ANALYTIC HARDY FIELDS

MATTHIAS ASCHENBRENNER AND LOU VAN DEN DRIES

Abstract. We show that maximal analytic Hardy fields are η_1 in the sense of Hausdorff. We also prove various embedding theorems about analytic Hardy fields. For example, the ordered differential field $\mathbb{T}$ of transseries is shown to be isomorphic to an analytic Hardy field.

Contents

Introduction

This is a follow-up on [6] where the main result is that for any Hardy field H and countable subsets $A < B$ of H there exists y in a Hardy field extension of H such that $A < y < B$. Equivalently, (the underlying ordered set of) any maximal Hardy field is η_1 in the sense of Hausdorff. In this result we do not require $H \subseteq \mathcal{C}^{\omega}$, and the glueing constructions in [6] do not give $y \in \mathcal{C}^{\omega}$, even if $H \subseteq \mathcal{C}^{\omega}$; see *Notations and Conventions* at the end of this introduction for the notation used here. We call a Hardy field H *smooth* if $H \subseteq \mathcal{C}^{\infty}$ and *analytic* if $H \subseteq \mathcal{C}^{\omega}$. By [5, Theorem 6.7.22, Proposition 7.2.7], maximal Hardy fields, maximal smooth Hardy fields, and maximal analytic Hardy fields are all elementarily equivalent to the ordered differential field $\mathbb{T}$ of transseries, and have no proper d-algebraic H-field extension with constant field $\mathbb{R}$. We shall tacitly use these facts throughout.

Some may view non-analytic Hardy fields as somewhat artificial, since Hardy fields that occur "in nature" are analytic. (See [21, 22] for Hardy fields $H \not\subseteq \mathcal{C}^{\infty}$, and [30] for Hardy fields $H \subseteq \mathcal{C}^{\infty}$, $H \not\subseteq \mathcal{C}^{\omega}$.) To conciliate this view and answer an obvious question we prove in Section 4 the analytic version of [6]:

Date: November 2023.

Theorem A. *If H is an analytic Hardy field with countable subsets $A < B$, then there exists $y \in \mathcal{C}^\omega$ in a Hardy field extension of H such that $A < y < B$.*

Equivalently, all maximal analytic Hardy fields are η_1. The theorem goes through for smooth Hardy fields with $y \in \mathcal{C}^\infty$ in the conclusion; this can be obtained by refining the glueing constructions from [6] (as was actually done in an early version of that paper at the cost of three extra pages). Here we take care of the smooth and analytic versions simultaneously. Compared to [6] the new tool we use is a powerful theorem due to Whitney on approximating any $\mathcal{C}^n$-function or $\mathcal{C}^\infty$-function by an analytic function, where the approximation also takes derivatives into account. From that we obtain an analogue for germs, namely Corollary 1.8, which in turn we use to derive Theorem A from various results in the non-analytic setting of [6].

In the course of establishing Theorem A we also revisit results on pc-sequences and on extensions of type (b) from [6]. In Section 4 this also leads to:

Theorem B. *If H is a maximal analytic Hardy field, then H is dense in any Hardy field extension of H.*

If all maximal analytic Hardy fields are maximal Hardy fields, which seems to us implausible, then of course the theorems above would be trivially true. Can a maximal analytic Hardy field ever be a maximal Hardy field? For all we know answering questions of this kind might involve set-theoretic assumptions like CH. In [6] and the present paper we ran into other set-theoretic issues of this kind, and in Section 8 we state some problems that arose this way.

Sections 5–7 prove embedding theorems about (not necessarily maximal) analytic Hardy fields. A special case of a result in Section 7: the ordered differential field $\mathbb{T}$ is isomorphic over $\mathbb{R}$ to an analytic Hardy field extension of $\mathbb{R}$.

Notations and conventions. We take these from [6, end of introduction], but for the convenience of the reader we list here what is most needed.

We let i, j, k, l, m, n range over $\mathbb{N} = \{0, 1, 2, \dots\}$. We let $\mathcal{C}$ be the ring of germs at $+\infty$ of continuous functions $(a, +\infty) \to \mathbb{R}$, $a \in \mathbb{R}$, and $\mathcal{C}^r$ for $r \in \mathbb{N} \cup \{\infty\}$ its subring of germs of r times continuously differentiable functions $(a, +\infty) \to \mathbb{R}$, $a \in \mathbb{R}$. Thus $\mathcal{C}^{<\infty} := \bigcap_n \mathcal{C}^n$ is a differential ring with the obvious derivation, and has $\mathcal{C}^\infty$ as a differential subring. A *Hardy field* is a differential subring of $\mathcal{C}^{<\infty}$ whose underlying ring is a field; it is naturally also an ordered and valued field, see [ADH, pp. 384–386]. We let $\mathcal{C}^\omega$ be the differential subring of $\mathcal{C}^\infty$ consisting of the germs of real analytic functions $(a, +\infty) \to \mathbb{R}$, $a \in \mathbb{R}$.

1. Whitney's Approximation Theorem

In this section we let $r \in \mathbb{N} \cup \{\infty\}$ and $a, b \in \mathbb{R}$. We shall use the one-variable case of an approximation theorem due to Whitney [37, Lemma 6] to upgrade various constructions of smooth functions to analytic functions. To formulate this theorem we introduce some notation. Let $U \subseteq \mathbb{R}$ be open and $S \subseteq U$ be nonempty. For f in $\mathcal{C}(U)$ we set

$$\|f\|_S := \sup\big\{|f(s)| : s \in S\big\} \in [0, \infty],$$

so for $f, g \in \mathcal{C}(U)$ and $\lambda \in \mathbb{R}$ (and the convention $0 \cdot \infty = \infty \cdot 0 = 0$) we have

$$\|f + g\|_S \leqslant \|f\|_S + \|g\|_S, \quad \|\lambda f\|_S = |\lambda| \cdot \|f\|_S, \quad \text{and} \quad \|fg\|_S \leqslant \|f\|_S \|g\|_S.$$

If $\emptyset \neq S' \subseteq S$ then $\|f\|_{S'} \leqslant \|f\|_S$. Next, let $f \in \mathcal{C}^m(U)$. We then put

$$\|f\|_{S;\,m} \ :=\ \max\big\{\|f\|_S, \dots, \|f^{(m)}\|_S\big\} \in [0,\infty].$$

Then again for $f, g \in \mathcal{C}(U)$ and $\lambda \in \mathbb{R}$ we have

$$\|f+g\|_{S;m} \leqslant \|f\|_{S;m} + \|g\|_{S;m}, \quad \|\lambda f\|_{S;m} = |\lambda| \cdot \|f\|_{S;m},$$

and

$$\|fg\|_{S;\,m} \ \leqslant\ 2^m \|f\|_{S;\,m} \|g\|_{S;\,m} \tag{1.1}$$

Let $f \in \mathcal{C}(U)$. For $U = \mathbb{R}$ we set $\|f\|_m := \|f\|_{\mathbb{R};\,m}$. For $k \leqslant m$ and $\emptyset \neq S' \subseteq S \subseteq U$ we have $\|f\|_{S';\,k} \leqslant \|f\|_{S;\,m}$. Moreover, $\|f\|_{S;\,m}$ does not change if S is replaced by its closure in U.

Theorem 1.1 (Whitney)**.** *Let (a_n), (b_n), (ε_n) be sequences in $\mathbb{R}$ and (r_n) in $\mathbb{N}$ such that $a_0 = b_0$, (a_n) is strictly decreasing, (b_n) is strictly increasing, and $\varepsilon_n > 0$, $r_n \leqslant r$ for all n. Set $I := \bigcup_n K_n$, where $K_n := [a_n, b_n]$. Then, for any $f \in \mathcal{C}^r(I)$, there exists $g \in \mathcal{C}^\omega(I)$ such that for all n we have $\|f-g\|_{K_{n+1}\setminus K_n;\,r_n} < \varepsilon_n$.*

For a self-contained proof of Theorem 1.1, see the appendix to this paper.

For $f \in \mathcal{C}^m_a$ and nonempty $S \subseteq [a,+\infty)$ we put $\|f\|_{S;\,m} := \|g\|_{S;\,m}$ where $g \in \mathcal{C}^m(U)$ is any extension of f to an open neighborhood $U \subseteq \mathbb{R}$ of $[a,+\infty)$. We shall use the following special case of Theorem 1.1:

Corollary 1.2. *Let $f \in \mathcal{C}^r_b$, and let (b_n) be a strictly increasing sequence in $\mathbb{R}$ such that $b_0 = b$ and $b_n \to \infty$ as $n \to \infty$, and let (ε_n) be a sequence in $\mathbb{R}^>$ and (r_n) be a sequence in $\mathbb{N}$ with $r_n \leqslant r$ for all n. Then there exists $g \in \mathcal{C}^\omega_b$ such that for all n we have $\|f-g\|_{[b_n,b_{n+1}];\,r_n} < \varepsilon_n$.*

Proof. Extend f to a function in $\mathcal{C}^r(I)$, also denoted by f, where $I := (a,+\infty)$, $a < b$, and take a strictly decreasing sequence (a_n) in $\mathbb{R}$ with $a_0 = b_0$ and $a_n \to a$ as $n \to +\infty$. Now apply Theorem 1.1. □

Here is a useful reformulation of Corollary 1.2:

Corollary 1.3. *Let $f \in \mathcal{C}^r_b$ and $\varepsilon \in \mathcal{C}_b$ be such that $\varepsilon > 0$ on $[b,+\infty)$. Then there exists $g \in \mathcal{C}^\omega_b$ such that $|(f-g)^{(k)}(t)| < \varepsilon(t)$ for all $t \geqslant b$ and $k \leqslant \min\{r, 1/\varepsilon(t)\}$.*

Proof. Take a strictly increasing sequence (b_n) in $\mathbb{R}$ with $b_0 = b$ and $b_n \to \infty$ as $n \to \infty$, and for each n, set

$$\varepsilon_n := \min\big\{\varepsilon(t) : t \in [b_n, b_{n+1}]\big\} \in \mathbb{R}^>, \quad r_n := \min\big\{r, \big\lfloor \|1/\varepsilon\|_{[b_n,b_{n+1}]}\big\rfloor\big\} \in \mathbb{N}.$$

Corollary 1.2 yields $g \in \mathcal{C}^\omega_b$ such that $\|f-g\|_{[b_n,b_{n+1}];\,r_n} < \varepsilon_n$ for all n. Then for $t \in [b_n, b_{n+1}]$ and $k \leqslant \min\{r, 1/\varepsilon(t)\}$ we have $k \leqslant r_n$ and so

$$|(f-g)^{(k)}(t)| \leqslant \|f-g\|_{[b_n,b_{n+1}];\,r_n} < \varepsilon_n \leqslant \varepsilon(t). \qquad \square$$

This leads to an improved version of [6, Lemma 2.5]:

Lemma 1.4. *Let $f, g \in \mathcal{C}_b$ be such that $f < g$ on $[b,+\infty)$. Then there exists $y \in \mathcal{C}^\omega_b$ such that $f < y < g$ on $[b,+\infty)$.*

Proof. Let $z := \frac{1}{2}(f+g) \in \mathcal{C}_b$ and $\varepsilon := \frac{1}{2}(g-f) \in \mathcal{C}_b$. Corollary 1.3 (with $r = 0$) then yields $y \in \mathcal{C}^\omega_b$ such that $|y-z| < \varepsilon$ on $[b,+\infty)$, so $f < y < g$ on $[b,+\infty)$. □

Thus we can replace "$\phi \in \mathcal{C}^\infty$" by "$\phi \in \mathcal{C}^\omega$" in the statements of Lemma 2.7 and Corollary 2.8 in [6]. Here is another consequence of Corollary 1.3:

Corollary 1.5. *Let $f \in \mathcal{C}^r$ and $\varepsilon \in \mathcal{C}$, $\varepsilon >_{\mathrm{e}} 0$. Then there exists $g \in \mathcal{C}^\omega$ such that for all $k \leqslant r$ we have $|(f-g)^{(k)}| <_{\mathrm{e}} \varepsilon$.*

Proof. Pick a and representatives of f in $\mathcal{C}^r_a$ and of ε in $\mathcal{C}_a$, also denoted by f, ε, with $\varepsilon > 0$ on $[a,+\infty)$. Take $\varepsilon^* \in \mathcal{C}_a$ with $0 < \varepsilon^* \leqslant \varepsilon$ on $[a,+\infty)$ and $\varepsilon^* \prec 1$. Corollary 1.3 applied to ε^* in place of ε yields $g \in \mathcal{C}^\omega_a$ such that $|(f-g)^{(k)}(t)| < \varepsilon^*(t)$ for all $t \geqslant a$ and $k \leqslant \min\{r, 1/\varepsilon^*(t)\}$. Given $k \leqslant r$, take $b \geqslant a$ such that $k \leqslant 1/\varepsilon^*(t)$ for all $t \geqslant b$; then $|(f-g)^{(k)}(t)| < \varepsilon(t)$ for such t. ☐

Our next goal is to prove a version of Corollary 1.5 for approximating germs in $\mathcal{C}^{<\infty}$ by germs in $\mathcal{C}^\omega$: see Corollary 1.8 below. First a lemma about glueing two approximations g_- and g_+ to a function f to make a single approximation g to f that combines properties of g_- and g_+:

Lemma 1.6. *Let $f \in \mathcal{C}_{a_0}$ and $a_0 \leqslant a < b$. Suppose f is of class $\mathcal{C}^n$ on $[a,+\infty)$ and of class $\mathcal{C}^{n+1}$ on $[b,+\infty)$. Let also functions $\varepsilon \in \mathcal{C}_{a_0}$ and $g_-, g_+ \in \mathcal{C}^\infty_{a_0}$ be given such that*

- *$\varepsilon > 0$ on $[a_0,+\infty)$;*
- *$|(f-g_-)^{(j)}| < \varepsilon$ on $[a,+\infty)$ for $j = 0,\dots,n$; and*
- *$|(f-g_+)^{(j)}| < \varepsilon$ on $[b,+\infty)$ for $j = 0,\dots,n+1$.*

Then, for any $\delta \in \mathbb{R}^>$, there is a function $g \in \mathcal{C}^\infty_{a_0}$ and a $b' > b$ such that:

(i) *$g = g_-$ on $[a_0,b]$ and $g = g_+$ on $[b',+\infty)$;*
(ii) *$|(f-g)^{(j)}| < (1+\delta)\varepsilon$ on $[a,+\infty)$ for $j = 0,\dots,n$; and*
(iii) *$|(f-g)^{(j)}| < \varepsilon$ on $[b',+\infty)$ for $j = 0,\dots,n+1$.*

Proof. Let $b' > b$, set $\beta := \alpha_{b,b'}$ as in [6, (3.4)], and $g := (1-\beta)g_- + \beta g_+$ on $[a_0,+\infty)$, so $g \in \mathcal{C}^\infty_{a_0}$. Let $\delta > 0$; we show that if $b'-b$ is sufficiently large, then g satisfies (i), (ii), (iii). It is clear that (i) holds, and so (iii) as well. Then the inequality in (ii) holds on $[a,b]$ and on $[b',+\infty)$, so it suffices to consider what happens on $[b,b']$. There we have for $j = 0,\dots,n$:

$$(f-g)^{(j)} \;=\; f^{(j)} - \big((1-\beta)g_-^{(j)} + \beta g_+^{(j)}\big) - \sum_{i=0}^{j-1} \binom{j}{i} \beta^{(j-i)} \big(g_+^{(i)} - g_-^{(i)}\big),$$

and

$$f^{(j)} - \big((1-\beta)g_-^{(j)} + \beta g_+^{(j)}\big) = (1-\beta)\big(f-g_-\big)^{(j)} + \beta\big(f-g_+\big)^{(j)},$$

so

$$\big|f^{(j)} - \big((1-\beta)g_-^{(j)} + \beta g_+^{(j)}\big)\big| \;\leqslant\; \max\big\{\big|(f-g_-)^{(j)}\big|, \big|(f-g_+)^{(j)}\big|\big\} \;<\; \varepsilon \text{ on } [b,b'].$$

By [6, (3.5)] we have reals $C_m \geqslant 1$ (independent of b') with $|\beta^{(m)}| \leqslant C_m/(b'-b)^m$. Hence for $j = 0,\dots,n$ we have on $[b,b']$:

$$\left|\sum_{i=0}^{j-1} \binom{j}{i} \beta^{(j-i)} \big(g_+^{(i)} - g_-^{(i)}\big)\right| \;\leqslant\; \sum_{i=0}^{j-1} \binom{j}{i} \frac{C_{j-i}}{(b'-b)^{j-i}} \, \big|g_+^{(i)} - g_-^{(i)}\big|$$

and $\big|g_+^{(i)} - g_-^{(i)}\big| < 2\varepsilon$ for $i = 0,\dots,n$. So for $b'-b$ so large that

$$\sum_{i=0}^{j-1} \binom{j}{i} \frac{C_{j-i}}{(b'-b)^{j-i}} \;<\; \delta/2,$$

condition (ii) is satisfied. (See also Figure 1.) □

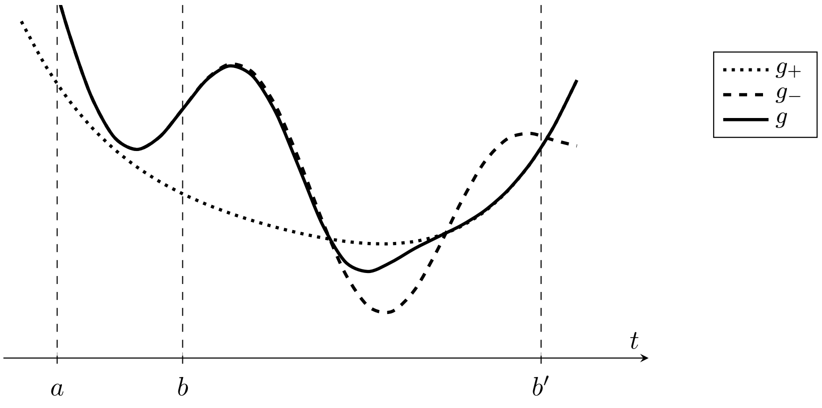

FIGURE 1

Proposition 1.7. *Suppose $f \in \mathcal{C}^{<\infty}$ and $\varepsilon \in \mathcal{C}$, $\varepsilon >_{\mathrm{e}} 0$. Then there exists $g \in \mathcal{C}^{\infty}$ such that $|(f-g)^{(n)}| <_{\mathrm{e}} \varepsilon$ for all n.*

Proof. Represent f and ε by continuous functions $[a_0, +\infty) \to \mathbb{R}$ ($a_0 \in \mathbb{R}$), also denoted by f and ε, such that $\varepsilon > 0$ on $[a_0, +\infty)$. Next, take a strictly increasing sequence (a_n) of real numbers starting with the already given a_0, such that $a_n \to \infty$ as $n \to \infty$, and f is of class $\mathcal{C}^n$ on $[a_n, +\infty)$, for each n. Then Corollary 1.3 gives for each n a function $g_n \in \mathcal{C}^{\infty}_{a_0}$ such that $|(f-g_n)^{(j)}| < \varepsilon/2$ on $[a_n, +\infty)$ for $j = 0, \dots, n$. All this remains true when increasing each a_n while keeping a_0 fixed and maintaining that (a_n) is strictly increasing. Now use the lemma above to construct g as required: first glue g_0 and g_1 and increase the a_n for $n \geqslant 1$, then glue the resulting function with g_2 and increase the a_n for $n \geqslant 2$, and so on, and arrange the product of the $(1+\delta)$-factors to be < 2. □

Now Corollary 1.5 (for $r = \infty$) and Proposition 1.7 yield:

Corollary 1.8. *For any germs $f \in \mathcal{C}^{<\infty}$ and $\varepsilon \in \mathcal{C}$ with $\varepsilon >_{\mathrm{e}} 0$, there exists a germ $g \in \mathcal{C}^{\omega}$ such that $|(f-g)^{(n)}| <_{\mathrm{e}} \varepsilon$ for all n.*

In the next section we apply Corollary 1.8 to bounded Hardy fields.

2. BOUNDED HARDY FIELDS

As in [5, Section 5.4] a set $H \subseteq \mathcal{C}$ is called *bounded* if for some $\phi \in \mathcal{C}$ we have $h \leqslant \phi$ for all $h \in H$, and *unbounded* otherwise. Every countable subset of $\mathcal{C}$ is bounded, cf. [5, remarks after Lemma 5.4.17]. As a consequence, the union of countably many bounded subsets of $\mathcal{C}$ is also bounded.

In the rest of this section H is a Hardy field. If H is bounded, then there is a $\phi \in \mathcal{C}$ with $\phi >_{\mathrm{e}} 0$ and $g \prec \phi$ for all $g \in H$, so $\varepsilon := 1/\phi \in \mathcal{C}^{\times}$ satisfies $\varepsilon >_{\mathrm{e}} 0$ and $\varepsilon \prec h$ for all $h \in H^{\times}$. By [5, Lemmas 5.4.18, 5.4.19] we have:

Lemma 2.1. *If H is bounded, then any* d*-algebraic Hardy field extension of H is bounded, and for any H-hardian $f \in \mathcal{C}^{<\infty}$, the Hardy field $H\langle f \rangle$ is bounded.*

Corollary 2.2. *If H is bounded and F is a Hardy field extension of H and* d*-algebraic over $H\langle S\rangle$ for some countable $S\subseteq F$, then F is bounded.*

Lemma 2.3. *Let $f,g\in\mathcal{C}^{<\infty}$ be such that f is H-hardian,* d*-transcendental over H, and $(f-g)^{(n)}\prec h$ for all $h\in H\langle f\rangle^{\times}$ and all n. Then g is H-hardian, and there is a unique isomorphism $H\langle f\rangle\to H\langle g\rangle$ of Hardy fields fields over H sending f to g.*

Proof. Let $P\in H\{Y\}^{\neq}$, $r:=\operatorname{order}P$, so $P(f)\in H\langle f\rangle^{\times}$. It suffices to show that then $P(f)\sim P(g)$. By Taylor expansion [ADH, p. 210], with $\boldsymbol{i}$ ranging over $\mathbb{N}^{1+r}$:

$$P(g)-P(f)=\sum_{|\boldsymbol{i}|\geqslant 1}P_{(\boldsymbol{i})}(f)(g-f)^{\boldsymbol{i}}\quad\text{where } P_{(\boldsymbol{i})}=\frac{P^{(\boldsymbol{i})}}{\boldsymbol{i}!}\in H\{Y\}.$$

If $|\boldsymbol{i}|\geqslant 1$, then $(g-f)^{\boldsymbol{i}}\prec h$ for all $h\in H\langle f\rangle^{\times}$, and hence $P_{(\boldsymbol{i})}(f)(g-f)^{\boldsymbol{i}}\prec P(f)$. Thus $P(g)-P(f)\prec P(f)$ as required. □

With Corollary 1.8 we now obtain analytic "copies" of certain H-hardian germs:

Corollary 2.4. *Suppose H is bounded and f in a Hardy field extension of H is* d*-transcendental over H. Then there is an H-hardian $g\in\mathcal{C}^{\omega}$ and an isomorphism $H\langle f\rangle\to H\langle g\rangle$ of Hardy fields over H sending f to g.*

Proof. By Lemma 2.1, the Hardy field $H\langle f\rangle$ is bounded, so we can take $\varepsilon\in\mathcal{C}^{\times}$ with $\varepsilon>_{\mathrm{e}}0$ and $\varepsilon\prec h$ for all $h\in H\langle f\rangle^{\times}$. Corollary 1.8 yields a $g\in\mathcal{C}^{\omega}$ such that $|(f-g)^{(n)}|\leqslant\varepsilon$ for all n, and so it remains to appeal to Lemma 2.3. □

We can now also strengthen [6, Theorem 5.1]:

Corollary 2.5. *Suppose $H\supseteq\mathbb{R}$ is Liouville closed, and $\phi\in\mathcal{C}$, $\phi>_{\mathrm{e}}H$. Then there is an H-hardian $z\in\mathcal{C}^{\omega}$ with $z>_{\mathrm{e}}\phi$.*

Proof. By [6, Theorem 5.1] we have an H-hardian $y\in\mathcal{C}^{\infty}$ with $y>_{\mathrm{e}}\phi+1$. Then y is d-transcendental over H and $H\langle y\rangle$ is bounded; cf. [5, Lemma 5.4.1] and [ADH, Lemma 16.6.10]. This yields $\varepsilon\in\mathcal{C}$ such that $\varepsilon>_{\mathrm{e}}0$ and $\varepsilon\prec h$ for all $h\in H\langle y\rangle^{\times}$. Now Corollary 1.8 gives $z\in\mathcal{C}^{\omega}$ with $|y^{(n)}-z^{(n)}|<_{\mathrm{e}}\varepsilon$ for all n. Then z is H-hardian by Lemma 2.3, and $z=y+(z-y)>_{\mathrm{e}}\phi$. □

Thus maximal Hardy fields, maximal $\mathcal{C}^{\infty}$-Hardy fields, and maximal $\mathcal{C}^{\omega}$-Hardy fields are unbounded. (Cf. also [5, Corollary 5.4.23 and succeeding remarks].) As [6, Theorem 5.1] gave rise to [6, Corollary 5.2], so Corollary 2.5 yields:

Corollary 2.6. *If H is a maximal analytic Hardy field, then $\operatorname{cf}(H)>\omega$, and thus*

$$\operatorname{ci}(H)\ =\ \operatorname{cf}(H^{<a})\ =\ \operatorname{ci}(H^{>a})\ >\ \omega\ \text{ for all } a\in H.$$

Likewise with "smooth" in place of "analytic".

Call a subset F of $\mathcal{C}$ **cofinal** if for each $\phi\in\mathcal{C}$ there exists $f\in F$ with $\phi\leqslant f$. If $F_1,F_2\subseteq\mathcal{C}$ and for all $f_1\in F_1$ there is an $f_2\in F_2$ with $f_1\leqslant f_2$, and F_1 is cofinal, then F_2 is cofinal. Clearly each cofinal subset of $\mathcal{C}$ is unbounded. The following strengthens [33, Theorem 7]:

Corollary 2.7. *Assume the Continuum Hypothesis* CH*: $2^{\aleph_0}=\aleph_1$. Then there is a cofinal analytic Hardy field.*

Proof. Put $\mathfrak{c} := 2^{\aleph_0}$, and let α, α', β range over ordinals $< \mathfrak{c}$. Choose an enumeration $(\phi_\alpha)_{\alpha<\mathfrak{c}}$ of $\mathcal{C}$. Suppose $\big((H_\alpha, h_\alpha)\big)_{\alpha<\beta}$ is a family of bounded analytic Hardy fields H_α, each with an element $h_\alpha \in H_\alpha$, such that

$$\alpha < \alpha' < \beta \ \Rightarrow\ H_\alpha \subseteq H_{\alpha'} \qquad \text{and} \qquad \alpha < \beta \ \Rightarrow\ \phi_\alpha <_{\mathrm{e}} h_\alpha. \tag{2.1}$$

Then $H := \bigcup_{\alpha<\beta} H_\alpha$ is an analytic Hardy field, and H is bounded, as the union of countably many bounded subsets of $\mathcal{C}$. By [5, Lemma 5.4.18], $H^* := \operatorname{Li}\big(H(\mathbb{R})\big)$ is also bounded. Take $\phi \in \mathcal{C}$ with $\phi >_{\mathrm{e}} H^*$ and $\phi \geqslant \phi_\beta$. Corollary 2.5 yields an H^*-hardian $h_\beta \in \mathcal{C}^\omega$ with $h_\beta >_{\mathrm{e}} \phi$. Then the analytic Hardy field $H_\beta := H^*\langle h_\beta\rangle$ is bounded by Lemma 2.1, contains H_α for all $\alpha < \beta$, and $\phi_\beta <_{\mathrm{e}} h_\beta$.

Now transfinite recursion yields a family $\big((H_\alpha, h_\alpha)\big)_{\alpha<\mathfrak{c}}$ where H_α is a bounded analytic Hardy field and $h_\alpha \in H_\alpha$ such that (2.1) holds with $\mathfrak{c}$ in place of β. Then $\bigcup_{\alpha<\mathfrak{c}} H_\alpha$ is a cofinal analytic Hardy field. □

See Corollary 4.14 below for a strengthening of Corollary 2.7.

Remark. Vera Fischer suggested replacing CH in Corollary 2.7 by $\mathfrak{b} = \mathfrak{d}$, which is strictly weaker than CH (provided of course that our base theory ZFC is consistent). Here $\mathfrak{b}$ and $\mathfrak{d}$ are so-called *cardinal characteristics of the continuum.* See [10, 2.1, 2.2] for their definitions, and [10, 2.4] for the inequalities $\aleph_1 \leqslant \mathfrak{b} \leqslant \mathfrak{d} \leqslant \mathfrak{c}$. *Martin's Axiom* (MA) implies $\mathfrak{b} = \mathfrak{d} = \mathfrak{c}$, see [10, 6.8, 6.9] and [32, Corollary 8]. If ZFC is consistent, then MA is strictly weaker than CH by [34]. It is easy to check from their definitions that $\mathfrak{b}$ is the least cardinality of an unbounded subset of $\mathcal{C}$, and $\mathfrak{d}$ is the least cardinality of a cofinal subset of $\mathcal{C}$. Replacing in the proof above $\mathfrak{c}$ by $\mathfrak{d}$ and taking $(\phi_\alpha)_{\alpha<\mathfrak{d}}$ to enumerate a cofinal subset of $\mathcal{C}$, the proof does indeed go through with $\mathfrak{b} = \mathfrak{d}$ instead of CH.

Hardy fields of countable cofinality. Hardy fields of countable cofinality are bounded. For later use we study here such Hardy fields in more detail. Given a valued differential field K, let C denote its constant field and Γ its value group.

Lemma 2.8. *Let K be a pre-H-field with $\Gamma \neq \{0\}$. Then $\operatorname{cf}(K) = \operatorname{cf}(\Gamma)$.*

Proof. Apply [ADH, 2.1.4] to the increasing surjection $f \mapsto -vf\colon K^> \to \Gamma$. □

Lemma 2.9. *Let Γ be an ordered abelian group, $\Gamma \neq \{0\}$. If $[\Gamma]$ has no largest element, then $\operatorname{cf}(\Gamma) = \operatorname{cf}([\Gamma])$; otherwise $\operatorname{cf}(\Gamma) = \omega$.*

Proof. If $[\Gamma]$ has no largest element, then [ADH, 2.1.4] applied to the increasing surjection $\gamma \mapsto [\gamma]\colon \Gamma^{\geqslant} \to [\Gamma]$ yields $\operatorname{cf}(\Gamma) = \operatorname{cf}([\Gamma])$. If $\gamma \in \Gamma^>$ is such that $[\gamma]$ is the largest element of $[\Gamma]$, then $\mathbb{N}\gamma$ is cofinal in Γ. □

Let G be an abelian group, with divisible hull $\mathbb{Q}G = \mathbb{Q} \otimes_{\mathbb{Z}} G$. Then $\operatorname{rank}_{\mathbb{Q}} G := \dim_{\mathbb{Q}} \mathbb{Q}G$ (a cardinal) is the *rational rank* of G. (NB: in [ADH, 1.7] we defined the rational rank of G to be ∞ if the $\mathbb{Q}$-linear space $\mathbb{Q}G$ is not finitely generated.)

Lemma 2.10. *Let $\{0\} \neq \Delta \subseteq \Gamma$ be an extension of ordered abelian groups such that $\operatorname{rank}_{\mathbb{Q}}(\Gamma/\Delta) \leqslant \aleph_0$. Then $\operatorname{cf}(\Gamma) \leqslant \operatorname{cf}(\Delta)$.*

Proof. By Lemma 2.9 we may assume that $[\Gamma]$ has no maximum. Let S be a well-ordered cofinal subset of $[\Delta]$ of order type $\operatorname{cf}([\Delta])$, so $|S| = \operatorname{cf}([\Delta])$. Then $\widetilde{S} := S \cup \big([\Gamma] \setminus [\Delta]\big)$ is cofinal in $[\Gamma]$, so $\operatorname{cf}(\Gamma) = \operatorname{cf}([\Gamma]) = \operatorname{cf}(\widetilde{S}) \leqslant |\widetilde{S}|$ by Lemma 2.9 and [ADH, 2.1.2]. Since $[\Gamma] \setminus [\Delta]$ is countable by [ADH, 2.3.9], we have $|\widetilde{S}| \leqslant \max\big\{|S|, \omega\big\} = \max\big\{\operatorname{cf}([\Delta]), \omega\big\}$. Now apply Lemma 2.9 to Δ in place of Γ. □

Lemma 2.11. *Let $K \subseteq L$ be an extension of pre-H-fields where* $\operatorname{rank}_{\mathbb{Q}}(\Gamma_L/\Gamma)$ *is countable, and suppose $\Gamma \neq \{0\}$ or L has very small derivation and archimedean residue field. Then* $\operatorname{cf}(L) \leqslant \operatorname{cf}(K)$.

Proof. If $\Gamma \neq \{0\}$, then by Lemmas 2.8 and 2.10 we have $\operatorname{cf}(L) = \operatorname{cf}(\Gamma_L) \leqslant \operatorname{cf}(\Gamma) = \operatorname{cf}(K)$. Suppose $\Gamma = \{0\}$, so L has very small derivation and archimedean residue field. Then K is archimedean, so $\operatorname{cf}(K) = \omega$, and Γ_L is countable, hence $\operatorname{cf}(\Gamma_L) \leqslant \omega$. Therefore, if $\Gamma_L \neq \{0\}$, then $\operatorname{cf}(L) = \operatorname{cf}(\Gamma_L) \leqslant \omega = \operatorname{cf}(K)$ by Lemma 2.8 applied to L in place of K, and if $\Gamma_L = \{0\}$, then $\operatorname{cf}(L) = \omega = \operatorname{cf}(K)$. □

By [ADH, 3.1.10] the hypothesis on $\operatorname{rank}_{\mathbb{Q}}(\Gamma_L/\Gamma)$ in Lemma 2.11 is satisfied if $\operatorname{trdeg}(L|K)$ is countable. Hence this lemma yields:

Corollary 2.12. *If F is a Hardy field extension of H such that* $\operatorname{trdeg}(F|H)$ *is countable, then* $\operatorname{cf}(F) \leqslant \operatorname{cf}(H)$. *Hence if* $\operatorname{trdeg}(H|C_H)$ *is countable, then* $\operatorname{cf}(H) = \omega$ (*and so H is bounded*).

In [6, Corollary 3.13] we showed that if $H \supseteq \mathbb{R}$ and $H^{>\mathbb{R}}$ has countable coinitiality, and H^{da} is the d-closure of H in a maximal Hardy field extension of H, then $(H^{\text{da}})^{>\mathbb{R}}$ also has countable coinitiality. The property of H having countable cofinality is equally robust:

Theorem 2.13. *Let E be a differentially algebraic Hardy field extension of H such that* $\exp\big(E(x)\big) \subseteq E(x)$. *Then* $\operatorname{cf}(E) \leqslant \operatorname{cf}(H)$, *with equality if* $\exp(H(x)) \subseteq H(x)$.

Here is an immediate consequence:

Corollary 2.14. *If H has countable cofinality, then so does* $\operatorname{Li}\big(H(\mathbb{R})\big)$ *as well as the* d*-closure of H in any maximal Hardy field extension of H.*

We precede the proof of Theorem 2.13 by a few lemmas.

Lemma 2.15. *Let K be a pre-*d*-valued field of H-type and* $\operatorname{dv}(K)$ *its* d*-valued hull, and suppose $\Gamma \neq \{0\}$. Then Γ is cofinal in* $\Gamma_{\operatorname{dv}(K)}$.

Proof. This is clear if $\Gamma = \Gamma_{\operatorname{dv}(K)}$. Otherwise $\Gamma_{\operatorname{dv}(K)} = \Gamma + \mathbb{Z}\alpha$ where $0 < n\alpha < \Gamma^{>}$ for all $n \geqslant 1$, by [ADH, 10.3.2], so Γ is cofinal in $\Gamma_{\operatorname{dv}(K)}$. □

Lemma 2.16. *H is cofinal in $H(\mathbb{R})$.*

Proof. This is clear if $H \subseteq \mathbb{R}$; assume $H \not\subseteq \mathbb{R}$. Let E be the H-field hull of H, taken as an H-subfield of the Hardy field extension $H(\mathbb{R})$ of H. Then H is cofinal in E (Lemma 2.15), so replacing H by E we arrange that H is an H-field. Now use that $\Gamma_{H(\mathbb{R})} = \Gamma_H \neq \{0\}$ by [ADH, 10.5.15 and remark preceding 4.6.16]. □

Lemma 2.17. *Let K be a pre-*d*-valued field of H-type with $\Gamma \neq \{0\}$. Let $s \in K$ and $y' = s$, y in a pre-*d*-valued extension of K of H-type. Then Γ is cofinal in* $\Gamma_{K(y)}$.

Proof. Set $L := K(y)$ and $M := \operatorname{dv}(L)$. Lemma 2.15 allows us to replace K by its d-valued hull inside M to arrange that K is d-valued. Using [ADH, 10.5.15 and remark preceding 4.6.16] we replace K by $K(C_M)$ to arrange also L to be d-valued with $C = C_L$. Finally, replacing K by its algebraic closure inside an algebraic closure of L we arrange K to be algebraically closed. We may assume $y \notin K$, so y is transcendental over K. Then

$$S \;:=\; \big\{v(s - a') : a \in K\big\} \;\subseteq\; \Gamma.$$

Assume for now that S has a maximum β. Then $\beta \notin (\Gamma^{\neq})'$ by [ADH, 10.2.5(i)], so $\beta = \max\Psi$ or β is a gap in K. If $\beta = \max\Psi$, then $\Gamma_L = \Gamma + \mathbb{Z}\alpha$ with $\Gamma^{<} < n\alpha < 0$ for all $n \geqslant 1$, so Γ is cofinal in Γ_L. Suppose β is a gap in K. Take $a \in K$ with $\beta = v(s-a')$ and set $z := y - a$, so $z' = s - a'$. We arrange $z \not\asymp 1$ by replacing a with $a + c$ for suitable $c \in C_L = C$. If $z \prec 1$, then [ADH, 10.2.1 and its proof] gives $\Gamma_L = \Gamma + \mathbb{Z}\alpha$ with $0 < n\alpha < \Gamma^{>}$ for all $n \geqslant 1$, so Γ is cofinal in Γ_L. If $z \succ 1$, then [ADH, 10.2.3 and its proof] gives likewise that Γ is cofinal in Γ_L.

If S does not have a largest element, then L is an immediate extension of K: this holds by [ADH, 10.2.6] if $S < (\Gamma^{>})'$; otherwise take $a \in K$ with $v(s - a') \in (\Gamma^{>})'$ and $y - a \not\asymp 1$, and apply [ADH, 10.2.4 and 10.2.5(iii)] to $s - a'$, $y - a$ in place of s, y, respectively. □

Lemma 2.17 and [5, Proposition 5.3.2(iv)] yield:

Lemma 2.18. *If $H \subseteq \mathbb{R}$, then $x^{\mathbb{N}}$ is cofinal in $H(x)$, and if $H \not\subseteq \mathbb{R}$, then H is cofinal in $H(x)$. Hence* $\operatorname{cf}(H) = \operatorname{cf}(H(x))$.

We can now give the proof of Theorem 2.13. First, replacing E, H by $E(x)$, $H(x)$, respectively, and using the last lemma, we arrange $x \in H$. Let S be a well-ordered cofinal subset of H of order type $\operatorname{cf}(H)$. Then $|S| = \operatorname{cf}(H)$, and $\widetilde{S} := \bigcup_n \exp_n(S)$ is a cofinal subset of E, by [5, Lemma 5.4.1]. Thus $\operatorname{cf}(E) = \operatorname{cf}(\widetilde{S}) \leqslant |\widetilde{S}| = |S| = \operatorname{cf}(H)$ as claimed. If $\exp(H) \subseteq H$, then $\widetilde{S} \subseteq H$, hence $\operatorname{cf}(E) = \operatorname{cf}(H)$. □

For use in Section 3 we also include the following cofinality result, which is immediate from [ADH, 10.4.5(i)]:

Lemma 2.19. *Let K be a* d*-valued field of H-type with divisible asymptotic couple (Γ, ψ), $\Gamma \neq \{0\}$, and let $s \in K$ be such that*

$$S := \{v(s - a^{\dagger}) : a \in K^{\times}\} < (\Gamma^{>})'$$

and S has no largest element. Let f be an element of an H-asymptotic field extension of K, transcendental over K, with $f^{\dagger} = s$. Then $[\Gamma] = [\Gamma_{K(f)}]$, so Γ is cofinal in $\Gamma_{K(f)}$.

3. Pseudoconvergence in Analytic Hardy Fields

We complement the material on pc-sequences from [6, Sections 3, 4] by criteria for germs in $\mathcal{C}^{<\infty}$ to be pseudolimits of pc-sequences in Hardy fields, and then use this to show that each pc-sequence of countable length in an analytic Hardy field has an analytic pseudolimit. *In this section H is a Hardy field.*

Revisiting pseudoconvergence in Hardy fields. Let $H \supseteq \mathbb{R}(x)$ be real closed with asymptotic integration and (f_ρ) a pc-sequence in H of d-transcendental type over H with pseudolimit f in a Hardy field extension of H. Then $Z(H, f) = \emptyset$ by [ADH, 11.4.13], so the valued field extension $H\langle f\rangle \supseteq H$ is immediate by [ADH, 11.4.7]. In [6] we only considered pc-sequences of countable length, but here we do not assume (f_ρ) has countable length (to be exploited in the proof of Theorem B). We begin by deriving a sufficient condition on $y \in \mathcal{C}^{<\infty}$ to be H-hardian with $f_\rho \rightsquigarrow y$. This will enable us to find such y in $\mathcal{C}^{\omega}$. (Another possible use is to find such y with oscillating $y - f$, so that $H\langle y\rangle$ and $H\langle f\rangle$ are "incompatible" Hardy field extensions of H.) To simplify notation, set $t := x^{-1}$. We first observe:

Lemma 3.1. *Let ϕ be active in H, $0<\phi\prec 1$, let $\derivdelta := \phi^{-1}\partial$ be the derivation of $(\mathcal{C}^{<\infty})^{\phi}$, and let $z\in\mathcal{C}^{<\infty}$ satisfy $z^{(i)}\prec t^j$ for all i, j. Then $\delta^k(z)\prec 1$ for all k.*

Proof. This is clear for $k=0$. Suppose $k\geqslant 1$. The identity (3.1) in [6] gives $\delta^k(z)=\phi^{-k}\sum_{j=1}^{k}R^k_j(-\phi^\dagger)z^{(j)}$ where $R^k_j(Z)\in\mathbb{Q}\{Z\}$ for $j=1,\dots,k$. This yields $\delta^k(z)\prec 1$ in view of $\phi^\dagger\preccurlyeq 1$ and $z^{(j)}\prec t^{2k}\prec\phi^k$ for $j=1,\dots,k$. □

Proposition 3.2. *Suppose $0\in v(f-H)$ and $y\in\mathcal{C}^{<\infty}$ is such that for all $\mathfrak{m}\in H^\times$ with $v\mathfrak{m}\in v(f-H)$ and all i, j, k we have*

$$y^{(i)}-f^{(i)}\ \prec\ \mathfrak{m}^j t^k\quad\text{in } \mathcal{C}^{<\infty}.$$

Then y is H-hardian and there is an isomorphism $H\langle y\rangle\to H\langle f\rangle$ of Hardy fields over H sending y to f (and thus $f_\rho\rightsquigarrow y$).

Proof. Let ϕ be active in H, $0<\phi\prec 1$, and $\delta=\phi^{-1}\partial$ the derivation of $(\mathcal{C}^{<\infty})^{\phi}$. Let also $h\in H$ and $\mathfrak{m}\in H^\times$ with $f-h\preccurlyeq\mathfrak{m}$, and put $z:=\frac{y-f}{\mathfrak{m}}$. By [5, Corollary 6.7.12] it suffices to show that then $\delta^k\big(\frac{y-h}{\mathfrak{m}}\big)\preccurlyeq 1$ for all k; equivalently, $\delta^k(z)\preccurlyeq 1$ for all k (thanks to $\frac{y-h}{\mathfrak{m}}-z=\frac{f-h}{\mathfrak{m}}\preccurlyeq 1$ and smallness of the derivation of $H\langle f\rangle^{\phi}$).

Claim 1: *Suppose $\mathfrak{m}\preccurlyeq 1$. Then $z^{(n)}\prec\mathfrak{m}^j t^k$ for all n, j, k.*

This holds for $n=0$ because $\mathfrak{m}z\prec\mathfrak{m}^{j+1}t^k$ for all j, k. Let $n\geqslant 1$ and assume inductively that $z^{(i)}\prec\mathfrak{m}^j t^k$ for $i=0,\dots,n-1$ and all j, k. Now $(\mathfrak{m}z)^{(n)}=y^{(n)}-f^{(n)}\prec\mathfrak{m}^j t^k$ for all j, k, and

$$(\mathfrak{m}z)^{(n)}\ =\ \mathfrak{m}^{(n)}z+\cdots+\mathfrak{m}z^{(n)}.$$

Since $\mathfrak{m}\preccurlyeq 1$, the inductive assumption gives $\mathfrak{m}^{(n-i)}z^{(i)}\prec\mathfrak{m}^j t^k$ for $i=0,\dots,n-1$ and all j, k, so $\mathfrak{m}z^{(n)}\prec\mathfrak{m}^j t^k$ for all j, k, and thus $z^{(n)}\prec\mathfrak{m}^j t^k$ for all j, k.

Claim 2: $\delta^k(z)\prec 1$ *for all k.*

If $\mathfrak{m}\preccurlyeq 1$, then this holds by Claim 1 and the lemma above. In general, take $h_1\in H$ with $f-h_1\prec f-h$ and $f-h_1\preccurlyeq 1$, and then $\mathfrak{m}_1\in H^\times$ with $f-h_1\asymp\mathfrak{m}_1$. By the special case just proved with h_1, $\mathfrak{m}_1$ in place of h, $\mathfrak{m}$ we have $\delta^k\big(\frac{y-f}{\mathfrak{m}_1}\big)\prec 1$ for all k. Now $z=\big(\frac{y-f}{\mathfrak{m}_1}\big)\big(\frac{\mathfrak{m}_1}{\mathfrak{m}}\big)$ and $\frac{\mathfrak{m}_1}{\mathfrak{m}}\prec 1$ (in H), so the claim follows using the Product Rule for the derivation of $(\mathcal{C}^{<\infty})^{\phi}$ and smallness of the derivation of H^{ϕ}. □

Here is a more useful variant for the case $0\notin v(f-H)$:

Proposition 3.3. *Suppose $0\notin v(f-H)$, and $y\in\mathcal{C}^{<\infty}$ is such that*

$$y^{(i)}-f^{(i)}\ \prec\ t^k\ \text{for all } i,\ k.$$

Then y is H-hardian and there is an isomorphism $H\langle y\rangle\to H\langle f\rangle$ of Hardy fields over H sending y to f.

Proof. Let ϕ, h, $\mathfrak{m}$, z be as in the proof of Proposition 3.2; as in that proof it suffices to show that $\delta^k(z)\preccurlyeq 1$ for all k. Now $\mathfrak{m}\succ 1$, hence with $\mathfrak{n}:=\mathfrak{m}^{-1}\in H^\times$ we have $z=\mathfrak{n}(y-f)$ and $\mathfrak{n}\prec 1$, so in view of $z^{(i)}=\mathfrak{n}^{(i)}(y-f)+\cdots+\mathfrak{n}(y-f)^{(i)}$ we obtain $z^{(i)}\prec t^j$ for all i, j, and Lemma 3.1 then yields $\delta^k(z)\prec 1$ for each k. □

Multiplicative conjugation gives a reduction to Proposition 3.3, except when (f_ρ) is a cauchy sequence, not just a pc-sequence. We shall exploit this several times.

Constructing analytic pseudolimits in Hardy field extensions. Let (f_ρ) be a pc-sequence in H. Corollary 3.2 from [6] says: if (f_ρ) has countable length, then (f_ρ) pseudoconverges in some Hardy field extension of H. Using Corollary 1.8 and Proposition 3.3 we now deduce smooth and analytic versions of this key fact.

Proposition 3.4. *Suppose H is an analytic Hardy field and (f_ρ) pseudoconverges in some Hardy field extension of H. Suppose also that H is bounded or (f_ρ) does not have width $\{\infty\}$ in the valued field H. Then (f_ρ) pseudoconverges in an analytic Hardy field extension of H. Likewise with "smooth" in place of "analytic".*

Proof. Assume H is analytic; the smooth case goes the same way. As in [6] we can pass from H to an extension of H and reduce to the case that $H \supseteq \mathbb{R}$, H is closed, and (f_ρ) has no pseudolimit in H. Take f in a Hardy field extension of H such that $f_\rho \rightsquigarrow f$. Then $f \notin H$, so f is d-transcendental over H. If H is bounded, then Corollary 2.4 yields an H-hardian $y \in \mathcal{C}^\omega$ with $f_\rho \rightsquigarrow y$.

Suppose (f_ρ) does not have width $\{\infty\}$. Then take $h \in H^\times$ with $v(hf - hf_\rho) < 0$ for all ρ. Take $\varepsilon \in \mathcal{C}$ such that $\varepsilon >_{\mathrm{e}} 0$ and $\varepsilon \prec t^k$ for all k, for example, $\varepsilon = \mathrm{e}^{-x}$. Now Corollary 1.8 gives $y \in \mathcal{C}^\omega$ such that $(hf)^{(i)} - y^{(i)} \prec t^k$ for all k. Then y is H-hardian and $hf_\rho \rightsquigarrow y$ by Proposition 3.3, hence $h^{-1}y \in \mathcal{C}^\omega$ is H-hardian and $f_\rho \rightsquigarrow h^{-1}y$. □

Corollary 3.5. *If H is an analytic Hardy field, then every pc-sequence in H of countable length pseudoconverges in an analytic Hardy field extension of H. Likewise with "smooth" in place of "analytic".*

Proof. Suppose (f_ρ) has countable length. Then (f_ρ) pseudoconverges in a Hardy field extension of H, by [6, Corollary 3.2]. Moreover, if (f_ρ) has width $\{\infty\}$, then $\big(v(f - f_\rho)\big)$ is cofinal in Γ_H, so $\operatorname{cf}(H) = \operatorname{cf}(\Gamma_H) = \omega$ by Lemma 2.8, hence H is bounded. Now use Proposition 3.4. □

Arguing as in the proof of [6, Corollary 4.8], using Corollary 3.5 instead of [6, Corollary 3.2], yields:

Corollary 3.6. *If H is a maximal analytic or maximal smooth Hardy field, then $\operatorname{ci}(H^{>\mathbb{R}}) > \omega$.*

In view of [6, Remark preceding Corollary 8.1], Corollaries 2.6, 3.5, 3.6 yield a version of [6, Corollary 8.1] for maximal analytic Hardy fields:

Corollary 3.7. *If H is a maximal analytic or maximal smooth Hardy field, then its H-couple (Γ, ψ) is countably spherically complete and*

$$\operatorname{cf}(\Gamma^{<}) = \operatorname{ci}(\Gamma^{>}) > \omega, \qquad \operatorname{ci}(\Gamma) = \operatorname{cf}(\Gamma) > \omega.$$

4. Proofs of Theorems A and B

We begin with revisiting Case (b) extensions, then prove Theorem A and use it to characterize the possible gaps in maximal analytic Hardy fields. We also determine the number of maximal analytic Hardy fields. Next we prove Theorem B, and finish this section with two subsections on dense pairs of closed H-fields.

Case (b) **extensions.** Let $H \supseteq \mathbb{R}$ be a Liouville closed Hardy field with H-couple (Γ,ψ) over $\mathbb{R}$. Suppose β in an H-couple (Γ^*,ψ^*) over $\mathbb{R}$ extending (Γ,ψ) falls under Case (b), that is, with $\big(\Gamma\langle\beta\rangle,\psi_\beta\big)$ the H-couple over $\mathbb{R}$ generated by β over (Γ,ψ) in (Γ^*,ψ^*):

(b) We have a sequence (α_i) in Γ and a sequence (β_i) in Γ^* that is $\mathbb{R}$-linearly independent over Γ, such that $\beta_0=\beta-\alpha_0$ and $\beta_{i+1}=\beta_i^\dagger-\alpha_{i+1}$ for all i, and such that $\Gamma\langle\beta\rangle=\Gamma\oplus\bigoplus_{i=0}^\infty\mathbb{R}\beta_i$.

Unlike in key parts of [6, Section 9] we do not assume β is of countable type over Γ, and this will be exploited in the proof of Theorem B. By [6, Corollary 8.15], β also falls under Case (b) with the same sequences (α_i), (β_i) when (Γ,ψ) and (Γ^*,ψ^*) are viewed as H-couples over $\mathbb{Q}$; see [6, Section 8] for the relevant definitions.

In the next proposition and its corollary we assume $y\in\mathcal{C}^{<\infty}$ is H-hardian, $y>0$, and vy realizes the same cut in Γ as β. Then by [6, Remark 8.21] we have a unique isomorphism over Γ of the H-couple over $\mathbb{Q}$ generated by $\Gamma\cup\{\beta\}$ in (Γ^*,ψ^*) with the H-couple of the Hardy field $H\langle y\rangle^{\mathrm{rc}}$ over $\mathbb{Q}$ sending β to vy. Moreover, if $z\in\mathcal{C}^{<\infty}$ is also H-hardian with $z>0$ and vz realizes the same cut in Γ as β, then we have a unique Hardy field isomorphism $H\langle y\rangle\to H\langle z\rangle$ over H sending y to z, by [6, Proposition 8.20]. The problem here is to find such z in $\mathcal{C}^\omega$ (in which case $H\langle z\rangle$ is smooth, respectively analytic, if H is). This can always be done, in view of Corollary 1.8 and the following:

Proposition 4.1. *There exists $\varepsilon\in\mathcal{C}$ such that $\varepsilon>_{\mathrm{e}}0$ and for all $z\in\mathcal{C}^{<\infty}$, if $(z-y)^{(i)}\prec\varepsilon$ for all i, then z is H-hardian and vz realizes the same cut in Γ as β.*

Proof. Let the sequences (α_i), (β_i) be as in (b). Take $f_i\in H^>$ with $vf_i=\alpha_i$, and recursively we set $y_0:=y/f_0$, $y_{i+1}:=y_i^\dagger/f_{i+1}$. Then vy_i realizes the same cut in Γ as β_i, and $H\langle y\rangle=H(y_0,y_1,y_2,\dots)$, by [6, proof of Proposition 8.20]. Next set $K:=\mathbb{R}\langle f_0,f_1,f_2,\dots\rangle$ and note that $K\langle y\rangle=K(y_0,y_1,y_2,\dots)$, using that the right hand side contains all y_n'. Suppose $z\in\mathcal{C}^{<\infty}$ is such that $(z-y)^{(i)}\prec f$ for all i and all $f\in K\langle y\rangle^\times$. Then z is K-hardian and d-transcendental over K by Lemma 2.3, and the proof of that lemma also shows that $P(y)\sim P(z)$ for all $P\in K\{Y\}^{\neq}$. Thus we have elements $z_i\in K\langle z\rangle$ defined recursively by $z_0:=z/f_0$, $z_{i+1}:=z_i^\dagger/f_{i+1}$, and then $y_i\sim z_i$ for all i. For the Hausdorff field $H_n:=H(y_0,\dots,y_n)$ we have $v(H_n^\times)=\Gamma\oplus\mathbb{Z}vy_0\oplus\cdots\oplus\mathbb{Z}vy_n$ by [6, proof of Proposition 8.20], and for $h\in H^\times$ and $i_0,\dots,i_n\in\mathbb{N}$ we have $hy_0^{i_0}\cdots y_n^{i_n}\sim hz_0^{i_0}\cdots z_n^{i_n}$ (in $\mathcal{C}$). Hence $z_0,\dots,z_n$ generate a Hausdorff field over H with an isomorphism $H_n\to H(z_0,\dots,z_n)$ over H sending y_i to z_i for $i=0,\dots,n$. These isomorphisms have therefore a common extension to an isomorphism $H\langle y\rangle\to H\langle z\rangle$ of Hardy fields over H. In particular, z is H-hardian, and vz realizes the same cut in Γ as β. Now by Lemma 2.2 and the remarks preceding it there exists $\varepsilon\in\mathcal{C}$ such that $\varepsilon>_{\mathrm{e}}0$ and $\varepsilon\prec f$ for all $f\in K\langle y\rangle^\times$, so any such ε has the desired property. □

Combining Corollary 1.8 with Proposition 4.1 yields:

Corollary 4.2. *There exists an H-hardian $z\in\mathcal{C}^\omega$ such that $z>0$ and vz realizes the same cut in Γ as β.*

We can now use [6, Theorem 9.2] to obtain an analytic strengthening of it:

Corollary 4.3. *Suppose β is of countable type over Γ and $\beta_i^\dagger<0$ for all i. Then for some H-hardian $z\in\mathcal{C}^\omega$: $z>0$ and vz realizes the same cut in Γ as β.*

Proof. [6, Theorem 9.2] gives H-hardian $y > 0$ such that vy realizes the same cut in Γ as β. Then Corollary 4.2 gives a z as required. $\square$

Proof of Theorem A. First an analytic/smooth version of [6, Lemma 9.1]:

Lemma 4.4. *Let H be a maximal analytic or maximal smooth Hardy field with H-couple (Γ, ψ) over $\mathbb{R}$. Then no element in any H-couple over $\mathbb{R}$ extending (Γ, ψ) has countable type over Γ.*

Proof. By Corollary 3.7, (Γ, ψ) is countably spherically complete, and both Γ and $\Gamma^{<}$ have uncountably cofinality. By [6, Lemma 8.11], any element of any H-couple over $\mathbb{R}$ extending (Γ, ψ) and of countable type over Γ falls under Case (b). Now argue as in the proof of [6, Lemma 9.1], using Corollary 4.3 in place of [6, Theorem 9.2], that there are no such elements. $\square$

Theorem A from the introduction and its smooth version follow from Corollary 3.5 and Lemma 4.4, just as the main theorem in [6] is derived in the beginning of [6, Section 9] from the non-smooth analogues of that corollary and lemma. We now use this to characterize gaps in maximal analytic Hardy fields: Corollary 4.9 below.

Characters of gaps in maximal Hardy fields. Let S be an ordered set (as in [6] this means *linearly ordered set*) and C a cut in S, that is a downward closed subset of S. We define the **character** of C (in S) to be the pair (α, β^*) where $\alpha := \operatorname{cf}(C)$ and β^* is the set $\beta := \operatorname{ci}(S \setminus C)$ equipped with the reversed ordering. We then also call C an (α, β^*)**-cut** (in S); see [25, §3.2]. The characters of the cuts $\emptyset$ and S in S are $\big(0, \operatorname{ci}(S)^*\big)$ and $\big(\operatorname{cf}(S), 0\big)$, respectively. Note that S is η_1 iff no cut in S has character (α, β^*) with $\alpha, \beta \leqslant \omega$. A **gap** $A < B$ **in** S is a pair (A, B) of subsets of S such that $A < B$ and there is no $s \in S$ with $A < s < B$. The character of such a gap $A < B$ is defined to be the character (α, β^*) of the cut $A^{\downarrow} = S \setminus B^{\uparrow}$ in S, and then $A < B$ is also called an (α, β^*)-gap in S.

Let G be an ordered abelian group. If $v \colon G \to S_\infty$ is a surjective convex valuation on G ([ADH, p. 99]) and $A < B$ is an (α, β^*)-gap in S where $\alpha, \beta \geqslant \omega$, then $\big(v^{-1}(A) \cap G^{<}, v^{-1}(B) \cap G^{<}\big)$ is a (α, β^*)-gap in G. If H is an ordered field, then $\operatorname{cf}(H^{<}) = \operatorname{ci}(H^{>}) = \operatorname{cf}(H)$, and the cuts $H^{<h}$ and $H^{\leqslant h}$ $(h \in H)$ in H have character $\big(\operatorname{cf}(H), 1\big)$ and $\big(1, \operatorname{cf}(H)^*\big)$, respectively.

Corollary 4.5. *Let H be a maximal Hardy field, or a maximal analytic Hardy field, or a maximal smooth Hardy field. Set $\kappa := \operatorname{ci}(H^{>\mathbb{R}})$. Then $\omega < \kappa \leqslant \mathfrak{c}$, and H has gaps of character (ω, κ^*), (κ, ω^*), and (κ, κ^*).*

Proof. Corollary 3.6 and [6, Corollary 4.8] give $\omega < \kappa \leqslant \mathfrak{c}$. The gaps $\mathbb{R} < H^{>\mathbb{R}}$ and $H^{<\mathbb{R}} < \mathbb{R}$ in H have character (ω, κ^*) and (κ, ω^*), respectively. To obtain a (κ, κ^*)-gap in H, take a coinitial sequence $(\ell_\rho)_{\rho<\kappa}$ in $H^{>\mathbb{R}}$ with $\ell_\rho \succ \ell_{\rho'}$ for all $\rho < \rho' < \kappa$. Put $\upgamma_\rho := \ell_\rho^\dagger > 0$, so $(1/\ell_\rho)' = (1/\ell_\rho)^\dagger/\ell_\rho = -\upgamma_\rho/\ell_\rho < 0$. Set $A := \{\upgamma_\rho/\ell_\rho : \rho < \kappa\}$, $B := \{\upgamma_\rho : \rho < \kappa\}$. Then $A < B$ is a (κ, κ^*)-gap in H, using that H has asymptotic integration. $\square$

Let now G be an ordered abelian group. Assume $G \neq \{0\}$ and $G^{>}$ has no smallest element, so $\operatorname{ci}(G^{>}) \geqslant \omega$. A gap $A < B$ in G is said to be **cauchy** if $A, B \neq \emptyset$, A has no largest element, B has no smallest element, and for each $\varepsilon \in G^{>}$ there are $a \in A$, $b \in B$ with $b - a < \varepsilon$. If $A < B$ is a cauchy gap in G, then so is $-B < -A$.

Lemma 4.6. *Let $A < B$ be a cauchy gap in G and (a_ρ) be an increasing cofinal well-indexed sequence in A. Then (a_ρ) is a divergent c-sequence in G.*

Proof. Let $\varepsilon \in G^>$ and take $a \in A$, $b \in B$ with $b - a < \varepsilon$. Take ρ_0 such that $a \leqslant a_\rho$ for all $\rho > \rho_0$. Then $0 < a_{\rho'} - a_\rho < b - a < \varepsilon$ for $\rho_0 < \rho < \rho'$. Hence (a_ρ) is a c-sequence in G, and there is no $a \in G$ with $a_\rho \to a$. □

Lemma 4.7. *Every cauchy gap in G has character $\big(\operatorname{cf}(G^<), \operatorname{cf}(G^<)^*\big)$. Moreover, G is complete iff G has no cauchy gap.*

Proof. The first claim follows from Lemma 4.6 and [ADH, 2.4.11]. It also follows from this lemma that if G is complete, then G has no cauchy gap. Conversely, suppose G has no cauchy gap. Let (a_ρ) be a c-sequence in G. For each $\varepsilon \in G^>$, take ρ_ε such that $|a_\rho - a_{\rho'}| < \varepsilon$ for all $\rho, \rho' \geqslant \rho_\varepsilon$, and set

$$A \ := \ \{a_{\rho_\varepsilon} - \varepsilon : \ \varepsilon \in G^>\}, \qquad B \ := \ \{a_{\rho_\delta} + \delta : \ \delta \in G^>\}.$$

Then $A < B$ and for all $\varepsilon \in G^>$ there are $a \in A$ and $b \in B$ with $b - a < \varepsilon$. But $A < B$ is no cauchy gap, so we have $g \in G$ with $A \leqslant g \leqslant B$. Then $a_\rho \to g$. □

Corollary 4.8. *Let H be a maximal, or maximal analytic, or maximal smooth Hardy field, and $\lambda := \operatorname{cf}(H)$. Then $\omega < \lambda \leqslant \mathfrak{c}$, and H has gaps of character $(0, \lambda^*)$, $(\lambda, 0)$, $(1, \lambda^*)$, $(\lambda, 1)$, and if H is not complete, then H has a (λ, λ^*)-gap.*

Proof. Lemma 2.8 yields $\lambda = \operatorname{cf}(\Gamma)$, so $\omega < \lambda \leqslant \mathfrak{c}$ by [6, Corollary 8.1] and Corollary 3.7. For the rest use the remarks before Corollary 4.5 and Lemma 4.7. □

The main result of [6], Theorem A, and Corollaries 4.5, 4.8 now give:

Corollary 4.9. *Assume* CH. *If H is a maximal Hardy field, or a maximal analytic Hardy field, or a maximal smooth Hardy field, then the characters of gaps in H are*

$$(0, \omega_1^*),\ (\omega_1, 0),\ (1, \omega_1^*),\ (\omega_1, 1),\ (\omega, \omega_1^*),\ (\omega_1, \omega^*),\ \text{and } (\omega_1, \omega_1^*).$$

The number of maximal analytic Hardy fields. Next analytic and smooth versions of Theorem 7.1 from [6]:

Corollary 4.10. *The number of maximal analytic Hardy fields is $2^{\mathfrak{c}}$ where $\mathfrak{c} = 2^{\aleph_0}$. Likewise with "smooth" in place of "analytic".*

Proof. We treat the number of maximal analytic Hardy fields; the smooth case is similar, using the smooth version of Theorem A. In the argument following the statement of [6, Proposition 7.4] we replace $\mathcal{H}$ by the set of all analytic Hardy fields $H \supseteq \mathbb{R}$ with $|{*H^{\text{te}}}| < \mathfrak{c}$. Thus modified, this argument shows that it is enough to prove that in [6, Proposition 7.4] we can choose f_0, f_1 to be analytic whenever the Hardy field H is analytic. For this we first note that if H in [6, Lemma 7.7] is analytic, then we can take y there to be analytic, by appealing to Theorem A instead of [6, Section 5 and Corollary 6.7]. Now argue as in the remarks following [6, Lemma 7.10] using this analytic version of [6, Lemma 7.7]. □

Corollary 7.8 of [6] has an analytic version with a similar proof:

Corollary 4.11. *If H is a maximal analytic Hardy field, then the ordered set ${*H^{\text{te}}}$ is η_1, and $|{*H^{\text{te}}}| = \mathfrak{c}$.*

We now improve Corollary 2.7: assuming CH, there are as many cofinal maximal analytic Hardy fields as there are maximal analytic Hardy fields, by Corollary 4.14.

Lemma 4.12. *Let $\phi \in \mathcal{C}^\omega$ be overhardian. Then there is a set $\mathcal{H}_\phi$ of analytic Hardy field extensions of $\mathbb{R}\langle\phi\rangle$ with $|\mathcal{H}_\phi| = 2^{\mathfrak{c}}$ such that for each $H \in \mathcal{H}_\phi$, $*\phi$ is the largest element of $*H^{\text{te}}$, and each Hardy field contains at most one $H \in \mathcal{H}_\phi$.*

Proof. By [6, Lemma 7.7] we have $*\mathbb{R}\langle\phi\rangle^{\text{te}} = \{*\phi\}$. Let $H \supseteq \mathbb{R}\langle\phi\rangle$ be an analytic Hardy field with $*\phi = \max *H^{\text{te}}$, and let $P < Q$ be a countable gap in $*H^{\text{te}}$ with $Q < *\phi$. Then [6, Proposition 7.4] and the argument in the proof of Corollary 4.10 yields analytic Hardy fields $H_0 = H\langle f_0\rangle$ and $H_1 = H\langle f_1\rangle$ without a common Hardy field extension such that for $j = 0, 1$, we have $f_j \in H_j^{\text{te}}$, $P < *f_j < Q \cup \{*\phi\}$, and $*H_j^{\text{te}} = *H^{\text{te}} \cup \{*f_j\}$ (thus $*\phi = \max *H_j^{\text{te}}$).

We now follow the argument after the statement of [6, Proposition 7.4], with $\mathcal{H}$ now the set of all analytic Hardy fields $H \supseteq \mathbb{R}\langle\phi\rangle$ such that $|*H^{\text{te}}| < \mathfrak{c}$ and $*\phi = \max *H^{\text{te}}$. For an ordinal λ we let 2^λ be the set of functions $\lambda \to \{0, 1\}$. With s ranging over $\bigcup_{\lambda<\mathfrak{c}} 2^\lambda$, we construct a tree (H_s) in $\mathcal{H}$ with $|*H^{\text{te}}| \leqslant |\lambda + 1|$ for $s \in 2^\lambda$, as follows. For $\lambda = 0$ the function s has empty domain and we take $H_s = \mathbb{R}\langle\phi\rangle$. If $s \in 2^\lambda$ $(\lambda < \mathfrak{c})$ and $H_s \in \mathcal{H}$ are given with $|*H_s^{\text{te}}| \leqslant |\lambda + 1|$, then [6, Lemma 7.2] provides a countable gap P, Q in $*H_s^{\text{te}} \setminus \{*\phi\}$, and we let $H_{s0}, H_{s1} \in \mathcal{H}$ be obtained from H_s as H_0, H_1 are obtained from H in the remark above. Suppose $\lambda < \mathfrak{c}$ is an infinite limit ordinal, $s \in 2^\lambda$, and that for every $\alpha < \lambda$ we are given $H_{s|\alpha} \in \mathcal{H}$ with $H_{s|\alpha} \subseteq H_{s|\beta}$ whenever $\alpha \leqslant \beta < \lambda$. Then we set $H_s := \bigcup_{\alpha<\lambda} H_{s|\alpha} \in \mathcal{H}$. Assuming also inductively that $|*H_{s|\alpha}^{\text{te}}| \leqslant |\alpha + 1|$ for all $\alpha < \lambda$, we have $|*H_s^{\text{te}}| \leqslant |\lambda| \cdot |\lambda + 1| = |\lambda + 1|$, as desired. This finishes the construction of our tree. Then for each $s \in 2^{\mathfrak{c}}$ we have an analytic Hardy field $H_s := \bigcup_{\lambda<\mathfrak{c}} H_{s|\lambda}$ such that if $s, s' \in 2^{\mathfrak{c}}$ are different, then H_s, $H_{s'}$ have no common Hardy field extension. Hence $\mathcal{H}_\phi := \{H_s : s \in 2^{\mathfrak{c}}\}$ has the required properties. □

Lemma 4.13. *Assume* CH. *Let H be a bounded analytic Hardy field. Then H extends to a cofinal analytic Hardy field.*

Proof. By Lemma 2.1 we can replace H by $\operatorname{Li}\big(H(\mathbb{R})\big)$ to arrange that $H \supseteq \mathbb{R}$ and H is Liouville closed. Next, take an enumeration $(\phi_\alpha)_{\alpha<\mathfrak{c}}$ of $\mathcal{C}$ with $\phi_0 >_{\text{e}} H$. Corollary 2.5 yields an H-hardian $h_0 \in \mathcal{C}^\omega$ with $h_0 >_{\text{e}} \phi_0$, and then the analytic Hardy field $H_0 := H\langle h_0\rangle$ is bounded by Lemma 2.1. Now a transfinite recursion as in the proof of Corollary 2.7, beginning with (H_0, h_0), yields a cofinal analytic Hardy field extension of H_0 and thus of H. □

Corollary 2.5 gives an overhardian $\phi \in \mathcal{C}^\omega$. For such ϕ and $\mathcal{H}_\phi$ as in Lemma 4.12, all $H \in \mathcal{H}_\phi$ are bounded. With Lemma 4.13 we can now improve Corollary 2.7:

Corollary 4.14. *Assuming* CH, *there are $2^{\mathfrak{c}}$ cofinal maximal analytic Hardy fields.*

Maximal analytic Hardy fields approximate maximal Hardy fields. A maximal analytic Hardy field is an $\infty\omega$-elementary substructure of any maximal Hardy field extension, by Corollary 7.5 below. Maximal analytic Hardy fields are also very close to maximal Hardy fields in another way:

Theorem 4.15. *Let H be a maximal analytic Hardy field or a maximal smooth Hardy field. Then H is dense in any Hardy field extension of H.*

Proof. We establish two claims:

Claim 1: If $f \in \mathcal{C}^{<\infty}$ is H-hardian and $H\langle f\rangle$ is an immediate extension of H, then H is dense in $H\langle f\rangle$.

To prove this, assume $f \in \mathcal{C}^{<\infty}$ is H-hardian, (f_ρ) is a divergent pc-sequence in H, and $f_\rho \rightsquigarrow f$. By [ADH, 16.0.3, Section 11.4], (f_ρ) is of d-transcendental type over H. If the sequence $\big(v(f - f_\rho)\big)$ is cofinal in $\Gamma := v(H^\times)$, then (f_ρ) is a cauchy sequence, and so H is indeed dense in $H\langle f\rangle$, by [5, Corollary 4.1.6]. Suppose $\big(v(f - f_\rho)\big)$ is not cofinal in Γ. Then we have $h \in H^\times$ such that $0 \notin v(hf - H)$, so Corollary 1.8 and Proposition 3.3 yield an H-hardian pseudolimit of (hf_ρ) in $\mathcal{C}^\omega$, contradicting the maximality of H. This proves Claim 1.

Claim 2: *For any Hardy field extension K of H we have $\Gamma_K = \Gamma$.*

Towards a contradiction, suppose K is a Hardy field extension of H and $\beta \in \Gamma_K \setminus \Gamma$. We arrange that K is Liouville closed. Let (Γ, ψ) and (Γ_K, ψ_K) be the H-couples of H and K over $\mathbb{R}$, respectively, and let $\big(\Gamma\langle\beta\rangle, \psi_\beta\big)$ be the H-couple over $\mathbb{R}$ generated by β over (Γ, ψ) in (Γ_K, ψ_K). There are several cases to consider, and we show that each is impossible.

First the case that $\big(\Gamma\langle\beta\rangle, \psi_\beta\big)$ is an immediate extension of (Γ, ψ). Then we have a divergent pc-sequence (γ_ρ) in (Γ, ψ) with $\gamma_\rho \rightsquigarrow \beta$. As in the beginning of [6, Section 8] we take $g_\rho \in H$ with $vg_\rho = \gamma_\rho$ so that $(g_\rho^\dagger)$ is a pc-sequence in H, and arguing as in loc. cit. (using $H^\dagger = H$) we see that $(g_\rho^\dagger)$ has no pseudolimit in H (because then (γ_ρ) would have one in Γ). Now take $g \in K$ with $vg = \beta$. Then $v(g^\dagger - g_\rho^\dagger) = (\beta - \gamma_\rho)^\dagger$, and the latter is eventually strictly increasing as a function of ρ, and so $g_\rho^\dagger \rightsquigarrow g^\dagger$. Moreover, $\big(v(g^\dagger - g_\rho^\dagger)\big)$ is not cofinal in Γ. As at the end of the proof of Claim 1, with $g^\dagger$ and $(g_\rho^\dagger)$ in the role of f and (f_ρ), this contradicts the maximality assumption on H.

Now by [4, Proposition 4.1] and the remark following its proof, the vector β falls under Case (a), or Case (b), or Case (c)_n for a certain n. In Case (a) we have $(\Gamma + \mathbb{R}\beta)^\dagger = \Gamma^\dagger$ and so $\Gamma\langle\beta\rangle = \Gamma + \mathbb{R}\beta$; but (Γ_K, ψ_K) is of Hahn type, hence $\big(\Gamma\langle\beta\rangle, \psi_\beta\big)$ is an immediate extension of (Γ, ψ), and we have just excluded that possibility. Case (c)_n gives an element $\beta_n \in \Gamma\langle\beta\rangle$ with $\beta_n^\dagger \notin \Gamma$ and $\beta_n^\dagger$ falling under Case (a), and so this is also impossible.

Finally, suppose β falls under Case (b). Take $y \in K^>$ with $vy = \beta$. Then Corollary 4.2 gives an H-hardian $z \in \mathcal{C}^\omega$ with vz realizing the same cut in Γ as β, contradicting the maximality assumption on H. This finishes the proof of Claim 2.

To finish the proof of the theorem, let $f \in \mathcal{C}^{<\infty}$ be H-hardian; it suffices to show that then H is dense in $H\langle f\rangle$. Now by Claim 2, $H\langle f\rangle$ is an immediate extension of H, and hence H is indeed dense in $H\langle f\rangle$ by Claim 1. $\square$

Question. Is every maximal Hardy field dense in every Hausdorff field extension?

Dense pairs of closed H-fields. Let $\mathcal{L} = \{0, 1, -, +, \cdot, \partial, \leqslant, \preccurlyeq\}$ be the language of ordered valued differential rings; cf. [ADH, p. 678]. We view each ordered valued differential field as an $\mathcal{L}$-structure in the natural way. We let $\mathcal{L}^2$ extend $\mathcal{L}$ by a new unary predicate symbol U. The $\mathcal{L}^2$-structures are presented as pairs (K, F) where K is an $\mathcal{L}$-structure and U names the subset F of K. Let T be the $\mathcal{L}$-theory of closed H-fields with small derivation. Recall from [ADH] that T is complete and model-complete.

Theorem 4.16. *The following requirements on $\mathcal{L}^2$-structures (K, F) axiomatize a complete $\mathcal{L}^2$-theory T^{d}:*

(1) *$K \models T$, that is, K is a closed H-field with small derivation;*

(2) *F is the underlying set of a closed H-subfield of K; and*
(3) *$F \neq K$ and F is dense in the ordered field K.*

Moreover, each $\mathcal{L}^2$-formula $\varphi(x)$ where $x = (x_1, \dots, x_m)$ is T^{d}-equivalent to a boolean combination of formulas of the form

$$\exists y_1 \cdots \exists y_n \big(U(y_1) \ \&\ \cdots\ \&\ U(y_n) \ \&\ \psi(x,y)\big) \tag{4.1}$$

where $\psi(x,y)$ with $y = (y_1, \dots, y_n)$ is an $\mathcal{L}$-formula.

This follows from Fornasiero's [20, Theorems 8.3 and 8.5], with details of how it follows to appear in [7]. We just note here that by this theorem the $\mathcal{L}^2$-theory T^{d} is decidable. Moreover, no pair $(K,F) \models T^{\mathrm{d}}$ induces "new structure" on F:

Corollary 4.17. *Let $(K,F) \models T^{\mathrm{d}}$, and let $S \subseteq K^m$ be A-definable in (K,F), where $A \subseteq F$. Then $S \cap F^m$ is A-definable in the $\mathcal{L}$-substructure F of K.*

Proof. By the theorem this reduces to the case where S is defined in (K,F) by a formula as in (4.1) where however $\psi(x,y)$ is now an $\mathcal{L}_A$-formula. Then $S \cap F^m$ is defined in F by the $\mathcal{L}_A$-formula $\exists y \psi(x,y)$. ☐

Note that if M is a maximal analytic or maximal smooth Hardy field and N a maximal Hardy field with $M \subseteq N$, $M \neq N$, then $(N,M) \models T^{\mathrm{d}}$ by Theorem 4.15. But strictly speaking, we do not know whether there exist such M, N.

To secure a model of the complete theory T^{d} we proceed as follows. Let F be an H-field. Then the completion F^{c} of the ordered valued differential field F is an H-field extension of F, and F is dense in F^{c}; see [ADH, 10.5.9]. If F is closed and of countable cofinality, then F^{c} is closed, by [ADH, 14.1.6], so if in addition F has small derivation and $F \neq F^{\mathrm{c}}$, then $(F^{\mathrm{c}}, F) \models T^{\mathrm{d}}$. Now $\mathbb{T}$ is not complete: set $\mathrm{e}_0 = x$ and $\mathrm{e}_{i+1} = \exp \mathrm{e}_i$ for all i; then $\big(\sum_{i=0}^{n} 1/\mathrm{e}_i\big)_{n=0}^{\infty}$ is a cauchy sequence in $\mathbb{T}$ but has no limit in $\mathbb{T}$. Therefore $(\mathbb{T}^{\mathrm{c}}, \mathbb{T}) \models T^{\mathrm{d}}$.

5. Analytic Hardy Fields of Countable Cofinality

Generalizing terminology introduced in [6, Section 8], call a valued abelian group **countably spherically complete** if every pc-sequence in it of length ω pseudoconverges in it. Any η_1-ordered abelian group with a convex valuation is countably spherically complete, by [ADH, 2.4.2]. Thus maximal analytic and maximal smooth Hardy fields are countably spherically complete. In the subsection below we use this fact to realize the completion of an analytic Hardy field of countable cofinality as an analytic Hardy field.

Completing analytic Hardy fields of countable cofinality. Let K be an asymptotic field. Equip the completion K^{c} of the valued field K with the unique extension of the derivation of K to a continuous derivation on K^{c}; cf. [ADH, 4.4.11, 9.1.5]. Then K^{c} is asymptotic by [ADH, 9.1.6], and if K is a pre-H-field (H-field, respectively), then so is K^{c} by [ADH, 10.5.9]. Let L be an asymptotic field extension of K such that Γ is cofinal in Γ_L. By [ADH, 3.2.20], the natural inclusion $K \to L$ extends uniquely to an embedding $K^{\mathrm{c}} \to L^{\mathrm{c}}$ of valued fields, and it is easily checked that this is an embedding of valued *differential* fields. If K is dense in L, then there is a unique valued field embedding $L \to K^{\mathrm{c}}$ over K, by [ADH, 3.2.13], and this is also an embedding of valued differential fields.

Lemma 5.1. *Let K be an $\upomega$-free H-asymptotic field whose value group Γ_K has countable cofinality. Let M be a newtonian H-asymptotic field with asymptotic integration, and suppose M is countably spherically complete. Then any embedding $K \to M$ extends to an embedding $K^{\mathrm{c}} \to M$.*

Proof. Let $\iota\colon K \to M$ be an embedding; we need to extend ι to an embedding $K^{\mathrm{c}} \to M$. The d-valued hull $L := \operatorname{dv}(K)$ of K is $\upomega$-free by [ADH, remark after 13.6.1], and $\Gamma_L = \Gamma_K$ by [ADH, 10.3.2(i)]. By [ADH, 14.2.5], M is d-valued; let ι_L be the extension of ι to an embedding $L \to M$. Using a remark before the lemma we see that it is enough to show that ι_L extends to an embedding $L^{\mathrm{c}} \to M$. Hence replacing K, ι by L, ι_L, we arrange K is d-valued. Take an immediate asymptotically d-algebraically maximal d-algebraic extension L of K; by a remark following the statement of [ADH, Theorem 14.0.1] such L exists and is $\upomega$-free and newtonian. Then L is a newtonization of K by [29, Theorem 3.5], so embeds into M over K. Passing to this newtonization we arrange that K is newtonian. Then K^{c} is $\upomega$-free by [ADH, 11.7.20] and newtonian by [ADH, 14.1.5].

Suppose $f \in K^{\mathrm{c}} \setminus K$. It suffices to show that then ι extends to an embedding $\iota_f\colon K\langle f\rangle \to M$. Here is why: $K\langle f\rangle$ is $\upomega$-free by the remark before [ADH, 11.7.20], so $K\langle f\rangle$ has a newtonization E in K^{c} by [29, Theorem B]; by the same remark E is $\upomega$-free; moreover, ι_f extends to an embedding $E \to M$. Hence we can transfinitely iterate this extension process to obtain an embedding $K^{\mathrm{c}} \to M$ extending ι.

To construct ι_f, take a c-sequence (f_ρ) in K with $f_\rho \to f$ (in K^{c}). By [ADH, 2.2.25] the index set of (f_ρ) has cofinality ω, so by passing to a cofinal subsequence we arrange (f_ρ) is a divergent pc-sequence in K of length ω and width $\{\infty\}$ such that $f_\rho \rightsquigarrow f$. Take $g \in M$ such that $\iota(f_\rho) \rightsquigarrow g$. Now K is asymptotically d-algebraically maximal by [29, Theorem A], so (f_ρ) is of d-transcendental type over K by [ADH, 11.4.8, 11.4.13], hence [ADH, 11.4.7] yields an embedding $K\langle f\rangle \to M$ extending ι and sending f to g. $\square$

Lemma 5.1 yields a pre-H-field version of it without the $\upomega$-free hypothesis on K:

Proposition 5.2. *Let K be a pre-H-field with $\operatorname{cf}(\Gamma_K) = \omega$ and M a countably spherically complete closed H-field. Then every embedding $K \to M$ extends to an embedding $K^{\mathrm{c}} \to M$.*

For this we arrange that M extends K and then have to find an embedding $K^{\mathrm{c}} \to M$ over K. Take any expansion $\boldsymbol{M}$ of M to a $\Lambda\Omega$-field and expand K to a pre-$\Lambda\Omega$-field $\boldsymbol{K}$ such that $\boldsymbol{K} \subseteq \boldsymbol{M}$, see [ADH, 16.3.21]. Then the proposition below applied to $\boldsymbol{M}$ in place of $\boldsymbol{L}$ yields an $\upomega$-free H-field extension K^* of K such that K is cofinal in K^* and an embedding $\iota^*\colon K^* \to M$ over K. Lemma 5.1 gives an extension of ι^* to an embedding $(K^*)^{\mathrm{c}} \to M$, and by a remark before that lemma this yields an embedding $K^{\mathrm{c}} \to M$ as required.

It remains to establish the following "cofinality" refinement of [ADH, 16.4.1]:

Proposition 5.3. *Let $\boldsymbol{K}$ be a pre-$\Lambda\Omega$-field with $\Gamma_{\boldsymbol{K}} \neq \{0\}$. Then there exists an $\upomega$-free $\Lambda\Omega$-field extension $\boldsymbol{K}^*$ of $\boldsymbol{K}$ such that;*

(i) $\operatorname{res}\boldsymbol{K}^*$ *is algebraic over* $\operatorname{res}\boldsymbol{K}$*;*
(ii) $\boldsymbol{K}$ *is cofinal in* $\boldsymbol{K}^*$*; and*
(iii) *any embedding of* $\boldsymbol{K}$ *into a Schwarz closed* $\Lambda\Omega$*-field* $\boldsymbol{L}$ *extends to an embedding* $\boldsymbol{K}^* \to \boldsymbol{L}$*.*

We revisit the proof of [ADH, 16.4.1], which consists of several lemmas and a corollary. Let $\boldsymbol{K} = (K, I, \Lambda, \Omega)$ be a pre-$\Lambda\Omega$-field with $\Gamma := \Gamma_K \neq \{0\}$.

Lemma 5.4. *Suppose K is grounded, or there exists $b \asymp 1$ in K such that $v(b')$ is a gap in K. Then $\boldsymbol{K}$ has an ω-free $\Lambda\Omega$-field extension $\boldsymbol{K}^*$ such that $\operatorname{res}\boldsymbol{K} = \operatorname{res}\boldsymbol{K}^*$, $\boldsymbol{K}$ is cofinal in $\boldsymbol{K}^*$, and any embedding of $\boldsymbol{K}$ into a $\Lambda\Omega$-field $\boldsymbol{L}$ closed under logarithms extends to an embedding $\boldsymbol{K}^* \to \boldsymbol{L}$.*

Proof. By Lemma 2.15, K is cofinal in the H-field hull $F := H(K)$ of K, and hence by Lemma 2.17, K is also cofinal in the H-field extension F_ω of F constructed in [ADH, 11.7]. Thus the lemma follows from the proof of [ADH, 16.4.2]. □

Lemma 5.5. *Suppose K has gap β and $v(b') \neq \beta$ for all $b \asymp 1$ in K. Then there exists a grounded pre-$\Lambda\Omega$-field extension $\boldsymbol{K}_1$ of $\boldsymbol{K}$ such that $\operatorname{res}\boldsymbol{K} = \operatorname{res}\boldsymbol{K}^*$, $\boldsymbol{K}$ is cofinal in $\boldsymbol{K}_1$, and any embedding of $\boldsymbol{K}$ into a $\Lambda\Omega$-field $\boldsymbol{L}$ closed under integration extends to an embedding $\boldsymbol{K}_1 \to \boldsymbol{L}$.*

Proof. Take $s \in K$ with $vs = \beta$. Following the proof of [ADH, 16.4.3], suppose $s \notin \mathrm{I}(K)$, and take K_1 as in Case 1 of that proof, so $K_1 = H(K)(y)$ where $y' = s$. Now use that $\Gamma_{H(K)} = \Gamma$, and that $\Gamma_{H(K)}$ is cofinal in Γ_{H_1} by Lemma 2.17. If $s \in \mathrm{I}(K)$ and K_1 is as in Case 2, then $K_1 = K(y)$ where $y' = s$, so again K is cofinal in K_1 by Lemma 2.17. □

These two lemmas yield a "cofinality" refinement of [ADH, 16.4.4]:

Corollary 5.6. *Suppose K does not have asymptotic integration. Then $\boldsymbol{K}$ has an ω-free $\Lambda\Omega$-field extension $\boldsymbol{K}^*$ such that $\operatorname{res}\boldsymbol{K}^* = \operatorname{res}\boldsymbol{K}$, $\boldsymbol{K}$ is cofinal in $\boldsymbol{K}^*$, and and any embedding of $\boldsymbol{K}$ into a $\Lambda\Omega$-field $\boldsymbol{L}$ closed under integration extends to an embedding $\boldsymbol{K}^* \to \boldsymbol{L}$.*

The next three lemmas are "cofinality" refinements of [ADH, 16.4.5, 16.4.6, 16.4.7] and take care of the case where K has asymptotic integration.

Lemma 5.7. *Assume K has asymptotic integration and is not λ-free. Then $\boldsymbol{K}$ extends to an ω-free $\Lambda\Omega$-field $\boldsymbol{K}^*$ such that $\operatorname{res}\boldsymbol{K}^* = (\operatorname{res}\boldsymbol{K})^{\mathrm{rc}}$, $\boldsymbol{K}$ is cofinal in $\boldsymbol{K}^*$, and any embedding of $\boldsymbol{K}$ into a Liouville closed $\Lambda\Omega$-field $\boldsymbol{L}$ extends to an embedding $\boldsymbol{K}^* \to \boldsymbol{L}$.*

Proof. As in the proof of [ADH, 16.4.5], it is enough, by Corollary 5.6, to show that $\boldsymbol{K}$ has a $\Lambda\Omega$-field extension $\boldsymbol{K}_1 = (K_1, \dots)$ with a gap such that $\operatorname{res}\boldsymbol{K}_1 = (\operatorname{res}\boldsymbol{K})^{\mathrm{rc}}$, K is cofinal in K_1, and any embedding of $\boldsymbol{K}$ into a Liouville closed $\Lambda\Omega$-field $\boldsymbol{L}$ extends to an embedding $\boldsymbol{K}_1 \to \boldsymbol{L}$. Take $\boldsymbol{K}_1$ as in the proof of [ADH, 16.4.5]. Put $E := H(K)^{\mathrm{rc}}$. Then $\Gamma_E = \mathbb{Q}\Gamma$, so K is cofinal in E. If E has a gap, then $K_1 = E$, and we are done. Suppose E has no gap. Then $K_1 = E(f)$ where $f \in K_1^\times$ and $\lambda := -f^\dagger \in K$, and $s := -\lambda$ creates a gap over E. By the proof of Case 2 in [ADH, 16.4.5], the hypothesis of Lemma 2.19 holds for E in place of K, so E is cofinal in K_1, and hence so is K. □

Lemma 5.8. *Suppose K is λ-free but not ω-free. Then $\boldsymbol{K}$ has an ω-free $\Lambda\Omega$-field extension $\boldsymbol{K}^*$ such that $\operatorname{res}\boldsymbol{K}^*$ is algebraic over $\operatorname{res}\boldsymbol{K}$, $\boldsymbol{K}$ is cofinal in $\boldsymbol{K}^*$, and any embedding of $\boldsymbol{K}$ into a Schwarz closed $\Lambda\Omega$-field $\boldsymbol{L}$ extends to an embedding of $\boldsymbol{K}^*$ into $\boldsymbol{L}$.*

Proof. Take $\omega \in K$ with $\omega_\rho \rightsquigarrow \omega$. Let $\boldsymbol{K}^*$ be as in the proof of [ADH, 16.4.6]. That proof shows that $\Omega = \omega(K)^{\downarrow}$ or $\Omega = K \setminus \sigma\big(\Gamma(K)\big)^{\uparrow}$. Suppose first that $\Omega = \omega(K)^{\downarrow}$. With $\boldsymbol{K}_\gamma = (K_\gamma, \dots)$ as in Case 1 of that proof, we have $K_\gamma = K\langle\gamma\rangle$ where $\gamma \neq 0$, $\sigma(\gamma) = \omega$, and $v\gamma$ is a gap in K_γ. The remarks before [ADH, 13.7.7] give $[\Gamma] = \big[\Gamma_{K_\gamma}\big]$, so K is cofinal in K_γ. Now follow the argument in Case 1 of loc. cit., using Corollary 5.6 instead of [ADH, 16.4.4]. If $\Omega = K \setminus \sigma\big(\Gamma(K)\big)^{\uparrow}$, then we argue as in Case 2 of loc. cit., using Lemma 5.7 instead of [ADH, 16.4.5]. □

Lemma 5.9. *Suppose K is ω-free. Then $\boldsymbol{K}$ has an ω-free $\Lambda\Omega$-field extension $\boldsymbol{K}^*$ such that $\operatorname{res}\boldsymbol{K}^* = \operatorname{res}\boldsymbol{K}$, $\boldsymbol{K}$ is cofinal in $\boldsymbol{K}^*$, and any embedding of $\boldsymbol{K}$ into a $\Lambda\Omega$-field $\boldsymbol{L}$ extends to an embedding of $\boldsymbol{K}^*$ into $\boldsymbol{L}$.*

Proof. Take $\boldsymbol{K}^* = (K^*, \dots)$ as in the proof of [ADH, 16.4.7]. Then $K^* = H(K)$, and by Lemma 2.15, K is cofinal in $H(K)$. □

This concludes the proof of Proposition 5.3 and of Proposition 5.2. Combining the latter with [6, Corollary 3.2] and Corollary 3.5 yields:

Corollary 5.10. *Let H be a Hardy field of countable cofinality and $M \supseteq H$ a maximal Hardy field. Then there is an embedding $H^{\mathrm{c}} \to M$ over H. Likewise if $M \supseteq H$ is a maximal analytic Hardy field or a maximal smooth Hardy field.*

As a consequence of Corollary 5.10, with $t := x^{-1}$, each maximal analytic Hardy field contains a Hardy field extending $\mathbb{R}(t)$ and isomorphic over $\mathbb{R}(t)$ to the ordered field $\mathbb{R}(\!(t)\!)$ of Laurent series over $\mathbb{R}$ equipped with the continuous $\mathbb{R}$-linear derivation given by $t' = -t^2$. (This may be viewed as a Hardy field version of Besicovitch's strengthening [9] of Borel's theorem on $\mathcal{C}^\infty$-functions with prescribed Taylor series [11].) In Corollary 7.10 we show that even the ordered differential field $\mathbb{T}$ of transseries, which vastly extends $\mathbb{R}(\!(t)\!)$, embeds into any given maximal analytic Hardy field. As a first step we accomplish this below for the H-subfield $\mathbb{T}_{\log}$ of $\mathbb{T}$ of logarithmic transseries (cf. [ADH, p. 722]). For this it is useful to have available some facts about short ordered sets, also needed in Section 6.

Short ordered sets. Let S be an ordered set. (As in [6], this means "linearly ordered set".) Let S^* denote S equipped with the reversed ordering. Then the following are equivalent:

(S1) all well-ordered subsets of S and of S^* are countable;
(S2) there are no embeddings of ω_1 into S or S^*;
(S3) $\operatorname{cf}(A), \operatorname{ci}(A) \leqslant \omega$ for all ordered subsets A of S.

Call S **short** if any of the equivalent conditions (S1)–(S3) holds; cf. [16, 1.7(i)] and [31, pp. 88, 170–171]. If S is short, then so are S^* and every ordered subset of S. If S is countable, then it is short; more generally, if S is a union of countably many short ordered subsets, then S is short. If $S \to S'$ is a surjective increasing map between ordered sets and S is short, then so is S'; similarly with "decreasing" instead of "increasing". Shortness enters our story via the following observation:

Lemma 5.11. *Let (G, S, v) be a valued abelian group where S is short, and let (a_ρ) be a pc-sequence in (G, S, v). Then some final segment of (a_ρ) has countable length.*

Proof. Put $s_\rho := v(a_{\rho+1} - a_\rho)$, where $\rho + 1$ is the successor of ρ. After deleting an initial segment of (a_ρ) we arrange that the sequence (s_ρ) in S is strictly increasing. Then the image of the index set of (a_ρ) under the embedding $\rho \mapsto s_\rho$ of ordered sets is a well-ordered subset of S and hence countable. □

Lemma 5.12. *If the order topology of S is second countable, then S is short.*

Proof. Suppose $i\colon \omega_1 \to S$ is strictly increasing. With λ ranging over the limit ordinals $< \omega_1$ we then have uncountably many nonempty pairwise disjoint open intervals $\big(i(\lambda), i(\lambda+2)\big)$ in S, so S is not second countable. An embedding $\omega_1 \to S^*$ yields the same conclusion. $\square$

In particular, the real line (the ordered set of real numbers) is short. (In fact, by [23, Theorem 2], each Borel ordered set is short.) The following observation is due to Hausdorff [24, p. 133] and Urysohn [36].

Lemma 5.13. *Suppose S is short and T is an η_1-ordered set. Then any embedding of an ordered subset of S into T extends to an embedding $S \to T$. In particular, there exists an embedding $S \to T$.*

Proof. Let A be an ordered subset of S and $i\colon A \to T$ an embedding. Suppose $s \in S \setminus A$. Then $\operatorname{cf}(A^{<s}), \operatorname{ci}(A^{>s}) \leqslant \omega$, so we have $t \in T$ with $i(A^{<s}) < t < i(A^{>s})$. Thus i extends to an embedding $A \cup \{s\} \to T$ sending s to t. Zorn does the rest. $\square$

Corollary 5.14. *Every η_1-ordered set has cardinality $\geqslant \mathfrak{c}$. There is an η_1-ordered set of cardinality $\mathfrak{c}$.*

Proof. For the first claim, apply Lemma 5.13 to $S =$ the real line. The second claim follows from the first together with [ADH, B.9.6]. $\square$

Combining Lemma 5.13 and Corollary 5.14 we obtain:

Corollary 5.15 (Urysohn [35, 36])**.** *Every short ordered set has cardinality $\leqslant \mathfrak{c}$.*

For (ordered) Hahn products, see [ADH, 2.2, 2.4]. Shortness of $\mathbb{R}$ is at the root of the following result due to Esterle [19, Lemme 2.2 and the remark after it]:

Lemma 5.16. *If S is short, then so is the Hahn product $H[S, \mathbb{R}]$.*

From Lemma 5.16 and the Hahn Embedding Theorem [ADH, 2.4.19] we obtain a characterization of short ordered abelian groups:

Lemma 5.17. *For an ordered abelian group Γ, the following are equivalent:*

(i) Γ *is short;*
(ii) *the ordered set* $[\Gamma]$ *is short;*
(iii) Γ *embeds into* $H[S, \mathbb{R}]$ *for some short* S.

Corollary 5.18. *Let $\Delta \subseteq \Gamma$ be an extension of ordered abelian groups. Then*

$$\Gamma \text{ is short} \iff \Delta \text{ and } [\Gamma] \setminus [\Delta] \text{ are short.}$$

In particular, if $\operatorname{rank}_{\mathbb{Q}}(\Gamma/\Delta) \leqslant \aleph_0$*, then Γ is short iff Δ is short, and if Δ is convex, then Γ is short iff Δ and Γ/Δ are short.*

Proof. The direction $\Rightarrow$ is clear from Lemma 5.17. For the converse, note that if Δ and $[\Gamma] \setminus [\Delta]$ are short, then so is $[\Gamma]$, and hence Γ as well by Lemma 5.17. Next, use that if $\operatorname{rank}_{\mathbb{Q}}(\Gamma/\Delta) \leqslant \aleph_0$, then $[\Gamma] \setminus [\Delta]$ is countable by [ADH, 2.3.9]. For convex Δ, see [ADH, p. 102]. $\square$

Lemma 5.19. *Let K be an ordered field equipped with a convex valuation whose residue field is archimedean. Then K is short iff its value group Γ is short.*

Proof. Suppose Γ is short. Then $\mathbb{Q}\Gamma$ is also short, by the previous corollary, and the real closure of $\operatorname{res} K$ remains archimedean; hence to show that K is short we may replace K by its real closure to arrange that K is real closed. Using [ADH, 3.3.32, 3.3.42, 3.5.1, 3.5.12] we obtain an ordered field embedding of K into the ordered Hahn field $\mathbb{R}(\!(t^{\Gamma})\!)$. The underlying ordered additive group of $\mathbb{R}(\!(t^{\Gamma})\!)$ is isomorphic with the ordered Hahn product $H[t^{\Gamma},\mathbb{R}]$; see [ADH, p. 114]. Hence K is short by Lemma 5.17. Conversely, if K is short then so is its ordered subset $K^{>}$ and then also the image Γ of $K^{>}$ under the decreasing map $f \mapsto vf\colon K^{>} \to \Gamma$. $\square$

Hence if an ordered field is short, then so is its real closure. If K as in Lemma 5.19 is short and L is an ordered field extension of K with a convex valuation that makes it an immediate extension of K, then L is short. (NB: the ordered fraction field of a short ordered integral domain may fail to be short [14, 15].) The following is from [18, §2.10]:

Corollary 5.20. *$\mathbb{T}$ is short.*

Proof. We recall some features of the construction of $\mathbb{T}$ from [ADH, Appendix A]. We have the ordered subfield $\mathbb{T}_{\exp} = \bigcup_m E_m$ of $\mathbb{T}$ where $E_m = \mathbb{R}[[G_m]]$ for certain ordered subgroups G_m of $\mathbb{T}^{>}$, with $G_0 = x^{\mathbb{R}}$ and $G_{m+1} = G_m \exp(A_m)$ for some subgroup A_m of the additive group of E_m, with G_m a convex subgroup of G_{m+1}. An easy induction on m shows that each E_m is short, and thus $\mathbb{T}_{\exp}$ is short. Now $\mathbb{T} = \bigcup_n (\mathbb{T}_{\exp}){\downarrow^n}$ where $f \mapsto f{\downarrow^n}$ is the nth compositional iterate of the automorphism $f \mapsto f{\downarrow} = f \circ \log x$ of the ordered field $\mathbb{T}$, hence $\mathbb{T}$ is also short. $\square$

Question. Is every d-algebraic H-field extension of a short H-field also short?

Next two algebraic variants of Lemma 5.13, attributed to Esterle in [16, 2.37]:

Lemma 5.21. *Let Δ be a short ordered abelian group and Γ a divisible η_1-ordered abelian group. Then any embedding of an ordered subgroup of Δ into Γ extends to an embedding $\Delta \to \Gamma$.*

Proof. Let Δ_0 be an ordered subgroup of Δ and $i\colon \Delta_0 \to \Gamma$ an embedding. The divisible hull $\mathbb{Q}\Delta \subseteq \Gamma$ of Δ is short, by Corollary 5.18. Replace Δ_0, Δ by $\mathbb{Q}\Delta_0$, $\mathbb{Q}\Delta$ (and i accordingly) to arrange Δ_0, Δ to be divisible. Given $\delta \in \Delta \setminus \Delta_0$, Lemma 5.13 yields $\gamma \in \Gamma$ with $i(\Delta_0^{<\delta}) < \gamma < i(\Delta_0^{>\delta})$, and then i extends to an embedding of the ordered subgroup $\Delta_0 \oplus \mathbb{Q}\delta$ of Δ into Γ sending δ to γ. Zorn does the rest. $\square$

In the same way, taking real closures instead of divisible hulls in the proof:

Lemma 5.22. *Any embedding of an ordered subfield of a short ordered field K into a real closed η_1-ordered field L extends to an embedding $K \to L$.*

Combining Corollary 5.20 and the previous lemma yields:

Corollary 5.23. *The ordered field $\mathbb{T}$ embeds into each real closed η_1-ordered field.*

Lemma 7.8 below is an analogue of Lemma 5.22 for H-fields with small derivation.

Realizing $\mathbb{T}_{\log}$ as an analytic Hardy field. This uses the following variant of Lemma 5.1 for embedding $\upomega$-free immediate extensions:

Lemma 5.24. *Let K be an H-asymptotic field with short value group, L an $\upomega$-free immediate extension of K, and M a newtonian H-asymptotic field with asymptotic integration. Suppose M is countably spherically complete. Then any embedding $K \to M$ extends to an embedding $L \to M$.*

Proof. Let $\iota\colon K \to M$ be an embedding; we shall extend ι to an embedding $L \to M$. Now L is pre-d-valued by [ADH, 10.1.3], and as $\operatorname{dv}(L)$ is $\boldsymbol{\omega}$-free by [ADH, remark after 13.6.1] and $\Gamma_{\operatorname{dv}(L)} = \Gamma$ by [ADH, 10.3.2(i)], we can replace L by $\operatorname{dv}(L)$ to arrange that L is d-valued. Then L has an immediate d-algebraic newtonian $\boldsymbol{\omega}$-free extension by [ADH, remark after 14.0.1], which is then a newtonization of L by [29, Theorem 3.5]. Replacing L by this newtonization we also arrange that L is newtonian. Using Zorn we further arrange that ι does not extend to any embedding into M of any valued differential subfield of L properly containing K. Note that K is $\boldsymbol{\omega}$-free by [ADH, remark preceding 11.7.20]. Now M is d-valued by [ADH, 14.2.5], hence so is K by the universal property of $\operatorname{dv}(K)$. Likewise, K is newtonian, by the semiuniversal property of the newtonization of K (which exists by the same arguments as we used for L). Hence K is asymptotically d-algebraically maximal by [29, Theorem A]. It remains to show that $K = L$. Suppose towards a contradiction that $f \in L \setminus K$. Take a divergent pc-sequence (f_ρ) in K with pseudolimit f. By Lemma 5.11 we arrange that (f_ρ) has length ω, and hence we can take $g \in M$ with $\iota(f_\rho) \rightsquigarrow g$. As in the proof of Lemma 5.1 we then obtain an embedding $K\langle f\rangle \to M$ extending ι and sending f to g, a contradiction. $\square$

Set $\ell_0 := x \in \mathbb{T}$ and $\ell_{n+1} := \log \ell_n$. Recall that $\mathbb{T}_{\log} = \bigcup_n \mathbb{R}[[\mathfrak{L}_n]]$ where $\mathfrak{L}_n := \ell_0^{\mathbb{R}} \cdots \ell_n^{\mathbb{R}}$ is the subgroup of the monomial group G^{LE} of $\mathbb{T}$ generated by the real powers of the ℓ_i $(i = 0, \dots, n)$. The ordered subgroup $\mathfrak{L} := \bigcup_n \mathfrak{L}_n$ of G^{LE} is divisible and short, $\mathbb{T}_{\log}$ is real closed, $\boldsymbol{\omega}$-free and an immediate H-field extension of its H-subfield $\mathbb{R}(\mathfrak{L})$. Identify $\mathbb{R}(\mathfrak{L})$ with an H-subfield of the analytic Hardy field $\operatorname{Li}\big(\mathbb{R}(x)\big)$ in the obvious way. From [6, Corollary 3.2], Corollary 3.5, and Lemma 5.24, we obtain:

Corollary 5.25. *The H-field $\mathbb{T}_{\log}$ embeds over $\mathbb{R}(\mathfrak{L})$ into any maximal Hardy field. Likewise with* maximal analytic *and with* maximal smooth *in place of* maximal*.*

By Corollary 5.10, every embedding $i\colon \mathbb{T}_{\log} \to M$ as in Corollary 5.25 extends to an embedding of the completion of $\mathbb{T}_{\log}$ into M. In the next section we show that i even extends to an embedding of *every* immediate H-field extension of $\mathbb{T}_{\log}$ into M.

Implications between "short", "countable cofinality", and "bounded". The first two notions are defined for ordered sets, and "bounded" is defined for subsets of $\mathcal{C}$. It is clear that for ordered sets,

$$\text{short} \implies \text{countable cofinality},$$

and that for Hausdorff fields,

$$\text{countable cofinality} \implies \text{bounded}.$$

These implications cannot be reversed for analytic Hardy fields: Let H be a maximal analytic Hardy field. Corollary 4.11 gives a sequence (h_λ) in H, indexed by the ordinals $\lambda \leqslant \omega_1$, such that all h_λ are transexponential and $*h_\lambda < *h_\mu$ for all $\lambda < \mu \leqslant \omega_1$. It follows that $\mathbb{R}\langle h_\lambda : \lambda < \omega_1\rangle$ is a bounded analytic Hardy field (bounded by h_{ω_1}) with cofinality ω_1, and so $\mathbb{R}\langle h_\lambda : \lambda \leqslant \omega_1\rangle$ is an analytic Hardy field of cofinality ω that is not short. (We thank Philip Ehrlich and Elliot Kaplan for a useful email discussion on this topic.)

6. Embeddings of Immediate Extensions

The goal of this section is to prove the following theorem, which partly generalizes Lemma 5.24 beyond the ω-free setting:

Theorem 6.1. *Let K be a short pre-H-field with archimedean residue field, and suppose K is ω-free or not λ-free. Let $\widehat{K}$ be an immediate pre-H-field extension of K and let M be a countably spherically complete closed H-field. Then every embedding $K \to M$ extends to an embedding $\widehat{K} \to M$.*

Using also [6, Corollary 3.2] and Corollary 3.5, this yields:

Corollary 6.2. *Let K be a short Hardy field which is ω-free or not λ-free, and let M be a maximal Hardy field extending K. Then every immediate Hardy field extension of K embeds into M over K. Likewise with "maximal analytic" as well as with "maximal smooth" in place of "maximal".*

The main steps towards the proof of Theorem 6.1 are Propositions 6.3 and 6.10 below. This requires us to revisit the topic of pre-$\Lambda\Omega$-fields once again.

We note also that by [2] and [ADH,10.5.8], each pre-H-field has an immediate strict pre-H-field extension that is spherically complete.

Immediate pairs of pre-$\Lambda\Omega$-fields. Here we generalize [ADH, 16.4.1] to certain pairs of pre-$\Lambda\Omega$-fields. A **pre-$\Lambda\Omega$-pair** is a pair $(\boldsymbol{K}, \widehat{\boldsymbol{K}})$ of pre-$\Lambda\Omega$-fields with $\boldsymbol{K} \subseteq \widehat{\boldsymbol{K}}$. Let $(\boldsymbol{K}, \widehat{\boldsymbol{K}})$ be a pre-$\Lambda\Omega$-pair, with $\boldsymbol{K} = (K, \dots)$ and $\widehat{\boldsymbol{K}} = (\widehat{K}, \dots)$. We call $(\boldsymbol{K}, \widehat{\boldsymbol{K}})$ a **$\Lambda\Omega$-pair** if both $\boldsymbol{K}$, $\widehat{\boldsymbol{K}}$ are $\Lambda\Omega$-fields, and we say that $(\boldsymbol{K}, \widehat{\boldsymbol{K}})$ is **immediate** if the valued field extension $K \subseteq \widehat{K}$ is immediate. We also call $(\boldsymbol{K}, \widehat{\boldsymbol{K}})$ **ω-free** if both K, $\widehat{K}$ are ω-free, and similarly for other properties of pre-H-fields. A pre-$\Lambda\Omega$-pair $(\boldsymbol{K}^*, \widehat{\boldsymbol{K}^*})$ **extends** $(\boldsymbol{K}, \widehat{\boldsymbol{K}})$ if $\boldsymbol{K} \subseteq \boldsymbol{K}^*$ and $\widehat{\boldsymbol{K}} \subseteq \widehat{\boldsymbol{K}^*}$.

Proposition 6.3. *Suppose $(\boldsymbol{K}, \widehat{\boldsymbol{K}})$ is an immediate pre-$\Lambda\Omega$-pair such that if K is ω-free (λ-free, respectively), then so is $\widehat{K}$. Then $(\boldsymbol{K}, \widehat{\boldsymbol{K}})$ extends to an immediate ω-free $\Lambda\Omega$-pair $(\boldsymbol{K}^*, \widehat{\boldsymbol{K}^*})$ such that $\operatorname{res} \boldsymbol{K}^*$ is algebraic over $\operatorname{res} \boldsymbol{K}$ and any embedding of $\boldsymbol{K}$ into a Schwarz closed $\Lambda\Omega$-field $\boldsymbol{L}$ extends to an embedding $\boldsymbol{K}^* \to \boldsymbol{L}$.*

$$\begin{tikzcd}
\boldsymbol{K}^* \arrow[r, "\subseteq", "\text{immediate}"'] & \widehat{\boldsymbol{K}^*} \\
\boldsymbol{K} \arrow[u, "\subseteq"'] \arrow[r, "\subseteq", "\text{immediate}"'] & \widehat{\boldsymbol{K}} \arrow[u, "\subseteq"]
\end{tikzcd}$$

Moreover, if $\boldsymbol{K}$ is short and $\operatorname{res} \boldsymbol{K}$ is archimedean, then we can choose such a pair $(\boldsymbol{K}^*, \widehat{\boldsymbol{K}^*})$ where $\boldsymbol{K}^*$ is also short.

As with Proposition 5.3 we adapt the proof of [ADH, 16.4.1]. We assume $(\boldsymbol{K}, \widehat{\boldsymbol{K}})$ is an immediate pre-$\Lambda\Omega$-pair and $\boldsymbol{K} = (K, I, \Lambda, \Omega)$, $\widehat{\boldsymbol{K}} = (\widehat{K}, \widehat{I}, \widehat{\Lambda}, \widehat{\Omega})$. We identify $H(K)$ in the usual way with an H-subfield of $H(\widehat{K})$, and for ungrounded K we tacitly use that the sequences (λ_ρ), (ω_ρ) in K also serve for $\widehat{K}$.

Lemma 6.4. *Suppose K is grounded, or there exists $b \asymp 1$ in K such that $v(b')$ is a gap in K. Then $(\boldsymbol{K}, \widehat{\boldsymbol{K}})$ extends to an immediate ω-free $\Lambda\Omega$-pair $(\boldsymbol{K}^*, \widehat{\boldsymbol{K}^*})$ such that $\operatorname{res} \boldsymbol{K}^* = \operatorname{res} \boldsymbol{K}$ and any embedding of $\boldsymbol{K}$ into a $\Lambda\Omega$-field $\boldsymbol{L}$ closed under logarithms extends to an embedding $\boldsymbol{K}^* \to \boldsymbol{L}$.*

Proof. Note that if K is grounded, then so is $\widehat{K}$, and any gap in K remains a gap in $\widehat{K}$. Put $E := H(K)$ and $F := H(\widehat{K})$, and note that the H-field extension $E \subseteq F$ is immediate by [ADH, 10.3.2 and remark preceding it]. Next take $e \in E$ with $e \succ 1$ and $v(e^\dagger) = \max \Psi_E = \max \Psi_F$. We now construct $K^* := E_\omega$ and $\widehat{K}^* := F_\omega$ as in [ADH, 11.7] with E, e and F, e in the role of F, f there, so $E_\omega = \bigcup_n E_n$, $F_\omega = \bigcup_n F_n$, $E_0 = E$, $F_0 = F$. We take care to do that in such a way that by induction on n using [ADH, 10.2.3 and its proof] we have for all n an immediate extension $E_n \subseteq F_n$ of grounded H-fields with a distinguished element $e_n \in E_n^\times$ such that $e_0 = e$, $e_n \succ 1$, $v(e_n^\dagger) = \max \Psi_{E_n} = \max \Psi_{F_n}$, and

$$E_{n+1} = E_n(e_{n+1}), \quad F_{n+1} = F_n(e_{n+1}), \quad e'_{n+1} = e_n^\dagger.$$

This yields an immediate extension $E_\omega \subseteq F_\omega$ of ω-free H-fields. Expanding $K^*, \widehat{K}^*$ uniquely to $\Lambda\Omega$-fields gives a pair $(\boldsymbol{K}^*, \widehat{\boldsymbol{K}}^*)$ with the required properties. □

Lemma 6.5. *Suppose K has gap β and $v(b') \neq \beta$ for all $b \asymp 1$ in K. Then $(\boldsymbol{K}, \widehat{\boldsymbol{K}})$ extends to an immediate grounded $\Lambda\Omega$-pair $(\boldsymbol{K}_1, \widehat{\boldsymbol{K}_1})$ such that $\operatorname{res} \boldsymbol{K}_1 = \operatorname{res} \boldsymbol{K}$ and any embedding of $\boldsymbol{K}$ into a $\Lambda\Omega$-field $\boldsymbol{L}$ closed under integration extends to an embedding $\boldsymbol{K}_1 \to \boldsymbol{L}$.*

Proof. Note that β is a gap in $\widehat{K}$, and $v(b') \neq \beta$ for all $b \asymp 1$ in $\widehat{K}$. By [ADH, 10.3.2 and remark preceding it], $H(K)$ is an immediate extension of K, and $H(\widehat{K})$ of $\widehat{K}$, so $H(\widehat{K})$ is an immediate extension of $H(K)$.

Take $s \in K$ with $vs = \beta$ and follow the proof of [ADH, 16.4.3]. Suppose $s \notin I$. Then also $s \notin \widehat{I}$. Take $\widehat{K}_1 = H(\widehat{K})(y)$ as in Case 1 of that proof applied to $\widehat{\boldsymbol{K}}$ in place of $\boldsymbol{K}$. We have the H-subfield $K_1 := H(K)(y)$ of $\widehat{K}_1$, and $\widehat{K}_1$ is an immediate extension of K_1 by [ADH, 10.2.2 and its proof]. Expanding K_1, $\widehat{K}_1$ uniquely to pre-$\Lambda\Omega$-fields gives a pair $(\boldsymbol{K}_1, \widehat{\boldsymbol{K}_1})$ with the required property. If $s \in I$, proceed as before, but following instead Case 2 of the proof of [ADH, 16.4.3] and with $H(K)$ and $H(\widehat{K})$ instead of K and $\widehat{K}$, using [ADH, 10.2.1 and its proof]. □

Corollary 6.6. *Suppose K does not have asymptotic integration. Then $(\boldsymbol{K}, \widehat{\boldsymbol{K}})$ extends to an immediate ω-free $\Lambda\Omega$-pair $(\boldsymbol{K}^*, \widehat{\boldsymbol{K}^*})$ such that $\operatorname{res} \boldsymbol{K}^* = \operatorname{res} \boldsymbol{K}$, and any embedding of $\boldsymbol{K}$ into a $\Lambda\Omega$-field $\boldsymbol{L}$ closed under integration extends to an embedding $\boldsymbol{K}^* \to \boldsymbol{L}$.*

In the next three lemmas we treat the case where K has asymptotic integration. For the first we adapt the proof of [ADH, 16.4.5] and use parts of it:

Lemma 6.7. *Assume K has asymptotic integration and is not λ-free. Then $(\boldsymbol{K}, \widehat{\boldsymbol{K}})$ extends to an immediate ω-free $\Lambda\Omega$-pair $(\boldsymbol{K}^*, \widehat{\boldsymbol{K}^*})$ such that $\operatorname{res} \boldsymbol{K}^* = (\operatorname{res} \boldsymbol{K})^{\mathrm{rc}}$, and every embedding of $\boldsymbol{K}$ into a Liouville closed $\Lambda\Omega$-field $\boldsymbol{L}$ extends to an embedding $\boldsymbol{K}^* \to \boldsymbol{L}$.*

Proof. By Corollary 6.6 it is enough to show that $(\boldsymbol{K}, \widehat{\boldsymbol{K}})$ extends to an immediate $\Lambda\Omega$-pair $(\boldsymbol{K}_1, \widehat{\boldsymbol{K}_1})$ with a gap such that $\operatorname{res} \boldsymbol{K}_1 = (\operatorname{res} \boldsymbol{K})^{\mathrm{rc}}$ and every embedding of $\boldsymbol{K}$ into a Liouville closed $\Lambda\Omega$-field $\boldsymbol{L}$ extends to an embedding $\boldsymbol{K}_1 \to \boldsymbol{L}$. Let

$$E := H(K)^{\mathrm{rc}} \subseteq F := H(\widehat{K})^{\mathrm{rc}}.$$

Then $\Gamma_E = \mathbb{Q}\Gamma$, and F is an immediate extension H-field extension of E. We distinguish two cases:

Case 1: *E has a gap.* Take $s \in E^\times$ and $n \geqslant 1$ such that vs is a gap in E and $s^n \in K$. Then E has exactly two $\Lambda\Omega$-cuts $(I_1, \Lambda_1, \Omega_1)$, $(I_2, \Lambda_2, \Omega_2)$, where $I_1 = \{y \in E : y \prec s\}$, $I_2 = \{y \in E : y \preccurlyeq s\}$, and F has exactly two $\Lambda\Omega$-cuts $(\widehat{I}_1, \widehat{\Lambda}_1, \widehat{\Omega}_1)$ and $(\widehat{I}_2, \widehat{\Lambda}_2, \widehat{\Omega}_2)$, with $\widehat{I}_1 = \{y \in F : y \prec s\}$, $\widehat{I}_2 = \{y \in F : y \preccurlyeq s\}$ (so $I_j = \widehat{I}_j \cap E$ for $j = 1, 2$). Take $\boldsymbol{K}_1$ as in Case 1 of the proof of [ADH, 16.4.5]: if $-s^\dagger \in \Lambda$, then $\boldsymbol{K}_1 := (E, I_1, \Lambda_1, \Omega_1)$, and if $-s^\dagger \notin \Lambda$, then $\boldsymbol{K}_1 := (E, I_2, \Lambda_2, \Omega_2)$. Similarly, if $-s^\dagger \in \Lambda$, then $\widehat{\boldsymbol{K}_1} := (F, \widehat{I}_1, \widehat{\Lambda}_1, \widehat{\Omega}_1)$, and if $-s^\dagger \notin \Lambda$, then $\widehat{\boldsymbol{K}_1} := (F, \widehat{I}_2, \widehat{\Lambda}_2, \widehat{\Omega}_2)$. Then $(\boldsymbol{K}_1, \widehat{\boldsymbol{K}_1})$ is an immediate $\Lambda\Omega$-pair with the desired property.

Case 2: *E has no gap.* Then E, F have asymptotic integration, and the sequence (λ_ρ) for K also serves for E and for F. Take $\lambda \in K$ such that $\lambda_\rho \rightsquigarrow \lambda$. Then $-\lambda$ creates a gap over E and over F by [ADH, 11.5.14]. Take an element $f \neq 0$ in some Liouville closed H-field extension of F such that $f^\dagger = -\lambda$. Then $F(f)$ is an H-field and $E(f)$ is an H-subfield of $F(f)$ with $\operatorname{res} E(f) = \operatorname{res} E = \operatorname{res} F = \operatorname{res} F(f)$. Moreover, vf is a gap in $F(f)$ and in $E(f)$, and $F(f)$ is an immediate extension of $E(f)$, by the remark after [ADH 11.5.14] and the uniqueness part of [ADH, 10.4.5]. Now $E(f)$ has exactly two $\Lambda\Omega$-cuts $(I_1, \Lambda_1, \Omega_1)$ and $(I_2, \Lambda_2, \Omega_2)$, where

$$I_1 = \big\{y \in E(f) : y \prec f\big\}, \qquad I_2 = \big\{y \in E(f) : y \preccurlyeq f\big\},$$

and $F(f)$ has exactly two $\Lambda\Omega$-cuts $(\widehat{I}_1, \widehat{\Lambda}_1, \widehat{\Omega}_1)$ and $(\widehat{I}_2, \widehat{\Lambda}_2, \widehat{\Omega}_2)$, with

$$\widehat{I}_1 = \big\{y \in F(f) : y \prec f\big\}, \qquad \widehat{I}_2 = \big\{y \in F(f) : y \preccurlyeq f\big\}.$$

Therefore $I_j = \widehat{I}_j \cap E(f)$ for $j = 1, 2$. We set $\boldsymbol{K}_1 := \big(E(f), I_1, \Lambda_1, \Omega_1\big)$ and $\widehat{\boldsymbol{K}_1} := \big(F(f), \widehat{I}_1, \widehat{\Lambda}_1, \widehat{\Omega}_1\big)$ if $\lambda \in \Lambda$, and $\boldsymbol{K}_1 := \big(E(f), I_2, \Lambda_2, \Omega_2\big)$, $\widehat{\boldsymbol{K}_1} := \big(F(f), \widehat{I}_2, \widehat{\Lambda}_2, \widehat{\Omega}_2\big)$ if $\lambda \notin \Lambda$. Then $\boldsymbol{K}_1 \subseteq \widehat{\boldsymbol{K}_1}$, and the immediate $\Lambda\Omega$-pair $(\boldsymbol{K}_1, \widehat{\boldsymbol{K}_1})$ is as required. $\square$

Lemma 6.8. *Suppose K is not ω-free and $\widehat{K}$ is λ-free. Then $(\boldsymbol{K}, \widehat{\boldsymbol{K}})$ extends to an immediate ω-free $\Lambda\Omega$-pair $(\boldsymbol{K}^*, \widehat{\boldsymbol{K}}^*)$ such that $\operatorname{res} \boldsymbol{K}^*$ is algebraic over $\operatorname{res} \boldsymbol{K}$ and any embedding $\boldsymbol{K} \to \boldsymbol{L}$ into a Schwarz closed $\Lambda\Omega$-field $\boldsymbol{L}$ extends to an embedding $\boldsymbol{K}^* \to \boldsymbol{L}$.*

Proof. We adapt and use the proof of [ADH, 16.4.6]. Take $\omega \in K$ with $\omega_\rho \rightsquigarrow \omega$. Then $\omega\big(\Lambda(K)\big)^\downarrow < \omega < \sigma\big(\Gamma(K)\big)^\uparrow$ and either $\Omega = \omega(K)^\downarrow$ or $\Omega = K \setminus \sigma\big(\Gamma(K)\big)^\uparrow$. Likewise with $\widehat{K}$ in place of K. Also $\omega \notin \omega(K)^\downarrow$, $\omega \notin \omega(\widehat{K})^\downarrow$. There are two cases:

Case 1: $\Omega = \omega(K)^\downarrow$. Then $\omega \notin \widehat{\Omega}$ and so $\widehat{\Omega} = \omega(\widehat{K})^\downarrow$. Take a pre-$H$-field extension $\widehat{K}_\gamma$ of $\widehat{K}$ as in Case 1 of the proof of [ADH, 16.4.6] with $\widehat{K}$ in place of K. Then $\operatorname{res} \widehat{K}_\gamma = \operatorname{res} \widehat{K} = \operatorname{res} K$. Put $K_\gamma := K\langle\gamma\rangle$, a pre-$H$-subfield of $\widehat{K}_\gamma$ with $\operatorname{res} K_\gamma = \operatorname{res} K$. Then $v\gamma$ is a gap in K_γ and in $\widehat{K}_\gamma$, so by [ADH, 13.7.6], $\widehat{K}_\gamma$ is an immediate extension of K_γ. Expanding K_γ to a pre-$\Lambda\Omega$-field $\boldsymbol{K}_\gamma$ as in Case 1 of the proof of [ADH, 16.4.6], and similarly expanding $\widehat{K}_\gamma$ to a pre-$\Lambda\Omega$-field $\widehat{\boldsymbol{K}_\gamma}$, we thus obtain the immediate pre-$\Lambda\Omega$-pair $(\boldsymbol{K}_\gamma, \widehat{\boldsymbol{K}_\gamma})$ extending $(\boldsymbol{K}, \widehat{\boldsymbol{K}})$. Take an immediate ω-free $\Lambda\Omega$-pair $(\boldsymbol{K}^*, \widehat{\boldsymbol{K}}^*)$ extending $(\boldsymbol{K}_\gamma, \widehat{\boldsymbol{K}_\gamma})$ as in Corollary 6.6 applied to $(\boldsymbol{K}_\gamma, \widehat{\boldsymbol{K}_\gamma})$ in place of $(\boldsymbol{K}, \widehat{\boldsymbol{K}})$. Then $(\boldsymbol{K}^*, \widehat{\boldsymbol{K}}^*)$ has the required property.

Case 2: $\Omega = K \setminus \sigma\big(\Gamma(K)\big)^\uparrow$. Then $\omega \in \Omega \subseteq \widehat{\Omega}$, so $\widehat{\Omega} = \widehat{K} \setminus \sigma\big(\Gamma(\widehat{K})\big)^\uparrow$. As in the proof of [ADH, 16.4.6] we obtain an immediate pre-H-field extension $\widehat{K}_\lambda := \widehat{K}(\lambda)$ of $\widehat{K}$ with $\lambda_\rho \rightsquigarrow \lambda$ and $\omega(\lambda) = \omega$. Put $K_\lambda := K(\lambda)$, an immediate pre-H-field extension of K. Expand K_λ to a pre-$\Lambda\Omega$-field $\boldsymbol{K}_\lambda$ as in Case 2 of the proof of [ADH, 16.4.6], and similarly expand $\widehat{K}_\lambda$ to a pre-$\Lambda\Omega$-field $\widehat{\boldsymbol{K}}_\lambda$. Then $\boldsymbol{K}_\lambda \supseteq \boldsymbol{K}$ and $\widehat{\boldsymbol{K}}_\lambda \supseteq \widehat{\boldsymbol{K}}$, and from $\lambda \notin \Lambda(K_\lambda)^{\downarrow}$ and $\lambda \in \big(\widehat{K}_\lambda \setminus \Delta(\widehat{K}_\lambda)^{\uparrow}\big) \cap K_\lambda$ we obtain $\widehat{\boldsymbol{K}}_\lambda \supseteq \boldsymbol{K}_\lambda$. Thus $(\boldsymbol{K}_\lambda, \widehat{\boldsymbol{K}}_\lambda)$ is an immediate pre-$\Lambda\Omega$-pair and extends $(\boldsymbol{K}, \widehat{\boldsymbol{K}})$. Take an immediate ω-free $\Lambda\Omega$-pair $(\boldsymbol{K}^*, \widehat{\boldsymbol{K}}^*)$ extending $(\boldsymbol{K}_\lambda, \widehat{\boldsymbol{K}}_\lambda)$ obtained from Lemma 6.7 applied to $(\boldsymbol{K}_\lambda, \widehat{\boldsymbol{K}}_\lambda)$ in place of $(\boldsymbol{K}, \widehat{\boldsymbol{K}})$. Then $(\boldsymbol{K}^*, \widehat{\boldsymbol{K}}^*)$ has the required property. ☐

Lemma 6.9. *Suppose $\widehat{K}$ is ω-free. Then $(\boldsymbol{K}, \widehat{\boldsymbol{K}})$ extends to an immediate ω-free $\Lambda\Omega$-pair $(\boldsymbol{K}^*, \widehat{\boldsymbol{K}}^*)$ such that any embedding of $\boldsymbol{K}$ into a $\Lambda\Omega$-field $\boldsymbol{L}$ extends to an embedding of $\boldsymbol{K}^*$ into $\boldsymbol{L}$.*

Proof. As $\widehat{K}$ is ω-free, so is K. By [ADH, 13.6.1], $H(K)$ is ω-free, and by [ADH, 10.3.2 and remark (a) before it], $H(K)$ is an immediate extension of K, and likewise with $\widehat{K}$ in place of K. Let $\boldsymbol{K}^*$, $\widehat{\boldsymbol{K}}^*$ be the unique expansions of $H(K)$, $H(\widehat{K})$, respectively, to $\Lambda\Omega$-fields. Then $(\boldsymbol{K}^*, \widehat{\boldsymbol{K}}^*)$ has the required properties, by the proof of [ADH, 16.4.7]. ☐

The first claim of Proposition 6.3 now follows. As to the shortness part, one checks that $\operatorname{rank}_{\mathbb{Q}}(\Gamma_{K^*}/\Gamma_K) \leqslant \aleph_0$ for $(\boldsymbol{K}^*, \widehat{\boldsymbol{K}}^*)$ as constructed above, hence if K is short and $\operatorname{res} K$ is archimedean, then K^* is short by Corollary 5.18 and Lemma 5.19. ☐

Immediate extensions and $\Lambda\Omega$-cuts. Let $K \subseteq \widehat{K}$ be an extension of pre-H-fields. Given a $\Lambda\Omega$-cut $(\widehat{I}, \widehat{\Lambda}, \widehat{\Omega})$ in $\widehat{K}$, we obtain the $\Lambda\Omega$-cut

$$(\widehat{I}, \widehat{\Lambda}, \widehat{\Omega}) \cap K := (\widehat{I} \cap K, \widehat{\Lambda} \cap K, \widehat{\Omega} \cap K)$$

in K. Recall from [ADH, remark before 16.3.19] that a pre-H-field has at least one and at most two $\Lambda\Omega$-cuts. In the rest of this subsection we assume that $K \subseteq \widehat{K}$ is immediate and (I, Λ, Ω) is a $\Lambda\Omega$-cut in K, and we ask when there is a $\Lambda\Omega$-cut $(\widehat{I}, \widehat{\Lambda}, \widehat{\Omega})$ in $\widehat{K}$ such that $(I, \Lambda, \Omega) = (\widehat{I}, \widehat{\Lambda}, \widehat{\Omega}) \cap K$.

Proposition 6.10. *The following are equivalent:*

(i) *There is a $\Lambda\Omega$-cut $(\widehat{I}, \widehat{\Lambda}, \widehat{\Omega})$ in $\widehat{K}$ with $(I, \Lambda, \Omega) = (\widehat{I}, \widehat{\Lambda}, \widehat{\Omega}) \cap K$;*

(ii) *K is not λ-free, or K is ω-free, or $\widehat{K}$ is λ-free, or $\Omega \neq \omega(K)^{\downarrow}$.*

This is a consequence of Lemmas 6.11–6.15 below, which also address the uniqueness of the $\Lambda\Omega$-cut in $\widehat{K}$ in part (i) of the proposition. For the next two labeled displays, let K be ungrounded. Then by [ADH, 11.8.14] we have

$$\text{(6.1)}\quad \Lambda(K)^{\downarrow} \;=\; \Lambda(\widehat{K})^{\downarrow} \cap K, \qquad \Delta(K)^{\uparrow} \;=\; \Delta(\widehat{K})^{\uparrow} \cap K, \quad \Gamma(K)^{\uparrow} \;=\; \Gamma(\widehat{K})^{\uparrow} \cap K,$$

and by [ADH, 11.8.14, remark before 11.8.21, and 11.8.29]:

$$\text{(6.2)}\quad \omega\big(\Lambda(\widehat{K})\big)^{\downarrow} \cap K \;=\; \omega\big(\Lambda(K)\big)^{\downarrow}, \qquad \big(\widehat{K} \setminus \sigma\big(\Gamma(\widehat{K})\big)^{\uparrow}\big) \cap K \;=\; K \setminus \sigma\big(\Gamma(K)\big)^{\uparrow}.$$

By [ADH, 11.8.2] we also have $\mathrm{I}(K) = \mathrm{I}(\widehat{K}) \cap K$ if K has asymptotic integration.

Lemma 6.11. *Suppose K does not have asymptotic integration or $\widehat{K}$ is ω-free. Then there is a unique $\Lambda\Omega$-cut $(\widehat{I}, \widehat{\Lambda}, \widehat{\Omega})$ in $\widehat{K}$ with $(I, \Lambda, \Omega) = (\widehat{I}, \widehat{\Lambda}, \widehat{\Omega}) \cap K$.*

Proof. Note that K has asymptotic integration iff $\widehat{K}$ has, and if K has a gap β and $v(a') \neq \beta$ for all $a \asymp 1$ in K, then β remains a gap in $\widehat{K}$ and $v(b') \neq \beta$ for all $b \asymp 1$ in $\widehat{K}$. If $\widehat{K}$ is ω-free, then so is K. Now use [ADH, 16.3.11–16.3.14]. ☐

Lemma 6.12. *Suppose K is $\upomega$-free, but $\widehat{K}$ is not. Then there are exactly two $\Lambda\Omega$-cuts $(\widehat{I},\widehat{\Lambda},\widehat{\Omega})$ in $\widehat{K}$ with $(I,\Lambda,\Omega)=(\widehat{I},\widehat{\Lambda},\widehat{\Omega})\cap K$.*

Proof. By [ADH, 16.3.14], (I,Λ,Ω) is the unique $\Lambda\Omega$-cut in K, so $(I,\Lambda,\Omega)=(\widehat{I},\widehat{\Lambda},\widehat{\Omega})\cap K$ for every $\Lambda\Omega$-cut $(\widehat{I},\widehat{\Lambda},\widehat{\Omega})$ in $\widehat{K}$. Moreover, $\widehat{K}$ has exactly two $\Lambda\Omega$-cuts, by [ADH, 16.3.16] if $\widehat{K}$ is $\uplambda$-free, and by [ADH, 16.3.17, 16.3.18] if not. $\square$

In particular, if K has no asymptotic integration or K is $\upomega$-free then we have a $\Lambda\Omega$-cut $(\widehat{I},\widehat{\Lambda},\widehat{\Omega})$ in $\widehat{K}$ with $(I,\Lambda,\Omega)=(\widehat{I},\widehat{\Lambda},\widehat{\Omega})\cap K$. The next lemmas deal with the case where K has asymptotic integration and K is not $\upomega$-free.

Lemma 6.13. *Suppose K has asymptotic integration and is not $\uplambda$-free. Then there is exactly one $\Lambda\Omega$-cut $(\widehat{I},\widehat{\Lambda},\widehat{\Omega})$ in $\widehat{K}$ with $(I,\Lambda,\Omega)=(\widehat{I},\widehat{\Lambda},\widehat{\Omega})\cap K$.*

Proof. Suppose first that 2Ψ has no supremum in Γ. Then by [ADH, 16.3.17] there are exactly two $\Lambda\Omega$-cuts (I_1,Λ_1,Ω_1), (I_2,Λ_2,Ω_2) in K, with $\Lambda_1=\Lambda(K)^{\downarrow}$, $\Lambda_2=K\setminus\Delta(K)^{\uparrow}$, and $\Lambda_1\neq\Lambda_2$. Similarly there are exactly two $\Lambda\Omega$-cuts $(\widehat{I}_1,\widehat{\Lambda}_1,\widehat{\Omega}_1)$, $(\widehat{I}_2,\widehat{\Lambda}_2,\widehat{\Omega}_2)$ in $\widehat{K}$, with $\widehat{\Lambda}_1=\Lambda(\widehat{K})^{\downarrow}$, $\widehat{\Lambda}_2=\widehat{K}\setminus\Delta(\widehat{K})^{\uparrow}$. Now use that by (6.1) we have $\Lambda(K)^{\downarrow}=\Lambda(\widehat{K})^{\downarrow}\cap K$ and $K\setminus\Delta(K)^{\uparrow}=\big(\widehat{K}\setminus\Delta(\widehat{K})^{\uparrow}\big)\cap K$. The case where 2Ψ has a supremum in Γ is similar, using [ADH, 16.3.18] instead of [ADH, 16.3.17]. $\square$

Lemma 6.14. *Suppose K is not $\upomega$-free and $\widehat{K}$ is $\uplambda$-free. Then there is exactly one $\Lambda\Omega$-cut $(\widehat{I},\widehat{\Lambda},\widehat{\Omega})$ in $\widehat{K}$ such that $(I,\Lambda,\Omega)=(\widehat{I},\widehat{\Lambda},\widehat{\Omega})\cap K$.*

Proof. By [ADH, 16.3.16], K being $\uplambda$-free, but not $\upomega$-free, it has exactly two $\Lambda\Omega$-cuts, namely $\big(\mathrm{I}(K),\Lambda(K)^{\downarrow},\omega\big(\Lambda(K)\big)^{\downarrow}\big)$ and $\big(\mathrm{I}(K),\Lambda(K)^{\downarrow},K\setminus\sigma\big(\Gamma(K)\big)^{\uparrow}\big)$, and similarly with $\widehat{K}$ in place of K. Now use (6.1) and (6.2). $\square$

Lemma 6.15. *Suppose K is $\uplambda$-free, but not $\upomega$-free, and $\widehat{K}$ is not $\uplambda$-free. Then there is a $\Lambda\Omega$-cut $(\widehat{I},\widehat{\Lambda},\widehat{\Omega})$ in $\widehat{K}$ such that $(I,\Lambda,\Omega)=(\widehat{I},\widehat{\Lambda},\widehat{\Omega})\cap K$ iff $\Omega\neq\omega(K)^{\downarrow}$, and in this case there are exactly two such $\Lambda\Omega$-cuts in $\widehat{K}$.*

Proof. By [ADH, 16.3.16] K has exactly two $\Lambda\Omega$-cuts (I_1,Λ_1,Ω_1), (I_2,Λ_2,Ω_2) where $I_1=I_2=\mathrm{I}(K)$, $\Lambda_1=\Lambda_2=\Lambda(K)^{\downarrow}$, $\Omega_1=K\setminus\sigma\big(\Gamma(K)\big)^{\uparrow}\neq\Omega_2=\omega(K)^{\downarrow}$. Now K is $\uplambda$-free, so 2Ψ has no supremum in Γ by [ADH, 9.2.17, 11.6.8], hence by [ADH, 16.3.17], $\widehat{K}$ has exactly two $\Lambda\Omega$-cuts $(\widehat{I}_1,\widehat{\Lambda}_1,\widehat{\Omega}_1)$, $(\widehat{I}_2,\widehat{\Lambda}_2,\widehat{\Omega}_2)$, where

$$\widehat{I}_1=\widehat{I}_2=\mathrm{I}(\widehat{K}),\quad \widehat{\Lambda}_1=\Lambda(\widehat{K})^{\downarrow},\quad \widehat{\Lambda}_2=\widehat{K}\setminus\Delta(\widehat{K})^{\uparrow},\quad \widehat{\Omega}_1=\widehat{\Omega}_2=\widehat{K}\setminus\sigma\big(\Gamma(\widehat{K})\big)^{\uparrow}.$$

Thus $(\widehat{I}_j,\widehat{\Lambda}_j,\widehat{\Omega}_j)\cap K=(I_1,\Lambda_1,\Omega_1)$ for $j=1,2$ by (6.2). This yields the lemma. $\square$

Proof of Theorem 6.1. Let K, $\widehat{K}$, M be as in the statement of the theorem, and let $i\colon K\to M$ be an embedding. If $\widehat{K}$ is $\upomega$-free, then Lemma 5.24 and [ADH, 10.5.8] give an extension of i to an embedding $\widehat{K}\to M$ as required.

In the rest of the proof we therefore assume that $\widehat{K}$ is not $\upomega$-free. If $\widehat{K}$ is $\uplambda$-free, then, taking $\upomega\in\widehat{K}$ with $\upomega_\rho\rightsquigarrow\upomega$, [ADH, 11.7.13] yields an immediate pre-H-field extension $\widehat{K}_{\uplambda}:=\widehat{K}(\uplambda)$ of $\widehat{K}$ with $\uplambda_\rho\rightsquigarrow\uplambda$ and $\omega(\uplambda)=\upomega$, so that replacing $\widehat{K}$ by $\widehat{K}_{\uplambda}$ we arrange that $\widehat{K}$ is not even $\uplambda$-free.

Suppose K is not $\uplambda$-free. Let $\boldsymbol{M}$ be the unique expansion of M to a $\Lambda\Omega$-field, and expand K to a pre-$\Lambda\Omega$-field $\boldsymbol{K}$ such that i is an embedding $\boldsymbol{K}\to\boldsymbol{M}$ of pre-$\Lambda\Omega$-fields. Proposition 6.10 yields an expansion of $\widehat{K}$ to a pre-$\Lambda\Omega$-field $\widehat{\boldsymbol{K}}$

such that $\boldsymbol{K} \subseteq \widehat{\boldsymbol{K}}$, and then Proposition 6.3 gives an immediate $\upomega$-free short $\Lambda\Omega$-pair $(\boldsymbol{K}^*, \widehat{\boldsymbol{K}}^*)$ extending $(\boldsymbol{K}, \widehat{\boldsymbol{K}})$ with $\operatorname{res} \boldsymbol{K}^*$ algebraic over $\operatorname{res} \boldsymbol{K}$ and an extension of i to an embedding $i^* : \boldsymbol{K}^* \to \boldsymbol{M}$. The case of $\upomega$-free $\widehat{K}$ treated earlier applied instead to $\widehat{K}^*$ now yields an extension of i^* to an embedding $\widehat{K}^* \to M$.

Next, suppose K is $\upomega$-free. Then the pc-sequence $(\uplambda_\rho)$ in K is of d-transcendental type over K, by [ADH, 13.6.3]. Take $\uplambda \in \widehat{K}$ such that $\uplambda_\rho \rightsquigarrow \uplambda$. Now K is short and M is countably spherically complete, so by Lemma 5.11 we have $\uplambda^* \in M$ with $i(\uplambda_\rho) \rightsquigarrow \uplambda^*$. By [ADH, 11.4.7, 11.4.13, 10.5.8] we obtain a unique extension of i to an embedding $j \colon K\langle\uplambda\rangle \to M$ such that $j(\uplambda) = \uplambda^*$. The case of non-$\uplambda$-free K applied instead to $K\langle\uplambda\rangle$ yields an extension of j to an embedding $\widehat{K} \to M$. □

7. Embeddings into Analytic Hardy Fields

Below we use Theorem A to derive results about back-and-forth equivalence and isomorphism for maximal analytic Hardy fields, as was done in [6, Section 10] for maximal Hardy fields. Let **No** be the ordered field of surreal numbers equipped with the derivation ∂_{BM} of Berarducci and Mantova [8]. Then **No** is a closed H-field, by [3]. Moreover, given an uncountable cardinal κ, the surreal numbers of length $< \kappa$ form an ordered differential subfield $\mathbf{No}(\kappa)$ of **No** with $\mathbf{No}(\kappa) \preccurlyeq \mathbf{No}$, by [3, Corollary 4.6]. As in the argument leading up to [6, Corollary 10.4], combining Theorem A and [6, Corollary 10.3] yields:

Corollary 7.1. *Let M be a maximal analytic or maximal smooth Hardy field. Then the ordered differential fields M and $\mathbf{No}(\omega_1)$ are back-and-forth equivalent. Hence $M \equiv_{\infty\omega} \mathbf{No}(\omega_1)$, and assuming* CH, *$M \cong \mathbf{No}(\omega_1)$.*

The ordered field $\mathbf{No}(\omega_1)$ is not complete: Set $a_\nu := \sum_{\mu<\nu} \omega^{-\mu}$ with μ, ν ranging over countable ordinals. Then (a_ν) is a cauchy sequence in $\mathbf{No}(\omega_1)$ without a limit in $\mathbf{No}(\omega_1)$. Thus, assuming CH, no real closed η_1-ordered field extension of $\mathbb{R}$ of cardinality $\mathfrak{c}$ is complete, in particular, no maximal Hardy field is complete. (This also follows from [16, Theorem 3.12(ii)]: if G is a complete η_1-ordered abelian group, then $|G| > \aleph_1$.)

Let K be an H-field with small derivation and constant field $\mathbb{R}$. Then [3, Theorem 3] yields an embedding $K \to \mathbf{No}$ of ordered differential fields. The argument in the proof of [3, Theorem 3] shows that if $\kappa > |K|$ is a regular cardinal, then we can choose ι so that $\iota(K) \subseteq \mathbf{No}(\kappa)$. If $\operatorname{trdeg}(K|\mathbb{R})$ is countable, then K actually embeds into $\mathbf{No}(\omega_1)$. This is a consequence of the next lemma, a variant of [6, Lemma 10.1]. For "very small derivation" see [5, Sections 1.1 and 7.3].

Lemma 7.2. *Let K be a pre-H-field with very small derivation, archimedean residue field, and* $\operatorname{trdeg}(K|C) \leqslant \aleph_0$. *Let L be a closed η_1-ordered H-field with small derivation and $C_L = \mathbb{R}$. Then K embeds into L.*

Proof. Passing to $H(K)$ we arrange that K is an H-field. Without loss, C_K is an ordered subfield of $\mathbb{R}$, and then adjoining new constants if necessary, we arrange $C_K = \mathbb{R}$. Take a closed H-field $\widehat{K}$ extending K. Next, take a countable set $S \subseteq K$ such that $K = \mathbb{R}\langle S\rangle$ and then a countable closed H-subfield $K_0 \supseteq S$ of $\widehat{K}$. Let E be a copy of the prime model of the theory of closed H-fields with small derivation inside K_0. Applying [6, Lemma 10.1] to an H-field embedding $E \to L$ with K_0 in place of K yields an H-field embedding $i \colon K_0 \to L$. Then i is the identity on $C_{K_0} \subseteq \mathbb{R}$, and then [ADH, 10.5.15, 10.5.16] yield an extension of i to an H-field embedding $K_0(\mathbb{R}) \to L$ that is the identity on $\mathbb{R}$, and the restriction of this embedding to K is a pre-H-field embedding $K \to L$. ☐

Using also Theorem A and its smooth version we obtain from Lemma 7.2:

Corollary 7.3. *Let K be as in Lemma 7.2 and let M be a maximal Hardy field. Then K embeds into M. Likewise if M is a maximal analytic Hardy field or a maximal smooth Hardy field.*

The following immediate consequence of the last corollary is worth recording:

Corollary 7.4. *Every Hardy field of countable transcendence degree over its constant field is isomorphic to an analytic Hardy field.*

The next corollary strengthens [5, Corollary 7.1.4]:

Corollary 7.5. *Let M be a maximal analytic or maximal smooth Hardy field, and let N be a maximal Hardy field with $M \subseteq N$. Then $M \preccurlyeq_{\infty\omega} N$.*

Proof. By Theorem A and its smooth version, M is η_1, and by Theorem A of [6], N is η_1. It remains to use [6, Lemma 10.5]. ☐

At the heart of the proof of [6, Lemma 10.1] is [ADH, 16.2.3] of which we now give a version with the cofinality hypothesis replaced by a shortness assumption:

Proposition 7.6. *Let E be an ω-free H-field and K be a closed short H-field extending E such that $C_E = C_K$. Let $i\colon E \to L$ be an embedding where L is a closed η_1-ordered H-field. Then i extends to an embedding $K \to L$.*

Proof. Suppose $E \neq K$; it is enough to show that i extends to an embedding of some ω-free H-subfield F of K into L, where F properly contains E.

Consider first the case $\Gamma_E^{<}$ is not cofinal in $\Gamma^{<}$. Then we have $y \in K^{>}$ such that $\Gamma_E^{<} < vy < 0$. Now E is short, so we have $y^* \in L^{>}$ such that $\Gamma_{iE}^{<} < vy^* < 0$. As in the proof of [ADH, 16.2.3] we then obtain an ω-free H-subfield F of K with $F \supseteq E\langle y\rangle$ and an extension of i to an embedding $F \to L$.

For the rest of the proof we assume $\Gamma_E^{<}$ is cofinal in $\Gamma^{<}$. Then every differential subfield of K containing E is an ω-free H-subfield of K.

Subcase 1: *E is not closed.* This goes like Subcase 1 in the proof of [ADH, 16.2.3].

Subcase 2: *E is closed, and $E\langle y\rangle$ is an immediate extension of E for some $y \in K\setminus E$.* For such y, Lemma 5.24 yields an extension of i to an embedding $E\langle y\rangle \to L$.

Subcase 3: *E is closed, and there is no $y \in K \setminus E$ such that $E\langle y\rangle$ is an immediate extension of E.* Take any $f \in K \setminus E$. Since E is short and L is η_1 we have $g \in L$ such that for all $a \in E$, $a < f \Leftrightarrow i(a) < g$. Now [ADH, 16.1.5] gives an H-field embedding $E\langle f\rangle \to L$ extending i which sends f to g. ☐

Corollary 7.7. *Let E, K, L, i be as in Proposition 7.6, with "$C_E = C_K$" replaced by "C_K is archimedean and $C_L = \mathbb{R}$". Then i extends to an embedding $K \to L$.*

Proof. The ordered field embedding $i|_{C_E}\colon C_E \to C_L = \mathbb{R}$ extends uniquely to an ordered field embedding $j\colon C_K \to C_L$. Now argue as in the proof of [ADH, 16.2.4], using Proposition 7.6 in place of [ADH, 16.2.3]. ☐

Proposition 7.6 leads to a version of Lemma 7.2 for short closed H-fields:

Lemma 7.8. *Let K be a closed short H-field with small derivation and archimedean constant field, and L a closed η_1-ordered H-field with small derivation and $C_L = \mathbb{R}$. Then K embeds into L.*

Proof. Take $x \in K$ with $x' = 1$. Now K has small derivation, so $x \succ 1$, the H-field $C_K(x)$ is grounded, and we have an embedding $i\colon C_K(x) \to L$ extending the unique ordered field embedding $C_K \to C_L$. By [ADH, 10.6.23] we have a Liouville closure E of $C_K(x)$ in K and i extends to an embedding $E \to L$. Moreover, E is ω-free, by [5, Lemma 1.4.18], so we can use Proposition 7.6. □

With $K = \mathbb{T}$ and L a maximal analytic Hardy field in Lemma 7.8 we conclude:

Corollary 7.9. *The ordered differential field $\mathbb{T}$ is isomorphic over $\mathbb{R}$ to an analytic Hardy field containing $\mathbb{R}$.*

We upgrade this as follows:

Corollary 7.10. *Let E be a pre-H-subfield of $\mathbb{T}$, M be a maximal Hardy field, and $i\colon E \to M$ be an embedding. Then i extends to an embedding $\mathbb{T} \to M$. Likewise with "maximal analytic" and with "maximal smooth" instead of "maximal".*

Proof. Expand E, M (uniquely) to pre-$\Lambda\Omega$-fields $\boldsymbol{E}$, $\boldsymbol{M}$, respectively, such that i is an embedding $\boldsymbol{E} \to \boldsymbol{M}$, and expand $\mathbb{T}$ (uniquely) to a pre-$\Lambda\Omega$-field $\boldsymbol{T}$. Then $\boldsymbol{E} \subseteq \boldsymbol{T}$ by [5, Lemma 7.1.1, Corollary 7.1.16]. Now [ADH, 16.4.1] yields an ω-free $\Lambda\Omega$-field $\boldsymbol{E}^*$ with $\boldsymbol{E} \subseteq \boldsymbol{E}^* \subseteq \boldsymbol{T}$ and an extension of i to an embedding $E^* \to M$, which in turn extends to an embedding $\mathbb{T} \to M$ by Corollary 7.7. □

Remark. If $\widehat{\mathbb{T}}$ is an immediate H-field extension of $\mathbb{T}$, then any embedding of $\mathbb{T}$ into a maximal Hardy field M extends to an embedding $\widehat{\mathbb{T}} \to M$, by Theorem 6.1. Likewise for M a maximal smooth or maximal analytic Hardy field. With $\mathbf{No}$ in place of M we can also take strong additivity into account. To see this recall from [3, Proposition 5.1 and subsequent remarks] that the unique strongly additive embedding $\iota\colon \mathbb{T} \to \mathbf{No}$ over $\mathbb{R}$ of exponential ordered fields which sends $x \in \mathbb{T}$ to $\omega \in \mathbf{No}$ is also an embedding of differential fields, with $\iota(\mathbb{T}) \subseteq \mathbf{No}(\omega_1)$ by [3, Proposition 5.2(3)]. By [3, Proposition 5.2(1)], $\iota(G^{\mathrm{LE}}) = \mathfrak{M} \cap \iota(\mathbb{T})$, where G^{LE} is the group of LE-monomials (cf. [ADH, p. 718]) and $\mathfrak{M}$ is the class of monomials in $\mathbf{No}$ (cf. [3, §1]), hence ι extends uniquely to a strongly additive ordered field embedding $\widehat{\iota}\colon \mathbb{R}[[G^{\mathrm{LE}}]] \to \mathbf{No}$. The derivation of $\mathbf{No}$ is strongly additive, so $\widehat{\iota}(\mathbb{R}[[G^{\mathrm{LE}}]]) = \mathbb{R}[[\iota(G^{\mathrm{LE}})]]$ is a differential subfield of $\mathbf{No}$. The derivation on $\mathbb{R}[[G^{\mathrm{LE}}]]$ that makes $\widehat{\iota}$ a differential field embedding is then the unique strongly additive derivation on $\mathbb{R}[[G^{\mathrm{LE}}]]$ extending the derivation of $\mathbb{T}$. It also makes $\mathbb{R}[[G^{\mathrm{LE}}]]$ a spherically complete immediate H-field extension of $\mathbb{T}$, so the result stated at the beginning of this extended remark applies to the H-field $\mathbb{R}[[G^{\mathrm{LE}}]]$ in the role of $\widehat{\mathbb{T}}$.

8. Some Set-Theoretic Issues

We finish with some questions of a set-theoretic nature that others might be better prepared to answer. We assume our base theory ZFC is consistent, and these are questions about relative consistency with ZFC.

1. Is it consistent that there are non-isomorphic maximal Hardy fields?
2. Is it consistent that no maximal Hardy field is isomorphic to $\mathbf{No}(\omega_1)$?

(3) Is it consistent that there is a complete maximal Hardy field?

Positive answers would mean (at least) that we cannot drop the assumption CH in some results we proved under this hypothesis. Note also that with CH we have $\operatorname{cf}(H) = \operatorname{ci}(H^{>\mathbb{R}}) = \omega_1$ for all maximal Hardy fields. This suggests:

(4) Is it consistent that $\operatorname{cf}(H_1) \neq \operatorname{cf}(H_2)$ for some maximal Hardy fields H_1, H_2? Same with $\operatorname{ci}(H_i^{>\mathbb{R}})$ instead of $\operatorname{cf}(H_i)$.
(5) Is it consistent that $\operatorname{cf}(H) \neq \operatorname{ci}(H^{>\mathbb{R}})$ for some maximal Hardy field H?
(6) Is it consistent that there is a maximal Hardy field H and a gap in H of character (α, β^*) with $\alpha, \beta \geqslant \omega$, not equal to one of (ω, κ^*), (κ, ω^*), (κ, κ^*), (λ, λ^*), where $\kappa := \operatorname{ci}(H^{>\mathbb{R}})$, $\lambda := \operatorname{cf}(H)$?

One can also ask these questions for maximal analytic Hardy fields and maximal smooth Hardy fields instead of maximal Hardy fields. We can even ask them for maximal Hausdorff fields (containing at least $\mathbb{R}$, say) instead of maximal Hardy fields. As with Corollary 2.7, might some weaker assumption like $\mathfrak{b} = \mathfrak{d}$ be enough for some results where we assumed CH?

If H is a Hardy field with $H^{>\mathbb{R}}$ closed under compositional inversion, then

$$h \mapsto h^{\text{inv}} \colon H^{>\mathbb{R}} \to H^{>\mathbb{R}}$$

is a strictly decreasing bijection, so $\operatorname{cf}(H) = \operatorname{ci}(H^{>\mathbb{R}})$. However, we don't know if there is a maximal Hardy field H with $H^{>\mathbb{R}}$ closed under compositional inversion.

Appendix. A Proof of Whitney's Approximation Theorem

For the convenience of the reader, we include here a proof of Theorem 1.1, adapting the exposition in [28, §1.6]. Throughout this appendix $r \in \mathbb{N} \cup \{\infty\}$ and $a, b \in \mathbb{R}$.

Recall that the support $\operatorname{supp} f$ of a function $f\colon \mathbb{R} \to \mathbb{R}$ is the closure in $\mathbb{R}$ of the set $\{t \in \mathbb{R} : f(t) \neq 0\}$. We begin with two lemmas, where $f \in \mathcal{C}^m(\mathbb{R})$ is such that $\operatorname{supp} f$ is bounded; let also λ range over $\mathbb{R}^>$. From the Gaussian integral $\int_{-\infty}^{\infty} \mathrm{e}^{-s^2}\, ds = \pi^{1/2}$ we get $(\lambda/\pi)^{1/2} \int_{-\infty}^{\infty} \mathrm{e}^{-\lambda s^2}\, ds = 1$. Consider $f_\lambda \colon \mathbb{R} \to \mathbb{R}$ given by

$$f_\lambda(t) := (\lambda/\pi)^{1/2} \int_{-\infty}^{\infty} f(s)\, \mathrm{e}^{-\lambda(s-t)^2}\, ds. \tag{A.1}$$

Note that we could have replaced here the bounds $-\infty$, ∞ in this integral by any a, b such that $\operatorname{supp}(f) \subseteq [a, b]$. A change of variables gives

$$f_\lambda(t) \ = \ (\lambda/\pi)^{1/2} \int_{-\infty}^{\infty} f(t-s)\, \mathrm{e}^{-\lambda s^2}\, ds.$$

As in [17, (8.12), Exercise 2(b)] one obtains that $f_\lambda \in \mathcal{C}^\infty(\mathbb{R})$ and for $k \leqslant m$:

$$f_\lambda^{(k)}(t) = (\lambda/\pi)^{1/2} \int_{-\infty}^{\infty} f^{(k)}(s)\, \mathrm{e}^{-\lambda(s-t)^2}\, ds = (\lambda/\pi)^{1/2} \int_{-\infty}^{\infty} f^{(k)}(t-s)\, \mathrm{e}^{-\lambda s^2}\, ds.$$

Moreover:

Lemma A.1. *f_λ extends to an entire function; in particular, $f_\lambda \in \mathcal{C}^\omega(\mathbb{R})$.*

Proof. Take $a < b$ such that $\operatorname{supp} f \subseteq [a, b]$ and consider $g\colon [a, b] \times \mathbb{C} \to \mathbb{C}$ given by $g(s, z) := f(s)\, \mathrm{e}^{-\lambda(s-z)^2}$. Then g is continuous, for each $s \in [a, b]$ the function $g(s, -)\colon \mathbb{C} \to \mathbb{C}$ is analytic, and $\partial g/\partial z\colon [a, b] \times \mathbb{C} \to \mathbb{C}$ is continuous. Hence $z \mapsto \int_a^b g(s, z)\, ds \colon \mathbb{C} \to \mathbb{C}$ is analytic by [17, (9.10), Exercise 3]. □

Lemma A.2. *$\|f_\lambda - f\|_m \to 0$ as $\lambda \to \infty$.*

Proof. For $k \leqslant m$ we have

$$\begin{aligned} f_\lambda^{(k)}(t) - f^{(k)}(t) &= (\lambda/\pi)^{1/2} \int_{-\infty}^{\infty} \big(f^{(k)}(t-s) - f^{(k)}(t)\big)\, \mathrm{e}^{-\lambda s^2}\, ds \\ &= (\lambda/\pi)^{1/2} \int_{-\infty}^{\infty} \big(f^{(k)}(s) - f^{(k)}(t)\big)\, \mathrm{e}^{-\lambda (s-t)^2}\, ds. \end{aligned}$$

Let $\varepsilon \in \mathbb{R}^{>}$ be given, and choose $\delta > 0$ such that

$$|f^{(k)}(s) - f^{(k)}(t)| \leqslant \varepsilon/2 \quad \text{whenever } |s-t| \leqslant \delta \text{ and } k \leqslant m.$$

For $s \leqslant t - \delta$ and for $s \geqslant t + \delta$ we have $\mathrm{e}^{-\lambda(s-t)^2} \leqslant \mathrm{e}^{-(\lambda/2)\delta^2}\, \mathrm{e}^{-(\lambda/2)(s-t)^2}$, so

$$\begin{aligned} \int_{-\infty}^{t-\delta} \mathrm{e}^{-\lambda(s-t)^2}\, ds + \int_{t+\delta}^{\infty} \mathrm{e}^{-\lambda(s-t)^2}\, ds &\leqslant \mathrm{e}^{-(\lambda/2)\delta^2} \int_{-\infty}^{\infty} \mathrm{e}^{-(\lambda/2)(s-t)^2}\, ds \\ &= \mathrm{e}^{-(\lambda/2)\delta^2} (2\pi/\lambda)^{1/2}. \end{aligned}$$

Set $M := \|f\|_m \in \mathbb{R}^{\geqslant}$. For $k \leqslant m$ we have

$$\int_{-\infty}^{\infty} \big(f^{(k)}(s) - f^{(k)}(t)\big)\, \mathrm{e}^{-\lambda(s-t)^2}\, ds = \int_{-\infty}^{t-\delta} (\dots)\, ds + \int_{t-\delta}^{t+\delta} (\dots)\, ds + \int_{t+\delta}^{\infty} (\dots)\, ds,$$

hence

$$\begin{aligned} |f_\lambda^{(k)}(t) - f^{(k)}(t)| &\leqslant (\lambda/\pi)^{1/2} \Bigg(M \int_{-\infty}^{t-\delta} \mathrm{e}^{-\lambda(s-t)^2}\, ds + \\ &\qquad (\varepsilon/2) \int_{-\infty}^{\infty} \mathrm{e}^{-\lambda(s-t)^2}\, ds + M \int_{t+\delta}^{\infty} \mathrm{e}^{-\lambda(s-t)^2}\, ds \Bigg) \\ &\leqslant (\varepsilon/2) + \sqrt{2} M\, \mathrm{e}^{-(\lambda/2)\delta^2}. \end{aligned}$$

Thus if λ is so large that $\sqrt{2}M\, \mathrm{e}^{-(\lambda/2)\delta^2} \leqslant \varepsilon/2$, then $\|f_\lambda - f\|_m \leqslant \varepsilon$. □

In the next lemma we let $U \subseteq \mathbb{R}$ be nonempty and open and let K range over nonempty compact subsets of U and m over the natural numbers $\leqslant r$.

Lemma A.3. *Let (f_n) be a sequence in $\mathcal{C}^r(U)$ which, for all K, m, is a cauchy sequence with respect to $\|\cdot\|_{K;m}$. Then there exists $f \in \mathcal{C}^r(U)$ such that for all K, m we have $\|f_n - f\|_{K;m} \to 0$ as $n \to \infty$.*

Proof. For all K, m, $(f_n^{(m)})$ is a cauchy sequence with respect to $\|\cdot\|_K$. Hence for each m we obtain an $f^m \in \mathcal{C}(U)$ such that for all K, $\|f_n^{(m)} - f^m\|_K \to 0$ as $n \to \infty$; cf. [17, (7.2.1)]. Set $f := f^0$. By induction on $m \leqslant r$ we show that $f \in \mathcal{C}^m(U)$ and $f^{(m)} = f^m$. This is clear for $m = 0$, so suppose $0 < m \leqslant r$ and $f \in \mathcal{C}^{m-1}(U)$, $f^{(m-1)} = f^{m-1}$. Let $a \in U$, and take $\varepsilon > 0$ such that $K := [a - \varepsilon, a + \varepsilon] \subseteq U$. Let $t \in K \setminus \{a\}$. Then for each n we have s_n with $|a - s_n| \leqslant |a - t|$ such that

$$f_n^{(m-1)}(t) - f_n^{(m-1)}(a) = f_n^{(m)}(s_n) \cdot (t - a).$$

Take a subsequence (s_{n_k}) of (s_n) and $s = s(t)$ with $\lim\limits_{k\to\infty} s_{n_k} = s$. Then $|a - s| \leqslant |a - t| \leqslant \varepsilon$ and

$$\lim_{k\to\infty} \big(f_{n_k}^{(m-1)}(t) - f_{n_k}^{(m-1)}(a)\big) = f^{m-1}(t) - f^{m-1}(a) = f^{(m-1)}(t) - f^{(m-1)}(a)$$

and $\lim_{k\to\infty} f^{(m)}_{n_k}(s_{n_k}) = f^m(s)$, since $\lim_{n\to\infty}\|f^{(m)}_n - f^m\|_K = 0$. Hence

$$f^{(m-1)}(t) - f^{(m-1)}(a) = f^m(s)\cdot(t-a)$$

where $f^m\big(s(t)\big) \to f^m(a)$ as $t \to a$, since f^m is continuous at a. $\square$

We now prove Theorem 1.1. Let (a_n), (b_n), (ε_n) be sequences in $\mathbb{R}$ and (r_n) in $\mathbb{N}$ such that $a_0 = b_0$, (a_n) is strictly decreasing, (b_n) is strictly increasing, and $\varepsilon_n > 0$, $r_n \leqslant r$ for all n. Set $I := \bigcup_n K_n$, where $K_n := [a_n, b_n]$, and let $f \in \mathcal{C}^r(I)$. *We need to show the existence of a* $g \in \mathcal{C}^\omega(I)$ *such that* $\|f-g\|_{K_{n+1}\setminus K_n;\, r_n} < \varepsilon_n$ *for each* n. Replacing ε_n by $\min\{\varepsilon_n, \frac{1}{n+1}\}$ and r_n by $\max\{r_0,\dots,r_n\}$ we first arrange that $\varepsilon_n \to 0$ as $n\to\infty$ and $r_n \leqslant r_{n+1}$ for all n. Set

$$L_n \ := \ K_{n+1}\setminus K_n \ = \ [a_{n+1}, a_n) \cup (b_n, b_{n+1}],$$

and take $\varphi_n \in \mathcal{C}^\infty(\mathbb{R})$ such that $\varphi_n = 0$ on a neighborhood of K_{n-1} (satisfied automatically for $n=0$, by convention), $\varphi_n = 1$ on a neighborhood of $\operatorname{cl}(L_n) = [a_{n+1}, a_n] \cup [b_n, b_{n+1}]$, and $\operatorname{supp}\varphi_n \subseteq K_{n+2}$. For example, for $n \geqslant 1$, $\alpha_{a,b} \in \mathcal{C}^\infty(\mathbb{R})$ as in [6, (3.4)], and sufficiently small positive $\varepsilon = \varepsilon(n)$, set

$$\alpha_n(t) := \begin{cases} \alpha_{a_{n+2}+\varepsilon,\, a_{n+1}-\varepsilon}(t) & \text{if } t \leqslant a_n, \\ 1 - \alpha_{a_n+\varepsilon,\, a_{n-1}-\varepsilon}(t) & \text{otherwise.} \end{cases}$$

and

$$\beta_n(t) := \begin{cases} \alpha_{b_{n-1}+\varepsilon,\, b_n-\varepsilon}(t) & \text{if } t \leqslant b_n, \\ 1 - \alpha_{b_{n+1}+\varepsilon,\, b_{n+2}-\varepsilon}(t) & \text{otherwise.} \end{cases}$$

and put $\varphi_n := \alpha_n + \beta_n$. (See Figure A.1.)

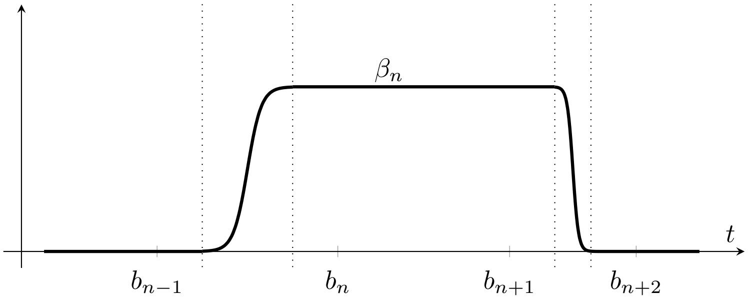

FIGURE A.1. The hump function β_n

With $M_n := 1 + 2^{r_n}\|\varphi_n\|_{r_n}$, choose $\delta_n \in \mathbb{R}^>$ so that for all n,

$$2\delta_{n+1} \leqslant \delta_n, \qquad \sum_{m=n}^{\infty} \delta_m M_{m+1} \leqslant \varepsilon_n/4 \tag{A.2}$$

Given $g \in \mathcal{C}(\mathbb{R})$ with bounded support and $\lambda \in \mathbb{R}^>$, let $I_\lambda(g) := g_\lambda$ be as in (A.1), with g in place of f. Next, let $f \in \mathcal{C}^r(I)$ be given. Then we inductively define sequences (λ_n) in $\mathbb{R}^>$ and (g_n) in $\mathcal{C}^\omega(\mathbb{R})$ as follows: Let $\lambda_m \in \mathbb{R}^>$ and $g_m \in \mathcal{C}^\omega$ for $m < n$; then consider the function $h_n \in \mathcal{C}^r(\mathbb{R})$ given by

$$h_n(t) \ := \ \begin{cases} \varphi_n(t)\cdot\big(f(t) - \big(g_0(t)+\dots+g_{n-1}(t)\big)\big) & \text{if } t \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\operatorname{supp} h_n \subseteq \operatorname{supp} \varphi_n$ is bounded. Put $g_n := I_{\lambda_n}(h_n) \in \mathcal{C}^{\omega}(\mathbb{R})$ where we take $\lambda_n \in \mathbb{R}^{>}$ such that $\|g_n - h_n\|_{r_n} < \delta_n$ (any sufficiently large λ_n will do, by Lemma A.2). So $\|g_{n+1} - h_{n+1}\|_{K_n;\, r_{n+1}} < \delta_{n+1}$, and since φ_{n+1} and thus also h_{n+1} vanish on a neighborhood of K_n, this yields

$$\|g_{n+1}\|_{K_n;\, r_{n+1}} < \delta_{n+1}. \tag{A.3}$$

Likewise, since $\varphi_n = 1$ on a neighborhood of $\operatorname{cl}(L_n)$,

$$\|f - (g_0 + \cdots + g_n)\|_{L_n;\, r_n} < \delta_n. \tag{A.4}$$

Also

$$\|g_{n+1} - h_{n+1}\|_{L_n;\, r_n} \leqslant \|g_{n+1} - h_{n+1}\|_{r_{n+1}} < \delta_{n+1},$$

and thus by (1.1) and (A.4):

$$\begin{aligned} \|g_{n+1}\|_{L_n;\, r_n} &\leqslant \|g_{n+1} - h_{n+1}\|_{L_n;\, r_n} + \|\varphi_{n+1} \cdot (f - (g_0 + \cdots + g_n))\|_{L_n;\, r_n} \\ &\leqslant \delta_{n+1} + 2^{r_n} \|\varphi_{n+1}\|_{L_n;\, r_n} \cdot \|f - (g_0 + \cdots + g_n)\|_{L_n;\, r_n} \\ &\leqslant \delta_{n+1} + M_{n+1}\delta_n. \end{aligned}$$

Moreover, by (A.3) and $r_n \leqslant r_{n+1}$ we have $\|g_{n+1}\|_{K_n;\, r_n} < \delta_{n+1}$. Hence by (A.2):

$$\|g_{n+1}\|_{K_{n+1};\, r_n} \leqslant \delta_{n+1} + M_{n+1}\delta_n + \delta_{n+1} \leqslant M_{n+1}\delta_n + \delta_n \leqslant 2\delta_n M_{n+1}. \tag{A.5}$$

Let $K \subseteq I$ be nonempty and compact, and let $m \leqslant r_n$ for some n. We claim that $(g_0 + \cdots + g_i)$ is a cauchy sequence with respect to $\|\cdot\|_{K;\, m}$. To see this, let $\varepsilon \in \mathbb{R}^{>}$ be given, and take n such that $K \subseteq K_{n+1}$ and $m \leqslant r_n$. Then by (A.2) and (A.5) we have for $j > i \geqslant n$:

$$\begin{aligned} \|g_{i+1} + \cdots + g_j\|_{K;\, m} &\leqslant \|g_{i+1}\|_{K;\, m} + \cdots + \|g_j\|_{K;\, m} \\ &\leqslant \|g_{i+1}\|_{K_{i+1};\, r_i} + \cdots + \|g_j\|_{K_j;\, r_{j-1}} \\ &\leqslant 2\delta_i M_{i+1} + \cdots + 2\delta_{j-1}M_j \leqslant \varepsilon_i/2. \end{aligned}$$

So Lemma A.3 yields a function $g\colon I \to \mathbb{R}$ such that $g(t) = \sum_{i=0}^{\infty} g_i(t)$ for all $t \in I$ and $g \in \mathcal{C}^{r_n}(I)$ for all n. In the same way, using (A.2) and (A.5) and denoting the restriction of g_i to I also by g_i, we obtain

$$\|g - (g_0 + \cdots + g_n)\|_{L_n;\, r_n} = \left\| \sum_{i=n+1}^{\infty} g_i \right\|_{L_n;\, r_n} \leqslant \varepsilon_n/2$$

and hence by (A.2) and (A.4):

$$\begin{aligned} \|f - g\|_{L_n;\, r_n} &\leqslant \|f - (g_0 + \cdots + g_n)\|_{L_n;\, r_n} + \|g - (g_0 + \cdots + g_n)\|_{L_n;\, r_n} \\ &\leqslant \delta_n + \tfrac{1}{2}\varepsilon_n < \varepsilon_n. \end{aligned}$$

To complete the proof we are going to choose sequences (g_n) and (λ_n) as above so that g is analytic. Now for $t \in \mathbb{R}$ we have

$$g_n(t) = (\lambda_n/\pi)^{1/2} \int_{-\infty}^{\infty} h_n(s)\, \mathrm{e}^{-\lambda_n(s-t)^2}\, ds = (\lambda_n/\pi)^{1/2} \int_{a_{n+2}}^{b_{n+2}} h_n(s)\, \mathrm{e}^{-\lambda_n(s-t)^2}\, ds$$

and g_n is the restriction to $\mathbb{R}$ of the entire function $\widehat{g}_n$ given by

$$\widehat{g}_n(z) = (\lambda_n/\pi)^{1/2} \int_{a_{n+2}}^{b_{n+2}} h_n(s)\, \mathrm{e}^{-\lambda_n(s-z)^2}\, ds \qquad (z \in \mathbb{C}).$$

(See the proof of Lemma A.1.) Put

$$\rho_n \;:=\; \tfrac{1}{2}\min\bigl\{(a_n-a_{n+1})^2,(b_{n+1}-b_n)^2\bigr\}\in\mathbb{R}^{>}$$

and

$$U_n \;:=\; \bigl\{z\in\mathbb{C}:\ a_{n+1}<\operatorname{Re} z<b_{n+1},\ \operatorname{Re}\bigl((z-a_{n+1})^2\bigr),\ \operatorname{Re}\bigl((z-b_{n+1})^2\bigr)>\rho_n\bigr\},$$

an open subset of $\mathbb{C}$ containing K_n such that $\operatorname{Re}\bigl((s-z)^2\bigr)>\rho_n$ for all $z\in U_n$ and $s\in\mathbb{R}\setminus K_{n+1}$. (Cf. Figure A.2.)

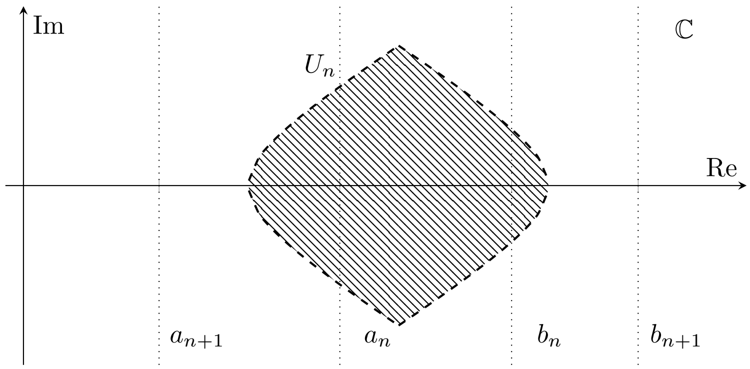

FIGURE A.2. The domain U_n

We also set

$$H_m \;:=\; 2(\lambda_m/\pi)^{1/2}\,\|h_m\|_{K_{m+2}}\,(b_{m+2}-a_{m+2})\in\mathbb{R}^{\geqslant}.$$

Recall that h_m only depends on the g_j with $j<m$. Fix a sequence (c_m) of positive reals such that $\sum_m c_m<\infty$. Then we can and do choose the sequences (g_m), (λ_m) so that in addition

$$H_m\exp(-\lambda_m/m)\;\leqslant\;c_m\quad\text{for all } m\geqslant 1.$$

Then

$$\sum_m H_m\exp(-\lambda_m\rho)\;<\infty\qquad\text{for all }\rho\in\mathbb{R}^{>}. \tag{A.6}$$

It is enough that for each n the series $\sum_m \widehat{g}_m$ converges uniformly on compact subsets of U_n, because then by [17, (9.12.1)] we have a holomorphic function

$$z\mapsto\sum_m\widehat{g}_m(z)\;:\;U:=\bigcup_n U_n\to\mathbb{C}$$

whose restriction to I is g. To prove such convergence, fix n and let $m\geqslant n+2$. Then $\operatorname{supp} h_m\subseteq K_{m+2}\setminus K_{m-1}\subseteq K_{m+2}\setminus K_{n+1}$. Hence $|\widehat{g}_m(z)|\leqslant H_m\,\mathrm{e}^{-\lambda_m\rho_n}$ for $z\in U_n$. Together with (A.6) this now yields that $\sum_m\widehat{g}_m$ converges uniformly on compact subsets of U_n. $\square$

In the remainder of this appendix we discuss how to control the domain of the holomorphic function $\widehat{g}$ in the proof of Theorem 1.1; this leads to improvements of Corollaries 1.2 and 1.3 which might be useful elsewhere: Corollaries A.6 and A.7

below. For the next corollary we are in the setting of that theorem and $f \in \mathcal{C}^r(I)$. With $\alpha \in \mathbb{R} \cup \{-\infty\}$ and $\beta \in \mathbb{R} \cup \{+\infty\}$ such that $I = (\alpha, \beta)$, put

$$V \ := \ \big\{z \in \mathbb{C} : \ \operatorname{Re}(z) \in I, \ |\operatorname{Im} z| < \operatorname{Re}(z) - \alpha, \ \beta - \operatorname{Re}(z)\big\},$$

an open subset of $\mathbb{C}$ containing I.

Corollary A.4. *Suppose* $a_n - a_{n+1} \to 0$ *and* $b_{n+1} - b_n \to 0$ *as* $n \to \infty$. *Then there is a holomorphic* $\widehat{g}\colon V \to \mathbb{C}$, *real-valued on* $\mathbb{R}$, *such that* $g := \widehat{g}|_I \in \mathcal{C}^\omega(I)$ *satisfies*

$$\|f - g\|_{K_{n+1}\setminus K_n;\, r_n} < \varepsilon_n, \ \textit{for all } n.$$

Proof. It suffices to show that the open set $U \subseteq \mathbb{C}$ in the proof of Theorem 1.1 contains V. Note that $\rho_n \to 0$ as $n \to \infty$. Let $z = x + yi \in V$ $(x, y \in \mathbb{R})$. Then

$$(x - a_{n+1})^2 - y^2 - \rho_n \to (x - \alpha)^2 - y^2 > 0 \text{ as } n \to \infty,$$

and thus $\operatorname{Re}\big((z - a_{n+1})^2\big) = (x - a_{n+1})^2 - y^2 > \rho_n$ for all sufficiently large n. Likewise, $\operatorname{Re}\big((z - b_{n+1})^2\big) > \rho_n$ for all sufficiently large n. Therefore $z \in U_n$ for sufficiently large n. □

Corollary A.5. *Suppose* $r \in \mathbb{N}$, $f \in \mathcal{C}^r(\mathbb{R})$, $\varepsilon \in \mathcal{C}(\mathbb{R})$, *and* $\varepsilon > 0$ *on* $\mathbb{R}$. *Then there is an entire function* $g\colon \mathbb{C} \to \mathbb{C}$ *such that* $|(f - g)^{(k)}| \leqslant \varepsilon$ *on* $\mathbb{R}$ *for all* $k \leqslant r$.

Proof. Set $b_n := \log(n + 1)$, $a_n := -b_n$, and $K_n := [a_n, b_n]$. Then $\bigcup_n K_n = \mathbb{R}$ and $a_n - a_{n+1} \to 0$ and $b_{n+1} - b_n \to 0$ as $n \to \infty$. Set $\varepsilon_n := \min\big\{\varepsilon(t) : t \in K_{n+1}\big\}$ and $r_n := r$. Then $V = \mathbb{C}$ and we apply Corollary A.4. □

Remark. Corollary A.5 is due to Carleman [13] for $r = 0$, to Kaplan [27] for $r = 1$, and to Hoischen [26, Satz 2] in general; see [12, Chapter VIII, pp. 273–276, 291]. In a similar way, Corollary A.4 also yields the $\mathcal{C}^\infty$-version of Corollary A.5 in [26, Satz 1]. For a multivariate version of these facts, see [1].

Given any a we now consider the open sector V_a in the complex plane given by

$$V_a \ := \ \big\{z \in \mathbb{C} : |\operatorname{Im}(z)| < \operatorname{Re}(z) - a\big\} \ = \ a + \big\{z \in \mathbb{C}^\times : -\tfrac{\pi}{4} < \arg z < \tfrac{\pi}{4}\big\}.$$

Corollary A.6. *Let* f, (b_n), (ε_n), (r_n) *be as in Corollary 1.2. Then there are* $a < b$ *and a holomorphic function* $\widehat{g}\colon V_a \to \mathbb{C}$, *real-valued on* $\mathbb{R}$, *such that* $g := \widehat{g}|_{\mathbb{R}^{\geqslant b}} \in \mathcal{C}^\omega_b$ *satisfies* $\|f - g\|_{[b_n, b_{n+1}];\, r_n} < \varepsilon_n$ *for all* n.

Proof. We first arrange that $b_{n+1} - b_n \to 0$ as $n \to \infty$. For this, let (b^*_m) be the strictly increasing sequence in $\mathbb{R}$ such that

$$\{b^*_0, b^*_1, \dots\} \ = \ \{b_0, b_1, \dots\} \cup \big\{b + \log 1, b + \log 2, \dots\big\},$$

for each m, set $\varepsilon^*_m := \varepsilon_n$, $r^*_m := r_n$ with n such that $[b^*_m, b^*_{m+1}] \subseteq [b_n, b_{n+1}]$, and replace (b_n), (ε_n), (r_n) by (b^*_m), (ε^*_m), (r^*_m). Now argue as in the proof of Corollary 1.2, using Corollary A.4 instead of Theorem 1.1. □

Now the proof of Corollary 1.3, using Corollary A.6 instead of Corollary 1.2, gives:

Corollary A.7. *Let* f, ε *be as in Corollary 1.3. Then there are* $a < b$ *and a holomorphic* $\widehat{g}\colon V_a \to \mathbb{C}$, *real-valued on* $\mathbb{R}$, *such that* $g := \widehat{g}|_{\mathbb{R}^{\geqslant b}} \in \mathcal{C}^\omega_b$ *satisfies* $|(f - g)^{(k)}(t)| < \varepsilon(t)$ *for all* $t \geqslant b$ *and* $k \leqslant \min\big\{r, 1/\varepsilon(t)\big\}$.

References

The citation [ADH] refers to our book

M. Aschenbrenner, L. van den Dries, J. van der Hoeven, *Asymptotic Differential Algebra and Model Theory of Transseries,* Annals of Mathematics Studies, vol. 195, Princeton University Press, Princeton, NJ, 2017.

[1] C. Andradas, E. Aneiros, A. Díaz-Cano, *An extension of Whitney approximation theorem,* in: M. Castrillón López et al. (eds.), *Contribuciones Matemáticas en Honor a Juan Tarrés,* pp. 17–25, Universidad Complutense de Madrid, Facultad de Ciencias Matemáticas, Madrid, 2012.

[2] M. Aschenbrenner, L. van den Dries, J. van der Hoeven, *Maximal immediate extensions of valued differential fields,* Proc. London Math. Soc. **117** (2018), no. 2, 376–406.

[3] M. Aschenbrenner, L. van den Dries, J. van der Hoeven, *The surreal numbers as a universal H-field,* J. Eur. Math. Soc. **21** (2019), 1179–1199.

[4] M. Aschenbrenner, L. van den Dries, J. van der Hoeven, *Revisiting closed asymptotic couples*, Proc. Edinb. Math. Soc. (2) **65** (2022), 530–555.

[5] M. Aschenbrenner, L. van den Dries, J. van der Hoeven, *Maximal Hardy fields*, manuscript, `arXiv:2304.10846`, 2023.

[6] M. Aschenbrenner, L. van den Dries, J. van der Hoeven, *Filling gaps in Hardy fields*, preprint, `arXiv:2308.02446`, 2023.

[7] M. Aschenbrenner, L. van den Dries, J. van der Hoeven, *More on dimension in transseries*, in preparation.

[8] A. Berarducci, V. Mantova, *Surreal numbers, derivations, and transseries,* J. Eur. Math. Soc. **20** (2018), 339–390.

[9] A. Besikowitsch, *Über analytische Funktionen mit vorgeschriebenen Werten ihrer Ableitungen,* Math. Z. **21** (1924), 111–118.

[10] A. Blass, *Combinatorial cardinal characteristics of the continuum,* in: M. Foreman, A. Kanamori (eds.), *Handbook of Set Theory,* Vol. 1, pp. 395–489, Springer, Dordrecht, 2010.

[11] E. Borel, *Sur quelques points de la théorie des fonctions,* Ann. Sci. École Norm. Sup. (3) **12** (1895), 9–55.

[12] R. Burckel, *An Introduction to Classical Complex Analysis. Vol. 1,* Pure and Applied Mathematics, vol. 82, Academic Press, Inc., New York-London, 1979.

[13] T. Carleman, *Sur un théorème de Weierstrass,* Ark. Mat. Astr. Fys. **20B** (1927), no. 4, 1–5.

[14] K. Ciesielski, *A short ordered commutative domain whose quotient field is not short,* Algebra Universalis **25** (1988), no. 1, 1–6.

[15] K. Ciesielski, 2^{2^ω} *nonisomorphic short ordered commutative domains whose quotient fields are long,* Proc. Amer. Math. Soc. **113** (1991), no. 1, 217–227.

[16] H. G. Dales, W. H. Woodin, *Super-Real Fields,* London Mathematical Society Monographs, New Series, vol. 14, Oxford University Press, New York, 1996.

[17] J. Dieudonné, *Foundations of Modern Analysis,* Pure Appl. Math., vol. 10, Academic Press, New York-London, 1960.

[18] L. van den Dries, A. Macintyre, D. Marker, *Logarithmic-exponential series,* Ann. Pure Appl. Logic **111** (2001), 61–113.

[19] J. Esterle, *Solution d'un problème d'Erdös, Gillman et Henriksen et application à l'étude des homomorphismes de $C(K)$,* Acta Math. (Hungarica) **30** (1977), 113–127.

[20] A. Fornasiero, *Dimensions, matroids, and dense pairs of first-order structures,* Ann. Pure Appl. Logic **162** (2011), 514–543.

[21] D. Gokhman, *Functions in a Hardy field not ultimately C^∞*, Complex Variables Theory Appl. **32** (1997), no. 1, 1–6.

[22] K. Grelowski, *Extending Hardy fields by non-$\mathcal{C}^\infty$-germs,* Ann. Polon. Math. **93** (2008), no. 3, 281–297.

[23] L. Harrington, S. Shelah, *Counting equivalence classes for co-κ-Souslin equivalence relations,* in: D. van Dalen et al. (eds.), *Logic Colloquium '80,* pp. 147–152, Studies in Logic and the Foundations of Mathematics, vol. 108, North-Holland Publishing Co., Amsterdam-New York, 1982.

[24] F. Hausdorff, *Untersuchungen über Ordnungstypen,* IV, V, Ber. Königl. Sächs. Gesell. Wiss. Leipzig Math.-Phys. Kl. **59** (1907), 84–159.

[25] E. Harzheim, *Ordered Sets,* Advances in Mathematics, vol. 7, Springer, New York, 2005.

[26] L. Hoischen, *Eine Verschärfung eines Approximationssatzes von Carleman,* J. Approximation Theory **9** (1973), 272–277.

[27] W. Kaplan, *Approximation by entire functions,* Michigan Math. J. **3** (1955), 43–52.

[28] R. Narasimhan, *Analysis on Real and Complex Manifolds,* 2nd ed., Advanced Studies in Pure Mathematics, vol. 1, Masson & Cie, Éditeurs, Paris; North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973.

[29] N. Pynn-Coates, *Newtonian valued differential fields with arbitrary value group*, Comm. Algebra **47** (2019), no. 7, 2766–2776.

[30] J.-P. Rolin, P. Speissegger, A. J. Wilkie, *Quasianalytic Denjoy-Carleman classes and o-minimality,* J. Amer. Math. Soc. **16** (2003), no. 4, 751–777.

[31] J. Rosenstein, *Linear Orderings,* Pure and Applied Mathematics, vol. 98, Academic Press, Inc., New York-London, 1982.

[32] M. E. Rudin, *Martin's axiom,* in: J. Barwise (ed.), *Handbook of Mathematical Logic,* pp. 491–501, Studies in Logic and the Foundations of Mathematics, vol. 90, North-Holland Publishing Co., Amsterdam, 1977.

[33] G. Sjödin, *Hardy-fields*, Ark. Mat. **8** (1970), no. 22, 217–237.

[34] R. M. Solovay, S. Tennenbaum, *Iterated Cohen extensions and Souslin's problem,* Ann. of Math. (2) **94** (1971), 201–245.

[35] P. Urysohn, *Un théorème sur la puissance des ensembles ordonnés,* Fund. Math. **5** (1923), 14–19.

[36] P. Urysohn, *Remarque sur ma note: "Un théorème sur la puissance des ensembles ordonnés",* Fund. Math. **6** (1924), 278.

[37] H. Whitney, *Analytic extension of differentiable functions defined on closed sets,* Trans. Amer. Math. Soc. **36** (1934), 63–89.

Kurt Gödel Research Center for Mathematical Logic, Universität Wien, 1090 Wien, Austria
Email address: `matthias.aschenbrenner@univie.ac.at`

Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, U.S.A.
Email address: `vddries@illinois.edu`