graded K-algebra R with the graded K-algebra $R[T]$. Then the canonical element of R^* is T, and for non-zero $f \in R$ of degree d we have

$$f^* = \sum_{i=0}^{d} f_{(i)} T^{d-i} \in (R^*)_{(d)},$$

and for $h = \sum_{i=0}^n h_i T^i \in R^*$ we get $h_* = \sum_{i=0}^n h_i \in R$.

2.7. **The opposite algebra and the enveloping algebra.** The **opposite algebra** of R is the K-algebra R^{op} whose underlying K-linear space is the same as that of R and whose multiplication operation $\cdot^{\mathrm{op}}$ is given by $a \cdot^{\mathrm{op}} b = b \cdot a$ for $a, b \in R$. The **enveloping algebra** of R is the K-algebra $R^{\mathrm{env}} := R \otimes_K R^{\mathrm{op}}$. There is a natural one-to-one correspondence between R-bimodules and left R^{env}-modules: every R-bimodule M also has a left R^{env}-module structure given by

$$(a \otimes b) \cdot f = afb \qquad \text{for } a \in R,\, b \in R^{\mathrm{op}}, \text{ and } f \in M,$$

and conversely, every left R^{env}-module M' also carries an R-bimodule structure with

$$af'b = (a \otimes b)f' \qquad \text{for } a \in R,\, b \in R^{\mathrm{op}}, \text{ and } f' \in M'.$$

There is a surjective morphism $\mu\colon R^{\mathrm{env}} \to R$ of left R^{env}-modules with

$$\mu(a \otimes b) = ab \qquad \text{for } a \in R,\, b \in R^{\mathrm{op}}.$$

For every n, acting component by component, μ induces a surjective morphism $(R^{\mathrm{env}})^n \to R^n$ of left R^{env}-modules, which we also denote by μ. Thus for every R-bisubmodule M of R^n we obtain a left R^{env}-submodule $\mu^{-1}(M)$ of $(R^{\mathrm{env}})^n$ containing $\ker \mu$, and the image

$\mu(M')$ of a left R^{env}-submodule M' of $(R^{\mathrm{env}})^n$ with $\ker\mu \subseteq M'$ is an R-bisubmodule of R^n. The kernel of μ is generated by

$$(f_1 \otimes 1, \dots, f_n \otimes 1) - (1 \otimes f_1, \dots, 1 \otimes f_n) \qquad (f_1, \dots, f_n \in R).$$

2.8. **Non-commutative polynomials.** In the following we let $X = (X_1, \dots, X_N)$ be a tuple of N distinct indeterminates over K and denote by X^* the free monoid generated by $\{X_1, \dots, X_N\}$. The free K-algebra $K\langle X\rangle = K\langle X_1, \dots, X_N\rangle$ generated by X (that is, the monoid algebra of X^* over K) has a natural grading

$$K\langle X\rangle = \bigoplus_d K\langle X\rangle_{(d)}$$

defined by the length of words in X^*. Let I be a two-sided ideal of $K\langle X\rangle$. The K-algebra $R = K\langle X\rangle/I$ is generated by the cosets $X_i + I$ ($i = 1, \dots, N$). Let T be an indeterminate over K distinct from $X_1, \dots, X_N$. We identify the Rees algebra $K\langle X\rangle^*$ of $K\langle X\rangle$ with the graded K-algebra $K\langle X\rangle[T]$ as explained in the previous subsections; similarly, the Rees algebra R^* of R will be identified with $K\langle X\rangle^*/I^* = K\langle X\rangle[T]/I^*$. For a non-zero $f \in K\langle X\rangle$ of degree d we define the homogeneous polynomial

$$f^{\mathrm{h}} := \sum_{i=0}^{d} f_{(i)} T^{d-i} \in K\langle X, T\rangle. \tag{2.2}$$

The two-sided ideal I^{h} of $K\langle X, T\rangle$ generated by f^{h} for non-zero $f \in I$ and the polynomials $X_i T - T X_i$ ($i = 1, \dots, N$) is homogeneous, and the natural K-linear map $K\langle X, T\rangle \to K\langle X\rangle[T]$ induces an isomorphism

$$K\langle X, T\rangle/I^{\mathrm{h}} \xrightarrow{\cong} R^* = K\langle X\rangle[T]/I^* \tag{2.3}$$

of graded K-algebras.

2.9. **Affine algebras.** *In the rest of this section, we let R be a finitely generated K-algebra and we fix a tuple $x = (x_1, \dots, x_N)$ of elements of R.* For a multi-index $\alpha = (\alpha_1, \dots, \alpha_N)$ put $x^\alpha := x_1^{\alpha_1} \cdots x_N^{\alpha_N}$. We say that the K-algebra R is **affine** with respect to x if the family $\{x^\alpha\}_\alpha$ is a monomial basis of the K-linear space R. (Note that then $x_1, \dots, x_N$ generate R as a K-algebra.) Usually, we obtain affine K-algebras by specifying a **commutation system** in $K\langle X\rangle$, that is, a family $\mathcal{R} = (R_{ij})_{1 \leqslant i < j \leqslant N}$ of $\binom{N}{2} = N(N-1)/2$ polynomials

$$R_{ij} = X_j X_i - c_{ij} X_i X_j - P_{ij}$$
$$\text{where } 0 \neq c_{ij} \in K \text{ and } P_{ij} \in \bigoplus_\alpha K X^\alpha \text{ for } 1 \leqslant i < j \leqslant N. \tag{2.4}$$

Let $\mathcal{R} = (R_{ij})$ be a commutation system and $I = I(\mathcal{R})$ be the two-sided ideal of $K\langle X\rangle$ generated by the polynomials R_{ij} ($1 \leqslant i < j \leqslant N$), and suppose $R = K\langle X > I$ with $x_i = X_i + I$ ($i = 1, \dots, N$). We say that the finitely presented K-algebra R is **defined by $\mathcal{R}$.** We construe $K\langle X\rangle$ as a filtered K-algebra via filtration by degree of polynomials in $K\langle X\rangle$, and we equip R with the filtration induced by the natural surjection $K\langle X\rangle \to K\langle X\rangle/I = R$. We call the filtration of R arising in this way the **standard filtration** of R (with respect to $x_1, \dots, x_N$). If R turns out to be affine, then the generators $x_1, \dots, x_N$ of the K-algebra R have degree 1.

The proposition below contains a useful criterion, due to Bergman [4], for verifying that $K\langle X\rangle/I$ is affine. Before we can state it, we need to introduce some further notation. For a word $w = X_{i_1} \cdots X_{i_m} \in X^*$ with $i_1, \dots, i_m \in \{1, \dots, N\}$ we define the **misordering**

index $i(w)$ of w as the number of pairs (k,l) with $1 \leqslant k < l \leqslant m$ and $i_k > i_l$. We define a (strict) ordering of X^* by setting $v \prec w$ if v is of smaller length than w or if v is a permutation of the symbols of w with $i(v) < i(w)$. Note that $\prec$ is only a partial ordering of X^* if $N > 0$. Given elements a and b of a ring, we put $[a,b] := ab - ba$.

Proposition 2.1. *Suppose $\mathcal{R} = (R_{ij})$ is a commutation system with R_{ij} as in* (2.4)*, such that $c_{ij} = 1$ for $1 \leqslant i < j \leqslant N$. Then the K-algebra defined by $\mathcal{R}$ is affine if and only if for $1 \leqslant i < j < k \leqslant N$, the polynomial*

$$[X_i, P_{jk}] + [X_j, [X_k, X_i]] + [X_k, P_{ij}]$$

is a K-linear combination of polynomials of the form $vQ_{rs}w$ where $v, w \in X^*$ and $1 \leqslant r < s \leqslant N$ such that $vX_sX_rw \prec X_kX_jX_i$.

Affineness of K-algebras may also be shown with Mora's theory [28] of Gröbner bases for two-sided ideals in $K\langle X\rangle$, which we won't discuss here; cf. [17, Theorem 1.11].

Examples 2.2. We mention some prominent examples for K-algebras which can easily be seen to be affine using Proposition 2.1:

(1) A K-algebra is called **semi-commutative** if for every pair f, g of its elements there is a non-zero $c \in K$ with $fg = cgf$. If $P_{ij} = 0$ for $1 \leqslant i < j \leqslant N$ in (2.4), then the K-algebra defined by $\mathcal{R}$ is affine and semi-commutative. If in addition $c_{ij} = 1$ for $1 \leqslant i < j \leqslant N$, then the K-algebra defined by $\mathcal{R}$ is naturally isomorphic to the K-algebra $K[x] = K[x_1, \dots, x_N]$ of commutative polynomials in the tuple of indeterminates $x = (x_1, \dots, x_N)$ with coefficients in K.

(2) The n-th Weyl algebra $A_n(K)$ over K is the K-algebra generated by $N = 2n$ generators $x_1, \dots, x_n, \partial_1, \dots, \partial_n$ subject to the relations

$$\begin{array}{lll} x_jx_i = x_ix_j, & \partial_j\partial_i = \partial_i\partial j & \text{for } 1 \leqslant i < j \leqslant n, \\ \partial_jx_i = x_i\partial_j & & \text{for } 1 \leqslant i, j \leqslant n,\ i \neq j, \\ \partial_ix_i = x_i\partial_i + 1 & & \text{for } 1 \leqslant i \leqslant n. \end{array}$$

Using Proposition 2.1 one sees easily that $A_n(K)$ is affine with respect to the generating tuple $(x, \partial) := (x_1, \dots, x_n, \partial_1, \dots, \partial_n)$. The standard filtration of $A_n(K)$ is also known as the Bernstein filtration of $A_n(K)$.

(3) Let $\mathfrak{g}$ be a Lie algebra over K of dimension n, and let $\{x_1, \dots, x_N\}$ be a basis of $\mathfrak{g}$. The universal enveloping algebra of $\mathfrak{g}$ is a K-algebra $U(\mathfrak{g})$ which contains $\mathfrak{g}$ as K-linear subspace and is generated by $x_1, \dots, x_N$ subject to the relations

$$x_jx_i = x_ix_j - [x_j, x_i]_{\mathfrak{g}} \quad \text{for } 1 \leqslant i < j \leqslant N.$$

The fact that $U(\mathfrak{g})$ is affine with respect to the tuple $(x_1, \dots, x_N)$ is known as the Poincaré-Birkhoff-Witt Theorem (cf. [4, Theorem 3.1]).

We say that a commutation system $\mathcal{R} = (R_{ij})$ as above is **quadric** if every polynomial P_{ij} has degree $\leqslant 2$, **linear** if every P_{ij} has degree $\leqslant 1$, and **homogeneous** if all R_{ij} are either zero or homogeneous (necessarily of degree 2). All examples of affine K-algebras given above are defined by linear commutation systems (and the semi-commutative even by homogeneous ones).

2.10. **Algebras of solvable type.** The definition below is due to Kandri-Rody and Weispfenning [17]. Recall that $\leqslant$ denotes an admissible ordering of $\mathbb{N}^N$.

Definition 2.3. The K-algebra R is said to be **of solvable type** with respect to the fixed admissible ordering $\leqslant$ of $\mathbb{N}^N$ and the tuple $x = (x_1, \dots, x_N) \in R^N$ if R is affine with respect to x, and for $1 \leqslant i < j \leqslant N$ there are $c_{ij} \in K$, $c_{ij} \neq 0$, and $p_{ij} \in R$ such that

$$x_j x_i = c_{ij} x_i x_j + p_{ij} \quad \text{and} \quad \operatorname{lm}(p_{ij}) < x_i x_j.$$

(Note that the c_{ij} and p_{ij} are then uniquely determined.)

The following fact is proved in [17, Lemma 1.4]:

Lemma 2.4. *Suppose R is of solvable type with respect to $\leqslant$ and x. Then*

$$\operatorname{lm}(f \cdot g) = \operatorname{lm}(f) * \operatorname{lm}(g) \qquad \text{for non-zero } f, g \in R.$$

In particular, R is an integral domain.

If R is semi-commutative, then R is of solvable type with respect to x and every admissible ordering of $\mathbb{N}^N$, and each homogeneous component $R_{(\alpha)}$ of R has the form $R_{(\alpha)} = Kx^\alpha$. By the preceding lemma we have:

Corollary 2.5. *Suppose R is of solvable type with respect to $\leqslant$ and x. Then the family $\{R_{(\leqslant\alpha)}\}_\alpha$ with*

$$R_{(\leqslant\alpha)} := \bigoplus_{\beta\leqslant\alpha} Kx^\beta$$

is a multi-filtration of R, and its associated multi-graded K-algebra $\operatorname{gr}_\leqslant R$ is semi-commutative with respect to $\leqslant$ and $\xi = (\xi_1, \dots, \xi_N)$, where $\xi_i := \operatorname{gr}_\leqslant x_i$ for $i = 1, \dots, N$. If $c_{ij} = 1$ for $1 \leqslant i < j \leqslant N$, then $\operatorname{gr}_\leqslant R = K[\xi]$ is commutative.

Here is a way of constructing K-algebras of solvable type [17, Theorem 1.7]:

Proposition 2.6. *Let $\mathcal{R} = (R_{ij})$ be a commutation system with R_{ij} as in* (2.4)*, let $I = I(\mathcal{R})$, and suppose $R = K\langle X\rangle/I$ with $x_i = X_i + I$ for $1 \leqslant i \leqslant N$. Then R is of solvable type with respect to the admissible ordering $\leqslant$ and the tuple $x = (x_1, \dots, x_N)$ of generators for R if and only if the following two conditions are satisfied:*

1. $\operatorname{lm}(P_{ij}) < \operatorname{lm}(X_i X_j)$ *for* $1 \leqslant i < j \leqslant N$*, and*
2. $I \cap \bigoplus_\alpha KX^\alpha = \{0\}$.

Remark 2.7. Suppose that R is affine with respect to $\leqslant$ and x, and let $\pi\colon K\langle X\rangle \to R$ be the surjective K-algebra homomorphism with $X_i \mapsto x_i$ for $i = 1, \dots, N$. Let $\mathcal{R} = (R_{ij})$ be a commutation system as in (2.4) satisfying condition (1) in Proposition 2.6 and with $\ker\pi$ containing $I = I(\mathcal{R})$. Then $I = \ker\pi$, so R is of solvable type with respect to $\leqslant$ and x. (To see this note that $\ker\pi \cap \bigoplus_\alpha KX^\alpha = \{0\}$ since R is affine; in particular, $I \cap \bigoplus_\alpha KX^\alpha = \{0\}$, hence $K\langle X\rangle = I \oplus \bigoplus_\alpha KX^\alpha$ by Proposition 2.6, and thus $I = \ker\pi$.)

Every K-algebra of solvable type arises as described in Proposition 2.6: Suppose $R = K\langle x\rangle$ is of solvable type as in Definition 2.3; let π be as in Remark 2.7, for $1 \leqslant i < j \leqslant N$ let P_{ij} be the unique polynomial in $\bigoplus_\alpha KX^\alpha$ such that $\pi(P_{ij}) = p_{ij}$, and define the commutation system $\mathcal{R} = (R_{ij})$ as in (2.4). Then clearly $\ker\pi$ contains $I = I(\mathcal{R})$. So $\ker\pi = I$ by the preceding remark, and π induces an isomorphism $K\langle X\rangle/I \to R$. Hence we may define properties of a K-algebra of solvable type in terms of the unique commutation system that defines it. For example, we say that a K-algebra of solvable type is **quadric** or **homogeneous** if its defining commutation system is quadric or homogeneous, respectively.

Condition (1) in the previous proposition automatically holds if $P_{ij} \in K$ for $1 \leqslant i < j \leqslant N$, or if $\leqslant$ is degree-compatible and $\deg P_{ij} < 2$ for $1 \leqslant i < j \leqslant N$. Hence the n-th Weyl algebra $A_n(K)$ over K is of solvable type with respect to the generating tuple (x, ∂) and *every* admissible ordering of $\mathbb{N}^{2n}$. Similarly, the universal enveloping algebra of an N-dimensional Lie algebra over K is of solvable type with respect to the generating tuple x and every admissible ordering of $\mathbb{N}^N$. The only commutative K-algebra of solvable type with respect to x is the commutative polynomial ring $K[x_1, \dots, x_N]$, which is of solvable type with respect to every admissible ordering of $\mathbb{N}^N$. All of those examples are quadric.

Lemma 2.8. *Suppose that $N > 0$ and x_N is in the center of R. Let $S = R/Rx_N$, and for $i = 1, \dots, N-1$ let y_i be the image of x_i under the natural surjection $R \to S$.*

(1) *If the K-algebra R is affine with respect to x, then S is affine with respect to $y = (y_1, \dots, y_{N-1})$.*
(2) *If R is of solvable type with respect to $\leqslant$ and the tuple x, then S is of solvable type with respect to the restriction of $\leqslant$ to $\mathbb{N}^{N-1}$ and y, and if in addition R is quadric (homogeneous), then S is quadric (homogeneous, respectively).*

Proof. Part (1) is clear. For (2), suppose R is of solvable type with respect to $\leqslant$ and x. Let $\mathcal{R} = (R_{ij})_{1\leqslant i<j\leqslant N}$ be the commutation system in $K\langle X\rangle$ defining R. Let $Y = (Y_1, \dots, Y_{N-1})$ be a tuple of distinct indeterminates over K. The commutation system $\mathcal{S} = (S_{ij})_{1\leqslant i<j<N}$ in $K\langle Y\rangle$ with $S_{ij} := R_{ij}(Y, 0)$ for $1 \leqslant i < j < N$ satisfies condition (1) in Proposition 2.6, and $I(\mathcal{S})$ is contained in the kernel of the K-algebra homomorphism $K\langle Y\rangle \to S$ with $Y_i \mapsto y_i$ for $i = 1, \dots, N-1$. Hence by (1) and Remark 2.7, S is of solvable type with respect to the restriction of $\leqslant$ to $\mathbb{N}^{N-1}$ and y. If $\mathcal{R}$ is quadric (homogeneous) then $\mathcal{S}$ clearly is quadric (homogeneous, respectively). □

In the rest of this section, $\pi\colon K\langle X\rangle \to R$ is the K-algebra homomorphism with $\pi(X_i) = x_i$ for $i = 1, \dots, N$. We also let $\mathcal{R} = (R_{ij})$ be a commutation system defining $R = K\langle x\rangle$, with R_{ij} as in (2.4)*, and we assume that R is of solvable type with respect to $\leqslant$ and $x = (x_1, \dots, x_N)$. We put $p_{ij} := \pi(P_{ij})$ for $1 \leqslant i < j \leqslant N$.*

The opposite K-algebra R^{op} of R is again a K-algebra of solvable type in a natural way. To see this define the "write oppositely automorphism" of the K-algebra $K\langle X\rangle$ by

$$(X_{i_1} \cdots X_{i_r})^{\mathrm{op}} = X_{i_r} \cdots X_{i_1} \qquad \text{for all } i_1, \dots, i_r \in \mathbb{N}.$$

Also set $\alpha^{\mathrm{op}} := (\alpha_N, \dots, \alpha_1)$ for every multi-index $\alpha = (\alpha_1, \dots, \alpha_N)$ and define the "opposite ordering" of $\mathbb{N}^N$ by

$$\alpha \leqslant^{\mathrm{op}} \beta \quad :\Longleftrightarrow \quad \alpha^{\mathrm{op}} \leqslant \beta^{\mathrm{op}} \qquad \text{for all multi-indices } \alpha, \beta.$$

Then $\mathcal{R}^{\mathrm{op}} := (R_{ij}^{\mathrm{op}})$ is a commutation system defining a K-algebra of solvable type with respect to $\leqslant^{\mathrm{op}}$ and $x^{\mathrm{op}} := (x_N, \dots, x_1)$, which can be naturally identified with R^{op}.

2.11. **The enveloping algebra of an algebra of solvable type.** The class of K-algebras of solvable type is closed under tensor products. More precisely, let $\leqslant'$ be an admissible ordering of $\mathbb{N}^{N'}$ (where $N' \in \mathbb{N}$), and let $\mathcal{R}' = (R'_{ij})$ be a commutation system in $K\langle Y\rangle = K\langle Y_1, \dots, Y_{N'}\rangle$, with

$$R'_{ij} = Y_j Y_i - c'_{ij} Y_i Y_j - P'_{ij} \qquad (1 \leqslant i < j \leqslant N')$$

where $0 \neq c'_{ij} \in K$ and $P'_{ij} \in \bigoplus_{\alpha'} KY^{\alpha'}$. (Here and below, α' ranges over $\mathbb{N}^{N'}$.) Let $R' = K\langle Y\rangle/I(\mathcal{R}')$, with natural surjection $\pi'\colon K\langle Y\rangle \to R'$, and let $y_j := \pi'(Y_j)$ for $j = 1, \dots, N'$ and $p'_{ij} := \pi'(P'_{ij})$ for $1 \leqslant i < j \leqslant N'$. Suppose that R' is of solvable type with respect to $\leqslant'$ and $y = (y_1, \dots, y_{N'})$. The K-algebra $S := R \otimes_K R'$ is generated by the $(N + N')$-tuple

$$(x_1 \otimes 1, \dots, x_N \otimes 1, 1 \otimes y_1, \dots, 1 \otimes y_{N'}). \tag{2.5}$$

We have the following (see [31, Proposition 1]):

Proposition 2.9. *The K-algebra $S = R \otimes_K R'$ is of solvable type with respect to the lexicographic product of the orderings $\leqslant$ and $\leqslant'$, and the $(N + N')$-tuple of generators* (2.5). *The commutator relations of S are*

$$\begin{aligned} (x_j \otimes 1)(x_i \otimes 1) &= c_{ij}(x_i \otimes 1)(x_j \otimes 1) + p_{ij} \otimes 1 \qquad (1 \leqslant i < j \leqslant N) \\ (x_i \otimes 1)(1 \otimes y_j) &= (1 \otimes y_j)(x_i \otimes 1) \qquad (1 \leqslant i \leqslant N,\ 1 \leqslant j \leqslant N') \\ (1 \otimes y_j)(1 \otimes y_i) &= c'_{ij}(1 \otimes y_i)(1 \otimes y_j) + 1 \otimes p'_{ij} \qquad (1 \leqslant i < j \leqslant N'). \end{aligned}$$

Hence if R and R' are quadric, then so is S.

By the above, $R^{\mathrm{env}} = R \otimes_K R^{\mathrm{op}}$ is an algebra of solvable type in a natural way, with respect to the admissible ordering $\leqslant^{\mathrm{env}}$ on $\mathbb{N}^{2N} = \mathbb{N}^N \times \mathbb{N}^N$ obtained by taking the lexicographic product of $\leqslant$ with itself. For every given n, the kernel of the left R^{env}-morphism $\mu\colon (R^{\mathrm{env}})^n \to R^n$ introduced in Section 2.7 is generated by the elements

$$\big((x^{\varepsilon_i} \otimes 1) - (1 \otimes x^{\varepsilon_i})\big)e_j \qquad (1 \leqslant i \leqslant N,\ 1 \leqslant j \leqslant n) \tag{2.6}$$

of $(R^{\mathrm{env}})^n$. Here

$$\varepsilon_1 = (1, 0, \dots, 0), \varepsilon_2 = (0, 1, 0, \dots, 0), \dots, \varepsilon_N = (0, \dots, 0, 1) \in \mathbb{N}^N,$$

and $e_1, \dots, e_n$ are the standard basis elements of the left R^{env}-module $(R^{\mathrm{env}})^n$. Hence if M is an R-bisubmodule of R^n generated by

$$f_i = (f_{i1}, \dots, f_{in}) \in R^n \qquad (i = 1, \dots, m),$$

then the corresponding left R^{env}-submodule $\mu^{-1}(M)$ of $(R^{\mathrm{env}})^n$ is generated by the elements in (2.6) and

$$(f_{11} \otimes 1, \dots, f_{1n} \otimes 1), \dots, (f_{m1} \otimes 1, \dots, f_{mn} \otimes 1).$$

2.12. **Quadric algebras of solvable type.** *In the rest of this section, R is assumed to be quadric.* We have $\mathrm{lm}(\pi(v)) = \mathrm{lm}(\pi(w))$ for all words $v, w \in \langle X \rangle$ which are rearrangements of each other (by Lemma 2.4). This observation is crucial for the proof of the next lemma, to be used in the following subsection:

Lemma 2.10. *For every d we have*

$$R_{(\leqslant d)} = \bigoplus_{|\alpha| \leqslant d} K x^\alpha.$$

Proof. We equip $\mathbb{N}^{N+1} = \mathbb{N}^N \times \mathbb{N}$ with the lexicographic product of the given admissible ordering $\leqslant$ of $\mathbb{N}^N$ and the usual ordering of $\mathbb{N}$. It suffices to show, by induction on pairs $(\alpha, i) \in \mathbb{N}^N \times \mathbb{N}$: every word $w \in \langle X \rangle$ with $\mathrm{lm}(\pi(w)) = x^\alpha$ and $i(w) = i$ belongs to $I(\mathcal{R}) + \bigoplus_{|\beta| \leqslant d} K X^\beta$ where $d =$ length of w. If $i(w) = 0$ then $w \in \bigoplus_{|\beta| \leqslant d} K X^\beta$, and there is nothing to show; so suppose $i(w) > 0$ (in particular, $d > 0$). Then there are $i, j \in \{1, \dots, N\}$ and $u, v \in \langle X \rangle$ with $i < j$, $w = u X_j X_i v$ and $i(u) = 0$. We have $u R_{ij} v \in I(\mathcal{R})$ and

$$w = c_{ij} u X_i X_j v + u P_{ij} v + u R_{ij} v.$$

We also have $\mathrm{lm}(\pi(uX_iX_jv)) = \mathrm{lm}(\pi(w))$ and $i(uX_iX_jv) = i(w) - 1$, and moreover $\mathrm{lm}(\pi(uP_{ij}v)) < \mathrm{lm}(\pi(w))$ and $\deg(uP_{ij}v) \leqslant d$ since $\mathcal{R}$ is quadric. Thus by inductive hypothesis, uX_iX_jv and $uP_{ij}v$ are elements of $I(\mathcal{R}) + \bigoplus_{|\beta|\leqslant d} KX^{\beta}$; hence so is w. $\square$

2.13. **Homogenization and homogeneous algebras of solvable type.** Let T be an indeterminate over K distinct from $X_1, \dots, X_N$. In the following we identify the Rees algebra R^* of R with the graded K-algebra $K\langle X, T\rangle / I(\mathcal{R})^{\mathrm{h}}$ via the isomorphism (2.3). Then the canonical element of R^* is $t = T + I(\mathcal{R})^{\mathrm{h}}$, and the K-algebra R^* is generated by $x_1^*, \dots, x_N^*, t \in (R^*)_{(1)}$, where $x_i^* = X_i + I(\mathcal{R})^{\mathrm{h}}$ is the homogenization of x_i ($i = 1, \dots, N$). Let $x^* := (x_1^*, \dots, x_N^*)$. By Lemma 2.10, for every d we have

$$(R^*)_{(d)} = \bigoplus_{|\alpha|\leqslant d} K\,(x^*)^{\alpha} t^{d-|\alpha|}.$$

In particular, the K-algebra R^* is affine with respect to (x^*, t). In fact:

Corollary 2.11. *The Rees algebra R^* of R is homogeneous of solvable type with respect to the lexicographic product $\leqslant^*$ of the admissible ordering $\leqslant$ of $\mathbb{N}^N$ and the usual ordering of $\mathbb{N}$, and the generating tuple (x^*, t).*

Proof. We construct a homogeneous commutation system $\mathcal{R}^{\mathrm{h}}$ in $K\langle X, T\rangle$ by enlarging the family $(R_{ij}^{\mathrm{h}})_{1\leqslant i<j\leqslant N}$ by the polynomials $X_iT - TX_i$ ($i = 1, \dots, N$). (See (2.2) for the definition of R_{ij}^{h}.) One sees easily (by choice of $\leqslant^*$) that $\mathcal{R}^{\mathrm{h}}$ satisfies condition (1) in Proposition 2.6. Clearly the surjective K-algebra homomorphism $K\langle X, T\rangle \to R^*$ with $X_i \mapsto x_i^*$ for $i = 1, \dots, N$ and $T \mapsto t$ sends every polynomial in $I(\mathcal{R}^{\mathrm{h}})$ to zero, hence induces an isomorphism $K\langle X, T\rangle / I(\mathcal{R}^{\mathrm{h}}) \to R^*$ by Remark 2.7. Thus R^* is of solvable type as claimed. $\square$

In the following, by abuse of notation, we denote the homogenization $x_i^* \in R^*$ of $x_i \in R$ also just by x_i, for $i = 1, \dots, N$. So the homogenization of $f \in R$ of degree d is

$$f^* = \sum_{\alpha} f_{\alpha} x^{\alpha} t^{d-|\alpha|} \in (R^*)_{(d)},$$

and for every multi-index α and $i \in \mathbb{N}$ the dehomogenization of the monomial $x^{\alpha}t^i$ is given by $(x^{\alpha}t^i)_* = x^{\alpha}$.

Examples 2.12.

(1) The Rees algebra of the commutative polynomial ring $K[x_1, \dots, x_N]$ is the commutative polynomial ring $K[x_1, \dots, x_N, t]$ equipped with its usual grading by (total) degree.

(2) If $R = A_n(K)$, then R^* is the graded K-algebra generated by $2n + 1$ generators $x_1, \dots, x_n, \partial_1, \dots, \partial_n, t$ subject to the homogeneous relations

$$\begin{array}{lll} x_jx_i = x_ix_j, & \partial_j\partial_i = \partial_i\partial j & \text{for } 1 \leqslant i < j \leqslant n, \\ \partial_jx_i = x_i\partial_j & & \text{for } 1 \leqslant i, j \leqslant n, i \neq j, \\ \partial_ix_i = x_i\partial_i + t^2 & & \text{for } 1 \leqslant i \leqslant n, \\ x_it = tx_i, & \partial_it = t\partial_i & \text{for } 1 \leqslant i \leqslant n. \end{array}$$

The Rees algebra of $A_n(K)$ is known as the homogenized Weyl algebra, cf. [32].

(3) Let $\mathfrak{g}$ be a Lie algebra over K with basis $\{b_1, \dots, b_N\}$. The Rees algebra of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ is the graded K-algebra generated by $x_1, \dots, x_N, t$ subject to the homogeneous relations

$$\begin{array}{ll} x_jx_i = x_ix_j + [b_j, b_i]_{\mathfrak{g}} \cdot t^2 & \text{for } 1 \leqslant i < j \leqslant N, \\ x_it = tx_i & \text{for } 1 \leqslant i \leqslant N. \end{array}$$

The Rees algebra of $U(\mathfrak{g})$ is also called the homogenized enveloping algebra of $\mathfrak{g}$ in [34].

The elements $y_1,\dots,y_N$, where $y_i = \operatorname{gr} x_i \in (\operatorname{gr} R)_{(1)}$ for $i=1,\dots,N$, generate the K-algebra $\operatorname{gr} R$. Moreover:

Corollary 2.13. *The associated graded algebra* $\operatorname{gr} R$ *of* R *is homogeneous of solvable type with respect to the given admissible ordering* $\leqslant$ *of* $\mathbb{N}^N$ *and the tuple* $y = (y_1,\dots,y_N)$*. Moreover, if* $\deg P_{ij} < 2$ *for* $1 \leqslant i < j \leqslant N$ *then* $\operatorname{gr} R$ *is semi-commutative, and* $\operatorname{gr} R$ *is commutative if and only if* $\deg P_{ij} < 2$ *and* $c_{ij} = 1$ *for* $1 \leqslant i < j \leqslant N$.

Proof. The first statement follows from Lemmas 2.8, (2) and 2.11, using the isomorphism (2.1). Suppose $\deg P_{ij} < 2$ for $1 \leqslant i < j \leqslant N$. Then $x_j x_i = c_{ij} x_i x_j + p_{ij}$ where $p_{ij} \in R_{(<2)}$, and hence $y_j y_i = c_{ij} y_i y_j$ in $\operatorname{gr} R$, for $1 \leqslant i < j \leqslant N$. Therefore $\operatorname{gr} R$ is semi-commutative, and commutative if and only if $c_{ij} = 1$ for $1 \leqslant i < j \leqslant N$. □

In each of the examples in 2.12, the associated graded algebra is commutative. We have only considered the homogenization of R with respect to the standard filtration of R; for other types of homogenizations see [5, Section 4.7].

Now assume that R is homogeneous. Then R is a graded K-algebra, equipped with the grading induced from $K\langle X\rangle$ by $\pi\colon K\langle X\rangle \to R$. By Lemma 2.10 we have

$$R_{(d)} = \bigoplus_{|\alpha|=d} K x^\alpha$$

for every d. Hence if $N > 0$ then

$$\dim_K R_{(d)} = \binom{N+d-1}{d} \qquad \text{for every } d. \tag{2.7}$$

Given a homogeneous K-linear subspace V of R, the **Hilbert function** $H_V\colon \mathbb{N} \to \mathbb{N}$ **of** V is defined by

$$H_V(d) := \dim_K V_{(d)} \qquad \text{for each } d.$$

Clearly if a homogeneous K-linear subspace V of R can be decomposed as a direct sum

$$V = \bigoplus_{i\in I} V_i$$

of a family $\{V_i\}_{i\in I}$ of homogeneous K-linear subspaces of R with $V_i \subseteq V$ for every $i \in I$, then for every d we have

$$H_V(d) = \sum_{i\in I} H_{V_i}(d),$$

where all but finitely many summands in the sum on the right hand side are zero. In many important cases, the Hilbert function $H_V(d)$ of V will agree with a polynomial function for sufficiently large values of d. If there exists a (necessarily unique) polynomial $P \in \mathbb{Q}[T]$ such that $H_V(d) = P(d)$ for all sufficiently large d, then we will denote this polynomial by P_V, and call it the **Hilbert polynomial of** V**.** In this case, the smallest $r \in \mathbb{N}$ such that $H_V(d) = P_V(d)$ for all $d \geqslant r$ is called the **regularity** of the Hilbert function H_V, which we denote here by $\sigma(V)$. For example, for $V = R$ and $N > 0$ we have

$$P_R = \frac{1}{(N-1)!}(T+N-1)\cdot(T+N-2)\cdots(T+1)$$

by (2.7), with $\sigma(R) = 0$. In a similar vein, for a finitely generated graded left R-module M, each of the homogeneous components $M_{(d)}$ has finite dimension as a K-linear space, and the function $H_M\colon \mathbb{N} \to \mathbb{N}$ defined by

$$H_M(d) := \dim_K M_{(d)} \qquad \text{for each } d$$

is called the **Hilbert function** of M; moreover, there exists a polynomial $P_M \in \mathbb{Q}[T]$ of degree $< N$ such that $H_M(d) = P_M(d)$ for d sufficiently large, called the **Hilbert polynomial of** R. The degree of P_M is one less than the Gelfand-Kirillov dimension of the graded left R-module M. (See, e.g., [5, Chapter 7].) In particular, if I is a homogeneous left ideal of R, then P_I exists and has degree $< N$, and $P_{R/I} = P_R - P_I$(if R/I is considered as a left R-module). We define the **regularity** $r(M)$ of H_M similarly to the regularity of H_V above.

3. Gröbner bases in Algebras of Solvable Type

In this section we let $R = K\langle x\rangle$ be a K-algebra of solvable type with respect to a fixed admissible ordering $\leqslant$ of $\mathbb{N}^N$ and a tuple $x = (x_1, \dots, x_N) \in R^N$.

3.1. **Left reduction.** Given $f, f', g \in R$, $g \neq 0$, we write $f \underset{g}{\longrightarrow} f'$ if there exist $c \in K$ and multi-indices α, β such that

$$\operatorname{lm}(x^\beta g) = x^\alpha \in \operatorname{supp} f, \quad \operatorname{lc}(cx^\beta g) = f_\alpha, \quad f' = f - cx^\beta g.$$

We say that an element f of R is **reducible** by a non-zero element g of R if $\operatorname{lm}(g)$ divides some monomial in the support $\operatorname{supp} f$ of f, that is, if $f \underset{g}{\longrightarrow} f'$ for some $f' \in R$. If R is homogeneous, $f, f', g \in R$, $g \neq 0$, and f, g are homogeneous with $f \underset{g}{\longrightarrow} f'$, then f' is also homogeneous.

Let G be a subset of R. We say that an element f of R is **reducible by** G if f is reducible by some non-zero $g \in G$; otherwise we call f **irreducible** by G. We write $f \underset{G}{\longrightarrow} f'$ if $f \underset{g}{\longrightarrow} f'$ for some $g \in G$. The reflexive-transitive closure of the relation $\underset{G}{\longrightarrow}$ is denoted by $\overset{*}{\underset{G}{\longrightarrow}}$. We say that $f_0 \in R$ is a G**-normal form** of $f \in R$ if $f \overset{*}{\underset{G}{\longrightarrow}} f_0$ and f_0 is irreducible by G. One may show that there is no infinite sequence $f_0, f_1, \dots$ in R with

$$f_0 \underset{G}{\longrightarrow} f_1 \underset{G}{\longrightarrow} f_2 \underset{G}{\longrightarrow} \cdots \underset{G}{\longrightarrow} f_m \underset{G}{\longrightarrow} \cdots,$$

hence every element of R has a G-normal form [17, Lemma 3.2]. If R is homogeneous and G consists entirely of homogeneous elements of R, then every homogeneous element of R has a homogeneous G-normal form.

3.2. **Gröbner bases of left ideals in** R**.** Let G be a finite subset of R. Note that if $f \overset{*}{\underset{G}{\longrightarrow}} f'$ $(f, f' \in R)$, then there exist $g_1, \dots, g_m \in G$ and $p_1, \dots, p_m \in R$ such that

$$f = p_1 g_1 + \cdots + p_m g_m + f', \qquad \operatorname{lm}(p_1 g_1), \dots, \operatorname{lm}(p_m g_m) \leqslant \operatorname{lm}(f).$$

In particular, if $f \overset{*}{\underset{G}{\longrightarrow}} 0$ then f is an element of the left ideal (G) of R generated by G. If $f \overset{*}{\underset{G}{\longrightarrow}} 0$ for every $f \in (G)$, then G is called a **Gröbner basis** (with respect to our admissible ordering $\leqslant$). The following proposition (for a proof of which see [17, Lemma 3.8]) gives equivalent conditions that help to identify Gröbner bases.

Proposition 3.1. *The following are equivalent:*

(1) *G is a Gröbner basis.*

(2) *Every non-zero element of* (G) *is reducible by* G.
(3) *Every element of* R *has a unique* G*-normal form.*
(4) *For every non-zero* $f \in (G)$ *there is a non-zero* $g \in G$ *with* $\operatorname{lm}(g) | \operatorname{lm}(f)$.

Given a left ideal I of R, we say that a subset G of I which is a Gröbner basis and which generates I is a **Gröbner basis of** I (with respect to $\leqslant$). Suppose now that G is a Gröbner basis of $I = (G)$. Given $f \in R$, we denote by $\operatorname{nf}_G(f)$ the unique G-normal form of f, so $f - \operatorname{nf}_G(f) \in I$. Moreover, if $f, g \in R$ have distinct G-normal forms, then $h := \operatorname{nf}_G(f) - \operatorname{nf}_G(g)$ is a non-zero element of R which is irreducible by G, so $h \notin I$ by the equivalence of (1) and (2) in Proposition 3.1 and thus $f - g \notin I$. Hence two elements f and g of R have the same G-normal form if and only if $f - g \in I$.

Corollary 3.2. *Suppose* G *is a Gröbner basis of* I. *Then the map*

$$f \mapsto \operatorname{nf}_G(f) \colon R \to R$$

is K*-linear, and its image* $\operatorname{nf}_G(R)$ *satisfies*

$$R = I \oplus \operatorname{nf}_G(R) \qquad \text{(internal direct sum of } K\text{-linear subspaces of } R\text{).}$$

A basis of the K*-linear space* $\operatorname{nf}_G(R)$ *is given by the set of all monomials of* R *not divisible (in* $(x^\diamond, *)$*) by some* $\operatorname{lm}(g)$ *with* $g \in G$, $g \neq 0$.

Proof. Let $f, f', g \in R$, $g \neq 0$, and $c \in K$, $c \neq 0$. If $f \xrightarrow[g]{} f'$ then $cf \xrightarrow[g]{} cf'$, and if $f \in R$ is G-irreducible, then so is cf. This yields $\operatorname{nf}_G(cf) = c \operatorname{nf}_G(f)$. Also, $h := \operatorname{nf}_G(f) + \operatorname{nf}_G(f')$ is G-irreducible and $h - (f + f') \in I$, hence $h = \operatorname{nf}_G(h) = \operatorname{nf}_G(f + f')$ by the remark preceding the corollary, and thus $\operatorname{nf}_G(f + f') = \operatorname{nf}_G(f) + \operatorname{nf}_G(f')$. This shows K-linearity of $f \mapsto \operatorname{nf}_G(f)$. The rest of the corollary is clear. □

By the previous corollary, $\operatorname{nf}_G(R)$ does not depend on G. In fact, employing the notation introduced in Section 1 we have $\operatorname{nf}_G(R) = \operatorname{nf}_M(R)$ where M is the K-linear subspace of R generated by

$$\operatorname{lm}(I) = \big\{ \operatorname{lm}(f) : 0 \neq f \in I \big\}.$$

The decomposition $R = I \oplus \operatorname{nf}_G(R)$ of R corresponds to the decomposition $\operatorname{gr}_\leqslant R = \operatorname{gr}_\leqslant I \oplus \operatorname{gr}_\leqslant M$ of the semi-commutative associated graded algebra $\operatorname{gr}_\leqslant R = K\langle \xi \rangle$ of R with respect to the fine multi-filtration. Here $\operatorname{gr}_\leqslant I = (\operatorname{gr}_\leqslant \operatorname{lm}(I))$, so if $\operatorname{gr}_\leqslant R$ is commutative, then $\operatorname{gr}_\leqslant I$ a monomial ideal of $\operatorname{gr}_\leqslant R$ in the usual sense of the word. The K-linear subspace $\operatorname{gr}_\leqslant \operatorname{nf}_G(R)$ of $\operatorname{gr}_\leqslant R$ is generated by the symbols $\operatorname{gr}_\leqslant x^\alpha = \xi^\alpha$ with $x^\alpha \in \operatorname{nf}_G(R)$.

Every left ideal I of R has a Gröbner basis. (Since being a Gröbner basis includes being finite, this means in particular that the ring R is left Noetherian.) To see this, note that $\operatorname{lm}(I)$ is an ideal of the commutative monoid of monomials of R (with multiplication $*$). Hence there is a finite set G of non-zero elements of I such that for every non-zero $f \in I$ we have $\operatorname{lm}(g) | \operatorname{lm}(f)$ for some $g \in G$; then G is a Gröbner basis of I. This argument is non-constructive; however, as observed in [17], by an adaptation of **Buchberger's algorithm** one can construct a Gröbner basis of I from a given finite set of generators of I in an effective way (up to computations in the field K and comparisons of multi-indices in $\mathbb{N}^N$ by the chosen admissible ordering $\leqslant$). The main ingredient of this algorithm is the following notion:

Definition 3.3. The S**-polynomial** of elements f and g of R is defined by

$$S(f, g) := d \operatorname{lc}(g) \cdot x^\alpha f - c \operatorname{lc}(f) \cdot x^\beta g,$$

where α and β are the unique multi-indices such that

$$x^\alpha * \mathrm{lm}(f) = x^\beta * \mathrm{lm}(g) = \mathrm{lcm}\big(\mathrm{lm}(f), \mathrm{lm}(g)\big),$$

and $c = \mathrm{lc}(x^\alpha f)$, $d = \mathrm{lc}(x^\beta g)$.

Now we can add the following equivalent condition to Proposition 3.1 (cf. [17, Theorem 3.11]):

Proposition 3.4.

$$G \text{ is a Gröbner basis} \quad \Longleftrightarrow \quad S(f,g) \xrightarrow[G]{*} 0 \text{ for all } f,g \in G.$$

Starting with a finite subset G_0 of R, Buchberger's algorithm successively constructs finite subsets

$$G_0 \subseteq G_1 \subseteq \cdots \subseteq G_k \subseteq \cdots$$

of elements of the left ideal $I = (G_0)$ as follows: Suppose that G_k has been constructed already. For every pair (f,g) of elements of G_k find a G_k-normal form $r(f,g)$ of $S(f,g)$. If all of these normal forms are zero, then $G := G_k$ is a Gröbner basis of I, by the previous proposition, and the algorithm terminates. Otherwise, we put

$$G_{k+1} := G_k \cup \big\{ r(f,g) : f,g \in G_k \big\}$$

and iterate the procedure. Dickson's Lemma guarantees that this construction eventually stops. (See [17] for details.)

Definition 3.5. One says that G is a **reduced Gröbner basis** of the left ideal I of R if

(1) G is a Gröbner basis of I;
(2) $\mathrm{lc}(g) = 1$ for every $g \in G$; and
(3) $g \in \mathrm{nf}_{G\setminus\{g\}}(R)$ for every $g \in G$.

Every left ideal I of R has a unique reduced Gröbner basis (see [17, Section 4]); hence we can speak of *the* reduced Gröbner basis of I. From a given Gröbner basis G, the reduced Gröbner basis of $I = (G)$ can be computed as follows: First, after shrinking G if necessary, we may assume that $0 \notin G$ and that $\mathrm{lm}(G)$ is a minimal set of generators of the monomial ideal generated by $\mathrm{lm}(I)$. Multiplying each element g of G by $\mathrm{lc}(g)^{-1}$ we may then further achieve that $\mathrm{lc}(g) = 1$ for every $g \in G$. Suppose $G = \{g_1, \dots, g_m\}$ with pairwise distinct $g_1, \dots, g_m$. Let $h_i := g_i - \mathrm{lm}(g_i)$ and put $g_i' := \mathrm{lm}(g_i) + \mathrm{nf}_G(h_i)$, for $i = 1, \dots, m$. Then $G' := \{g_1', \dots, g_m'\}$ is the reduced Gröbner basis of I.

In summary, Gröbner bases of left ideals in R share properties similar to Gröbner bases of ideals in the commutative polynomial rings over K, with slight differences; most notably, a collection of monomials in R is not automatically a Gröbner basis for the left ideal it generates [17, p. 17].

3.3. **Gröbner bases of two-sided ideals in R.** It is possible to also define a notion of Gröbner basis for two-sided ideals of R:

Proposition 3.6. *Let G be a finite subset of R. The following statements are equivalent:*

(1) *G is a Gröbner basis, and the two-sided ideal of R generated by G agrees with the left ideal (G) of R generated by G.*
(2) *G is a Gröbner basis, and $gx_i \in (G)$ for every $g \in G$ and $i = 1, \dots, N$.*
(3) *For every non-zero element f of the two-sided ideal of R generated by G there exists a non-zero $g \in G$ with $\mathrm{lm}(g) | \mathrm{lm}(f)$.*

If a finite subset G of R satisfies one of the equivalent conditions in this proposition (proved in [17, Theorem 5.4]), then G is called a **two-sided Gröbner basis** (with respect to $\leqslant$). If I is a two-sided ideal of R, then a subset G of I is called a **Gröbner basis** of I (with respect to $\leqslant$) if G is a two-sided Gröbner basis which also generates the two-sided ideal I. The following observation (from [31]) allows one to reduce the computation of two-sided Gröbner bases in R to the computation of one-sided Gröbner bases in the enveloping algebra of R:

Proposition 3.7. *Let I be a two-sided ideal of R, and let G be a Gröbner basis of the left ideal $\mu^{-1}(I)$ of R^{env}. Then $\mu(G)$ is a Gröbner basis of I.*

3.4. **Gröbner bases in homogeneous algebras of solvable type.** In this subsection R is assumed to be homogeneous. From Buchberger's algorithm and earlier remarks we immediately obtain:

Corollary 3.8. *The reduced Gröbner basis of each homogeneous left ideal of R consists of homogeneous elements of R.*

The following lemma is essentially due to Macaulay. Here and below, the cardinality of a finite set S is denoted by $\#S$.

Lemma 3.9. *Suppose V is a homogeneous K-linear subspace of R. Then*

$$H_V(d) = \#\operatorname{lm}(V_{(d)}) \qquad \text{for every } d.$$

Proof. Put $W := V_{(d)}$, and for each $w \in \operatorname{lm}(W)$ choose $f_w \in W$ with $\operatorname{lm}(f_w) = w$. We claim that the f_w form a basis of the K-linear space W. Clearly, the f_w are K-linearly independent. Let W' be the K-linear subspace of W generated by the f_w, and suppose for a contradiction that $W' \neq W$. Take $f \in W \setminus W'$ such that $w := \operatorname{lm}(f)$ is minimal, with respect to our admissible ordering $\leqslant$, and put $g := f - \left(\frac{\operatorname{lc}(f)}{\operatorname{lc}(f_w)}\right) f_w \in W'$. Then $\operatorname{lm}(g) < w$, thus $g \in W'$ and hence $f \in W'$, a contradiction. $\square$

Let now I be a homogeneous left ideal of R with Gröbner basis G. The K-linear subspace $M := \operatorname{nf}_G(R)$ of R is generated by monomials of R, hence is homogeneous, with $R = I \oplus M$. Therefore, the Hilbert function of R/I can be expressed as:

$$H_{R/I}(d) = H_R(d) - H_I(d) = H_M(d) = \#\operatorname{lm}(M_{(d)}) \quad \text{for every } d.$$

3.5. **Gröbner bases and dehomogenization.** In this subsection we assume that R is quadric (so R^* is of solvable type as explained in Section 2.13). We collect a few facts concerning leading terms, reductions, and S-polynomials with respect to dehomogenization:

Lemma 3.10. *Let $f, f', g \in R^*$ be homogeneous, $g \neq 0$. Then*

1. $\operatorname{lm}(f_*) = (\operatorname{lm} f)_*$, $\operatorname{lc}(f_*) = \operatorname{lc}(f)$*;*
2. *if* $f \xrightarrow[g]{} f'$*, then* $f_* \xrightarrow[g_*]{} f'_*$*;*
3. $\big(S(f, f')\big)_* = S(f_*, f'_*)$.

Proof. For (1), let α and β be multi-indices and $i, j \in \mathbb{N}$. Then $(x^\alpha t^i)_* = x^\alpha$ and $(x^\beta t^j)_* = x^\beta$, so if $\deg(x^\alpha t^i) = \deg(x^\beta t^j)$, then $(x^\alpha t^i)_* = (x^\beta t^j)_*$ implies $i = j$, hence

$$x^\alpha t^i \leqslant^* x^\beta t^j \quad \Longleftrightarrow \quad (x^\alpha t^i)_* \leqslant (x^\beta t^j)_*.$$

This observation immediately yields (1). For (2), suppose $f \xrightarrow[g]{} f'$, and let α, β be multi-indices, $i, j \in \mathbb{N}$, and $c \in K$ such that

$$\operatorname{lm}(x^\beta t^j g) = x^\alpha t^i \in \operatorname{supp} f, \quad \operatorname{lc}(cx^\beta t^j g) = f_{(\alpha,i)}, \quad f' = f - cx^\beta t^j g.$$

Then $(f')_* = f_* - cx^\beta g_*$, and $\operatorname{lm}(x^\beta g_*) = x^\alpha$ by (1). Since f is homogeneous, we have $(f_*)_\alpha = f_{(\alpha,i)}$, so $x^\alpha \in \operatorname{supp} f_*$ and $\operatorname{lc}(cx^\beta g_*) = (f_*)_\alpha$. Thus $f_* \xrightarrow[g_*]{} f'_*$. For (3), let α, β be multi-indices and $i, j \in \mathbb{N}$ such that

$$x^\alpha t^i * \operatorname{lm}(f) = x^\beta t^j * \operatorname{lm}(f') = \operatorname{lcm}\big(\operatorname{lm}(f), \operatorname{lm}(f')\big),$$

and $c = \operatorname{lc}(x^\alpha t^i f)$, $d = \operatorname{lc}(x^\beta t^j f')$. Then

$$S(f, f') = d \operatorname{lc}(f') \cdot x^\alpha t^i f - c \operatorname{lc}(f) \cdot x^\beta t^j f',$$

hence

$$\big(S(f, f')\big)_* = d \operatorname{lc}(f') \cdot x^\alpha f_* - c \operatorname{lc}(f) \cdot x^\beta f'_*.$$

By (1) we also have

$$x^\alpha * \operatorname{lm}(f_*) = x^\beta * \operatorname{lm}(f'_*) = \operatorname{lcm}\big(\operatorname{lm}(f_*), \operatorname{lm}(f'_*)\big)$$

and $c = \operatorname{lc}(x^\alpha f_*)$, $d = \operatorname{lc}(x^\beta f'_*)$. This yields (3). $\square$

The following corollary often allows us to reduce questions about arbitrary Gröbner bases to a homogeneous situation:

Corollary 3.11. *Let I be a left ideal of R, and let G be a generating set for I. Let J be the left ideal of R^* generated by all g^* with $g \in G$, and let H be a Gröbner basis of J with respect to $\leqslant^*$ consisting of homogeneous elements of R^*. Then $H_* = \{h_* : h \in H\}$ is a Gröbner basis of I with respect to $\leqslant$.*

Proof. We have $I = J_* = (H)_* = (H_*)$, and by parts (2) and (3) of the previous lemma $S(f, g) \xrightarrow[H_*]{*} 0$ for all $f, g \in H_*$. Hence H_* is a Gröbner basis of I. $\square$

Remark 3.12. In the situation of the previous corollary, if H is reduced, then H_* is not necessarily reduced. For example, suppose $R = K[x]$, the commutative polynomial ring in a single indeterminate x over K, and $G = \{x^2, x + x^2\}$. Then $R^* = K[x, t]$ where t is an indeterminate distinct from x, and $J = (x^2, xt + x^2) = (xt, x^2)$. So $H = \{xt, x^2\}$ is the reduced Gröbner basis of J; but $H_* = \{x, x^2\}$ is not reduced.

3.6. **Gröbner bases and the associated graded algebra.** Our algebra R of solvable type comes equipped with two multi-filtrations: the standard filtration on the one hand, and the "fine multi-filtration" defined in Corollary 2.5 on the other. In both cases, under mild assumptions on R, the associated graded algebra of R is an ordinary commutative polynomial ring over K. (Corollaries 2.5 and 2.13.) Thus it might be tempting to try and deduce Theorem 0.1 from the main result of [9] using "filtered-graded transfer". Indeed, the following is proved in [23]:

Proposition 3.13. *Suppose $\leqslant$ is degree-compatible. Let I be a left ideal of R. If G is a Gröbner basis of I, then*

$$\operatorname{gr} G := \{\operatorname{gr} g : 0 \neq g \in G\}$$

is a Gröbner basis of the left ideal $\operatorname{gr} I$ *of* $\operatorname{gr} R$ *consisting of homogeneous elements. Conversely, if H is a Gröbner basis of* $\operatorname{gr} I$ *consisting of homogeneous elements and G is a finite subset of I with* $\operatorname{gr} G = H$*, then G is a Gröbner basis of I.*

Proposition 3.13 breaks down if $\leqslant$ is not degree-compatible:

Example 3.14. Suppose $R = K[x, y]$ is the commutative polynomial ring in two indeterminates x and y over K, and consider the ideal $I = (f_1, f_2, f_3)$ of R, where

$$f_1 = xy, \quad f_2 = x - y^2, \quad f_3 = x^2.$$

Then $G = \{f_1, f_2, f_3\}$ is not a Gröbner basis of I with respect to the lexicographic ordering of $\mathbb{N}^2$ (so $y^n < x$ for every n), since

$$S(f_1, f_2) = xy - y(x - y^2) = y^3$$

is irreducible by G. However, $\operatorname{gr} G$ is a Gröbner basis of $\operatorname{gr} I$ with respect to the degree-lexicographic ordering of $\mathbb{N}^2$. (To see this use Proposition 3.13 and verify that G is a Gröbner basis with respect to this ordering.)

Nevertheless, this proposition does seem to offer an easy way towards Theorem 0.1 in the special case where $\leqslant$ is degree-compatible and $\operatorname{gr} R$ is commutative. In this case we have $\operatorname{gr} R = K[y_1, \dots, y_N]$ where $y_i = \operatorname{gr} x_i$ for $i = 1, \dots, N$. Unfortunately, however, if the non-zero elements $f_1, \dots, f_n$ of R generate a left ideal I of R, then $\operatorname{gr} f_1, \dots, \operatorname{gr} f_n$ in general do not generate $\operatorname{gr} I$, as the following example from [23] shows:

Example 3.15. Suppose $R = A_2(K)$ is the second Weyl algebra, and let $I = (f_1, f_2)$ where

$$f_1 = x_1\partial_1, \quad f_2 = x_2(\partial_1)^2 - \partial_1.$$

Then $\operatorname{gr} f_1 = \operatorname{gr} x_1\partial_1$, $\operatorname{gr} f_2 = \operatorname{gr} x_2 \operatorname{gr}(\partial_1)^2$ do not generate $\operatorname{gr} I$. In fact, $\{\partial_1\}$ is a Gröbner basis for I with respect to the degree-lexicographic ordering of $\mathbb{N}^4$.

It seems even less likely to be able to reduce the proof of Theorem 0.1 to the associated graded algebra $\operatorname{gr}_{\leqslant} R$ of R equipped with the fine multi-filtration, since for every subset G of R, the set $\operatorname{gr}_{\leqslant} G = \{\operatorname{gr}_{\leqslant} g : 0 \neq g \in G\}$ simply consists of monomials.

3.7. **Decomposition of left ideals.** In this subsection we let I be a left ideal of R. For $f \in R$ we put

$$(I : f) := \{g \in R : gf \in I\},$$

a left ideal of R. If R, f and the left ideal I are homogeneous, then so is the left ideal $(I : f)$ of R. For $f_1, f_2 \in R$ we also write $(f_1 : f_2) := ((f_1) : f_2)$.

Lemma 3.16. *Let $f \in R$, and let G be a Gröbner basis of $(I : f)$. Then*

$$I + (f) = I \oplus \operatorname{nf}_G(R)f.$$

Proof. Let $h \in I + (f)$. Then we can write $h = a + bf$ with $a \in I$ and $b \in R$. Let $c := \operatorname{nf}_G(b)$; then $b - c \in (I : f)$ and

$$h = \big(a + (b - c)f\big) + cf,$$

where the first summand is in I and the second in $\operatorname{nf}_G(R)f$. This shows $I + (f) = I + \operatorname{nf}_G(R)f$; moreover, clearly $I \cap \operatorname{nf}_G(R)f = \{0\}$ by construction. $\square$

The previous lemma leads to a decomposition of I into K-linear subspaces of the form $S = \operatorname{nf}_G(R)f$ for certain $f \in R$ and Gröbner bases G as follows: Take $f_1, \dots, f_n \in R$, $n > 0$, such that $I = (f_1, \dots, f_n)$, and for $i = 2, \dots, n$ let G_i be a Gröbner basis of $\big((f_1, \dots, f_{i-1}) : f_i\big)$; then

$$I = (f_1) \oplus \operatorname{nf}_{G_2}(R)f_2 \oplus \cdots \oplus \operatorname{nf}_{G_n}(R)f_n.$$

Example 3.17. Suppose $R = A_1(K)$ is the first Weyl algebra, so $R = K\langle x, \partial\rangle$ with the relation $\partial x - x\partial = 1$, and let $I = (f_1, f_2)$ where $f_1 = \partial$ and $f_2 = x$. Then in fact $I = R$, and the above decomposition procedure yields

$$R = (f_1) \oplus \mathrm{nf}_{G_2}(R) f_2 = (\partial) \oplus K\partial \cdot x \oplus K[x] \cdot x.$$

Indeed, it is not hard to check that $G_2 = \{\partial^2, x\partial - 1\}$ is the reduced Gröbner basis of the left ideal $(f_1 : f_2)$ of R, with $\mathrm{nf}_{G_2}(R) = K\partial \oplus K[x]$. In particular $\partial \notin (f_1 : f_2)$; this is slightly counterintuitive, since it is always true that $(I : f) \supseteq I$ in the commutative world.

4. Cones and Cone Decompositions

In the first subsection we summarize (and, hopefully, somewhat clarify) the algorithmic core of Dubé's approach dealing with cone decompositions of monomial ideals. Afterwards, we show how to define and construct cone decompositions of homogeneous left ideals. Here, we have to adapt Dubé's ideas to deal with non-commutativity. We only give proofs selectively, and refer to [9] for complete details.

4.1. **Monomial cone decompositions.** In this subsection we let R be a K-linear space and $\{x^\alpha\}_\alpha$ be a monomial basis of R. Let M be a K-linear subspace of R spanned by monomials, and let $\mathcal{D}$ be a finite set of pairs (w, y) where w is a monomial in $x^\diamond$ and y is a subset of x. We define the **degree of** $\mathcal{D}$ as

$$\deg \mathcal{D} := \max \big\{ \deg w : (w, y) \in \mathcal{D} \big\} \in \mathbb{N} \cup \{\infty\},$$

where $\max \varnothing = \infty$ by convention. We also set

$$\mathcal{D}^+ := \big\{ (w, y) \in \mathcal{D} : y \neq \varnothing \big\}.$$

We say that $\mathcal{D}$ is a **cone decomposition of** M if $C(w, y) \subseteq M$ for every $(w, y) \in \mathcal{D}$ and

$$M = \bigoplus_{(w,y)\in\mathcal{D}} C(w, y),$$

and $\mathcal{D}$ is a **monomial cone decomposition** if $\mathcal{D}$ is a cone decomposition of some K-linear subspace of R. In the literature, "monomial cone decompositions" of finitely generated commutative graded K-algebras are also known as "Stanley decompositions" (since they were first introduced in an article [35] by Stanley). In this paper we stay with the perhaps more descriptive terminology introduced by Dubé in [9].

Lemma 4.1. *Suppose $\mathcal{D}$ is a monomial cone decomposition of a monomial ideal I of R. Let F be the minimal set of generators of I. Then for each $w \in F$ there is some $y \subseteq x$ with $(w, y) \in \mathcal{D}$.*

Proof. Let $w \in F$. Since $\mathcal{D}$ is a monomial cone decomposition of I, there is some $(w', y) \in \mathcal{D}$ with $w \in C(w', y)$, so $w = w' * a$ for some $a \in y^\diamond$. Since $w' \in I$, we can also write $w' = w'' * b$ for some $w'' \in F$ and $b \in x^\diamond$. So $w = w' * a = w'' * b * a$, hence $b * a = 1$ due to minimality of F, and $w = w' = w''$. □

In [36, 24], algorithms are given which, upon input of a finite list of generators of a monomial ideal I of R, produce a monomial cone decomposition for the natural complement $\mathrm{nf}_I(R)$ of I in R. In fact, Dubé specified an algorithm which does much more, as we describe next. As before, M is a K-linear subspace of R generated by monomials, and I is a monomial ideal of R.

Definition 4.2. We say that a pair of monomial cone decompositions $(\mathcal{P}, \mathcal{Q})$ **splits** M **relative to** I if

(1) $\mathcal{P} \cup \mathcal{Q}$ is a cone decomposition of M,
(2) $C(w, y) \subseteq I$ for all $(w, y) \in \mathcal{P}$,
(3) $C(w, y) \cap I = \{0\}$ for all $(w, y) \in \mathcal{Q}$.

It is easy to see that if $(\mathcal{P}, \mathcal{Q})$ is a pair of monomial cone decompositions which splits M relative to I, then $\mathcal{P}$ is a monomial cone decomposition of $M \cap I$ and $\mathcal{Q}$ is a monomial cone decomposition of $\mathrm{nf}_I(M)$.

Algorithm 1 accomplishes a basic task: it gives a procedure for splitting a monomial cone relative to I. The computation of a generating set F_1 for the monomial ideal $(I : w * x_i) = ((I : w) : x_i)$ in this algorithm is carried out by Algorithm 2: if the monomial ideal I is generated by $v_1, \dots, v_n \in x^\diamond$, then $(I : x_i)$ is generated by $w_1, \dots, w_n$ where

$$w_j = \begin{cases} v_j & \text{if } x_i \text{ does not divide } v_j, \\ w_j = v_j/x_i & \text{otherwise,} \end{cases}$$

where v_j/x_i denotes the monomial in $x^\diamond$ satisfying $v_j = (v_j/x_i) * x_i$.

Input: $w \in x^\diamond$, $y \subseteq x$, and a finite set F of generators for $(I : w)$;
Output: $\mathtt{SPLIT}(w, y, F) = (\mathcal{P}, \mathcal{Q})$, where $(\mathcal{P}, \mathcal{Q})$ splits the monomial cone $C(w, y)$ relative to the monomial ideal I of R;

if $1 \in F$ **then return** $\big(\{(w, y)\}, \varnothing\big)$;
if $F \cap y^\diamond = \varnothing$ **then return** $\big(\varnothing, \{(w, y)\}\big)$;
else
  choose $z \subseteq y$ maximal such that $F \cap z^\diamond = \varnothing$;
  choose $i \in \{1, \dots, N\}$ such that $x_i \in y \setminus z$;
  $(\mathcal{P}_0, \mathcal{Q}_0) := \mathtt{SPLIT}(w, y \setminus \{x_i\}, F)$; (*)
  $F_1 := \mathtt{QUOTIENT}(F, x_i)$;
  $(\mathcal{P}_1, \mathcal{Q}_1) := \mathtt{SPLIT}(w * x_i, y, F_1)$; (**)
  return $(\mathcal{P}_0 \cup \mathcal{P}_1, \mathcal{Q}_0 \cup \mathcal{Q}_1)$;
end

Algorithm 1: Splitting a monomial cone relative to I.

Input: a finite set F of generators for a monomial ideal I of R, and $i \in \{1, \dots, N\}$;
Output: $\mathtt{QUOTIENT}(F, x_i) = F'$, where F' is a finite set of generators of the monomial ideal $(I : x_i)$ of R;

$F' := \varnothing$;
while $F \neq \varnothing$ **do**
  choose $v \in F$;
  if $x_i | v$ **then** $F' := F' \cup \{v/x_i\}$;
  else
    $F' := F' \cup \{v\}$;
  end
  $F := F \setminus \{v\}$;
end

Algorithm 2: Computing a a set of generators for $(I : x_i)$.

Let $w \in x^\diamond$, $y \subseteq x$, and F be a set of generators for $(I : w)$. One checks:

Lemma 4.3.

(1) $C(w,y) \subseteq I \iff 1 \in F$;
(2) $C(w,y) \cap I = \{0\} \iff F \cap y^\diamond = \varnothing$.

Algorithm 1 proceeds by recursively decomposing the cone $C(w,y)$ as

$$C(w,y) = C(w, y \setminus \{x_i\}) \oplus C(w * x_i, y) \qquad (x_i \in y).$$

The lemma above shows that the base case is handled correctly. We refer to [9, Lemmas 4.3 and 4.4] for a detailed proof of the termination and correctness of Algorithm 1. The output of Algorithm 1 has a convenient property:

Definition 4.4. We say that a monomial cone decomposition $\mathcal{D}$ is d**-standard** if

(1) $\deg(w) \geqslant d$ for all $(w,y) \in \mathcal{D}^+$;
(2) for every $(w,y) \in \mathcal{D}^+$ and d' with $d \leqslant d' \leqslant \deg(w)$ there is some $(w',y') \in \mathcal{D}^+$ with $\deg(w') = d'$ and $\#y' \geqslant \#y$.

Proposition 4.5. *Let* $(\mathcal{P}, \mathcal{Q}) = \texttt{SPLIT}(w,y,F)$. *Then* $\mathcal{Q}$ *is* $\deg(w)$*-standard.*

In the proof of this proposition we use the following lemma:

Lemma 4.6. *Let* $(\mathcal{P}, \mathcal{Q}) = \texttt{SPLIT}(w,y,F)$.

(1) *For every* $(v',y') \in \mathcal{Q}$ *we have* $F \cap (y')^\diamond = \varnothing$ *and* $y' \subseteq y$.
(2) *For every* $y' \subseteq y$ *with* $F \cap (y')^\diamond = \varnothing$ *there exists* $y'' \subseteq y$ *with* $(w,y'') \in \mathcal{Q}$ *and* $\#y'' \geqslant \#y'$.

Proof. We prove part (1) by induction on the number of recursive calls in Algorithm 1 needed to compute $(\mathcal{P}, \mathcal{Q})$. The base case (no recursive calls) is obvious. If $(v',y') \in \mathcal{Q}_0$, then $F \cap (y')^\diamond = \varnothing$ and $y' \subseteq y \setminus \{x_i\} \subseteq y$ follows by inductive hypothesis. Suppose $(v',y') \in \mathcal{Q}_1$; then by inductive hypothesis we obtain $F_1 \cap (y')^\diamond = \varnothing$ and $y' \subseteq y$. By the way that F_1 is computed from F in Algorithm 2, every element of F is divisible by some element of F_1; hence $F \cap (y')^\diamond = \varnothing$.

We show part (2) by induction on $\#y - \#y'$. If $y' = y$, then the algorithm returns $\mathcal{Q} = \{(w,y)\}$, satisfying the condition in (2). Otherwise, we have $\#z \geqslant \#y'$ by maximality of z. Hence by inductive hypothesis applied to $(\mathcal{P}_0, \mathcal{Q}_0) = \texttt{SPLIT}(w, y \setminus \{x_i\}, F)$, there exists $y'' \subseteq y \setminus \{x_i\}$ such that $(w,y') \in \mathcal{Q}_0$ and $\#y'' \geqslant \#z$. $\square$

We now show Proposition 4.5 by the number of recursions in Algorithm 1 needed to compute $(\mathcal{P}, \mathcal{Q})$. If $\mathcal{Q}$ is empty or a singleton, then the conclusion of the proposition holds trivially. Inductively, assume that $\mathcal{Q}_0$ is $\deg(w)$-standard and $\mathcal{Q}_1$ is $(\deg(w)+1)$-standard. Let $(v',y') \in \mathcal{Q}^+$ and d with $\deg(w) \leqslant d \leqslant \deg(v')$ be given; we need to show that there exists a pair $(v'',y'') \in \mathcal{Q}$ with $\deg(v'') = d$ and $\#y'' \geqslant \#y'$. This is clear by inductive hypothesis if $(v',y') \in \mathcal{Q}_0$ or if $d \geqslant \deg(w)+1$. By Lemma 4.6 there exists $y'' \subseteq y$ with $(w,y'') \in \mathcal{Q}$ and $\#y'' \geqslant \#y'$, covering the case that $d = \deg(w)$. $\square$

Applied to $w = 1$, $y = x$, and $F =$ a set of generators for I, Algorithm 1 produces a pair $(\mathcal{P}, \mathcal{Q})$ consisting of a monomial cone decomposition $\mathcal{P}$ of I and a monomial cone decomposition $\mathcal{Q}$ of $\mathrm{nf}_I(R)$. We now analyze this situation in more detail. In the next lemma and its corollary, we suppose $I \neq R$, we let F be a set of generators of I, and let $(\mathcal{P}, \mathcal{Q}) = \texttt{SPLIT}(1, x, F)$.

Lemma 4.7. *Let* $F_{\min} \subseteq F$ *be the minimal set of generators for* I. *Then for every* $v \in F_{\min}$, *the set* $\mathcal{Q}$ *contains a pair* (v',y') *with* $\deg(v') = \deg(v) - 1$.

Proof. Let $v \in F_{\min}$. By Lemma 4.1 we have $(v, y) \in \mathcal{P}$ for some $y \subseteq x$. Since $1 \notin F$, the pair (v, y) arrived in $\mathcal{P}$ during the computation of $\texttt{SPLIT}(1, x, F)$ by means of a recursive call of the form $\texttt{SPLIT}(v, y, F')$ where F' is a set of generators for $(I : v)$. We have $v \in I$, and thus $1 \in F'$. This shows that the recursive call must have been made in (**), because the parameter F is passed on unchanged by the recursive call in (*). The call (**) occurred during the computation of some $\texttt{SPLIT}(v', y, F'')$ where v' satisfies $v = v' * x_i$ for some i, and F'' is a finite set of generators for $(I : v')$. Part (2) of Lemma 4.6 now yields the existence of $y' \subseteq y$ such that $(v', y') \in \mathcal{Q}$. $\square$

The preceding lemma immediately implies:

Corollary 4.8. *The set of all* $w \in F$ *with* $\deg(w) \leqslant 1 + \deg(\mathcal{Q})$ *generates* I.

Remark 4.9. In [24] we find an algorithm which, given a finite list F of generators for a monomial ideal I of R, computes a *Stanley filtration,* that is, a list of pairs

$$\big((w(1), y(1)), \dots, (w(m), y(m))\big),$$

each consisting of a monomial $w(j)$ and a subset $y(j)$ of x, such that for, $j = 1, \dots, m$, the set

$$\big\{(w(1), y(1)), \dots, (w(j), y(j))\big\}$$

is a cone decomposition of $\operatorname{nf}_{I(j)}(R)$ where

$$I(j) := I + C\big(w(j+1), x\big) + \dots + C\big(w(m), x\big).$$

It is easy to see (since Algorithm 1 and Algorithm 3.4 in [24] pursue similar "divide and conquer" strategies) that, for $(\mathcal{P}, \mathcal{Q}) = \texttt{SPLIT}(1, x, F)$, the pairs in $\mathcal{Q}$ can be ordered to form a Stanley decomposition.

4.2. **Cone decompositions of homogeneous ideals.** In the rest of this section, we let R be a K-algebra of solvable type with respect to $x = (x_1, \dots, x_N) \in R^N$ and a fixed admissible ordering $\leqslant$ of $\mathbb{N}^N$. Note that in general (unless R is commutative), a monomial ideal of R is *not* a left ideal of the algebra R. However, let I be a proper left ideal of R; then the K-linear subspace M of R generated by $\operatorname{lm}(I)$ is a monomial ideal of R. Moreover, let G be a Gröbner basis of I; then $\operatorname{lm}(I)$ is generated by $\operatorname{lm}(G)$, and $\operatorname{nf}_M(R) = \operatorname{nf}_G(R)$. The central outcome of the discussion in the previous subsection is:

Theorem 4.10. *The homogeneous* K*-linear subspace* $\operatorname{nf}_G(R)$ *of* R *has a standard monomial cone decomposition. More precisely, let* $(\mathcal{P}, \mathcal{Q}) = \texttt{SPLIT}(1, x, F)$ *where* $F = \operatorname{lm}(G)$*. Then* $\mathcal{Q}$ *is a standard monomial cone decomposition of* $\operatorname{nf}_G(R)$*. Moreover, the set of all* $g \in G$ *with* $\deg(g) \leqslant 1 + \deg \mathcal{Q}$ *is still a Gröbner basis of* $I = (G)$.

In this subsection we establish an analogous decomposition result (Corollary 4.18 below) for I in place of $\operatorname{nf}_G(R)$, provided R and I are homogeneous; thus: *until the end of this section we assume that* R *is homogeneous.* We first need to define the type of cones used in our decompositions: A **cone** of R is defined by a triple (w, y, h), where $w \in x^\diamond$, $y \subseteq x$, and $h \in R$ is homogeneous:

$$C(w, y, h) := C(w, y)h = \big\{gh : g \in C(w, y)\big\} \subseteq R.$$

Both monomial and general cones are homogeneous K-linear subspaces of R, and a monomial cone is a special case of a cone: $C(w, y) = C(w, y, 1)$. Note, however, that $C(1, y, w) \neq C(w, y)$ in general. We introduced this definition of cone in order to be able to speak about cone decompositions of (not necessarily monomial) ideals in the noncommutative setting.

A first important observation (immediate from Lemma 3.9) is that the Hilbert function of $C(w,y,h)$ depends only on the degrees of h and w and the cardinality of y:

Lemma 4.11. *Let $h \in R$ be non-zero and homogeneous, and $w \in x^\diamond$. Then*

$$H_{C(w,\varnothing,h)}(d) = \begin{cases} 0 & \text{if } d \neq \deg(w)+\deg(h), \\ 1 & \text{if } d = \deg(w)+\deg(h), \end{cases}$$

and for non-empty $y \subseteq x$:

$$H_{C(w,y,h)}(d) = \begin{cases} 0 & \text{if } d < \deg(w)+\deg(h), \\ \binom{d-\deg(w)-\deg(h)+\#y-1}{\#y-1} & \text{if } d \geqslant \deg(w)+\deg(h). \end{cases}$$

Let M be a homogeneous K-linear subspace of R, and let $\mathcal{D}$ be a finite set of triples (w,y,h) where w a monomial in $x^\diamond$, y is a subset of x, and h is a non-zero homogeneous element of R. We define the **degree of** $\mathcal{D}$ as

$$\deg \mathcal{D} := \max\big\{ \deg(w)+\deg(h) : (w,y,h) \in \mathcal{D}\big\} \in \mathbb{N} \cup \{\infty\},$$

where $\max \varnothing = \infty$ by convention. We also set

$$\mathcal{D}^+ := \big\{(w,y,h) \in \mathcal{D} : y \neq \varnothing\big\}.$$

We say that $\mathcal{D}$ is a **cone decomposition of** M if $C(w,y,h) \subseteq M$ for every $(w,y,h) \in \mathcal{D}$ and

$$M = \bigoplus_{(w,y,h)\in\mathcal{D}} C(w,y,h).$$

and $\mathcal{D}$ is simply a **cone decomposition** if $\mathcal{D}$ is a cone decomposition of some homogeneous K-linear subspace of R. By abuse of language we will also say that a cone decomposition $\mathcal{D}$ is **monomial** if $h=1$ for all $(w,y,h) \in \mathcal{D}$. Suppose now that $\mathcal{D}$ is a cone decomposition of M. Then for every d we have

$$H_M(d) = \sum_{(w,y,h)\in\mathcal{D}} H_{C(w,y,h)}(d).$$

By Lemma 4.11, the Hilbert polynomial of M exists, and is determined by the cones in $\mathcal{D}^+$: for $d > \deg(\mathcal{D})$ we have

$$\begin{aligned} H_M(d) &= \sum_{(w,y,h)\in\mathcal{D}^+} H_{C(w,y,h)}(d) \\ &= \sum_{(w,y,h)\in\mathcal{D}^+} \binom{d-\deg(w)-\deg(h)+\#y-1}{\#y-1} = P_M(d) \end{aligned} \tag{4.1}$$

and hence for every $d \geqslant \deg(\mathcal{D}^+)$:

$$H_M(d) = P_M(d) + \#\big\{(w,y,h) \in \mathcal{D}\setminus\mathcal{D}^+ : \deg(w)+\deg(h) = d\big\}.$$

By the above, we have $\sigma(M) \leqslant \deg(\mathcal{D})+1$ for every cone decomposition $\mathcal{D}$ of M. (Here $\sigma(M)$ denotes the regularity of the Hilbert function of M as defined in Section 2.13.)

The following is an adaptation of Definition 4.4:

Definition 4.12. We say that a cone decomposition $\mathcal{D}$ is d-**standard** if

1. $\deg(w)+\deg(h) \geqslant d$ for all $(w,y,h) \in \mathcal{D}^+$;
2. for every $(w,y,h) \in \mathcal{D}^+$ and d' with $d \leqslant d' \leqslant \deg(w)+\deg(h)$ there is some $(w',y',h') \in \mathcal{D}^+$ with $\deg(w')+\deg(h') = d'$ and $\#y' \geqslant \#y$.

We also say that $\mathcal{D}$ is **standard** if $\mathcal{D}$ is 0-standard.

If $\mathcal{D}^+ = \varnothing$ then $\mathcal{D}$ is d-standard for every d, whereas if $\mathcal{D}^+ \neq \varnothing$ and $\mathcal{D}$ is d-standard, then necessarily

$$d = \min\big\{ \deg(w) + \deg(h) : (w,y,h) \in \mathcal{D}^+ \text{ for some } y \subseteq x \big\}.$$

If $\mathcal{D}$ is d-standard for some d, then we let $d_{\mathcal{D}}$ denote the smallest d such that $\mathcal{D}$ is d-standard (so $d_{\mathcal{D}} = 0$ if $\mathcal{D}^+ = \varnothing$).

Examples 4.13. The empty set is a standard cone decomposition of the trivial K-linear subspace $\{0\}$ of R. If $h \in R$ is non-zero and homogeneous, and $y \subseteq x$, then $\{(1,y,h)\}$ is a $\deg(h)$-standard cone decomposition of $C(1,y,h)$. In particular, $\{(1,x,1)\}$ is a standard cone decomposition of $R = C(1,x)$.

The following properties are straightforward:

Lemma 4.14.

(1) *Suppose M_1 and M_2 are homogeneous K-linear subspaces of M with $M = M_1 \oplus M_2$, and let $\mathcal{E}_1$, $\mathcal{E}_2$ be cone decompositions of M_1 and M_2, respectively. Then $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$ is a cone decomposition of M. If $\mathcal{E}_1$ and $\mathcal{E}_2$ are d-standard, then so is $\mathcal{E}$.*

(2) *Suppose $\mathcal{D}$ is a d-standard cone decomposition of M, and let $f \in R$ be non-zero homogeneous. Then $\mathcal{D}f := \big\{(w,y,hf) : (w,y,h) \in \mathcal{D}\big\}$ is a $(d + \deg f)$-standard cone decomposition of Mf.*

The lemma below shows how the degrees of cone decompositions of K-linear subspaces decomposing the K-linear space R are linked:

Lemma 4.15. *Let M_1, M_2 be K-linear subspaces of R with $R = M_1 \oplus M_2$. For $i = 1, 2$, let $\mathcal{D}_i$ be a cone decomposition of M_i, which is d_i-standard for some d_i. Then*

$$\max\{\deg \mathcal{D}_1, \deg \mathcal{D}_2\} = \max\{\deg \mathcal{D}_1^+, \deg \mathcal{D}_2^+\}.$$

Proof. We have

$$H_{M_1}(d) + H_{M_2}(d) = H_R(d) = \binom{d+N-1}{N-1} \qquad \text{for every } d \tag{4.2}$$

and thus

$$P_{M_1} + P_{M_2} = \binom{T+N-1}{N-1}. \tag{4.3}$$

For $d \geqslant \max\{\deg \mathcal{D}_1^+, \deg \mathcal{D}_2^+\}$ and $i = 1, 2$, we have

$$H_{M_i}(d) = P_{M_i}(d) + \#\big\{(w,y,h) \in \mathcal{D}_i \setminus \mathcal{D}_i^+ : \deg(w) + \deg(h) = d\big\}.$$

Hence, by (4.2) and (4.3), neither $\mathcal{D}_1$ nor $\mathcal{D}_2$ contains a triple (w,y,h) with $y = \varnothing$ and $\deg(w) + \deg(h) \geqslant \max\big\{\deg(\mathcal{D}_1^+), \deg(\mathcal{D}_2^+)\big\}$. It follows that for $i = 1, 2$ we have

$$\deg(\mathcal{D}_i) \leqslant \max\big\{\deg(\mathcal{D}_i \setminus \mathcal{D}_i^+), \deg(\mathcal{D}_i^+)\big\} \leqslant \max\big\{\deg(\mathcal{D}_1^+), \deg(\mathcal{D}_2^+)\big\}$$

as required. □

Given $w \in x^\diamond$ as well as $y \subseteq x$ and a non-zero homogeneous $h \in R$, define

$$\mathcal{C}(w,y,h) := \big\{(w,\varnothing,h)\big\} \cup \big\{\big(w * x_i, y \cap \{x_j : j \geqslant i\}, h\big) : x_i \in y\big\}.$$

It is easy to check that $\mathcal{C}(w,y,h)$ is a $(1+\deg h)$-standard cone decomposition of the cone $C(w,y,h)$.

Lemma 4.16. *If M has a d-standard cone decomposition, then M has a d'-standard cone decomposition for every $d' \geqslant d$.*

Proof. If $\mathcal{D}$ is a d-standard cone decomposition of M with $\mathcal{D}^+ = \varnothing$, then $\mathcal{D}$ is d'-standard for all d'. Therefore, suppose $\mathcal{D}$ is a d-standard cone decomposition of M with $\mathcal{D}^+ \neq \varnothing$; it is enough to show that then M has a $(d+1)$-standard cone decomposition. Put

$$\mathcal{E} := \big\{(w, y, h) \in \mathcal{D} : \deg(w) + \deg(h) = d\big\}.$$

Then trivially $\mathcal{E}$ is d-standard and, since $\mathcal{D}$ is d-standard, $\mathcal{D} \setminus \mathcal{E}$ is $(d+1)$-standard. Now put

$$\mathcal{E}' := \bigcup_{(w,y,h)\in\mathcal{E}} \mathcal{C}(w, y, h).$$

Then $\mathcal{E}'$ is a $(d+1)$-standard cone decomposition of the homogeneous K-linear subspace $\bigoplus_{(w,y,h)\in\mathcal{E}} C(w, y, h) \subseteq M$ of R. Hence, $\mathcal{E}' \cup (\mathcal{D} \setminus \mathcal{E})$ is a $(d+1)$-standard cone decomposition of M. $\square$

Corollary 4.17. *Let $M_1, \dots, M_r \subseteq M$ be homogeneous K-linear subspaces of R with $M = M_1 \oplus \cdots \oplus M_r$. If each M_i has a d_i-standard cone decomposition, then M has a d-standard cone decomposition where $d = \max\{d_1, \dots, d_r\}$.*

Combining Theorem 4.10 with Corollary 4.17 we obtain:

Corollary 4.18. *Let $I = (f_1, \dots, f_n)$ be a left ideal of R where $f_1, \dots, f_n \in R$ are non-zero and homogeneous, and suppose $n > 0$. Let $d_i = \deg(f_i)$ for $i = 1, \dots, n$, and $d = \max\{d_1, \dots, d_n\}$. Then there is a K-linear subspace M of I with $I = (f_1) \oplus M$, which admits a d-standard cone decomposition $\mathcal{D}$.* (*Hence $\{(1, x, f_1)\} \cup \mathcal{D}$ is a d-standard cone decomposition of I.*)

Proof. For $i = 2, \dots, n$ let G_i be a Gröbner basis of $((f_1, \dots, f_{i-1}) : f_i)$. Then

$$I = (f_1) \oplus M \qquad \text{for } M := \mathrm{nf}_{G_2}(R)f_2 \oplus \cdots \oplus \mathrm{nf}_{G_n}(R)f_n,$$

as in the remark after Lemma 3.16. The principal left ideal (f_1) has a d_1-standard cone decomposition $\{(1, x, f_1)\}$ (Example 4.13). For each $i = 2, \dots, n$ let $\mathcal{D}_i$ be a standard monomial cone decomposition of $\mathrm{nf}_{G_i}(R)$ guaranteed by Theorem 4.10; then

$$\mathcal{D}_i f_i = \big\{(w, y, f_i) : (w, y) \in \mathcal{D}_i\big\}$$

is a d_i-standard cone decomposition of $\mathrm{nf}_{G_i}(R)f_i$ by Lemma 4.14, (2). The claim now follows from Corollary 4.17. $\square$

4.3. **Macaulay constants and exact cone decompositions.** What is stated in this subsection generalizes the corresponding concepts in Section 6 of [9].

Let $\mathcal{D}$ be a cone decomposition which is d-standard for some d. For every i we define the cone decomposition

$$\mathcal{D}_i := \big\{(w, y, h) \in \mathcal{D} : \#y \geqslant i\big\}.$$

Then we have

$$\mathcal{D} = \mathcal{D}_0 \supseteq \mathcal{D}^+ = \mathcal{D}_1 \supseteq \cdots \supseteq \mathcal{D}_N \supseteq \mathcal{D}_{N+1} = \varnothing.$$

We define the **Macaulay constants** $b_0, \dots, b_{N+1}$ of $\mathcal{D}$ as follows:

$$b_i := \min\big\{d_{\mathcal{D}}, 1 + \deg \mathcal{D}_i\big\} = \begin{cases} d_{\mathcal{D}} & \text{if } \mathcal{D}_i = \varnothing \\ 1 + \deg \mathcal{D}_i & \text{otherwise.} \end{cases}$$

From the definition it follows that $b_0 \geqslant \ldots \geqslant b_{N+1} = d_{\mathcal{D}}$. The integer b_0 is an upper bound for the regularity $\sigma(M)$ of H_M. The name of the constants is due to the fact that

Macaulay proved that if R is commutative and I a homogeneous ideal of R, then for $d \geqslant b_0$ we have

$$H_{R/I}(d) = \binom{d - b_{N+1} + N}{N} - 1 - \sum_{i=1}^{N} \binom{d - b_i + i - 1}{i}$$

for certain integers $b_0 \geqslant \cdots \geqslant b_{N+1} \geqslant 0$, which turn out to be the Macaulay constants of a special type of monomial cone decomposition of $\mathrm{nf}_G(R)$ (for an arbitrary Gröbner basis G of I), which we now define in general:

Definition 4.19. A cone decomposition $\mathcal{D}$ is called **exact** if $\mathcal{D}$ is d-standard for some d and for every degree d', $\mathcal{D}^+$ contains at most one triple (w, y, h) with $\deg(w) + \deg(h) = d'$.

Exact cone compositions have a strong rigidity property:

Lemma 4.20. *Let $\mathcal{D}$ be an exact cone decomposition with Macaulay constants b_i. Then for each $i = 1, \dots, N$ and each d with $b_{i+1} \leqslant d < b_i$ there is exactly one $(w, y, h) \in \mathcal{D}^+$ such that $\deg(w) + \deg(h) = d$, and for this triple we have $\#y = i$.*

Proof. Suppose $i \in \{1, \dots, N\}$ and d satisfy $b_{i+1} \leqslant d < b_i$. Let $(w', y', h') \in \mathcal{D}$ be such that $\#y' \geqslant i$ and $\deg(w') + \deg(h') = b_i - 1$. Then, since $\mathcal{D}$ is $d_{\mathcal{D}}$-standard, there exists $(w, y, h) \in \mathcal{D}$ with $\deg(w) + \deg(h) = d$ and $\#y \geqslant \#y' \geqslant i$. We have $\#y = i$, since otherwise $(w, y, h) \in \mathcal{D}_{i+1}$ with $\deg(w) + \deg(h) = d \geqslant b_{i+1} > \deg \mathcal{D}_{i+1}$, contradicting the definition of b_{i+1}. By exactness of $\mathcal{D}$, (w, y, h) is the only triple in $\mathcal{D}^+$ with $\deg(w) + \deg(h) = d$. $\square$

The next lemma allows one to split triples in cone decompositions in order to achieve exactness:

Lemma 4.21. *Let $\mathcal{D}$ be a d-standard cone decomposition of the K-linear subspace M of R, and let $(w, y, h), (v, z, g) \in \mathcal{D}$ such that*

$$\deg(w) + \deg(h) = \deg(v) + \deg(g), \qquad \#z \geqslant \#y > 0.$$

Let $x_i \in y$ be arbitrary. Then

$$\mathcal{D}' := \big(\mathcal{D} \setminus \big\{(w, y, h)\big\}\big) \cup \big\{(w, y \setminus \{x_i\}, h), (w * x_i, y, h)\big\}$$

is also a d-standard cone decomposition of M.

Proof. We have

$$C(w, y, h) = C(w, y \setminus \{x_i\}, h) \oplus C(w * x_i, y, h).$$

Therefore $\mathcal{D}'$ remains a cone decomposition of M, and it is easy to check that $\mathcal{D}'$ is d-standard. $\square$

By a straightforward adaptation of Algorithms `SHIFT` and `EXACT` in [9], and using Lemma 4.21 above in place of Lemma 6.2 of [9] in the verification of their correctness, one obtains:

Theorem 4.22. *There exists an algorithm that, given a d-standard cone decomposition $\mathcal{D}$ of a K-linear subspace M of R, produces an exact d-standard decomposition $\mathcal{D}'$ of M, whose Macaulay constant b_0 satisfies $b_0 \geqslant 1 + \deg(\mathcal{D})$.*

Let now $\mathcal{D}$ be an exact cone decomposition of a K-linear subspace M of R. Then by (4.1) and Lemma 4.20 we have

$$P_M(T) = \sum_{i=1}^{N} \sum_{j=b_{i+1}}^{b_i - 1} \binom{T - j + i - 1}{i - 1}.$$

One may show that this sum can be converted to

$$P_M(T) = \binom{T - b_{N+1} + N}{N} - 1 - \sum_{i=1}^{N} \binom{T - b_i + i - 1}{i},$$

and once b_{N+1} has been fixed, the coefficients $b_1, \dots, b_N$ uniquely determine the polynomial P_M; see [9, p. 768–769]; also, b_0 is the smallest $r \geqslant b_1$ such that $H_M(d) = P_M(d)$ for all $d \geqslant r$. In particular, the Macaulay constants $b_0 \geqslant b_1 \geqslant \cdots \geqslant b_{N+1} = 0$ of an exact *standard* cone decomposition $\mathcal{D}$ of M do not depend on our choice of $\mathcal{D}$, and the Hilbert function of M is uniquely determined by $b_0, \dots, b_N$. Since every K-linear subspace M which admits a standard cone decomposition also has an exact standard cone decomposition (by the previous theorem), we may, in this case, simply talk of the **Macaulay constants** $b_0, \dots, b_N$ **of** M. All this applies to $M = \mathrm{nf}_G(R)$ where G is a Gröbner basis of a left ideal of R; hence, by Theorems 4.10 and 4.22 we obtain:

Corollary 4.23. *Let G be the reduced Gröbner basis of a left ideal of R, and let $b_0, \dots, b_N$ be the Macaulay constants of $\mathrm{nf}_G(R)$. Then $\deg(g) \leqslant b_0$ for every $g \in G$.*

5. Proof of Theorem 0.1 and its Corollaries

Let R be a K-algebra of solvable type with respect to $x = (x_1, \dots, x_N)$ and an admissible ordering $\leqslant$ of $\mathbb{N}^N$.

5.1. **Degree bounds for Gröbner bases.** Let I be a left ideal of R generated by nonzero elements $f_1, \dots, f_n \in R$, where $n > 0$, and let d be the maximum of the degree of $f_1, \dots, f_n$. The central result of this section is:

Proposition 5.1. *Suppose the algebra R and the generators $f_1, \dots, f_n$ of I are homogeneous. Then the elements of the reduced Gröbner basis of I have degree at most*

$$D(N-1, d) = 2\left(\frac{d^2}{2} + d\right)^{2^{N-2}}.$$

Proof. We let t range over $\mathbb{N}$. Let $\mathcal{D}$ be a standard exact cone decomposition of $\mathrm{nf}_G(R)$ with Macaulay constants $b_0 \geqslant \cdots \geqslant b_{N+1} = 0$, where G is the reduced Gröbner basis of I. After reordering the $f_1, \dots, f_n$ we may assume that f_1 has degree d. Let $\mathcal{E}$ be a d-standard exact cone decomposition of a K-linear subspace M of I such that $I = (f_1) \oplus M$ (by Corollary 4.18 and Theorem 4.22), with Macaulay coefficients $a_0 \geqslant \cdots \geqslant a_{N+1} = d$. The computations in [9, Section 8] show that

$$a_j + b_j \leqslant D(N-j, d) \qquad \text{for } j = 1, \dots, N-2.$$

In particular $a_1 + b_1 \leqslant D := D(N-1, d)$. The d-standard cone decomposition $\mathcal{E} \cup \{(1, x, f_1)\}$ of I has the same Macaulay constants a_i as $\mathcal{E}$. Hence by Lemma 4.15 (applied to $M_1 = \mathrm{nf}_G(R)$, $M_2 = I$) we have

$$\max\{a_0, b_0\} = \max\{a_1, b_1\} \leqslant D.$$

Corollary 4.23 now yields the proposition. □

Remark 5.2. Suppose the hypothesis of the previous proposition holds. Implicit in the proof above, there is a uniform bound for the regularity of the Hilbert function of the left R-module R/I:

$$\sigma(R/I) \leqslant D(N-1, d).$$

A similar doubly-exponential bound for $\sigma(R/I)$ was obtained in [8]. In the case where R is a commutative polynomial ring, the regularity of the Hilbert function $\sigma(M)$ of a finitely generated R-module M is closely related to the Castelnuovo-Mumford regularity $\operatorname{reg}(M)$ of M. For example (see [7, 2.1]), in this case we have

$$\sigma(R/I) \leqslant \operatorname{reg}(R/I) = \operatorname{reg}(I) - 1.$$

There does exist a doubly-exponential bound on $\operatorname{reg}(I)$ in terms of N and d, valid independently of the characteristic of K (see [6]):

$$\operatorname{reg}(I) \leqslant (2d)^{2^{N-2}}.$$

It would be interesting to see whether this bound can also be deduced using the methods of the present paper.

We next address the inhomogeneous case:

Corollary 5.3. *Suppose R is quadric. Then there exists a Gröbner basis G of I with the following property: for every $g \in G$ we can write*

$$g = y_{g,1} f_1 + \cdots + y_{g,n} f_n$$

where $y_{g,i} \in R$ with

$$\deg(y_{g,i} f_i) \leqslant D(N,d) = 2\left(\frac{d^2}{2} + d\right)^{2^{N-1}} \qquad \text{for } i = 1, \dots, n.$$

Proof. By the proposition above, the reduced Gröbner basis H with respect to $\leqslant^*$ of the left ideal of R^* generated by $f_1^*, \dots, f_n^*$ consists of homogeneous elements of degree at most $D(N,d)$. Moreover, for every $h \in H$ there are homogeneous $z_{h,1}, \dots, z_{h,n} \in R$ such that

$$h = z_{h,1} f_1^* + \cdots + z_{h,n} f_n^*$$

and

$$\deg(z_{h,i} f_i^*) \leqslant \deg(h) \leqslant D(N,d) \qquad \text{for } i = 1, \dots, n.$$

Corollary 3.11 shows that $G := H_*$ is a Gröbner basis of I with respect to $\leqslant$, and for every $h \in H$ we have

$$h_* = y_{h_*,1} f_1 + \cdots + y_{h_*,n} f_n$$

with $y_{h_*,i} := (z_{h,i})_*$ and

$$\deg(y_{h_*,i} f_i) \leqslant \deg(z_{h,i} f_i^*) \leqslant D(n,d) \qquad \text{for } i = 1, \dots, n,$$

as required. □

The previous corollary yields Theorem 0.1. Next we show the first part of Corollary 0.4:

Corollary 5.4. *Suppose R is quadric. Every two-sided ideal of R generated by elements of degree at most d has a two-sided Gröbner basis consisting of elements of degree at most $D(2N,d)$.*

Proof. We may assume that $d > 0$. Suppose J is the two-sided ideal of R generated by $f_1, \dots, f_n$. Let $\mu\colon R^{\mathrm{env}} \to R$ be as in Section 2.7. By the discussion in Section 2.11, the left ideal $\mu^{-1}(J)$ of R^{env} is generated by the elements

$$f_1 \otimes 1, \dots, f_n \otimes 1, x^{\varepsilon_1} \otimes 1 - 1 \otimes x^{\varepsilon_1}, \dots, x^{\varepsilon_N} \otimes 1 - 1 \otimes x^{\varepsilon_N},$$

each of which has degree at most d. Hence by the previous corollary, $\mu^{-1}(J)$ has a Gröbner basis G (with respect to $\leqslant^{\mathrm{env}}$) consisting of elements of R^{env} of degree at most $D(2N,d)$.

By Proposition 3.7, $\mu(G)$ is a Gröbner basis of J whose elements obey the same degree bound. $\square$

Before we are able to compute a degree bound for reduced Gröbner bases which is also valid in the inhomogeneous situation, we need to study the complexity of reduction sequences.

5.2. **Degree bounds for normal forms.** Let A be a non-singular $N \times N$-matrix with non-negative integer entries, $D > 1$ an integer, and $\mathrm{wt} = \mathrm{wt}_{D,A}$. For non-zero $f \in R$ we set

$$\mathrm{wt}(f) := \max_{\alpha \in \mathrm{supp}(f)} \mathrm{wt}(\alpha),$$

and we let $w(0) := 0$. Then for all $f, g \in R$ we have

$$\deg(f) \leqslant \mathrm{wt}(f) \leqslant ||A|| \deg(f) \frac{D^N - 1}{D - 1} \tag{5.1}$$

by (1.2). Also

$$\mathrm{wt}(f+g) \leqslant \max\big\{\mathrm{wt}(f), \mathrm{wt}(g)\big\}, \qquad \mathrm{wt}(cf) = \mathrm{wt}(f) \text{ for non-zero } c \in K.$$

Suppose now that A represents $\leqslant$. Then

$$\mathrm{wt}(f) = \mathrm{wt}\big(\mathrm{lm}(f)\big) \qquad \text{if } \deg(f) < \frac{D}{||A||}. \tag{5.2}$$

We will need a variant of Lemma 2.4 (with an analogous proof). In the lemma below, we assume that the commutator relations between x_i and x_j in R are expressed as in Definition 2.3.

Lemma 5.5. *Suppose that* $\mathrm{wt}(p_{ij}) < \mathrm{wt}(x_i x_j)$ *for* $1 \leqslant i < j \leqslant N$. *Then for all* α, β *we have*

$$x^\alpha \cdot x^\beta = c x^{\alpha+\beta} + r \qquad \text{where } c \in K,\ c \neq 0, \text{ and } \mathrm{wt}(r) < \mathrm{wt}(x^{\alpha+\beta}),$$

in particular $\mathrm{wt}(x^\alpha \cdot x^\beta) = \mathrm{wt}(x^\alpha) + \mathrm{wt}(x^\beta)$.

Proof. We proceed by induction on the non-negative integer $\omega = \mathrm{wt}(x^{\alpha+\beta})$. If $\omega = 0$, then $\alpha = \beta = 0$, and there is nothing to show. So assume that $\omega > 0$, and we have shown the claim for all α, β with $\mathrm{wt}(x^{\alpha+\beta}) < \omega$. Note that this implies $\mathrm{wt}(fg) \leqslant \mathrm{wt}(f) + \mathrm{wt}(g)$ for all $f, g \in R$ with $\mathrm{wt}(f), \mathrm{wt}(g) < \omega$. Suppose α, β satisfy $\mathrm{wt}(x^{\alpha+\beta}) = \omega$. Put

$$m_\alpha := \min\{n : x_n | x^\alpha\}, \qquad n_\alpha := \max\{n : x_n | x^\alpha\},$$

and similarly we define m_β and n_β. If $n_\alpha \leqslant m_\beta$, then clearly $x^\alpha x^\beta = x^{\alpha+\beta}$, and we are done. Hence assume now that $n_\alpha > m_\beta$. We distinguish three cases:

Case 1: $m_\alpha \leqslant m_\beta$. Write $x^\alpha = x_{m_\alpha} x^{\alpha'}$ where $\mathrm{wt}(x^{\alpha'}) < \mathrm{wt}(x^\alpha)$. Then $x^{\alpha+\beta} = x_{m_\alpha} x^{\alpha'+\beta}$. By inductive hypothesis there is a non-zero $c' \in K$ such that $x^{\alpha'} x^\beta = c' x^{\alpha'+\beta} + r'$ with $\mathrm{wt}(r') < \mathrm{wt}(\alpha') + \mathrm{wt}(\beta)$. Thus

$$x^\alpha x^\beta = x_{m_\alpha}(c' x^{\alpha'+\beta} + r') = c' x_{m_\alpha} x^{\alpha'+\beta} + x_{m_\alpha} r' = c' x^{\alpha+\beta} + x_{m\alpha} r'$$

where $\mathrm{wt}(x_{m_\alpha} r') < \omega$.

Case 2: $n_\alpha \leqslant n_\beta$. This is treated similarly to Case 1, writing $x^\beta = x^{\beta'} x_{n_\beta}$.

Case 3: $m_\alpha > m_\beta$ *and* $n_\alpha > n_\beta$. Put $i := m_\beta$, $j := n_\alpha$, and write $x^\alpha = x^{\alpha'} x_i$ and $x^\beta = x_j x^{\beta'}$. Then we have

$$x^\alpha x^\beta = x^{\alpha'} x_j x_i x^{\beta'} = x^{\alpha'}(c_{ij} x_i x_j + p_{ij}) x^{\beta'} = c_{ij} x^{\alpha'} x_i x_j x^{\beta'} + x^{\alpha'} p_{ij} x^{\beta'}.$$

By assumption we have $\mathrm{wt}(p_{ij}) < \mathrm{wt}(x_i) + \mathrm{wt}(x_j)$, and so by inductive hypothesis $\mathrm{wt}(x^{\alpha'} p_{ij} x^{\beta'}) < \omega$. Moreover, the inductive hypothesis also yields

$$x^{\alpha'} x_i = c' x^{\alpha'} * x_i + r', \qquad x_j x^{\beta'} = c'' x_j * x^{\beta'} + r''$$

where $\mathrm{wt}(r') < \mathrm{wt}(\alpha') + \mathrm{wt}(x_i)$ and $\mathrm{wt}(r'') < \mathrm{wt}(x_j) + \mathrm{wt}(\beta')$. By assumption in this case, we have $x^{\alpha'} * x_i = x_i x^{\alpha'}$ and $x_j * x^{\beta'} = x^{\beta'} x_j$. Hence

$$\begin{aligned} x^{\alpha'} x_i x_j x^{\beta'} &= (c' x_i x^{\alpha'} + r')(c'' x^{\beta'} x_j + r'') \\ &= c'c''(x_i x^{\alpha'} x^{\beta'} x_j) + c'(x_i x^{\alpha'}) r'' + c'' r' (x^{\beta'} x_j) + r' r'' \end{aligned}$$

where the last three summands have weight smaller than ω. By inductive hypothesis again, we write

$$x^{\alpha'} x^{\beta'} = d x^{\alpha'+\beta'} + s$$

where $d \in K$ is non-zero and $\mathrm{wt}(s) < \mathrm{wt}(\alpha') + \mathrm{wt}(\beta')$. This yields

$$c'c''(x_i x^{\alpha'} x^{\beta'} x_j) = c'c'' d (x_i x^{\alpha'+\beta'} x_j) + x_i s x_j = c'c'' d \cdot x^{\alpha+\beta} + x_i s x_j$$

where $\mathrm{wt}(x_i s x_j) < \omega$ by inductive hypothesis. $\square$

We can now show:

Lemma 5.6. *Under the same hypothesis as the previous lemma, let G be a subset of R each of whose elements has degree less than $\frac{D}{||A||}$, and let $f, h \in R$. If $f \xrightarrow[G]{*} h$, then there are $g_1, \dots, g_m \in G$ and $p_1, \dots, p_m \in R$ with*

$$f - h = p_1 g_1 + \cdots + p_m g_m$$

and

$$\mathrm{wt}(p_1 g_1), \dots, \mathrm{wt}(p_m g_m) \leqslant \mathrm{wt}(f).$$

Proof. We let g range over G. We proceed on Noetherian induction on the well-founded relation $\xrightarrow[G]{}$. Suppose $f \xrightarrow[g]{} f' \xrightarrow[G]{*} h$. Then there exists $c \in K$ and α, β such that

$$\mathrm{lm}(x^\beta g) = x^\alpha \in \operatorname{supp} f, \quad \mathrm{lc}(c x^\beta g) = f_\alpha, \quad f' = f - c x^\beta g.$$

Now by the previous lemma and (5.2) applied to g, we have

$$\mathrm{wt}(c x^\beta g) \leqslant \mathrm{wt}(x^\beta) + \mathrm{wt}(g) = \mathrm{wt}(x^\beta) + \mathrm{wt}\left(\mathrm{lm}(g)\right) = \mathrm{wt}(x^\alpha) \leqslant \mathrm{wt}(f)$$

and thus $\mathrm{wt}(f') \leqslant \mathrm{wt}(f)$. By inductive hypothesis, there are $g_1', \dots, g_n' \in G$ and $p_1', \dots, p_n' \in R$ with

$$f' - h = \sum_{i=1}^{n} p_i' g_i \quad \text{and} \quad \mathrm{wt}(p_i' g) \leqslant \mathrm{wt}(f') \text{ for every } i.$$

Hence

$$f - h = (f - f') + (f' - h) = \sum_{i=1}^{n+1} p_i g_i$$

where $p_i := p_i'$, $g_i := g_i'$ for $i = 1, \dots, n$ and $p_{n+1} := c x^\beta$, $g_{n+1} := g$ satisfy $\mathrm{wt}(p_i g_i) \leqslant \mathrm{wt}(f)$ for every i, as required. $\square$

Corollary 5.7. *Suppose $D > 2\,||A||$ and $D > \deg(p_{ij})$ for $1 \leqslant i < j \leqslant N$, and let G be a subset of R each of whose elements has degree less than $\frac{D}{||A||}$, and $f, h \in R$. If $f \xrightarrow[G]{*} h$, then there are $g_1, \dots, g_m \in G$ and $p_1, \dots, p_m \in R$ with*

$$f - h = p_1 g_1 + \cdots + p_m g_m$$

and

$$\deg(p_1 g_1), \dots, \deg(p_m g_m), \deg(h) \leqslant ||A||\,\deg(f)\,\frac{D^N - 1}{D - 1}.$$

Proof. Since $D > 2\,||A||$ and $D > \deg(p_{ij})$, we have $\mathrm{wt}(p_{ij}) < \mathrm{wt}(x_i x_j)$, for $1 \leqslant i < j \leqslant N$, by (5.2). The claim now follows from the previous lemma and (5.1) □

If $\leqslant$ is degree-compatible, then the estimate in the corollary above can be improved, and the additional assumptions on R and G removed: Let G be a subset of R, $f, h \in R$; if $f \xrightarrow[G]{*} h$, then Noetherian induction as in the proof of Corollary 5.7 yields easily that there are $g_1, \dots, g_m \in G$ and $p_1, \dots, p_m \in R$ such that

$$f - h = p_1 g_1 + \cdots + p_m g_m$$

and

$$\mathrm{lm}(p_1 g_1), \dots, \mathrm{lm}(p_m g_m), \mathrm{lm}(h) \leqslant \mathrm{lm}(f).$$

Since our admissible ordering is degree-compatible, we have

$$\deg(p_1 g_1), \dots, \deg(p_m g_m), \deg(h) \leqslant \deg(f).$$

5.3. **Degree bounds for reduced Gröbner bases.** *In the rest of this section we assume that R is quadric.* The auxiliary results from the previous subsection allow us to show Corollary 0.2:

Corollary 5.8. *Suppose that the admissible ordering $\leqslant$ can be represented by rational weights. Then there is a constant C, which only depends on $\leqslant$, with the following property: the reduced Gröbner basis of every left ideal of R generated by elements of degree at most d consists of elements of degree at most $\big(C \cdot D(N,d)\big)^{N+1}$.*

Proof. We may assume $d > 0$. Put $D := 2\,||A||\,\lceil D(N,d)\rceil$. Let I be a left ideal of R generated by elements of degree at most d. Choose a Gröbner basis $G = \{g_1, \dots, g_m\}$ of I with $\deg(g_i) \leqslant D(N,d)$ for $i = 1, \dots, m$. (Corollary 5.3.) After pruning G, we may assume that $\mathrm{lm}(G)$ is a minimal set of generators for the monomial ideal of R generated by $\mathrm{lm}(I)$, and after normalizing each g_i, that $\mathrm{lc}(g_i) = 1$ for every i. Set $h_i := g_i - \mathrm{lm}(g_i)$ for every i. By Corollary 5.7 we have

$$\deg \mathrm{nf}_G(h_i) \leqslant ||A||\,\deg(g_i)\,\frac{D^N - 1}{D - 1} \leqslant D\,\frac{D^N - 1}{D - 1} \leqslant D^{N+1}$$

for every i. Then $G' := \{g_1', \dots, g_m'\}$ where $g_i' := \mathrm{lm}(g_i) + \mathrm{nf}_G(h_i)$ for every i is a reduced Gröbner basis of I with $\deg g_i' \leqslant D^{N+1}$ for every i. □

For degree-compatible admissible orderings one obtains in a similar way:

Corollary 5.9. *Suppose that the admissible ordering $\leqslant$ is degree-compatible. Then the reduced Gröbner basis of every left ideal of R generated by elements of degree at most d consists of elements of degree at most $D(N,d)$.*

5.4. **Ideal membership.** Now we turn to degree bounds for solutions to linear equations. In particular, we'll show Corollary 0.3.

Proposition 5.10. *Suppose $\leqslant$ can be represented by rational weights. Then there is a constant C, only depending on $\leqslant$ which satisfies the following: if $f \in I = (f_1, \dots, f_n)$ where $f_1, \dots, f_n \in R$ are of degree at most d, then there there are $y_1, \dots, y_n \in R$ of degree at most $\deg(f) \cdot \big(C \cdot D(N,d)\big)^N$ with*

$$f = y_1 f_1 + \cdots + y_n f_n.$$

Proof. We may assume $d > 0$. Put $D := 2\,||A||\,\lceil D(N,d)\rceil$. Let $f_1, \dots, f_n \in R$ have degree at most d, and choose a Gröbner basis G of $I = (f_1, \dots, f_n)$ with the property stated in Corollary 5.3. Let $f \in I$. Then by Corollary 5.7 there are $g_1, \dots, g_m \in G$ and $p_1, \dots, p_m \in R$ such that

$$f = p_1 g_1 + \cdots + p_m g_m$$

and

$$\deg(p_1 g_1), \dots, \deg(p_m g_m) \leqslant ||A||\ \deg(f)\,\frac{D^N - 1}{D - 1}.$$

Write each g_i as

$$g_i = y_{i,1} f_1 + \cdots + y_{i,n} f_n$$

where $y_{i,j} \in R$ satisfies $\deg(y_{i,j} f_j) \leqslant D(N,d)$. Then

$$f = y_1 f_1 + \cdots + y_n f_n$$

where each $y_j := \sum_i p_i y_{i,j}$ has degree at most

$$\deg(f) \cdot \left(||A||\,\frac{D^N - 1}{D - 1} + D \right),$$

and this yields the claim. □

In a similar way we obtain:

Proposition 5.11. *Suppose $\leqslant$ is degree-compatible, let $f_1, \dots, f_n \in R$ be of degree at most d, and $f \in R$. If*

$$f = y_1 f_1 + \cdots + y_n f_n$$

for some $y_1, \dots, y_n \in R$, there are such $y_1, \dots, y_n \in R$ of degree at most $\deg(f) + D(N,d)$.

In the rest of this section, we restrict ourselves to the case that the admissible ordering $\leqslant$ is degree-compatible. The next corollary is the second part of Corollary 0.4:

Corollary 5.12. *Let $f_1, \dots, f_n \in R$ be of degree at most d, and let $f \in R$. If the equation*

$$f = y_1 f_1 z_1' + \cdots + y_n f_n z_n'$$

has a solution $(y_1, \dots, y_n, z_1, \dots, z_n) \in R^{2n}$, then this equation also has such a solution where

$$\deg(y_i), \deg(z_i) \leqslant \deg(f) + D(2N, d) \qquad \textit{for } i = 1, \dots, n.$$

Proof. Apply the previous proposition to R^{env} and

$$f_1 \otimes 1, \dots, f_n \otimes 1, x^{\varepsilon_1} \otimes 1 - 1 \otimes x^{\varepsilon_1}, \dots, x^{\varepsilon_N} \otimes 1 - 1 \otimes x^{\varepsilon_N}$$

in place of R and $f_1, \dots, f_n$, respectively. (See Section 2.11.) □

5.5. **Generators for syzygy modules.** Below, the left R-module of left syzygies of a tuple $f = (f_1, \dots, f_n) \in R^n$ is denoted by $\mathrm{Syz}(f)$ (a submodule of the free left R-module R^n).

Suppose $G = \{g_1, \dots, g_m\}$ is a Gröbner basis in R. For $1 \leqslant i < j \leqslant m$ let α_{ij} and β_{ij} be the unique multi-indices such that

$$x^{\alpha_{ij}} * \mathrm{lm}(g_i) = x^{\beta_{ij}} * \mathrm{lm}(g_j) = \mathrm{lcm}\big(\mathrm{lm}(g_i), \mathrm{lm}(g_j)\big)$$

and

$$c_{ij} := \mathrm{lc}(x^{\alpha_{ij}} g_i), \qquad d_{ij} := \mathrm{lc}(x^{\beta_{ij}} g_j).$$

By Proposition 3.4, each S-polynomial

$$S(g_i, g_j) = d_{ij}\,\mathrm{lc}(g_j) x^{\alpha_{ij}} g_i - c_{ij}\,\mathrm{lc}(g_i) x^{\beta_{ij}} g_j$$

admits a representation of the form

$$S(g_i, g_j) = \sum_{k=1}^{m} p_{ijk} g_k, \quad \mathrm{lm}(p_{ijk} g_k) \leqslant \mathrm{lm}\, S(g_i, g_j) \qquad (p_{ijk} \in R).$$

Now consider the vectors

$$s_{ij} := d_{ij}\,\mathrm{lc}(g_j) x^{\alpha_{ij}} e_i - c_{ij}\,\mathrm{lc}(g_i) x^{\beta_{ij}} e_j - \sum_k p_{ijk} e_k \qquad (1 \leqslant i < j \leqslant m)$$

in R^m. Here $e_1, \dots, e_m$ denotes the standard basis of the free left R-module R^m. Each s_i is a left syzygy of $(g_1, \dots, g_m)$; in fact (see [17, Theorem 3.15]):

Theorem 5.13. *The syzygies s_{ij} (where $1 \leqslant i < j \leqslant m$) generate the left R-module* $\mathrm{Syz}(g_1, \dots, g_m)$.

We denote the set of $m \times n$-matrices with entries in R by $R^{m \times n}$. The $n \times n$-identity matrix is denoted by I_n. The following transformation rule for left syzygies is easy to verify:

Lemma 5.14. *Let $f = (f_1, \dots, f_n)^{\mathrm{tr}} \in R^n$ and $g = (g_1, \dots, g_m)^{\mathrm{tr}} \in R^m$, and suppose $A \in R^{m \times n}$, $B \in R^{n \times m}$ such that $g = Af$ and $f = Bg$. Let M be a matrix whose rows generate* $\mathrm{Syz}(g)$. *Then the rows of the matrix*

$$\left[\begin{array}{c} MA \\ \hline I_n - BA \end{array}\right]$$

generate $\mathrm{Syz}(f)$.

We now use these facts in the proof of:

Proposition 5.15. *Let $f = (f_1, \dots, f_n)^{\mathrm{tr}} \in R^n$ be of degree at most d. Then* $\mathrm{Syz}(f)$ *can be generated by elements of degree at most* $3D(N, d)$.

Proof. Let $g = (g_1, \dots, g_m)^{\mathrm{tr}} \in R^m$ be such that $G = \{g_1, \dots, g_m\}$ is a Gröbner basis of the left ideal of R generated by $f_1, \dots, f_n$ as in Corollary 5.3. Then there are $A \in R^{m \times n}$ of degree at most $D(N, d)$ and $B \in R^{n \times m}$ of degree at most d such that $g = Af$ and $f = Bg$. Now each S-polynomial $S(g_i, g_j)$ has degree at most $2D(N, d)$; hence there exists a matrix M of degree at most $D(N, d)$ whose rows generate $\mathrm{Syz}(g)$. Since $\deg(MA) \leqslant 3D(N, d)$ and $\deg(AB) \leqslant D(N, d) + d \leqslant 3D(N, d)$, the claim now follows from the previous lemma. ☐

References

1. Apel, J., *A relationship between Gröbner bases of ideals and vector modules of G-algebras,* in: Bokut, L. A. et al. (eds.), *Proceedings of the International Conference on Algebra,* Part 2 (Novosibirsk, 1989), 195–204, Contemp. Math., **131**, Part 2, Amer. Math. Soc., Providence, RI, 1992.
2. ______, *Computational ideal theory in finitely generated extension rings,* Theoret. Comput. Sci. **244** (2000), no. 1–2, 1–33.
3. Aschenbrenner, M., *Bounds and definability in polynomial rings,* Q. J. Math. **56** (2005), no. 3, 263–300.
4. Bergman, G., *The diamond lemma for ring theory,* Adv. Math. **29** (1978), 178–218.
5. Bueso, J., Gómez-Torrecillas, J., and Verschoren, A., *Algorithmic Methods in Non-Commutative Algebra. Applications to Quantum Groups,* Mathematical Modelling: Theory and Applications **17**, Kluwer Academic Publishers, Dordrecht, 2003.
6. Caviglia, G., and Sbarra, E., *Characteristic-free bounds for the Castelnuovo-Mumford regularity,* Compos. Math. **141** (2005), no. 6, 1365–1373.
7. Chardin, M., and Moreno-Socías, G., *Regularity of lex-segment ideals: some closed formulas and applications,* Proc. Amer. Math. Soc. **131** (2003), no. 4, 1093–1102 (electronic).
8. Chistov, A., and Grigoriev D., *Complexity of Janet basis of a D-module,* preprint (2007).
9. Dubé, T., *The structure of polynomial ideals and Gröbner bases,* SIAM J. Comput. **19** (1990), no. 4, 750–775.
10. Dubé, T., Mishra, B., and Yap, C., *Admissible orderings and bounds for Gröbner bases normal form algorithm,* Technical Report **88**, Courant Institute of Mathematical Sciences, Robotics Laboratory, New York University, 1986.
11. Galligo, A., *Some algorithmic questions on ideals of differential operators,* in: Caviness, B. (ed.), *EUROCAL '85,* vol. 2, Proceedings of the European Conference on Computer Algebra held in Linz, April 1–3, 1985, 413–421, Lecture Notes in Comput. Sci. **204**, Springer-Verlag, Berlin, 1985.
12. Grabmeier, J. et al., *Computer Algebra Handbook: Foundations, Applications, Systems,* Springer-Verlag, Berlin, 2003.
13. Grigoriev, D., *Complexity of solving systems of linear equations over the rings of differential operators,* in: Mora, T. and Traverso, C., *Effective Methods in Algebraic Geometry,* Papers from the symposium (MEGA-90) held in Castiglioncello, April 17–21, 1990, 195–202, Progress in Mathematics **94**, Birkhäuser Boston, Inc., Boston, MA, 1991.
14. Grigoriev, D. and Schwarz, F., *Loewy and primary decompositions of $\mathcal{D}$-modules,* Adv. in Appl. Math. **38** (2007), no. 4, 526–541.
15. Hausdorf, M. and Seiler, W. M., and Steinwandt, R., *Involutive bases in the Weyl algebra*, J. Symbolic Comput. **34** (2002), no. 3, 181–198.
16. Hermann, G., *Die Frage der endlich vielen Schritte in der Theorie der Polynomideale,* Math. Ann. **95** (1926), 736–788.
17. Kandri-Rody, A., and Weispfenning, V., *Non-commutative Gröbner bases in algebras of solvable type,* J. Symbolic Comput. **9** (1990), no. 1, 1–26.
18. Kredel, H., and Weispfenning, V., *Parametric Gröbner bases in rings of solvable type,* in: *Proc. IV. International Conference on Computer Algebra in Physical Research,* Joint Institute for Nuclear Research Dubna, USSR, May 1990, 236–244, World Scientific, Singapore, 1991.
19. Latyshev, V. N., *A combinatorial complexity of Gröbner bases,* J. Math. Sci. (New York) **102** (2000), no. 3, 4134–4138.
20. Lazard, D., *Gröbner bases, Gaussian elimination and resolution of systems of algebraic equations,* in: van Hulzen, J. A. (ed.), *Computer Algebra* (London, 1983), 146–156, Lecture Notes in Comput. Sci., 162, Springer-Verlag, Berlin, 1983.
21. Levandovskyy, V., and Schönemann, H., *PLURAL—a computer algebra system for noncommutative polynomial algebras,* in: *Proceedings of the 2003 International Symposium on Symbolic and Algebraic Computation,* 176–183 (electronic), Association for Computing Machinery, New York, 2003.
22. Li, H., *Non-commutative Gröbner Bases and Filtered-Graded Transfer,* Lecture Notes in Mathematics **1795**, Springer-Verlag, Berlin, 2002.
23. Li, H., and Wu, Y., *Filtered-graded transfer of Groebner basis computation in solvable polynomial algebras,* Comm. Algebra **28** (2000), no. 1, 15–32.
24. Maclagan, D., and Smith, G., *Uniform bounds on multigraded regularity,* J. Algebraic Geometry **14** (2005), 137–164.
25. Marker, D., *Model Theory,* Graduate Texts in Mathematics, vol. 217, Springer-Verlag, New York, 2002.
26. Mayr, E. W., and Meyer, A. R., *The complexity of the word problems for commutative semigroups and polynomial ideals*, Adv. in Math., **46** (1982), no. 3, 305–329.

27. Möller, H. M. and Mora, F., *Upper and lower bounds for the degree of Groebner bases*, Lecture Notes in Comput. Sci., **174** (1984), 172–183.
28. Mora, T., *An introduction to commutative and non-commutative Gröbner bases,* Theoret. Comput. Sci. **134** (1994), no. 1, 131–173.
29. Ōaku, T., and Shimoyama, T., *A Gröbner basis method for modules over rings of differential operators,* J. Symbolic Comput. **18** (1994), no. 3, 223–248.
30. Robbiano, L., *Term orderings on the polynomial ring,* in: Caviness, B., *EUROCAL '85,* Vol. 2 (Linz, 1985), 513–517, Lecture Notes in Comput. Sci., 204, Springer, Berlin, 1985.
31. Román, M. G. and Romń, S. G., *Gröbner bases and syzygies on bimodules over PBW algebras,* J. Symbolic Comput. **40** (2005), no. 3, 1039–1052.
32. Saito, M., Sturmfels, B., and Takayama, N., *Gröbner Deformations of Hypergeometric Differential Equations,* Algorithms and Computation in Mathematics **6**, Springer-Verlag, Berlin, 2000.
33. Seidenberg, A., *Constructions in algebra,* Trans. Amer. Math. Soc. **197** (1974), 273–313.
34. Smith, S., *Quantum groups: an introduction and survey for ring theorists,* in: Montgomery, S., and Small, L. (eds.), *Non-commutative Rings,* Survey lectures from the microprogram held in Berkeley, California, July 10–21, 1989, 131–178, Math. Sci. Res. Inst. Publ. **24**, Springer-Verlag, New York, 1992.
35. Stanley, R. P., *Linear Diophantine equations and local cohomology,* Invent. Math. **68** (1982), 175–193.
36. Sturmfels, B., and White, N., *Computing combinatorial decompositions of rings,* Combinatorica **11** (1991), 275–293.
37. Takayama, N., *Gröbner basis and the problem of contiguous relations,* Japan J. Appl. Math. **6** (1989), no. 1, 147–160.
38. Weispfenning, V., *Some bounds for the construction of Gröbner bases,* in: Beth, T., and Clausen, M. (eds.), *Applicable Algebra, Error-correcting Codes, Combinatorics and Computer Algebra* (*Karlsruhe, 1986*)*,* 195–201, Lecture Notes in Comput. Sci. **307**, Springer-Verlag, Berlin, 1988.

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