DIFFERENTIAL TRANSCENDENCE OF ITERATIVE LOGARITHMS

MATTHIAS ASCHENBRENNER

Abstract. We show that the iterative logarithm of the power series $e^z - 1$ is differentially transcendental over the ring of convergent power series.

1. Introduction

Let $p \geqslant 1$ and

$$f = z + \sum_{n \geqslant p+1} f_n z^n \in \mathbb{C}[[z]] \qquad (f_n \in \mathbb{C},\ f_{p+1} \neq 0) \tag{1.1}$$

be a formal power series with complex coefficients. In the theory of analytic iteration and conjugate power series, *Julia's equation*

$$y \circ f = y \cdot f' \tag{1.2}$$

plays a central role. (See, e.g., [8], [13], and [14, Section 8.5].) The functional equation (1.2) has a unique solution $\widetilde{y} \in \mathbb{C}[[z]]$ of the form

$$\widetilde{y} = f_{p+1} z^{p+1} + \sum_{n \geqslant p+2} y_n z^n \qquad (y_n \in \mathbb{C}), \tag{1.3}$$

and the general solution of (1.2) is given by $y = c \cdot \widetilde{y}$ where $c \in \mathbb{C}$. (See, e.g., Lemma 3.2 below.) The power series y_0 is called the *iterative logarithm* of f, denoted here by $\operatorname{itlog}(f)$. Conversely, equating coefficients of z^{2p+j-1} on both sides of (1.2), one sees easily that given a formal power series $\widetilde{y}$ as in (1.3) there exists a unique f as in (1.1) with $\operatorname{itlog}(f) = \widetilde{y}$. Thus, also setting $\operatorname{itlog}(z) := 0$, we obtain a bijective correspondence

$$z + z^2\mathbb{C}[[z]] \to z^2\mathbb{C}[[z]] \colon f \mapsto \operatorname{itlog}(f).$$

A somewhat more conceptual route to this bijection leads via the Lie algebra of the (infinite-dimensional) matrix group of iteration matrices (see, e.g., [1]). Using this approach one also sees fairly easily that corresponding properties of the matrix exponential imply that for $f, g \in z + z^2\mathbb{C}[[z]]$ which commute (i.e., $f \circ g = g \circ f$) one has

$$\operatorname{itlog}(f \circ g) = \operatorname{itlog}(f) + \operatorname{itlog}(g),$$

and hence

$$\operatorname{itlog}(f^{[k]}) = k \operatorname{itlog}(f) \qquad \text{for every } k \in \mathbb{Z},$$

where $f^{[0]} := z$ and for $n > 0$, $f^{[n]}$ denotes the nth compositional iterate of f and $f^{[-n]}$ is the compositional inverse of $f^{[n]}$. (This explains the terminology of the "iterative logarithm," which was introduced by Écalle [8].)

Partially supported by a grant from the National Science Foundation.
August 2010

If f as in (1.1) is convergent, then only in exceptional circumstances is $\operatorname{itlog}(f)$ also convergent. For example, by a theorem of Erdős and Jabotinsky [9] in combination with results of Baker [2] and Szekeres [21], it is known that if f is the Taylor series at 0 of a meromorphic function on the whole complex plane which is regular at 0, then $\operatorname{itlog}(f)$ always has radius of convergence 0 except when

$$f = \frac{z}{1-cz} \qquad (c \in \mathbb{C}),$$

in which case $\operatorname{itlog}(f) = cz^2$. (However, Écalle [7] has shown that $\operatorname{itlog}(f)$ is always Borel summable.)

Similarly, $\operatorname{itlog}(f)$ rarely satisfies a non-trivial algebraic differential equation. Let R be a subring of $\mathbb{C}[[z]]$ containing the polynomial ring $\mathbb{C}[z]$, and assume R is closed under differentiation. A formal power series $y \in \mathbb{C}[[z]]$ is said to be *differentially algebraic over* R if it satisfies an equation

$$P(y, y', \dots, y^{(n)}) = 0$$

where P is a non-zero polynomial in $n+1$ indeterminates with coefficients in R, and y is said to be *differentially transcendental over* R otherwise. If one simply speaks of y being *differentially algebraic* respectively *differentially transcendental,* then $R = \mathbb{C}[z]$ is understood. Many formal power series arising naturally in number theory are differentially algebraic (see [16]). If $y = \sum_{n \geqslant 0} y_n z^n \in \mathbb{C}[[z]]$ is differentially algebraic then its coefficient sequence (y_n) satisfies a certain kind of (in general, non-linear) recurrence relation [17, pp. 186–194]. Of particular importance in combinatorial enumeration is the class of *D-finite* (also called *holonomic*) power series [20, Chapter 6]. These are the differentially algebraic power series whose coefficient sequence satisfies a homogeneous *linear* recurrence relation of finite degree with polynomial coefficients. Equivalently [20, Proposition 6.4.3] a formal power series $y \in \mathbb{C}[[z]]$ is D-finite if and only if y satisfies a non-trivial *linear* differential equation

$$a_0 y + a_1 y' + \cdots + a_n y^{(n)} = 0 \qquad (a_i \in \mathbb{C}[z],\ a_n \neq 0).$$

A family $\mathcal{F}$ of elements of $\mathbb{C}[[z]]$ is said to be *uniformly differentially algebraic* (or *coherent*) if there is a non-zero polynomial P in $n+2$ indeterminates with constant complex coefficients (for some n) such that $P(z, y, y', \dots, y^{(n)}) = 0$ for every $y \in \mathcal{F}$. Boshernitzan and Rubel [6] showed that the iterative logarithm of a given power series $f \in z + z^2\mathbb{C}[[z]]$ is differentially algebraic if and only if the family $f^{[1]}, f^{[2]}, \dots$ of iterates of f is uniformly differentially algebraic. Bergweiler [4], on the other hand, showed that the iterates of the Taylor series at 0 of a transcendental entire function are not uniformly differentially algebraic.

Put together, these results imply that *if* $f \in z + z^2\mathbb{C}[[z]]$ *is transcendental with infinite radius of convergence, then* $\operatorname{itlog}(f)$ *is both differentially transcendental and has radius of convergence* 0*.* Obviously, the question of a common generalization of these facts arises:

Question. Let $\mathbb{C}\{z\}$ denote the ring of convergent power series in the indeterminate z, i.e., the subring of $\mathbb{C}[[z]]$ consisting of those power series which converge absolutely in a neighborhood of 0. Let $f \in z + z^2\mathbb{C}\{z\}$ be transcendental with infinite radius of convergence. Is $\operatorname{itlog}(f)$ differentially transcendental over $\mathbb{C}\{z\}$?

To answer this question in full generality might be quite difficult. In this note we consider a test case and show:

Theorem 1.1. *The power series*

$$\operatorname{itlog}(e^z-1)=\frac{1}{2}z^2-\frac{1}{12}z^3+\frac{1}{48}z^4-\frac{1}{180}z^5+\frac{11}{8640}z^6-\frac{1}{6720}z^7+\cdots$$

is differentially transcendental over $\mathbb{C}\{z\}$.

The power series $\operatorname{itlog}(e^z-1)$ is of interest (to this author) since it is the exponential generating function of a sequence

$$0,1,-\frac{1}{2},\frac{1}{2},-\frac{2}{3},\frac{11}{12},-\frac{3}{4},-\frac{11}{6},\frac{29}{4},\frac{493}{12},-\frac{2711}{6},-\frac{12406}{15},\frac{2636317}{60},\dots$$

of rational numbers which recently arose both in a conjecture made by Shadrin and Zvonkine [19] (and proved in [1]) in connection with a generating series for Hurwitz numbers, and also in another context (joint work of the author with van den Dries and van der Hoeven on asymptotic differential algebra). We do not know whether the *ordinary* generating function of this sequence is differentially transcendental (over $\mathbb{C}[z]$, let alone over $\mathbb{C}\{z\}$).

Our strategy for the proof of Theorem 1.1 is to argue by contradiction: assuming that $y=\operatorname{itlog}(e^z-1)$ is differentially algebraic over $\mathbb{C}\{z\}$, let P be a non-zero polynomial in $n+1$ indeterminates (for some n) with coefficients in $\mathbb{C}\{z\}$ such that $P(y,y',\dots,y^{(n)})=0$. We deduce (in Section 3) that such P, if chosen to be of *minimal complexity* (as defined in Section 2), has to have a very specific shape, entailing that certain functional equations have non-trivial solutions in $\mathbb{C}\{z\}$. On the other hand, in Section 4 we argue, by refining the elementary technique of Lewin [15], based on growth properties of entire functions and used in his proof that $\operatorname{itlog}(e^z-1)\notin\mathbb{C}\{z\}$, that these equations only admit trivial solutions in convergent power series, leading to the desired contradiction.

Our arguments also allow us to reduce the general case of the question above to the consideration of (perhaps novel) variations of Julia's equation (see Section 5):

Proposition 1.2. *Let* $f\in z+z^2\mathbb{C}[[z]]$, $f\neq z$, *have infinite radius of convergence, and suppose that there are no integers* $k>0$ *and non-zero* $y\in z\mathbb{C}\{z\}$ *such that*

$$y\circ f=\frac{y}{1-(f''/f')y}\cdot(f')^k \quad \textit{or} \quad y\circ f=\frac{y}{1-S(f)y}\cdot(f')^k,$$

where

$$S(f)=(f''/f')'-\frac{1}{2}(f''/f')^2=f^{(3)}/f'-\frac{3}{2}(f''/f')^2.$$

Then $\operatorname{itlog}(f)$ *is differentially transcendental over* $\mathbb{C}\{z\}$.

We conjecture that the condition in this proposition is always satisfied (and hence the question posed above has a positive answer). More generally, we conjecture: *Let* $f\in z+z^2\mathbb{C}\{z\}$ *have infinite radius of convergence, and let* $g\in\mathbb{C}\{z\}$ *be the Taylor series at* 0 *of a meromorphic function on* $\mathbb{C}$. *Then there is no non-zero* $y\in z\mathbb{C}\{z\}$ *satisfying*

$$y\circ f=\frac{y}{1-gy}\cdot(f')^k \qquad (k>0).$$

Notation. Throughout the paper, m, n range over the set $\mathbb{N}=\{0,1,2,\dots\}$ of natural numbers.

2. Differential Polynomials

A **differential ring** is a commutative ring R equipped with a **derivation** of R, i.e., a map $\partial\colon R \to R$ which is additive ($\partial(f+g) = \partial(f) + \partial(g)$ for all $f, g \in R$) and satisfies the Leibniz Rule ($\partial(f \cdot g) = f \cdot \partial(g) + \partial(f) \cdot g$ for all $f, g \in R$). We also write y' instead of $\partial(y)$ and similarly $y^{(n)}$ instead of $\partial^n(y)$, where ∂^n is the nth iterate of ∂. A subring of R which is closed under ∂ is called a **differential subring** of R.

Let Y be a differential indeterminate over the differential ring R. Then $R\{Y\}$ denotes the ring of differential polynomials in Y over R. As ring, $R\{Y\}$ is just the polynomial ring $R[Y, Y', Y'', \dots]$ in the distinct indeterminates $Y^{(n)}$ over R, where as usual we write $Y = Y^{(0)}$, $Y' = Y^{(1)}$, $Y'' = Y^{(2)}$. We consider $R\{Y\}$ as the differential ring whose derivation, extending the derivation of R and also denoted by ∂, is given by $\partial(Y^{(n)}) = Y^{(n+1)}$ for every n. For $P(Y) \in R\{Y\}$ and y an element of a differential ring containing R as a differential subring, we let $P(y)$ be the element of that extension obtained by substituting $y, y', \dots$ for $Y, Y', \dots$ in P, respectively. We say that an element y of a differential ring extension of R is **differentially algebraic over** R if there is some $P \in R\{Y\}$, $P \neq 0$, such that $P(y) = 0$, and if y is not differentially algebraic over R, then y is said to be **differentially transcendental over** R**.** Clearly to be algebraic over R means in particular to be differentially algebraic over R.

For any $(r+1)$-tuple $\boldsymbol{i} = (i_0, \dots, i_r)$ of natural numbers put

$$Y^{\boldsymbol{i}} := Y^{i_0} (Y')^{i_1} \cdots (Y^{(r)})^{i_r}.$$

We also set

$$|\boldsymbol{i}| := i_0 + \cdots + i_r, \qquad \|\boldsymbol{i}\| := i_1 + 2i_2 + \cdots + r i_r.$$

Let $P \in R\{Y\}$. The smallest $r \in \mathbb{N}$ such that $P \in R[Y, Y', \dots, Y^{(r)}]$ is called the **order** of the differential polynomial P. Let $r = \operatorname{order}(P)$, and let $\boldsymbol{i} = (i_0, \dots, i_r)$ range over $\mathbb{N}^{1+r}$. We denote by $P_{\boldsymbol{i}} \in R$ the coefficient of $Y^{\boldsymbol{i}}$ in P; then

$$P(Y) = \sum_{\boldsymbol{i}} P_{\boldsymbol{i}} Y^{\boldsymbol{i}}.$$

We also define the **support** of P as

$$\operatorname{supp} P := \big\{ \boldsymbol{i} \in \mathbb{N}^{1+r} : P_{\boldsymbol{i}} \neq 0 \big\},$$

and for each $i \in \mathbb{N}$ we define

$$\operatorname{supp}_{r,i} P := \big\{ \boldsymbol{i} \in \operatorname{supp} P : i_r = i \big\}.$$

For $P \neq 0$, by the **complexity** of P we mean the triple $(r, d, s) \in \mathbb{N}^3$ where $r = \operatorname{order}(P)$, d is the degree of P in the indeterminate $Y^{(r)}$, and s is the number of elements of the set $\operatorname{supp}_{r,d} P$. In this context we order $\mathbb{N}^3$ lexicographically.

3. A Criterion for Differential Transcendence over a Subring

In the following we view $\mathbb{C}[[z]]$ as a differential ring with the derivation $\frac{d}{dz}$; then $\mathbb{C}\{z\}$ is a differential subring of $\mathbb{C}[[z]]$. Moreover, if $y \in \mathbb{C}\{z\}$ then $y \circ f \in \mathbb{C}\{z\}$ for all $f \in z\mathbb{C}\{z\}$ and $y/f \in \mathbb{C}\{z\}$ for all non-zero $f \in \mathbb{C}\{z\}$ with $\operatorname{ord}(y) \geqslant \operatorname{ord}(f)$. Here and below, given a non-zero power series $f \in \mathbb{C}[[z]]$ with $f \in z^n\mathbb{C}[[z]] \setminus z^{n+1}\mathbb{C}[[z]]$ we let $\operatorname{ord}(f) = n$, and we set $\operatorname{ord}(0) := +\infty > \mathbb{N}$. More generally, given a subring R of $\mathbb{C}[[z]]$ and $f \in z\mathbb{C}[[z]]$, we say that R is **closed under substitution of** f if R is closed under the $\mathbb{C}$-algebra endomorphism $y \mapsto y \circ f$ of $\mathbb{C}[[z]]$, and we say that R is

closed under division in $\mathbb{C}[[z]]$ if for all $y, f \in R$ with $f \neq 0$ and $\operatorname{ord}(y) \geqslant \operatorname{ord}(f)$ we have $y/f \in R$.

We set $\phi = e^z - 1 \in z\mathbb{Q}[[z]]$. As a first step in the proof of Theorem 1.1, we want to give a criterion for solutions of Julia's equation (1.2) with $f = \phi$ to be algebraic over certain kinds of differential subrings of $\mathbb{C}[[z]]$:

Proposition 3.1. *Let R be a differential subring of $\mathbb{C}[[z]]$ which contains $\mathbb{C}[e^{\pm z}]$ and which is closed under substitution of ϕ and closed under division. Suppose that for every integer $k > 0$, R does not contain non-zero solutions y of the functional equations*

$$y \circ \phi = y \cdot e^{kz} \tag{3.1}$$

and

$$y \circ \phi = \frac{y}{1-y} \cdot e^{kz}, \qquad y \in z\mathbb{C}[[z]]. \tag{3.2}$$

Then every $h \in \mathbb{C}[[z]]$ which is differentially algebraic over R and satisfies Julia's equation $h \circ \phi = h \cdot e^z$ is algebraic over R.

We denote by $\left[{j \atop i}\right]$ the Stirling numbers of the first kind (the number of permutations of a j-element set having i disjoint cycles). Towards a proof of this proposition, we first note that an easy induction on n, using the familiar recurrence relation for the Stirling numbers of the first kind (see [10, (6.8)]), shows that for every $h \in \mathbb{C}[[z]]$ and every n we have

$$(h^{(n)} \circ \phi) \cdot e^{nz} = \sum_{m=0}^{n} (-1)^{n-m} \begin{bmatrix} n \\ m \end{bmatrix} (h \circ \phi)^{(m)}.$$

Let now $h \in \mathbb{C}[[z]]$ and suppose that $h \circ \phi = h \cdot e^z$. Then we further have

$$(h^{(n)} \circ \phi) \cdot e^{nz} = e^z \sum_{k=0}^{n} \left(\sum_{m=k}^{n} (-1)^{n-m} \binom{m}{k} \begin{bmatrix} n \\ m \end{bmatrix} \right) h^{(k)}. \tag{3.3}$$

The coefficients of the $h^{(k)}$ in this sum are given by the entries of the bi-infinite triangular matrix

$$(H_{kn}) := \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ & 1 & 1 & -1 & 2 & -6 & \cdots \\ & & 1 & 0 & -1 & 5 & \cdots \\ & & & 1 & -2 & 5 & \cdots \\ & & & & 1 & -5 & \cdots \\ & & & & & 1 & \cdots \\ & & & & & & \ddots \end{pmatrix}.$$

It is easy to see that

$$H_{n,n+1} = (n+1)\left(1 - \frac{n}{2}\right) \qquad \text{for every } n.$$

In particular, for each $n > 0$ there is some $k < n$ such that $H_{kn} \neq 0$ (namely, $k = n-1$ if $k \neq 3$ and $k = 1$ if $n = 3$).

Let now R be a differential subring of $\mathbb{C}[[z]]$ closed under substitution of ϕ and containing $\mathbb{C}[e^{\pm z}]$, and let H denote the endomorphism of the ring $R\{Y\}$ extending the endomorphism $y \mapsto y \circ \phi$ of R such that

$$Y^{(n)} \mapsto e^{(1-n)z} \sum_{k=0}^{n} H_{kn}\, Y^{(k)} \qquad \text{for every } n.$$

For every $P \in R\{Y\}$ we then have

$$P(h) \circ \phi = H(P)(h).$$

Suppose R satisfies all hypotheses of Proposition 3.1, and assume that h is differentially algebraic over R. Let $P \in R\{Y\}$ be non-zero of lowest complexity (r, d, s) (with respect to the lexicographic ordering of $\mathbb{N}^3$) such that $P(h) = 0$. We want to deduce that then $r = 0$ (hence h is algebraic over R), so for a contradiction assume $r > 0$ (and hence $d > 0$). Below we let $\boldsymbol{i}$, $\boldsymbol{j}$ range over $\mathbb{N}^{1+r}$. For every $\boldsymbol{i}$ we have

$$H(P_{\boldsymbol{i}} Y^{\boldsymbol{i}}) = (P_{\boldsymbol{i}} \circ \phi) e^{(|\boldsymbol{i}| - \|\boldsymbol{i}\|)z} Y^{\boldsymbol{i}} + \text{terms of degree} < i_r \text{ in } Y^{(r)}$$

and thus

$$H(P) = \left(\sum_{\boldsymbol{j} \in \operatorname{supp}_{r,d} P} (P_{\boldsymbol{j}} \circ \phi) e^{(|\boldsymbol{j}| - \|\boldsymbol{j}\|)z} Y^{\boldsymbol{j}} \right) + \text{terms of degree} < d \text{ in } Y^{(r)}.$$

Take $\boldsymbol{j} \in \operatorname{supp}_{r,d} P$, and define

$$Q := (P_{\boldsymbol{j}} \circ \phi) e^{(|\boldsymbol{j}| - \|\boldsymbol{j}\|)z}\, P - P_{\boldsymbol{j}}\, H(P) \in R\{Y\}.$$

Then the differential polynomial Q has smaller complexity than P and also satisfies $Q(h) = 0$; hence $Q = 0$ by choice of P and thus

$$\operatorname{supp} P = \operatorname{supp} H(P). \tag{3.4}$$

As we shall see, the fact that $Q = 0$ also severely restricts the shape of P. We first observe that by the displayed formula for $H(P)$ above, $Q = 0$ yields

$$(P_{\boldsymbol{j}} \circ \phi) e^{(|\boldsymbol{j}| - \|\boldsymbol{j}\|)z} P_{\boldsymbol{i}} = P_{\boldsymbol{j}} e^{(|\boldsymbol{i}| - \|\boldsymbol{i}\|)z} (P_{\boldsymbol{i}} \circ \phi)$$

for every $\boldsymbol{i} \in \operatorname{supp}_{r,d} P$. For every such $\boldsymbol{i}$, setting $y := P_{\boldsymbol{i}}/P_{\boldsymbol{j}}$ if $\operatorname{ord}(P_{\boldsymbol{j}}) \leqslant \operatorname{ord}(P_{\boldsymbol{i}})$ and $y := P_{\boldsymbol{j}}/P_{\boldsymbol{i}}$ if $\operatorname{ord}(P_{\boldsymbol{j}}) > \operatorname{ord}(P_{\boldsymbol{i}})$, we therefore obtain a non-zero solution $y \in R$ of (3.1) with $k = \pm(|\boldsymbol{j}| - |\boldsymbol{i}| + \|\boldsymbol{i}\| - \|\boldsymbol{j}\|)$. Our assumption on the solutions of (3.1) in R yields $k \leqslant 0$; the case $k < 0$ can be excluded on formal grounds, cf. Lemma 3.2 below. Therefore we have $k = 0$, that is, $|\boldsymbol{i}| - \|\boldsymbol{i}\| = |\boldsymbol{j}| - \|\boldsymbol{j}\|$. Since $\boldsymbol{j} \in \operatorname{supp}_{r,d} P$ was arbitrary, we thus have shown:

$$|\boldsymbol{i}| - \|\boldsymbol{i}\| = |\boldsymbol{j}| - \|\boldsymbol{j}\| \quad \text{for all } \boldsymbol{i}, \boldsymbol{j} \in \operatorname{supp}_{r,d} P. \tag{3.5}$$

Next, we concentrate on the terms of $Y^{(r)}$-degree $d-1$ in $H(P)$. Writing

$$H(P) = \sum_{i=0}^{d} H(P)_i\, (Y^{(r)})^i \qquad \text{where } H(P)_i \in R[Y, Y', \dots, Y^{(r-1)}],$$

and for every $\boldsymbol{i} = (i_0, \dots, i_r)$ setting $\boldsymbol{i}' := (i_0, \dots, i_{r-1})$, we have

$$H(P)_d = \sum_{\boldsymbol{j} \in \operatorname{supp}_{r,d} P} (P_{\boldsymbol{j}} \circ \phi) e^{(|\boldsymbol{j}| - \|\boldsymbol{j}\|)z} Y^{\boldsymbol{j}'}$$

and

$$H(P)_{d-1} = \sum_{\boldsymbol{j}\in\operatorname{supp}_{r,d} P} \sum_{k=0}^{r-1} dH_{kr}\,(P_{\boldsymbol{j}}\circ\phi)e^{(|\boldsymbol{j}|-\|\boldsymbol{j}\|)z} Y^{\boldsymbol{j}'}Y^{(k)} + \sum_{\boldsymbol{i}\in\operatorname{supp}_{r,d-1} P} (P_{\boldsymbol{i}}\circ\phi)e^{(|\boldsymbol{i}|-\|\boldsymbol{i}\|)z} Y^{\boldsymbol{i}'}.$$

Using (3.5) one easily shows:

Claim 1. *For each $\boldsymbol{i}$ there exists at most one pair $(\boldsymbol{j},k)$ where $\boldsymbol{j}\in\operatorname{supp}_{r,d} P$ and $k\in\{0,\dots,r-1\}$ such that $Y^{\boldsymbol{i}'}=Y^{\boldsymbol{j}'}Y^{(k)}$.*

Moreover, (3.4) implies:

Claim 2. *Let $\boldsymbol{j}\in\operatorname{supp}_{r,d} P$ and $k<r$ with $H_{kr}\neq 0$. Then there is some $\boldsymbol{i}\in\operatorname{supp}_{r,d-1} P$ such that $Y^{\boldsymbol{i}'}=Y^{\boldsymbol{j}'}Y^{(k)}$.*

(If not, then $Y^{\boldsymbol{j}'}Y^{(k)}$ appears with the non-zero coefficient $dH_{kr}\,(P_{\boldsymbol{j}}\circ\phi)e^{(|\boldsymbol{j}|-\|\boldsymbol{j}\|)z}$ in $H(P)_{d-1}$, by Claim 1 and the displayed formula for $H(P)_{d-1}$, and hence also appears with a non-zero coefficient in P_{d-1}, since $\operatorname{supp} P=\operatorname{supp} H(P)$, a contradiction.)

Take $\boldsymbol{j}\in\operatorname{supp}_{r,d} P$ and $k\in\{0,\dots,r-1\}$ with $H_{kr}\neq 0$ and $\boldsymbol{i}\in\operatorname{supp}_{r,d-1} P$ such that $Y^{\boldsymbol{i}'}=Y^{\boldsymbol{j}'}Y^{(k)}$. Note that $|\boldsymbol{i}|=|\boldsymbol{j}|$ and $\|\boldsymbol{i}\|=\|\boldsymbol{j}\|+k-r$. Now the vanishing of Q yields the identity

$$(P_{\boldsymbol{j}}\circ\phi)e^{(|\boldsymbol{j}|-\|\boldsymbol{j}\|)z}\,P_{\boldsymbol{i}} = P_{\boldsymbol{j}}\left(c\,(P_{\boldsymbol{j}}\circ\phi)e^{(|\boldsymbol{j}|-\|\boldsymbol{j}\|)z} + (P_{\boldsymbol{i}}\circ\phi)e^{(|\boldsymbol{i}|-\|\boldsymbol{i}\|)z}\right),$$

where $c:=dH_{kr}\neq 0$. Suppose $\operatorname{ord} P_{\boldsymbol{i}}\geqslant \operatorname{ord} P_{\boldsymbol{j}}$. Setting $y:=P_{\boldsymbol{i}}/(cP_{\boldsymbol{j}})\in R$ we then obtain

$$y\circ\phi = e^{(k-r)z}\cdot(y-1).$$

Substituting $z=0$ gives a contradiction. Now suppose $\operatorname{ord} P_{\boldsymbol{i}}<\operatorname{ord} P_{\boldsymbol{j}}$. Then setting $y:=cP_{\boldsymbol{j}}/P_{\boldsymbol{i}}\in R$ we obtain $\operatorname{ord} y>0$ and

$$(1-y)\cdot(y\circ\phi) = y\cdot e^{(r-k)z}.$$

This contradicts our assumptions on the solutions of (3.2) in R. These two contradictions finish the proof of Proposition 3.1. □

Before we continue, it is perhaps worth pointing out that (3.1) and (3.2) always have non-zero *formal* solutions. As in the introduction, we write $f_n=\frac{1}{n!}\frac{d^n f}{dz^n}(0)$ for the nth coefficient of the power series $f\in\mathbb{C}[[z]]$.

Lemma 3.2. *Let $f\in z+z^{p+1}\mathbb{C}[[z]]$ with $f_{p+1}\neq 0$ and $k\in\mathbb{Z}$, $k\neq 0$. If $k>0$, then the functional equation*

$$y\circ f = y\cdot(f')^k \tag{3.6}$$

has a unique solution $y\in z^{k(p+1)}\mathbb{C}[[z]]$ such that $y_{k(p+1)}=(f_{p+1})^k$, namely $y=\operatorname{itlog}(f)^k$, and the general solution of (3.6) is given by $y=c\cdot\operatorname{itlog}(f)^k$ where $c\in\mathbb{C}$. If $k<0$, then the only solution $y\in\mathbb{C}[[z]]$ to (3.6) is $y=0$.

Proof. Write $f'=1+z^pg$ where $g\in\mathbb{C}[[z]]$; then $(f')^k=1+z^ph$ where $h_0=kg_0=k(p+1)f_{p+1}$. Suppose the formal power series

$$y=\sum_{n\geqslant 0} y_n z^n\in\mathbb{C}[[z]]$$

with indeterminate coefficients $y_n \in \mathbb{C}$ satisfies (3.6). Substituting into (3.6) and subtracting y on both sides of the equation one obtains

$$\sum_{n\geqslant 1}\sum_{i=1}^{n}\binom{n}{i} y_n z^{n-i} r^i = \sum_{n\geqslant 0}\left(\sum_{i=0}^{n} h_{i+p} y_{n-i}\right) z^{n+p}$$

where

$$r := f - z = \sum_{n\geqslant p+1} f_n z^n \in z^{p+1}\mathbb{C}[[z]].$$

Comparing the coefficients of z^{p+j}, where $j \geqslant 0$, on both sides of this equation yields

$$j y_j f_{p+1} + A_j = k(p+1) f_{p+1} y_j + B_j$$

where A_j, B_j are linear combinations of y_i with $0 \leqslant i < j$. This easily yields the lemma. □

For (3.2) we show, slightly more generally:

Lemma 3.3. *Let $f \in z + z^{p+1}\mathbb{C}[[z]]$ with $f_{p+1} \neq 0$ and $g \in \mathbb{C}[[z]]$ with $\operatorname{ord}(g) \geqslant p \geqslant 1$. Then there exists a unique $y \in z^p\mathbb{C}[[z]]$ such that $y_p = \frac{1}{2}(p f_{p+1} - g_p)$ and*

$$y \circ f = \frac{y}{1-y} \cdot (1+g). \tag{3.7}$$

Proof. Set

$$y = \sum_{n\geqslant p} y_n z^n \in z^p\mathbb{C}[[z]]$$

with indeterminate coefficients y_n. Substituting into (3.7) and subtracting y on both sides we obtain

$$\sum_{n\geqslant p}\sum_{i=1}^{n}\binom{n}{i} y_n z^{n-i} r^i = \sum_{n\geqslant 0}\left(\sum_{i=0}^{n} g_{i+p} y_{n-i+p}\right) z^{n+2p} + (y^2 + y^3 + \cdots)(1+g)$$

where $r := f - z$. We now equate the coefficients of z^{p+j}, where $j \geqslant p$, on both sides of this equation. On the left-hand side we obtain $j y_j f_{p+1} + A_j$ where A_j is a linear combination of y_i with $p \leqslant i < j$ (with coefficients depending only on the f_n). The coefficient of z^{p+j} in the first summand on the right-hand side is $g_p y_j + B_j$ where B_j is a linear combination of y_i with $p \leqslant i < j$ (with coefficients depending only on the g_n). The coefficient of z^{p+j} in $y^m g$, where $m \geqslant 2$, is given by

$$\sum_{\substack{i_0+\cdots+i_m=j\\ i_1,\dots,i_m\geqslant p}} g_{p+i_0} y_{i_1}\cdots y_{i_m}$$

and hence is a polynomial in y_i with $p \leqslant i < j$. The coefficient of z^{p+j} in y^m, where $m \geqslant 2$, is

$$\sum_{\substack{i_1+\cdots+i_m=p+j\\ i_1,\dots,i_m\geqslant p}} y_{i_1}\cdots y_{i_m}.$$

If $m \geqslant 3$ then this sum only involves y_i with $p \leqslant i < j$, and if $m = 2$ then this sum has the form $2y_j y_p +$ quadratic form in y_i with $p \leqslant i < j$. In summary, we obtain the equations

$$j y_j f_{p+1} = g_p y_j + 2 y_p y_j + C_j \qquad (j \geqslant p) \tag{3.8}$$

where C_j is a polynomial in y_i with $p \leqslant i < j$ (depending only on the f_n and g_n) with $C_p = 0$. For $j = p$ this equation for y_p reads

$$py_p f_{p+1} = g_p y_p + 2(y_p)^2,$$

with zeros $y_p = 0$ and $y_p = \frac{1}{2}(pf_{p+1} - g_p)$. Moreover, if we have $y_p = \frac{1}{2}(pf_{p+1} - g_p)$ then (3.8) simplifies to $y_j \cdot (j-p) f_{p+1} = C_j$, hence in this case, the y_j with $j > p$ are uniquely determined. This yields the lemma. □

Since $\mathbb{C}\{z\}$ is well-known to be algebraically closed in $\mathbb{C}[[z]]$ (a consequence of Puiseux's Theorem), Proposition 3.1 and the following proposition imply Theorem 1.1:

Proposition 3.4. *Let $k > 0$ be an integer. Then the only $y \in \mathbb{C}\{z\}$ satisfying (3.1) or (3.2) is $y = 0$.*

The proof of this proposition is given in the next section.

4. Two Functional Equations

To prove Proposition 3.4, we follow the geometric argument of Lewin [15] showing that $\operatorname{itlog}(\phi) \notin \mathbb{C}\{z\}$. Of course, the part of Proposition 3.4 which concerns (3.1) also follows from Lemma 3.2 and the general fact, indicated in the introduction, that $\operatorname{itlog}(f)$ always has radius of convergence 0 for $f \in z + z^2\mathbb{C}\{z\}$, $f \neq z$. However, the argument of [15] will, with some modifications, also apply to (3.2), therefore we first show how to deal with (3.1) by this method. Lewin first observed the following elementary properties of the entire function $e^z - 1$:

Lemma 4.1. *Let $z \in \mathbb{C}$ with $-\pi < \operatorname{Im} z \leqslant \pi$, and $w = e^z - 1$. Then*

(1) *if $\operatorname{Re} z \leqslant 0$, then $|w| < |z|$; and*
(2) *if $\operatorname{Re} z > 0$, then either $\operatorname{Re} w \leqslant 0$ or $|w| > |z|$.*

Let now h be a non-constant holomorphic function defined in a neighborhood U of 0 in $\mathbb{C}$, and $V \subseteq U$ a neighborhood of 0 in $\mathbb{C}$ mapped into U under $z \mapsto e^z - 1$, and suppose h satisfies the functional equation

$$h(e^z - 1) = g(h(z)) \cdot e^{kz} \qquad \text{for every } z \in V, \tag{4.1}$$

where $k \in \mathbb{Z}$ and $g\colon h(V) \to \mathbb{C}$ is an injective holomorphic map. (In view of (3.1) and (3.2), of course, we are mainly interested in $g(z) = z$ and $g(z) = \frac{z}{1-z}$.) Let $\varrho > 0$ be the radius of convergence of the Taylor series of h at 0. Lemma 4.1 above implies that $\varrho = \infty$. To see how, suppose $\varrho < \infty$, and let z_0 be a singular point of h on the circle of convergence of h around 0. Since h is periodic with period $2\pi i$ (if V is large enough) we may assume that $-\pi < \operatorname{Im} z_0 \leqslant \pi$. Thus if $\operatorname{Re} z_0 \leqslant 0$ then $w_0 = e^{z_0} - 1$ satisfies $|w_0| < |z_0|$ by Lemma 4.1, (1) and thus lies inside the circle of convergence of h around 0; however, w_0 is also a singularity of h, by (4.1), a contradiction. Suppose $\operatorname{Re} z_0 > 0$, and let $w_0 \in \mathbb{C}$ such that $e^{w_0} - 1 = z_0$ and $-\pi < \operatorname{Im} w_0 \leqslant \pi$. Then $1 < |z_0 + 1| = |e^{w_0}| = e^{\operatorname{Re} w_0}$, hence $\operatorname{Re} w_0 > 0$ and thus $|w_0| < |z_0|$ by part (2) of Lemma 4.1, whereas by (4.1), w_0 is a singular point of h; this is again a contradiction. These contradictions show that $\varrho = \infty$.

This argument and the following lemma together already demonstrate that (3.1) has only the trivial solution in $\mathbb{C}\{z\}$:

Lemma 4.2. *Let $h\colon \mathbb{C} \to \mathbb{C}$ be an entire function and $k \in \mathbb{Z}$, $k \neq 0$ such that*

$$h(e^z - 1) = h(z) \cdot e^{kz} \qquad \text{for every } z \in \mathbb{C}. \tag{4.2}$$

Then $h \equiv 0$.

Before we give a proof of this lemma, we establish some auxiliary facts (in a generality which is sufficient to later aid in dealing with (3.1) as well).

Lemma 4.3. *Let $V\colon \mathbb{R}^{>0} \to \mathbb{R}$ be a convex function and $s_0 \in \mathbb{R}^{>0}$ such that for some real constants b, c, d and C with $c > b > 0$ and $c > 1$ we have*

$$V(cs) \leqslant bV(s) + ds + C \qquad \text{for all } s \geqslant s_0. \tag{4.3}$$

Then there are $A, B \in \mathbb{R}$ such that

$$V(s) \leqslant As + B \qquad \text{for all } s \geqslant s_0. \tag{4.4}$$

Proof. Choose $A > 0$ such that both

$$K := A(c - b) - d > 0$$

and

$$A(c-1)s_0 \geqslant (b-1)V(s_0) + ds_0 + C.$$

Set $B := V(s_0) - As_0$. Then the last inequality yields

$$(b-1)B = (b-1)V(s_0) - (b-1)As_0 \leqslant Ks_0 - C$$

and hence

$$bB + C \leqslant B + Kc^n s_0 \qquad \text{for every } n. \tag{4.5}$$

An easy induction on n now shows $V(c^n s_0) \leqslant A(c^n s_0) + B$ for every n: the case $n = 0$ holds by choice of B, and if the inequality in question has been shown for a given value of n, then

$$\begin{aligned} V(c^{n+1}s_0) &\leqslant d(c^n s_0) + bV(c^n s_0) + C \qquad \text{(by (4.3))} \\ &\leqslant d(c^n s_0) + bA(c^n s_0) + bB + C \qquad \text{(by inductive hypothesis)} \\ &= (Ac - K)c^n s_0 + bB + C \\ &\leqslant Ac^{n+1}s_0 + B \qquad \text{(by (4.5))}. \end{aligned}$$

Hence (4.4) holds for all s of the form $s = c^n s_0$ (for some n). Convexity of V yields that then (4.4) also holds for all s with $c^n s_0 < s < c^{n+1}s_0$ for some n. Hence (4.4) holds for all $s \geqslant s_0$. □

This lemma, in combination with Hadamard's Three Circle Theorem, has a useful consequence. Given an entire function f we denote as usual by

$$M(r, h) = \max_{|z|=r} |h(z)| \qquad (r > 0)$$

the maximum modulus of h on the circle $|z| = r$ (an increasing function of r).

Corollary 4.4. *Let h be an entire function, and suppose that for some real constants b, c, d, C with $c > 0$ and $C > 0$ we have*

$$M(r, h) \leqslant C\big(M(c \log r, h)\big)^b r^d \qquad \text{for all sufficiently large } r > 0.$$

Then h is a polynomial.

Proof. This is clear if $b \leqslant 0$: if $M(r,h) \leqslant 1$ for all sufficiently large $r > 0$, then h is constant, while if $b \leqslant 0$ and $M(r,h) > 1$ for sufficiently large $r > 0$ then the hypothesis of the lemma implies $M(r,h) \leqslant C\,r^d$ for sufficiently large $r > 0$, hence h is a polynomial by Liouville's Theorem. Thus, from now on we assume $b > 0$. Increasing c if necessary, we then further reduce to the case that $c > 1$ and $c > b$. Put $V(s) = \log M(r,h)$ where $r = e^s$. By Hadamard's theorem, $s \mapsto V(s)$ is a convex function, and by our hypothesis

$$V(cs) = \log M(r^c, h) \leqslant \log C + b \log M(c^2 \log r, h) + cds \leqslant \\ \log C + b\log M(r,h) + cds = \log C + bV(s) + cds$$

for sufficiently large $s > 0$. Thus the previous lemma applies to V, with $\log C$ in place of C and cd in place of d. Phrased in terms of $M(r,h)$ the lemma yields the existence of constants $\alpha, \beta > 0$ such that $M(r,h) \leqslant \beta r^\alpha$ for sufficiently large r. Hence h is a polynomial. □

By a classical result of Pólya [18] (see also [11, Theorem 2.9]) there exists a constant $c \in (0,1)$ such that for all entire functions f, g with $g(0) = 0$,

$$M(r, f \circ g) \geqslant M\left(cM(\tfrac{r}{2}, g), f\right) \quad \text{for all } r > 0. \tag{4.6}$$

We now show Lemma 4.2. So suppose h is an entire function satisfying (4.2), where $k \in \mathbb{Z}$, $k \neq 0$. Note that h is periodic (with period $2\pi i$), so if h is a polynomial, then h is constant and hence 0. Thus it is enough to show that h is a polynomial. Let r, s range over $\mathbb{R}^{>0}$, and write $M(r) = M(r,h)$. By Pólya's inequality and (4.2) we have

$$M(c(e^{r/2} - 1)) \leqslant M\left(cM(\tfrac{r}{2}, e^z - 1)\right) \leqslant M(r, h(e^z - 1)) \\ \leqslant M(r) \cdot M(r, e^{kz}) = M(r) \cdot e^{|k|r} \qquad \text{for all } r.$$

Since $M(r)$ is an increasing function of r, we have $M(e^{r/4}) \leqslant M(c(e^{r/2} - 1))$ for sufficiently large r, and hence

$$M(s) \leqslant M(4 \log s) \cdot s^{4|k|} \qquad \text{for sufficiently large } s.$$

By the corollary above we see that h is a polynomial as desired. □

Now we turn to the functional equation (3.2). The analogue of Lemma 4.2 in this case is:

Lemma 4.5. *Suppose $h\colon \mathbb{C} \to \mathbb{C}$ is an entire function with $h(0) = 0$, and $k \in \mathbb{Z}$ such that*

$$h(e^z - 1) = \frac{h(z)}{1 - h(z)} e^{kz} \qquad \text{for every } z \in \mathbb{C}. \tag{4.7}$$

Then $h \equiv 0$.

For the proof, it is convenient, in addition to $M(r,f)$, to employ the Nevanlinna characteristic

$$T(r,f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta \qquad (r > 0)$$

of an entire function f. Here $\log^+ x = \log x$ for every real $x > 1$ and $\log^+ x := 0$ if $0 \leqslant x \leqslant 1$. Some basic properties of T include, for all entire functions f and g and all $r > 0$:

$$T(r, fg) \leqslant T(r,f) + T(r,g)$$

and

$$T\left(r, \frac{1}{f-a}\right) \leqslant T(r,f) + O(1) \qquad \text{for } a \in \mathbb{C} \setminus f(\mathbb{C}).$$

Moreover, for all entire f we have

$$T(r,f) \leqslant \log^+ M(r,f) \leqslant 3T(2r,f) \qquad \text{for all } r > 0, \tag{4.8}$$

see [11, Theorem 1.6]. There also do exist analogues of Pólya's inequality concerning the Nevanlinna characteristic of composite functions (instead of the maximum modulus); see, e.g., [5]. However, the Pólya inequality (4.6) in combination with (4.8) suffices for our purposes.

We are now ready to show Lemma 4.5. Let h be an entire function vanishing at the origin and $k \in \mathbb{Z}$ satisfying (4.7). As in the proof of Lemma 4.2 it is enough to show that h is a polynomial. If h is bounded, then h is constant, and we are done; hence we can assume that h is unbounded; in particular $\log^+ M(r,h) = \log M(r,h)$ for sufficiently large $r > 0$. Since $T(r, e^{kz}) = kr/\pi$, by (4.7) we have

$$T(r, h(e^z - 1)) \leqslant 2\, T(r,h) + \frac{kr}{\pi} + O(1).$$

Using (4.6) and (4.8) this yields

$$\log M\big(c(e^{r/4} - 1), h\big) \leqslant \log M\left(\frac{r}{2}, h(e^z - 1)\right) \leqslant 6 \log M(r,h) + \frac{3kr}{\pi} + O(1)$$

where $c \in (0,1)$. Hence

$$\log M(s,h) \leqslant 6 \log M(8 \log s, h) + \frac{24k}{\pi} \log s + O(1)$$

for sufficiently large s, or equivalently, for some constant $C > 0$,

$$M(s,h) \leqslant C \left(M(8 \log s, h)\right)^6 s^{24k/\pi} \qquad \text{for sufficiently large } s.$$

Corollary 4.4 now shows that h is a polynomial. □

5. The General Case

In this final section we make a few remarks about how one could go about answering the question posed in the introduction about differential transcendence of $\operatorname{itlog}(f)$ over $\mathbb{C}\{z\}$ for general $f \in z + z^2\mathbb{C}\{z\}_\infty$, following the strategy employed above in the case $f = \phi = e^z - 1$.

We first consider a generalization of the transformation formula (3.3). In the following we fix $f \in z + z^2\mathbb{C}[[z]]$. Note that $f' \in 1 + z\mathbb{C}[[z]]$ is a unit in $\mathbb{C}[[z]]$. Set $G_{mn} := 0$ if $m > n$ or $m = 0 < n$, and $G_{00} := (f')^{-1}$, and define G_{mn} with $0 < m \leqslant n$ by the recurrence

$$G_{m,n+1} = (1-2n) G_{mn} f'' + (G'_{mn} + G_{m-1,n}) f'.$$

An easy induction on n shows that then for each $h \in \mathbb{C}[[z]]$ and n we have

$$(h^{(n)} \circ f) \cdot (f')^{2n-1} = G_{1n}(f)\,(h \circ f)' + G_{2n}(f)\,(h \circ f)'' + \cdots + G_{nn}(f)\,(h \circ f)^{(n)}.$$

Organizing the G_{mn} into a triangular matrix we obtain:

$$G := (G_{mn})_{m,n} = \begin{pmatrix} (f')^{-1} & 0 & 0 & 0 & \cdots \\ & 1 & -f'' & 3(f'')^2 - f'f^{(3)} & \cdots \\ & & f' & -3f'f'' & \cdots \\ & & & (f')^2 & \cdots \\ & & & & \ddots \end{pmatrix}.$$

Note that $G_{nn} = (f')^{n-1}$ for each n and

$$G_{n,n+1} = -\frac{n(n+1)}{2} f''(f')^{n-1} \qquad \text{for every } n. \tag{5.1}$$

Now set

$$H_{kn} = \sum_{m=k}^{n} \binom{m}{k} f^{(m-k+1)} G_{mn} \qquad \text{for } k = 0, \dots, n.$$

So if we define the triangular matrix

$$B := (B_{km}) = \begin{pmatrix} f' & f'' & f^{(3)} & f^{(4)} & \cdots \\ & f' & 2f'' & 3f^{(3)} & \cdots \\ & & f' & 3f'' & \cdots \\ & & & f' & \cdots \\ & & & & \ddots \end{pmatrix}$$

$$\text{where } B_{km} = \binom{m}{k} f^{(m-k+1)} \text{ for } m \geqslant k,$$

then

$$B \cdot G = \begin{pmatrix} 1 & f'' & f'f^{(3)} - (f'')^2 & (f')^2 f^{(4)} - 4f'f''f^{(3)} + 3(f'')^2 & \cdots \\ & f' & f'f'' & -3f'(f'')^2 + 2(f')^2 f^{(3)} & \cdots \\ & & (f')^2 & 0 & \cdots \\ & & & (f')^3 & \cdots \\ & & & & \ddots \end{pmatrix}.$$

Setting $C = (C_{mn})_{m,n}$ with $C_{nn} = (f')^{-n}$ for every n and $C_{mn} = 0$ for $m \neq n$ we finally let

$$H = (H_{kn}) := B \cdot G \cdot C =$$
$$\begin{pmatrix} 1 & f''/f' & f^{(3)}/f' - (f''/f')^2 & [\cdots] & \cdots \\ & 1 & f''/f' & -3(f''/f')^2 + 2f^{(3)}/f' & \cdots \\ & & 1 & 0 & \cdots \\ & & & 1 & \cdots \\ & & & & \ddots \end{pmatrix}.$$

Note that (5.1) implies

$$H_{n,n+1} = (n+1)\left(1 - \frac{n}{2}\right) f''/f' \qquad \text{for every } n.$$

Also note that

$$\frac{1}{2} H_{13} = f^{(3)}/f' - \frac{3}{2}(f''/f')^2$$

is the Schwarzian derivative $S(f)$ of f. It is well-known that if $g \in \mathbb{C}\{z\}$ and $S(g) = 0$ then g is a fractional linear transformation: $g = \frac{az+b}{cz+d}$ for some $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$ (see, e.g., [12, Section 10.1]). Thus:

Lemma 5.1. *Suppose $f \neq z$ has infinite radius of convergence. Then for each $n > 0$ there is some $k < n$ such that $H_{kn} \neq 0$, of the form $H_{kn} = C\,(f''/f')$ or $H_{kn} = C\,S(f)$ where $C \in \mathbb{Z}$.*

Suppose now that $h \in \mathbb{C}[[z]]$ and f satisfy Julia's equation $h \circ f = h \cdot f'$. Then for every n

$$(h^{(n)} \circ f) \cdot (f')^{2n-1} = H_{0n}(f)\,h + H_{1n}(f)\,h' + \cdots + H_{nn}(f)\,h^{(n)}.$$

Let R be a differential subring of $\mathbb{C}[[z]]$ closed under substitution of f which contains $\mathbb{C}[f', (f')^{-1}]$, and denote the R-algebra automorphism of $R\{Y\}$ with

$$Y^{(n)} \mapsto (f')^{1-2n}\big(H_{0n}\,Y + H_{1n}\,Y' + \cdots + H_{nn}\,Y^{(n)}\big) \qquad \text{for every } n$$

also by H. Then for every $P \in K\{Y\}$ we have $P(h) \circ f = H(P)(h)$. Formally arguing as in the proof of Proposition 3.4, using Lemma 5.1 at the appropriate places, we therefore arrive at the following generalization of this proposition:

Proposition 5.2. *Suppose $f \neq z$ has infinite radius of convergence, and let R be a differential subring of $\mathbb{C}[[z]]$ which contains $\mathbb{C}[f', (f')^{-1}]$ and which is closed under substitution of f and closed under division. Suppose that for no integer $k > 0$, R contains non-zero solutions y of the functional equations*

$$y \circ f = y \cdot (f')^k, \tag{5.2}$$

$$y \circ f = \frac{y}{1 - (f''/f')y} \cdot (f')^k, \qquad y \in z\mathbb{C}[[z]], \tag{5.3}$$

$$y \circ f = \frac{y}{1 - S(f)y} \cdot (f')^k, \qquad y \in z\mathbb{C}[[z]]. \tag{5.4}$$

Then every $h \in \mathbb{C}[[z]]$ which is differentially algebraic over R and satisfies Julia's equation $h \circ f = h \cdot f'$ is algebraic over R.

Thus, in order to answer the question posed in the introduction in full generality, it suffices to show that given a power series $f \neq z$ with infinite radius of convergence and a positive integer k, neither of the functional equations (5.2), (5.3) nor (5.4) has a non-zero convergent solution y. For (5.2), this follows from Lemma 3.2 and the fact (mentioned in the introduction) that $\operatorname{itlog}(f) \notin \mathbb{C}\{z\}$. This shows Proposition 1.2.

References

1. M. Aschenbrenner, *Logarithms of iteration matrices, and proof of a conjecture by Shadrin and Zvonkine,* manuscript (2010).
2. I. N. Baker, *Fractional iteration near a fixpoint of multiplier* 1, J. Austral. Math. Soc. **4** (1964), 143–148.
3. ———, *Zusammensetzungen ganzer Funktionen,* Math. Z. **69** (1958), 121–163.
4. W. Bergweiler, *Solution of a problem of Rubel concerning iteration and algebraic differential equations,* Indiana Univ. Math. J. **44** (1995), no. 1, 257–268.
5. ———, *On the Nevanlinna characteristic of a composite function,* Complex Variables Theory Appl. **10** (1988), no. 2-3, 225–236.
6. M. Boshernitzan and L. Rubel, *Coherent families of polynomials,* Analysis **6** (1986), no. 4, 339–389.

7. J. Écalle, *Sommations de séries divergentes en théorie de l'itération des applications holomorphes,* C. R. Acad. Sci. Paris Sér. A-B **282** (1976), no. 4, Aii, A203–A206.
8. ———, *Théorie itérative: introduction à la théorie des invariants holomorphes,* J. Math. Pures Appl. (9) **54** (1975), 183–258.
9. P. Erdős and E. Jabotinsky, *On analytic iteration,* J. Analyse Math. **8** (1960/1961), 361–376.
10. R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics,* 2nd ed., Addison-Wesley Publishing Company, Reading, MA, 1994.
11. W. Hayman, *Meromorphic Functions,* Oxford Mathematical Monographs, Clarendon Press, Oxford, 1964.
12. E. Hille, *Ordinary Differential Equations in the Complex Domain,* Dover Publications, Inc., Mineola, NY, 1997.
13. E. Jabotinsky, *Analytic iteration,* Trans. Amer. Math. Soc. **108** (1963), 457–477.
14. M. Kuczma, B. Choczewski, and G. Roman, *Iterative Functional Equations,* Encyclopedia of Mathematics and its Applications, vol. 32, Cambridge University Press, Cambridge, 1990.
15. M. Lewin, *An example of a function with non-analytic iterates,* J. Austral. Math. Soc. **5** (1965), 388–392.
16. L. Lipshitz and L. Rubel, *A gap theorem for power series solutions of algebraic differential equations,* Amer. J. Math. **108** (1986), no. 5, 1193–1213.
17. K. Mahler, *Lectures on Transcendental Numbers,* Lecture Notes in Mathematics, vol. 546, Springer-Verlag, Berlin-New York, 1976.
18. G. Pólya, *On an integral function of an integral function,* J. London Math. Soc. **1** (1926), 12–15.
19. S. Shadrin and D. Zvonkine, *Changes of variables in ELSV-type formulas,* Michigan Math. J. **55** (2007), no. 1, 209–228.
20. R. Stanley, *Enumerative Combinatorics,* vol. 2, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999.
21. G. Szekeres, *Fractional iteration of entire and rational functions,* J. Austral. Math. Soc. **4** (1964), 129–142.

University of California, Los Angeles, California, USA
E-mail address: matthias@math.ucla.edu