CONSTRUCTING ω-FREE HARDY FIELDS

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To the memory of Michael Boshernitzan (1950–2019)

Abstract. We show that every Hardy field extends to an ω-free Hardy field. This result relates to classical oscillation criteria for second-order homogeneous linear differential equations. It is essential in [12], and here we apply it to answer questions of Boshernitzan, and to generalize a theorem of his.

Introduction

We let a, b, c range over $\mathbb{R}$ in this introduction. Let $f\colon [a,+\infty) \to \mathbb{R}$ be continuous, and consider the second-order linear differential equation

$$(*) \qquad Y'' + fY \;=\; 0.$$

A (real) *solution* to $(*)$ is a $\mathcal{C}^2$-function $y\colon [a,+\infty) \to \mathbb{R}$ such that $y'' + fy = 0$, and such a solution is either (identically) zero, or its zero set as a subspace of $[a,+\infty)$ is discrete. Some equations $(*)$ have oscillating solutions. Here, a continuous function $g\colon [a,+\infty) \to \mathbb{R}$ *oscillates* if $g(t) = 0$ for arbitrarily large $t \geqslant a$, and $g(t) \neq 0$ for arbitrarily large $t \geqslant a$. Every oscillating solution to $(*)$ has arbitrarily large isolated zeros, whereas each nonzero non-oscillating solution to $(*)$ has only finitely many zeros. We say that f *generates oscillation* if $(*)$ has an oscillating solution. In this case, by Sturm [60], *every* nonzero solution to $(*)$ oscillates. This is really a property of the germ of f at $+\infty$: for $b \geqslant a$, f generates oscillation iff $f|_{[b,+\infty)}$ does. Below "germ" means "germ at $+\infty$" and "oscillates" and "generates oscillation" will hold by convention for the germ of f iff it holds for f.

There is an extensive literature giving criteria on f to generate oscillation (see for example [37, 61, 64]), some of which have their root in another fundamental result of Sturm, his Comparison Theorem: for continuous $g : [a,+\infty) \to \mathbb{R}$,

if f generates oscillation and $f \leqslant g$ on $[a,+\infty)$, then g generates oscillation.

To see this theorem in action, let $\ell_0 := x$ be the germ of the identity function on $\mathbb{R}$ and inductively define the germs $\ell_1, \ell_2, \dots$ by $\ell_{n+1} := \log \ell_n$ for each n. Also set

$$\gamma_n \;:=\; \frac{1}{\ell_0 \cdots \ell_n}, \qquad \omega_n \;:=\; \gamma_0^2 + \gamma_1^2 + \cdots + \gamma_n^2 \;=\; \frac{1}{\ell_0^2} + \frac{1}{(\ell_0\ell_1)^2} + \cdots + \frac{1}{(\ell_0 \cdots \ell_n)^2}.$$

Computation shows that the germ $y = \gamma_n^{-1/2}$ satisfies the equation $y'' + (\omega_n/4)y = 0$, so $\omega_n/4$ doesn't generate oscillation. More precisely, a germ

$$\frac{\omega_n + c\gamma_n^2}{4} \;=\; \frac{1}{4}\left(\frac{1}{\ell_0^2} + \frac{1}{(\ell_0\ell_1)^2} + \cdots + \frac{1}{(\ell_0 \cdots \ell_{n-1})^2} + \frac{c+1}{(\ell_0 \cdots \ell_n)^2}\right)$$

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generates oscillation if and only if $c > 0$. (See [14, Chapter 6, Theorem 10] or Corollary 7.8 below.) In the rest of this introduction $\mathcal{C}$ is the ring of germs (at $+\infty$) of continuous real-valued functions whose domain is a subset of $\mathbb{R}$ containing some interval $(a, +\infty)$, and f, g range over $\mathcal{C}$. We partially order $\mathcal{C}$ by:

$$f \leqslant g \;:\Longleftrightarrow\; f(t) \leqslant g(t), \text{ eventually,}$$

where tacitly f and g also denote representatives of their germs, and "eventually" means "for all sufficiently large real t". So if $f \leqslant \omega_n/4$ for some n, then $(*)$ has no oscillating solutions, whereas if $f \geqslant (\omega_n + c\gamma_n^2)/4$ for some n and $c > 0$, then f generates oscillation. For $n = 0$ this was first noted by A. Kneser [39]: if $f \leqslant 1/4x^2$, then f does not generate oscillation, but if $f \geqslant (1+c)/4x^2$ for some $c > 0$, then f generates oscillation. The general case is implicit in Riemann-Weber [63, p. 63], and was rediscovered by various authors [31, 36, 51].

Now $\omega_n \leqslant \omega_{n+1} \leqslant \omega_{n+1} + \gamma_{n+1}^2 \leqslant \omega_n + c\gamma_n^2$ for all n and all $c > 0$, and it is not difficult to obtain a germ ω such that $\omega_n \leqslant \omega \leqslant \omega_n + \gamma_n^2$ for all n, hence the Riemann-Weber criterion is inconclusive for $f = \omega/4$. (One can even take such ω to be the germ of an analytic function $(a, +\infty) \to \mathbb{R}$, by our Example 7.12.) However, Hartman [31] observed that if f is the germ of a *logarithmico-exponential function* (*LE-function*) in the sense of Hardy [26, 28], then this criterion applies:

(H) $\quad$ *f does not generate oscillation* $\quad \Longleftrightarrow \quad$ *$f \leqslant \omega_n/4$ for some n.*

Roughly speaking, LE-functions are the real-valued functions obtained in finitely many steps from real constants and the variable x using addition, multiplication, division, exponentiation, and taking logarithms. Examples: every rational function in $\mathbb{R}(x)$, each power function x^r ($r \in \mathbb{R}$), each iterated logarithm ℓ_n, $\mathrm{e}^{\sqrt{\ell_1}/\ell_2}$, etc. Hardy showed that each LE-function, defined on some interval $(a, +\infty) \to \mathbb{R}$, has eventually constant sign, and so the germs at $+\infty$ of such functions form what Bourbaki [20] later called a *Hardy field*: a subfield H of the ring $\mathcal{C}$ consisting of germs of continuously differentiable functions $(a, +\infty) \to \mathbb{R}$ such that the germ of the derivative of the function is also in H (so H is a differential field).

The Hardy field H_{LE} of LE-functions has good uses (see [21, 30, 50]), but overall is too small for many analytic purposes: for example, every $h \in H_{\mathrm{LE}}$ is *differentially algebraic over* $\mathbb{R}$ (that is, satisfies a differential equation $P\big(h, h', \dots, h^{(n)}\big) = 0$ with P a nonzero polynomial over $\mathbb{R}$ in $1 + n$ indeterminates), and yet H_{LE} contains no antiderivative of e^{x^2} (Liouville [47, 48]). Boshernitzan [18, Theorem 17.7] (see also Corollary 7.9) generalized Hartman's result and showed that the equivalence (H) continues to hold provided f is merely assumed to be *hardian* (i.e., contained in some Hardy field) and to be differentially algebraic over $\mathbb{R}$. In [18, Conjecture 17.11] he conjectured a version of (H) with the increasing sequence (ω_n) replaced by the decreasing sequence $(\omega_n + \gamma_n^2)$, and "for some" replaced accordingly by "for all": if f is hardian and differentially algebraic over $\mathbb{R}$, then

(B) $\quad$ *f does not generate oscillation* $\quad \Longleftrightarrow \quad$ *$f \leqslant (\omega_n + \gamma_n^2)/4$ for all n.*

Corollary 7.10 below establishes this conjecture along the way to our main result. Important here is (cf. Theorem 5.25 and Corollary 5.28) that if f is hardian and differentially algebraic over $\mathbb{R}$, then there is an n such that $\ell_n \leqslant g$ for all positive infinite g in $\mathbb{R}\langle f \rangle := \mathbb{R}(f, f', f'', \dots)$ (= the Hardy field generated by f over $\mathbb{R}$); here g *is positive infinite* ($g \in \mathcal{C}$) means that $g(t) \to +\infty$ as $t \to +\infty$.

The germ f being hardian and not generating oscillation has nice consequences: for example, each Hardy field containing such an f extends to one which contains a fundamental system of solutions of $(*)$, and a Hardy-type inequality with weight f holds. (See Proposition 6.1 and Remarks 3.16, 6.15, respectively.) Hence it is desirable to have versions of (H) and (B) for arbitrary hardian f. Many natural functions, for example the restrictions of Euler's Γ-function and Riemann's ζ-function to $(1, +\infty)$, have non-oscillating germs that are hardian but are not differentially algebraic over $\mathbb{R}$ [56]. More trouble are hardian germs ω as above, that is, $\omega_n \leqslant \omega \leqslant \omega_n + \gamma_n^2$ for all n, as in our Example 7.12. In this paper we show that nevertheless, *versions of* (H) *and* (B) *can be restored if we extend the relevant Hardy field and prolong the sequence* (ℓ_n) *of iterated logarithms accordingly.*

To make this precise, let H be a Hardy field. The partial ordering $\leqslant$ on $\mathcal{C}$ restricts to a total ordering on H, making H an ordered field. We also assume: $H \supseteq \mathbb{R}(x)$ and H is *log-closed,* that is, $\log h \in H$ for all $h > 0$ in H. (This holds for $H = H_{\text{LE}}$, and every Hardy field extends to one with these properties.) Then $H \supseteq \mathbb{R}(\ell_0, \ell_1, \ell_2, \dots)$ and we extend the sequence $\ell_0, \ell_1, \ell_2, \dots$ by transfinite recursion to a sequence (ℓ_ρ) of positive infinite elements of H, indexed by all ordinals ρ less than some infinite limit ordinal κ, as follows: $\ell_{\rho+1} := \log \ell_\rho$, and if λ is an infinite limit ordinal such that all ℓ_ρ with $\rho < \lambda$ have already been chosen, then we pick ℓ_λ to be any positive infinite element of H such that $\ell_\lambda \leqslant \ell_\rho$ for all $\rho < \lambda$, if there is such an element; otherwise we put $\kappa := \lambda$. Given $z \in H$ we set $\omega(z) := -(2z' + z^2)$; then for $y \in H \setminus \{0\}$ we have $y'' + fy = 0$ iff $z := 2y'/y$ satisfies the (Riccati) equation $\omega(z) = 4f$. We now define

$$\gamma_\rho := \ell_\rho'/\ell_\rho, \qquad \omega_\rho := \omega(-\gamma_\rho'/\gamma_\rho).$$

We have $\gamma_n := \ell_n'/\ell_n = 1/(\ell_0 \cdots \ell_n)$, and taking $y := 1/\sqrt{\gamma_n}$ we obtain

$$z := 2y'/y = -\gamma_n'/\gamma_n = \gamma_0 + \cdots + \gamma_n, \qquad \omega(z) = \gamma_0^2 + \gamma_1^2 + \cdots + \gamma_n^2.$$

(Note that these γ_n, ω_n agree with the γ_n, ω_n given earlier.) In the beginning of Section 7 we show that the sequences (ω_ρ) and $(\omega_\rho + \gamma_\rho^2)$ in H are strictly increasing and strictly decreasing, respectively, and that $\omega_\lambda < \omega_\mu + \gamma_\mu^2$ for all indices λ, μ.

Using results from [18, 55] it is easy (see Corollary 7.1 and subsequent comment) to extend any Hardy field to an H as above (that is, $H \supseteq \mathbb{R}(x)$ and H is log-closed) such that the following variant of (H) holds for all $f \in H$:

(H*) $\quad$ *f does not generate oscillation* $\iff$ *$f \leqslant \omega_\rho/4$ for some ρ.*

Restoring (B) requires the concept of a Hardy field being ω-*free*. This (first-order) concept was introduced in a more general setting in our book [ADH, 11.7], where it was shown to be very robust. We repeat the formal definition in Section 1, and Corollary 7.3 says that H is ω-free if and only if for all $f \in H$ the equivalence (H*) holds as well as the following equivalence:

(B*) $\quad$ *f does not generate oscillation* $\iff$ *$f \leqslant (\omega_\rho + \gamma_\rho^2)/4$ for all ρ.*

Keeping in mind that H ranges over log-closed Hardy fields containing $\mathbb{R}(x)$, here is the main result of this paper, already announced in [7, 8, 9]:

Theorem. *Every Hardy field is contained in some ω-free H.*

A more precise version is given by Theorem 7.14. There are H that are not ω-free, but those with a natural origin usually are. (For example, by Hartman's and Boshernitzan's oscillation criteria, H_{LE} is ω-free.) The proof of Theorem 7.14 takes nevertheless considerable effort.

After the preliminary Section 1 we give in Section 2 basic definitions and facts about germs of one-variable (real- or complex-valued) functions, and in Section 3 we collect the main facts we need about second-order linear differential equations. In Section 4 we introduce Hardy fields in more detail and review some extension results due to Boshernitzan [17, 18, 19] and Rosenlicht [55]. In Section 5 we discuss upper and lower bounds on the growth of hardian germs from [18, 19, 56], and Section 6 focusses on second-order linear differential equations over Hardy fields. In Section 7 we review ω-freeness, prove the theorem above, and some refinements.

By Zorn, each Hardy field is contained in one which is *maximal*, that is, which has no proper Hardy field extension. By the theorem above, maximal Hardy fields are ω-free. This is a first important step towards showing that they are *H-closed fields*, in the terminology of [8]. This requires serious further work, which is in [12].

In Section 8 we show that our main theorem by itself, combined with results about ω-freeness from [ADH], already has some applications. First, it yields a positive answer to a question about maximal Hardy fields posed by Boshernitzan [19, §7]:

Corollary 1. *Every maximal Hardy field contains a positive infinite germ ℓ_ω such that $\ell_\omega \leqslant \ell_n$ for all n.*

This corollary is actually much weaker than [10, Corollary 4.8], which however ultimately relies on deeper results from [12] that in turn depend on Theorem 7.14.

In the remainder of this introduction we let H range over arbitrary Hardy fields. The intersection $\mathrm{E}(H)$ of all maximal Hardy fields that contain H is a Hardy field that is log-closed land properly contains H_{LE}. These Hardy fields $\mathrm{E}(H)$ were studied in detail by Boshernitzan [17, 18], who proved, among other things, that the sequence (ℓ_n) is coinitial in the set of positive infinite elements of $\mathrm{E}(\mathbb{Q}) = \mathrm{E}(H_{\text{LE}})$. Theorem 8.6 generalizes this fact as follows:

Corollary 2. *If $H \supseteq \mathbb{R}(x)$ is log-closed and ω-free, then any log-sequence (ℓ_ρ) in H as above is coinitial in the set $\mathrm{E}(H)^{>\mathbb{R}}$ of positive infinite elements of $\mathrm{E}(H)$.*

Notations and conventions. We generally follow the conventions from [ADH]. In particular, m, n range over the set $\mathbb{N} = \{0, 1, 2, \dots\}$ of natural numbers. Given an additively written abelian group A we set $A^{\neq} := A \setminus \{0\}$. By convention, the ordering of an ordered abelian group or ordered field is *total*. For an ordered abelian group A and $b \in A$ we put $A^{>b} := \{a \in A : a > b\}$ and $A^{>} := A^{>0}$, and likewise with $\geqslant$, $<$, or $\leqslant$ in place of $>$. Rings are associative with identity 1 (and almost always commutative). For a commutative ring R we let $R^\times$ be the multiplicative group of units of R.

1. Preliminaries on Asymptotic Fields

In this section we first collect some basic definitions from [ADH] needed throughout this paper. We then review some general facts on iterated logarithmic derivatives, iterated exponentials, and the asymptotic behavior of "large" solutions of algebraic differential equations in H-asymptotic fields. We do not need these facts to achieve our main objective, but they will be used at a few points for applications and corollaries; see Sections 5 and 8.

Differential rings and fields. Let R be a *differential ring*, that is, a commutative ring R containing (an isomorphic copy of) $\mathbb{Q}$ as a subring equipped with a derivation $\partial\colon R \to R$. Then $C_R := \ker \partial$ is a subring of R, called the ring of constants of R, and $\mathbb{Q} \subseteq C_R$. If R is a field, then so is C_R. A *differential field* is a differential ring that happens to be a field. When the derivation ∂ of R is clear from the context and $a \in R$, then we denote $\partial(a), \partial^2(a), \dots, \partial^n(a), \dots$ by $a', a'', \dots, a^{(n)}, \dots$, and for $a \in R^\times$ we set $a^\dagger := a'/a$ (the *logarithmic derivative* of a), so $(ab)^\dagger = a^\dagger + b^\dagger$ for $a, b \in R^\times$.

We have the differential ring $R\{Y\} = R[Y, Y', Y'', \dots]$ of differential polynomials in a differential indeterminate Y over R. We say that $P = P(Y) \in R\{Y\}$ has *order* at most $r \in \mathbb{N}$ if $P \in R[Y, Y', \dots, Y^{(r)}]$; in this case $P = \sum_{\boldsymbol{i}} P_{\boldsymbol{i}} Y^{\boldsymbol{i}}$, as in [ADH, 4.2], with $\boldsymbol{i}$ ranging over tuples $(i_0, \dots, i_r) \in \mathbb{N}^{1+r}$, $Y^{\boldsymbol{i}} := Y^{i_0}(Y')^{i_1} \cdots (Y^{(r)})^{i_r}$, coefficients $P_{\boldsymbol{i}}$ in R, and $P_{\boldsymbol{i}} \neq 0$ for only finitely many $\boldsymbol{i}$.

For $P \in R\{Y\}$ and $a \in R$ we let $P_{\times a} := P(aY)$. For $\phi \in R^\times$ we let R^ϕ be the *compositional conjugate* of R by ϕ: the differential ring with the same underlying ring as R but with derivation $\phi^{-1}\partial$ (usually denoted by δ) instead of ∂. We have an R-algebra isomorphism $P \mapsto P^\phi\colon R\{Y\} \to R^\phi\{Y\}$ such that $P^\phi(y) = P(y)$ for all $y \in R$; see [ADH, 5.7].

Let K be a differential field. Then $K\{Y\}$ is an integral domain, and the differential fraction field of $K\{Y\}$ is denoted by $K\langle Y\rangle$. Let y be an element of a differential field extension L of K. We let $K\{y\}$ be the differential subring of L generated by y over K, and let $K\langle y\rangle$ be the differential fraction field of $K\{y\}$ in L. We say that y is *differentially algebraic* over K if $P(y) = 0$ for some $P \in K\{Y\}^{\neq}$; otherwise y is called *differentially transcendental* over K. As usual in [ADH], the prefix "d" abbreviates "differentially", so "d-algebraic" means "differentially algebraic". We say that L is d-*algebraic* over K if each $y \in L$ is d-algebraic over K. See [ADH, 4.1] for more on this. We set $K^\dagger := (K^\times)^\dagger$, a subgroup of the additive group of K.

Valued fields. For a field K we have $K^\times = K^{\neq}$, and a (Krull) valuation on K is a surjective map $v\colon K^\times \to \Gamma$ onto an ordered abelian group Γ (additively written) satisfying the usual laws, and extended to $v\colon K \to \Gamma_\infty := \Gamma \cup \{\infty\}$ by $v(0) = \infty$, where the ordering on Γ is extended to a total ordering on Γ_∞ by $\gamma < \infty$ for all $\gamma \in \Gamma$. A *valued field* K is a field (also denoted by K) together with a valuation ring $\mathcal{O}$ of that field, and the corresponding valuation $v\colon K^\times \to \Gamma$ on the underlying field is such that $\mathcal{O} = \{a \in K : va \geqslant 0\}$ as explained in [ADH, 3.1].

Let K be a valued field with valuation ring $\mathcal{O}$ and valuation $v\colon K^\times \to \Gamma$. Then $\mathcal{O}$ is a local ring with maximal ideal $\smallo := \{a \in K : va > 0\}$. In this paper K always has *equicharacteristic zero*, that is, the residue field $\operatorname{res}(K) := \mathcal{O}/\smallo$ of K has characteristic zero. In asymptotic differential algebra, sometimes the following notations are more natural: with a, b ranging over K, set

$$a \asymp b \;:\Leftrightarrow\; va = vb, \qquad a \preccurlyeq b \;:\Leftrightarrow\; va \geqslant vb, \qquad a \prec b \;:\Leftrightarrow\; va > vb,$$
$$a \succcurlyeq b \;:\Leftrightarrow\; b \preccurlyeq a, \qquad a \succ b \;:\Leftrightarrow\; b \prec a, \qquad a \sim b \;:\Leftrightarrow\; a - b \prec a.$$

It is easy to check that if $a \sim b$, then $a, b \neq 0$ and $a \asymp b$, and that $\sim$ is an equivalence relation on $K^\times$. Let L be a valued field extension of K; then the relations $\asymp$, $\preccurlyeq$, etc. on L restrict to the corresponding relations on K, and we identify in the usual way the value group of K with an ordered subgroup of the value group of L and $\operatorname{res}(K)$ with a subfield of $\operatorname{res}(L)$. Such a valued field extension is called *immediate* if for every $a \in L^\times$ there is a $b \in K^\times$ with $a \sim b$. We use *pc-sequence* to abbreviate *pseudocauchy sequence*, and $a_\rho \rightsquigarrow a$ indicates that (a_ρ) is a pc-sequence pseudoconverging to a; here the a_ρ and a lie in a valued field understood from the context, see [ADH, 2.2, 3.2].

A binary relation $\preccurlyeq$ on a field K for which there is a valuation v on K such that $a \preccurlyeq b \Leftrightarrow va \geqslant vb$ for each $a, b \in K$ is called a *dominance relation* on K. See [ADH, 3.1] for an axiomatization of dominance relations.

Valued differential fields. As in [ADH], a *valued differential field* is a valued field of equicharacteristic zero together with a derivation, generally denoted by ∂, on the underlying field. The derivation ∂ of a valued differential field K is said to be *small* if $\partial\smallo \subseteq \smallo$; then $\partial\mathcal{O} \subseteq \mathcal{O}$ [ADH, 4.4.2], so ∂ induces a derivation on $\operatorname{res}(K)$ making the residue map $\mathcal{O} \to \operatorname{res}(K)$ into a morphism of differential rings. A valued differential field K in this paper is usually an *asymptotic field*, that is, for all nonzero $f, g \prec 1$ in K we have: $f \preccurlyeq g \Longleftrightarrow f' \preccurlyeq g'$. Every compositional conjugate of an asymptotic field is asymptotic.

Let K be an asymptotic field, with constant field $C = C_K$ and valuation ring $\mathcal{O}$. Then $C \subseteq \mathcal{O}$, and we say that K is d-*valued* if for all $f \in K$ with $f \asymp 1$ there is a $c \in C$ with $f \sim c$. Let $\operatorname{I}(K)$ be the $\mathcal{O}$-submodule of K generated by $\partial\mathcal{O}$. Then K is called *pre*-d-*valued* if $\operatorname{I}(K) \cap (K^\times)^\dagger = (\mathcal{O}^\times)^\dagger$. (This is not exactly the definition from [ADH, 10.1], but equivalent to it.) Pre-d-valued fields are exactly the valued differential subfields of d-valued fields, by [3, 4.4].

We associate to K its *asymptotic couple* (Γ, ψ), where $\psi\colon \Gamma^{\neq} \to \Gamma$ is given by

$$\psi(vg) \ = \ v(g^\dagger) \ \text{ for } g \in K^\times \text{ with } vg \neq 0.$$

We put $\Psi := \psi(\Gamma^{\neq})$. If we want to stress the dependence on K, we write (Γ_K, ψ_K) and Ψ_K instead of (Γ, ψ) and Ψ, respectively. An *asymptotic couple* (without mentioning any asymptotic field) is a pair (Γ, ψ) consisting of an ordered abelian group Γ and a map $\psi\colon \Gamma^{\neq} \to \Gamma$ subject to natural axioms obeyed by the asymptotic couples of asymptotic fields, see [ADH, 6.5]. We extend $\psi\colon \Gamma^{\neq} \to \Gamma$ to a map $\Gamma_\infty \to \Gamma_\infty$ by $\psi(0) := \psi(\infty) := \infty$. If (Γ, ψ) is understood from the context and $\gamma \in \Gamma$ we write $\gamma^\dagger$ and γ' instead of $\psi(\gamma)$ and $\gamma + \psi(\gamma)$, respectively. An *H-asymptotic couple* is an asymptotic couple (Γ, ψ) such that for all $\gamma, \delta \in \Gamma$ we have: $0 < \gamma \leqslant \delta \Rightarrow \psi(\gamma) \geqslant \psi(\delta)$. An asymptotic field whose asymptotic couple is H-asymptotic is called an *H-asymptotic field* (or an asymptotic field of *H-type*).

Let (Γ, ψ) be an asymptotic couple and $\Psi := \psi(\Gamma^{\neq})$. Then $\gamma \in \Gamma$ is said to be a *gap* in (Γ, ψ) if $\Psi < \gamma < (\Gamma^{>})'$. (There is at most one such γ.) We also say that (Γ, ψ) is *grounded* if Ψ has a largest element, and (Γ, ψ) has *asymptotic integration* if $(\Gamma^{\neq})' = \Gamma$. An asymptotic field is said to have a gap if its asymptotic couple does, and likewise with “grounded” or “asymptotic integration” in place of “has a gap”. See [ADH, 9.1, 9.2] for more on this. An element ϕ of an asymptotic field K is said to be *active in* K if $\phi \succcurlyeq f^\dagger$ for some $f \not\asymp 1$ in $K^\times$; in that case the derivation $\phi^{-1}\partial$ of the compositional conjugate K^ϕ is small, cf. [ADH, 11.1].

Next two concepts from [ADH, 11.6, 11.7] that may seem technical but that are key to understanding subtler aspects of Hardy fields. Let K be an asymptotic field.

We say that K is $\boldsymbol{\lambda}$*-free* if K is H-asymptotic and ungrounded, and for all $f \in K$ there exists $g \succ 1$ in K such that $f - g^{\dagger\dagger} \succcurlyeq g^{\dagger}$. We say that K is $\boldsymbol{\omega}$*-free* if K is H-asymptotic, ungrounded, and for all $f \in K$ there exists $g \succ 1$ in K such that $f - \omega(g^{\dagger\dagger}) \succcurlyeq (g^{\dagger})^2$, where $\omega(z) := -2z' - z^2$ for $z \in K$. This notion of $\boldsymbol{\omega}$-freeness is clearly first-order in the logical sense. (We do not use $\boldsymbol{\lambda}$-freeness or $\boldsymbol{\omega}$-freeness before Section 6.)

Ordered differential fields. An *ordered differential field* is a differential field K with an ordering on K making K an ordered field. Likewise, an *ordered valued differential field* is a valued differential field K equipped an ordering on K making K an ordered field (no relation between derivation, valuation, or ordering being assumed). Let K be an ordered differential field. Then we have the convex subring

$$\mathcal{O} := \big\{g \in K : |g| \leqslant c \text{ for some } c \in C\big\},$$

which is a valuation ring of K and has maximal ideal

$$\smallo = \big\{g \in K : |g| < c \text{ for all positive } c \in C\big\}.$$

We call K an *H-field* if for all $f \in K$ with $f > C$ we have $f' > 0$, and $\mathcal{O} = C + \smallo$. We view such an H-field K as an ordered valued differential field with its valuation given by $\mathcal{O}$. *Pre-H-fields* are the ordered valued differential subfields of H-fields. Every pre-H-field is H-asymptotic, and each H-field is d-valued of H-type. See [ADH, 10.5] for basic facts about (pre-)H-fields. An H-field K is said to be *Liouville closed* if K is real closed and for all $f, g \in K$ there exists $y \in K^{\times}$ with $y' + fy = g$. Every H-field extends to a Liouville closed one; see [ADH, 10.6].

In the rest of this section K is an H-asymptotic field, and f, g range over K.

Iterated logarithmic derivatives. Let (Γ, ψ) be an H-asymptotic couple. We let γ range over Γ, and we denote by

$$[\gamma] = \big\{\delta \in \Gamma : |\gamma| \leqslant n|\delta| \text{ and } \delta \leqslant n|\gamma| \text{ for some } n \geqslant 1\big\}$$

the archimedean class of γ; cf. [ADH, 2.4]. We define $\gamma^{\langle n \rangle} \in \Gamma_{\infty}$ inductively by $\gamma^{\langle 0 \rangle} := \gamma$ and $\gamma^{\langle n+1 \rangle} := \psi(\gamma^{\langle n \rangle})$. The following is [2, Lemma 5.2]; for the convenience of the reader we include a proof:

Lemma 1.1. *Suppose that $0 \in (\Gamma^{<})'$, $\gamma \neq 0$, and $n \geqslant 1$. If $\gamma^{\langle n \rangle} < 0$, then $\gamma^{\langle i \rangle} < 0$ for $i = 1, \dots, n$ and $[\gamma] > [\gamma^{\dagger}] > \cdots > [\gamma^{\langle n-1 \rangle}] > [\gamma^{\langle n \rangle}]$.*

Proof. By [ADH, 9.2.9] we have $(\Gamma^{>})' \subseteq \Gamma^{>}$, so the case $n = 1$ follows from [ADH, 9.2.10(iv)]. Assume inductively that the lemma holds for a certain value of $n \geqslant 1$, and suppose $\gamma^{\langle n+1 \rangle} < 0$. Then $\gamma^{\langle n \rangle} \neq 0$, so we can apply the case $n = 1$ to $\gamma^{\langle n \rangle}$ instead of γ and get $[\gamma^{\langle n \rangle}] > [\gamma^{\langle n+1 \rangle}]$. By the inductive assumption the remaining inequalities will follow from $\gamma^{\langle n \rangle} < 0$. From $0 \in (\Gamma^{<})'$ we obtain an element 1 of $\Gamma^{>}$ with $0 = (-1)' = -1 + 1^{\dagger}$. Suppose $\gamma^{\langle n \rangle} \geqslant 0$. Then $\gamma^{\langle n \rangle} \in \Psi$, thus $0 < \gamma^{\langle n \rangle} < 1 + 1^{\dagger} = 1 + 1$ and so $[\gamma^{\langle n \rangle}] \leqslant [1]$. Hence $0 > \gamma^{\langle n+1 \rangle} \geqslant 1^{\dagger} = 1$, a contradiction. □

Suppose now that (Γ, ψ) is the asymptotic couple of K and $y \in K^{\times}$. In [ADH, p. 213], we defined the nth iterated logarithmic derivative of y: $y^{\langle 0 \rangle} := y$, and recursively, if $y^{\langle n \rangle} \in K$ is defined and nonzero, then $y^{\langle n+1 \rangle} := (y^{\langle n \rangle})^{\dagger}$, while otherwise $y^{\langle n+1 \rangle}$ is not defined. (Thus if $y^{\langle n \rangle}$ is defined, then so are $y^{\langle 0 \rangle}, \dots, y^{\langle n-1 \rangle}$.)

If $(vy)^{\langle n\rangle} \neq \infty$, then $y^{\langle n\rangle}$ is defined and $v(y^{\langle n\rangle}) = (vy)^{\langle n\rangle} \in \Gamma$. Recall from [ADH, p. 383] that for $f, g \neq 0$,

$$f \prec\!\!\prec g \;:\Leftrightarrow\; f^\dagger \prec g^\dagger, \quad f \preceq\!\!\preceq g \;:\Leftrightarrow\; f^\dagger \preccurlyeq g^\dagger, \quad f \asymp\!\!\asymp g \;:\Leftrightarrow\; f^\dagger \asymp g^\dagger,$$

hence, assuming also $f, g \not\asymp 1$,

$$f \prec\!\!\prec g \;\Rightarrow\; [vf] < [vg], \qquad [vf] \leqslant [vg] \;\Rightarrow\; f \preceq\!\!\preceq g.$$

In the rest of this section we are given $x \succ 1$ in K with $x' \asymp 1$. Then $0 \in (\Gamma^<)'$, so from the previous lemma we obtain:

Corollary 1.2. *If $y \in K^\times$, $y \not\asymp 1$, $n \geqslant 1$, and $(vy)^{\langle n\rangle} < 0$, then $y^{\langle i\rangle} \succ 1$ for $i = 1, \dots, n$ and $[vy] > \big[v(y^\dagger)\big] > \cdots > \big[v(y^{\langle n-1\rangle})\big] > \big[v(y^{\langle n\rangle})\big]$.*

Let $\boldsymbol{i} = (i_0, \dots, i_n) \in \mathbb{Z}^{1+n}$ and $y \in K^\times$ be such that $y^{\langle n\rangle}$ is defined; we put

$$y^{\langle \boldsymbol{i}\rangle} \;:=\; (y^{\langle 0\rangle})^{i_0} \cdots (y^{\langle n\rangle})^{i_n} \in K.$$

If $y^{\langle n\rangle} \neq 0$, then $\boldsymbol{i} \mapsto y^{\langle \boldsymbol{i}\rangle} \colon \mathbb{Z}^{1+n} \to K^\times$ is a group morphism. Suppose now that $y \in K^\times$, $(vy)^{\langle n\rangle} < 0$, and $\boldsymbol{i} = (i_0, \dots, i_n) \in \mathbb{Z}^{1+n}$, $\boldsymbol{i} \neq 0$, and $m \in \{0, \dots, n\}$ is minimal with $i_m \neq 0$. Then by Corollary 1.2, $\big[v(y^{\langle \boldsymbol{i}\rangle})\big] = \big[v(y^{\langle m\rangle})\big]$. Thus if $y \succ 1$, we have the equivalence $y^{\langle \boldsymbol{i}\rangle} \succ 1 \;\Leftrightarrow\; i_m \geqslant 1$. If K is equipped with an ordering making it a pre-H-field and $y \succ 1$, then $y^\dagger > 0$, so $y^{\langle i\rangle} > 0$ for $i = 1, \dots, n$, and thus $\operatorname{sign} y^{\langle \boldsymbol{i}\rangle} = \operatorname{sign} y^{i_0}$.

Iterated exponentials. *In this subsection we assume that Ψ is downward closed.* For $f \succ 1$ we have $f' \succ f^\dagger$, so we can and do choose $\mathrm{E}(f) \in K^\times$ such that $\mathrm{E}(f) \succ 1$ and $\mathrm{E}(f)^\dagger \asymp f'$, hence $f \prec \mathrm{E}(f)$ and $f \prec\!\!\prec \mathrm{E}(f)$. Moreover, if $f, g \succ 1$, then

$$f \prec g \quad\Longleftrightarrow\quad \mathrm{E}(f) \prec\!\!\prec \mathrm{E}(g).$$

For $f \succ 1$ define $\mathrm{E}_n(f) \in K^{\succ 1}$ inductively by

$$\mathrm{E}_0(f) \;:=\; f, \qquad \mathrm{E}_{n+1}(f) \;:=\; \mathrm{E}\big(\mathrm{E}_n(f)\big),$$

and thus by induction

$$\mathrm{E}_n(f) \;\prec\; \mathrm{E}_{n+1}(f) \quad \text{and} \quad \mathrm{E}_n(f) \;\prec\!\!\prec\; \mathrm{E}_{n+1}(f) \qquad \text{for all } n.$$

In the rest of this subsection $f \succcurlyeq x$, and y ranges over elements of H-asymptotic extensions of K. The proof of the next lemma is like that of [4, Lemma 1.3(2)].

Lemma 1.3. *If $y \succcurlyeq \mathrm{E}_{n+1}(f)$, $n \geqslant 1$, then $y \neq 0$ and $y^\dagger \succcurlyeq \mathrm{E}_n(f)$.*

Proof. If $y \succcurlyeq \mathrm{E}_2(f)$, then $y \neq 0$, and using $\mathrm{E}_2(f) \succ 1$ we obtain

$$y^\dagger \;\succcurlyeq\; \mathrm{E}_2(f)^\dagger \;\asymp\; \mathrm{E}(f)' \;=\; \mathrm{E}(f)\,\mathrm{E}(f)^\dagger \;\asymp\; \mathrm{E}(f) f' \;\succcurlyeq\; \mathrm{E}(f),$$

Thus the lemma holds for $n = 1$. In general, $\mathrm{E}_{n-1}(f) \succcurlyeq f \succcurlyeq x$, hence the lemma follows from the case $n = 1$ applied to $\mathrm{E}_{n-1}(f)$ in place of f. □

An obvious induction on n using Lemma 1.3 shows: if $y \succcurlyeq \mathrm{E}_n(f)$, then $(vy)^{\langle n\rangle} \leqslant vf < 0$. We shall use this fact without further reference.

Lemma 1.4. *If $y \succcurlyeq \mathrm{E}_{n+1}(f)$, then $y^{\langle n\rangle}$ is defined and $y^{\langle n\rangle} \succcurlyeq \mathrm{E}(f)$.*

Proof. First note that if $y \neq 0$, $n \geqslant 1$, and $(y^\dagger)^{\langle n-1\rangle}$ is defined, then $y^{\langle n\rangle}$ is defined and $y^{\langle n\rangle} = (y^\dagger)^{\langle n-1\rangle}$. Now use induction on n and Lemma 1.3. □

Lemma 1.5. *If $y \succcurlyeq \mathrm{E}_n(f^2)$, then $y^{\langle n\rangle}$ is defined and $y^{\langle n\rangle} \succcurlyeq f$, with $y^{\langle n\rangle} \succ f$ if $f \succ x$.*

Proof. This is clear if $n = 0$, so suppose $y \succcurlyeq \mathrm{E}_{n+1}(f^2)$. Then by Lemma 1.4 (applied with f^2 in place of f) we have $y^{\langle n\rangle} \succcurlyeq \mathrm{E}(f^2) \succ 1$, so

$$y^{\langle n+1\rangle} \;=\; (y^{\langle n\rangle})^{\dagger} \;\succcurlyeq\; \mathrm{E}(f^2)^{\dagger} \;\asymp\; (f^2)' \;=\; 2ff' \;\succcurlyeq\; f,$$

with $y^{\langle n+1\rangle} \succ f$ if $f \succ x$, as required. $\square$

Corollary 1.6. *Suppose $y \succcurlyeq \mathrm{E}_n(f^2)$, and let $\boldsymbol{i} \in \mathbb{Z}^{1+n}$ be such that $\boldsymbol{i} > 0$ lexicographically. Then $y^{\langle n\rangle}$ is defined and $y^{\langle \boldsymbol{i}\rangle} \succcurlyeq f$, with $y^{\langle \boldsymbol{i}\rangle} \succ f$ if $f \succ x$.*

Proof. By Lemma 1.5, $y^{\langle n\rangle}$ is defined with $y^{\langle n\rangle} \succcurlyeq f$, and $y^{\langle n\rangle} \succ f$ if $f \succ x$. Let $m \in \{0,\dots,n\}$ be minimal such that $i_m \neq 0$; so $i_m \geqslant 1$. If $m = n$ then $y^{\langle \boldsymbol{i}\rangle} = (y^{\langle n\rangle})^{i_n} \succcurlyeq y^{\langle n\rangle}$, hence $y^{\langle \boldsymbol{i}\rangle} \succcurlyeq f$, with $y^{\langle \boldsymbol{i}\rangle} \succ f$ if $f \succ x$. Suppose $m < n$. Then $y \succcurlyeq \mathrm{E}_{m+1}(f^2)$ and hence $y^{\langle m\rangle} \succcurlyeq \mathrm{E}(f^2)$ by Lemma 1.4. Also, $f \preccurlyeq f^2 \prec\!\!\prec \mathrm{E}(f^2)$, thus $y^{\langle m\rangle} \succ\!\!\succ f$. The remarks following Corollary 1.2 now yield $y^{\langle \boldsymbol{i}\rangle} \succ f$. $\square$

Asymptotic behavior of $P(y)$ for large y. In this subsection $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ range over $\mathbb{N}^{1+n}$. Let $P_{\langle \boldsymbol{i}\rangle} \in K$ be such that $P_{\langle \boldsymbol{i}\rangle} = 0$ for all but finitely many $\boldsymbol{i}$ and $P_{\langle \boldsymbol{i}\rangle} \neq 0$ for some $\boldsymbol{i}$, and set $P := \sum_{\boldsymbol{i}} P_{\langle \boldsymbol{i}\rangle} Y^{\langle \boldsymbol{i}\rangle} \in K\langle Y\rangle$. So if $P \in K\{Y\}$, then $P = \sum_{\boldsymbol{i}} P_{\langle \boldsymbol{i}\rangle} Y^{\langle \boldsymbol{i}\rangle}$ is the logarithmic decomposition of the differential polynomial P as defined in [ADH, 4.2]. If y is an element in a differential field extension L of K such that $y^{\langle n\rangle}$ is defined, then we put $P(y) := \sum_{\boldsymbol{i}} P_{\langle \boldsymbol{i}\rangle} y^{\langle \boldsymbol{i}\rangle} \in L$ (and for $P \in K\{Y\}$ this has the usual value). Let $\boldsymbol{j}$ be lexicographically maximal such that $P_{\langle \boldsymbol{j}\rangle} \neq 0$, and choose $\boldsymbol{k}$ so that $P_{\langle \boldsymbol{k}\rangle}$ has minimal valuation. If $P_{\langle \boldsymbol{k}\rangle}/P_{\langle \boldsymbol{j}\rangle} \succ x$, set $f := P_{\langle \boldsymbol{k}\rangle}/P_{\langle \boldsymbol{j}\rangle}$; otherwise set $f := x^2$. Then $f \succ x$ and $f \succcurlyeq P_{\langle \boldsymbol{i}\rangle}/P_{\langle \boldsymbol{j}\rangle}$ for all $\boldsymbol{i}$. The following is a more precise version of [ADH, 16.6.10] and [38, (8.8)]:

Proposition 1.7. *Suppose Ψ is downward closed, and y in an H-asymptotic extension of K satisfies $y \succcurlyeq \mathrm{E}_n(f^2)$. Then $y^{\langle n\rangle}$ is defined and $P(y) \sim P_{\langle \boldsymbol{j}\rangle} y^{\langle \boldsymbol{j}\rangle}$.*

Proof. Let $\boldsymbol{i} < \boldsymbol{j}$. We have $f \succ x$, so $y^{\langle \boldsymbol{j}-\boldsymbol{i}\rangle} \succ f \succcurlyeq P_{\langle \boldsymbol{i}\rangle}/P_{\langle \boldsymbol{j}\rangle}$ by Corollary 1.6. Hence $P_{\langle \boldsymbol{j}\rangle} y^{\langle \boldsymbol{j}\rangle} \succ P_{\langle \boldsymbol{i}\rangle} y^{\langle \boldsymbol{i}\rangle}$. $\square$

From Corollary 1.2, Lemma 1.5, and Proposition 1.7 we obtain:

Corollary 1.8. *Suppose Ψ is downward closed and y in an H-asymptotic extension of K satisfies $y \succ K$. Then y is* d*-transcendental over K, and for all n, $y^{\langle n\rangle}$ is defined, $y^{\langle n\rangle} \succ K$, and $y^{\langle n+1\rangle} \prec\!\!\prec y^{\langle n\rangle}$. The H-asymptotic extension $K\langle y\rangle$ of K has residue field $\operatorname{res} K\langle y\rangle = \operatorname{res} K$ and value group $\Gamma_{K\langle y\rangle} = \Gamma \oplus \bigoplus_n \mathbb{Z}v(y^{\langle n\rangle})$ (internal direct sum), and $\Gamma_{K\langle y\rangle}$ contains Γ as a convex subgroup.*

Suppose now that K is equipped with an ordering making it a pre-H-field. From Proposition 1.7 we recover [4, Theorem 3.4] in slightly stronger form:

Corollary 1.9. *Suppose y lies in a Liouville closed H-field extension of K. If $y \succcurlyeq \mathrm{E}_n(f^2)$, then $y^{\langle n\rangle}$ is defined and $\operatorname{sign} P(y) = \operatorname{sign} P_{\langle \boldsymbol{j}\rangle} y^{j_0}$. In particular, if $y^{\langle n\rangle}$ is defined and $P(y) = 0$, then $y \prec \mathrm{E}_n(f^2)$.*

Example. Suppose $P \in K\{Y\}$. Using [ADH, 4.2, subsection on logarithmic decomposition] we obtain $j_0 = \deg P$, and the logarithmic decomposition

$$P(-Y) \;=\; \sum_{\boldsymbol{i}} P_{\langle \boldsymbol{i}\rangle} (-1)^{i_0} Y^{\langle \boldsymbol{i}\rangle}.$$

If $\deg P$ is odd, and $y > 0$ lies in a Liouville closed H-field extension of K such that $y \succcurlyeq \mathrm{E}_n(f^2)$, then

$$\operatorname{sign} P(y) \ = \ \operatorname{sign} P_{\langle j \rangle}, \qquad \operatorname{sign} P(-y) \ = \ -\operatorname{sign} P_{\langle j \rangle} \ = \ -\operatorname{sign} P(y).$$

2. Germs of Continuous Functions

Hardy fields consist of germs of one-variable differentiable real-valued functions. In this section we first consider the ring $\mathcal{C}$ of germs of *continuous* real-valued functions, and its complex counterpart $\mathcal{C}[\mathrm{i}]$. With an eye towards applications to Hardy fields, we pay particular attention to extending subfields of $\mathcal{C}$.

Germs. As in [ADH, 9.1] we let $\mathcal{G}$ be the ring of germs at $+\infty$ of real-valued functions whose domain is a subset of $\mathbb{R}$ containing an interval $(a, +\infty)$, $a \in \mathbb{R}$; the domain may vary and the ring operations are defined as usual. If $g \in \mathcal{G}$ is the germ of a real-valued function on a subset of $\mathbb{R}$ containing an interval $(a, +\infty)$, $a \in \mathbb{R}$, then we simplify notation by letting g also denote this function if the resulting ambiguity is harmless. With this convention, given a property P of real numbers and $g \in \mathcal{G}$ we say that $P\big(g(t)\big)$ *holds eventually* if $P\big(g(t)\big)$ holds for all sufficiently large real t. Thus for $g \in \mathcal{G}$ we have $g = 0$ iff $g(t) = 0$ eventually (and so $g \neq 0$ iff $g(t) \neq 0$ for arbitrarily large t). Note that the multiplicative group $\mathcal{G}^\times$ of units of $\mathcal{G}$ consists of the $f \in \mathcal{G}$ such that $f(t) \neq 0$, eventually. We identify each real number r with the germ at $+\infty$ of the function $\mathbb{R} \to \mathbb{R}$ that takes the constant value r. This makes the field $\mathbb{R}$ into a subring of $\mathcal{G}$. Given $g, h \in \mathcal{G}$, we set

$$g \leqslant h \quad :\Longleftrightarrow \quad g(t) \leqslant h(t), \text{ eventually.} \tag{2.1}$$

This defines a partial ordering $\leqslant$ on $\mathcal{G}$ which restricts to the usual ordering of $\mathbb{R}$.

Let $g, h \in \mathcal{G}$. Then $g, h \geqslant 0 \Rightarrow g + h,\ g \cdot h,\ g^2 \geqslant 0$, and $g \geqslant r \in \mathbb{R}^> \Rightarrow g \in \mathcal{G}^\times$. We define $g < h :\Leftrightarrow g \leqslant h$ and $g \neq h$. Thus if $g(t) < h(t)$, eventually, then $g < h$; the converse is not generally valid.

Continuous germs. We call a germ $g \in \mathcal{G}$ *continuous* if it is the germ of a continuous function $(a, +\infty) \to \mathbb{R}$ for some $a \in \mathbb{R}$, and we let $\mathcal{C} \supseteq \mathbb{R}$ be the subring of $\mathcal{G}$ consisting of the continuous germs $g \in \mathcal{G}$. We have $\mathcal{C}^\times = \mathcal{G}^\times \cap \mathcal{C}$; thus for $f \in \mathcal{C}^\times$, we have $f(t) \neq 0$, eventually, hence either $f(t) > 0$, eventually, or $f(t) < 0$, eventually, and so $f > 0$ or $f < 0$. More generally, if $g, h \in \mathcal{C}$ and $g(t) \neq h(t)$, eventually, then $g(t) < h(t)$, eventually, or $h(t) < g(t)$, eventually. We let x denote the germ at $+\infty$ of the identity function on $\mathbb{R}$, so $x \in \mathcal{C}^\times$.

The ring $\mathcal{C}[\mathrm{i}]$. In analogy with $\mathcal{C}$ we define its complexification $\mathcal{C}[\mathrm{i}]$ as the ring of germs at $+\infty$ of $\mathbb{C}$-valued continuous functions whose domain is a subset of $\mathbb{R}$ containing an interval $(a, +\infty)$, $a \in \mathbb{R}$. It has $\mathcal{C}$ as a subring. Identifying each complex number c with the germ at $+\infty$ of the function $\mathbb{R} \to \mathbb{C}$ that takes the constant value c makes $\mathbb{C}$ also a subring of $\mathcal{C}[\mathrm{i}]$ with $\mathcal{C}[\mathrm{i}] = \mathcal{C} + \mathcal{C}i$, justifying the notation $\mathcal{C}[\mathrm{i}]$. The "eventual" terminology for germs $f \in \mathcal{C}$ (like "$f(t) \neq 0$, eventually") is extended in the obvious way to germs $f \in \mathcal{C}[\mathrm{i}]$. Thus for $f \in \mathcal{C}[\mathrm{i}]$ we have: $f(t) \neq 0$, eventually, if and only if $f \in \mathcal{C}[\mathrm{i}]^\times$. In particular $\mathcal{C}^\times = \mathcal{C}[\mathrm{i}]^\times \cap \mathcal{C}$.

Let $\Phi\colon U \to \mathbb{C}$ be a continuous function where $U \subseteq \mathbb{C}$, and let $f \in \mathcal{C}[\mathrm{i}]$ be such that $f(t) \in U$, eventually; then $\Phi(f)$ denotes the germ in $\mathcal{C}[\mathrm{i}]$ with $\Phi(f)(t) = \Phi\big(f(t)\big)$, eventually. For example, taking $U = \mathbb{C}$, $\Phi(z) = \mathrm{e}^z$, we obtain for $f \in \mathcal{C}[\mathrm{i}]$ the germ $\exp f = \mathrm{e}^f \in \mathcal{C}[\mathrm{i}]$ with $(\mathrm{e}^f)(t) = \mathrm{e}^{f(t)}$, eventually. Likewise, for $f \in \mathcal{C}$

with $f(t) > 0$, eventually, we have the germ $\log f \in \mathcal{C}$. For $f \in \mathcal{C}[i]$ we have $\overline{f} \in \mathcal{C}[i]$ with $\overline{f}(t) = \overline{f(t)}$, eventually; the map $f \mapsto \overline{f}$ is an automorphism of the ring $\mathcal{C}[i]$ with $\overline{\overline{f}} = f$ and $f \in \mathcal{C} \Leftrightarrow \overline{f} = f$. For $f \in \mathcal{C}[i]$ we also have $\operatorname{Re} f, \operatorname{Im} f, |f| \in \mathcal{C}$ with $f(t) = (\operatorname{Re} f)(t) + (\operatorname{Im} f)(t)i$ and $|f|(t) = |f(t)|$, eventually.

Asymptotic relations on $\mathcal{C}[i]$. Although $\mathcal{C}[i]$ is not a valued field, it will be convenient to equip $\mathcal{C}[i]$ with the asymptotic relations $\preccurlyeq$, $\prec$, $\sim$ (which are defined on any valued field [ADH, 3.1]) as follows: for $f, g \in \mathcal{C}[i]$,

$$\begin{aligned} f \preccurlyeq g \quad &:\Longleftrightarrow \quad \text{there exists } c \in \mathbb{R}^{>} \text{ such that } |f| \leqslant c|g|, \\ f \prec g \quad &:\Longleftrightarrow \quad g \in \mathcal{C}[i]^{\times} \text{ and } \lim_{t\to\infty} f(t)/g(t) = 0 \\ &\Longleftrightarrow \quad g \in \mathcal{C}[i]^{\times} \text{ and } |f| \leqslant c|g| \text{ for all } c \in \mathbb{R}^{>}, \\ f \sim g \quad &:\Longleftrightarrow \quad g \in \mathcal{C}[i]^{\times} \text{ and } \lim_{t\to\infty} f(t)/g(t) = 1 \\ &\Longleftrightarrow \quad f - g \prec g. \end{aligned}$$

If $h \in \mathcal{C}[i]$ and $1 \preccurlyeq h$, then $h \in \mathcal{C}[i]^{\times}$. Also, for $f, g \in \mathcal{C}[i]$ and $h \in \mathcal{C}[i]^{\times}$ we have

$$f \preccurlyeq g \;\Leftrightarrow\; fh \preccurlyeq gh, \qquad f \prec g \;\Leftrightarrow\; fh \prec gh, \qquad f \sim g \;\Leftrightarrow\; fh \sim gh.$$

The binary relation $\preccurlyeq$ on $\mathcal{C}[i]$ is reflexive and transitive, and $\sim$ is an equivalence relation on $\mathcal{C}[i]^{\times}$. Moreover, for $f, g, h \in \mathcal{C}[i]$ we have

$$f \prec g \;\Rightarrow\; f \preccurlyeq g, \qquad f \preccurlyeq g \prec h \;\Rightarrow\; f \prec h, \qquad f \prec g \preccurlyeq h \;\Rightarrow\; f \prec h.$$

Note that $\prec$ is a transitive binary relation on $\mathcal{C}[i]$. For $f, g \in \mathcal{C}[i]$ we also set

$$f \asymp g :\Leftrightarrow\; f \preccurlyeq g \;\&\; g \preccurlyeq f, \qquad f \succcurlyeq g :\Leftrightarrow\; g \preccurlyeq f, \qquad f \succ g :\Leftrightarrow\; g \prec f,$$

so $\asymp$ is an equivalence relation on $\mathcal{C}[i]$, and $f \sim g \Rightarrow f \asymp g$. Thus for $f, g, h \in \mathcal{C}[i]$,

$$f \preccurlyeq g \;\Rightarrow\; fh \preccurlyeq gh, \quad f \preccurlyeq h \;\&\; g \preccurlyeq h \;\Rightarrow\; f + g \preccurlyeq h, \quad f \preccurlyeq 1 \;\&\; g \prec 1 \;\Rightarrow\; fg \prec 1,$$

hence

$$\mathcal{C}[i]^{\preccurlyeq} \;:=\; \big\{f \in \mathcal{C}[i] : f \preccurlyeq 1\big\} \;=\; \big\{f \in \mathcal{C}[i] : |f| \leqslant n \text{ for some } n\big\}$$

is a subalgebra of the $\mathbb{C}$-algebra $\mathcal{C}[i]$ and

$$\mathcal{C}[i]^{\prec} \;:=\; \big\{f \in \mathcal{C}[i] : f \prec 1\big\} \;=\; \Big\{f \in \mathcal{C}[i] : \lim_{t\to\infty} f(t) = 0\Big\}$$

is an ideal of $\mathcal{C}[i]^{\preccurlyeq}$. The group of units of $\mathcal{C}[i]^{\preccurlyeq}$ is

$$\mathcal{C}[i]^{\asymp} \;:=\; \big\{f \in \mathcal{C}[i] : f \asymp 1\big\} \;=\; \big\{f \in \mathcal{C}[i] : 1/n \leqslant |f| \leqslant n \text{ for some } n \geqslant 1\big\}$$

and has the subgroup

$$\mathbb{C}^{\times}\big(1 + \mathcal{C}[i]^{\prec}\big) \;=\; \Big\{f \in \mathcal{C}[i] : \lim_{t\to\infty} f(t) \in \mathbb{C}^{\times}\Big\}.$$

We set $\mathcal{C}^{\preccurlyeq} := \mathcal{C}[i]^{\preccurlyeq} \cap \mathcal{C}$, and similarly with $\prec$, $\asymp$ in place of $\preccurlyeq$.

Lemma 2.1. *Let $f, g, f^*, g^* \in \mathcal{C}[i]^{\times}$ with $f \sim f^*$ and $g \sim g^*$. Then $1/f \sim 1/f^*$ and $fg \sim f^*g^*$. Moreover, $f \preccurlyeq g \Leftrightarrow f^* \preccurlyeq g^*$, and similarly with $\prec$, $\asymp$, or $\sim$ in place of $\preccurlyeq$.*

This follows easily from the observations above. For later reference we also note:

Lemma 2.2. *Let $f, g \in \mathcal{C}^{\times}$ be such that $1 \prec f \preccurlyeq g$; then $\log |f| \preccurlyeq \log |g|$.*

Proof. Clearly $\log|g| \succ 1$. Take $c \in \mathbb{R}^{>}$ such that $|f| \leqslant c|g|$. Then $\log|f| \leqslant \log c + \log|g|$ where $\log c + \log|g| \sim \log|g|$; hence $\log|f| \preccurlyeq \log|g|$. □

Lemma 2.3. *Let $f, g, h \in \mathcal{C}^{\times}$ be such that $f - g \prec h$ and $(f-h)(g-h) = 0$. Then $f \sim g$.*

Proof. Take $a \in \mathbb{R}$ and representatives $(a, +\infty) \to \mathbb{R}$ of f, g, h, denoted by the same symbols, such that for each $t > a$ we have $f(t), g(t), h(t) \neq 0$, and $f(t) = h(t)$ or $g(t) = h(t)$. Let $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon \leqslant 1$ be given, and choose $b \geqslant a$ such that $|f(t) - g(t)| \leqslant \frac{1}{2}\varepsilon|h(t)|$ for all $t > b$. Set $q := f/g$ and let $t > b$; we claim that then $|q(t) - 1| \leqslant \varepsilon$. This is clear if $g(t) = h(t)$, so suppose otherwise; then $f(t) = h(t)$, and $|1 - 1/q(t)| \leqslant \frac{1}{2}\varepsilon \leqslant \frac{1}{2}$. In particular, $0 < q(t) \leqslant 2$ and so $|1 - q(t)| = |1 - 1/q(t)| \cdot q(t) \leqslant \varepsilon$ as claimed. □

Subfields of $\mathcal{C}$. Let H be a *Hausdorff field*, that is, a subring of $\mathcal{C}$ that happens to be a field; see [8]. Then H has the subfield $H \cap \mathbb{R}$. If $f \in H^{\times}$, then $f(t) \neq 0$ eventually, hence either $f(t) < 0$ eventually or $f(t) > 0$ eventually. The partial ordering of $\mathcal{G}$ from (2.1) thus restricts to a total ordering on H making H an ordered field in the usual sense of that term. By [17, Propositions 3.4 and 3.6]:

Proposition 2.4. *Let H^{rc} consist of the $y \in \mathcal{C}$ with $P(y) = 0$ for some $P \in H[Y]^{\neq}$. Then H^{rc} is the unique real closed Hausdorff field that extends H and is algebraic over H. In particular, H^{rc} is a real closure of the ordered field H.*

Boshernitzan [17] assumes $H \supseteq \mathbb{R}$ for this result, but this is not really needed in the proof, much of which already occurs in Hausdorff [35].

Note that $H[\mathrm{i}]$ is a subfield of $\mathcal{C}[\mathrm{i}]$, and by Proposition 2.4 and [ADH, 3.5.4], the subfield $H^{\mathrm{rc}}[\mathrm{i}]$ of $\mathcal{C}[\mathrm{i}]$ is an algebraic closure of the field H. If $f \in \mathcal{C}[\mathrm{i}]$ is integral over H, then so is $\overline{f}$, and hence so are the elements $\operatorname{Re} f = \frac{1}{2}(f + \overline{f})$ and $\operatorname{Im} f = \frac{1}{2\mathrm{i}}(f - \overline{f})$ of $\mathcal{C}$ [ADH, 1.3.2]. Thus $H^{\mathrm{rc}}[\mathrm{i}]$ consists of the $y \in \mathcal{C}[\mathrm{i}]$ with $P(y) = 0$ for some $P \in H[Y]^{\neq}$.

The ordered field H has a convex subring

$$\mathcal{O} \;=\; \big\{f \in H : \;|f| \leqslant n \text{ for some } n\big\} \;=\; \mathcal{C}^{\preccurlyeq} \cap H,$$

which is a valuation ring of H, and we consider H accordingly as a valued ordered field. The maximal ideal of $\mathcal{O}$ is $\smallo = \mathcal{C}^{\prec} \cap H$. The residue morphism $\mathcal{O} \to \operatorname{res}(H)$ restricts to an ordered field embedding $H \cap \mathbb{R} \to \operatorname{res}(H)$, which is bijective if $\mathbb{R} \subseteq H$. Restricting the binary relations $\preccurlyeq$, $\prec$, $\sim$ from the previous subsection to H gives exactly the asymptotic relations $\preccurlyeq$, $\prec$, $\sim$ on H that it comes equipped with as a valued field. By [ADH, 3.5.15],

$$\mathcal{O} + \mathcal{O}\mathrm{i} \;=\; \big\{f \in H[\mathrm{i}] : |f| \leqslant n \text{ for some } n\big\} \;=\; \mathcal{C}[\mathrm{i}]^{\preccurlyeq} \cap H[\mathrm{i}]$$

is the unique valuation ring of $H[\mathrm{i}]$ whose intersection with H is $\mathcal{O}$. In this way we consider $H[\mathrm{i}]$ as a valued field extension of H. The maximal ideal of $\mathcal{O} + \mathcal{O}\mathrm{i}$ is $\smallo + \smallo\mathrm{i} = \mathcal{C}[\mathrm{i}]^{\prec} \cap H[\mathrm{i}]$. The asymptotic relations $\preccurlyeq$, $\prec$, $\sim$ on $\mathcal{C}[\mathrm{i}]$ restricted to $H[\mathrm{i}]$ are exactly the asymptotic relations $\preccurlyeq$, $\prec$, $\sim$ on $H[\mathrm{i}]$ that $H[\mathrm{i}]$ has as a valued field, with $f \asymp |f|$ in $\mathcal{C}[\mathrm{i}]$ for all $f \in H[\mathrm{i}]$. In particular, the binary relation $\preccurlyeq$ on $\mathcal{C}[\mathrm{i}]$ restricts to a dominance relation on each subfield of $H[\mathrm{i}]$ (see [ADH, 3.1.1]). Let K be a subfield of $\mathcal{C}[\mathrm{i}]$. We note that the following are equivalent:

(1) The binary relation $\preccurlyeq$ on $\mathcal{C}[\mathrm{i}]$ restricts to a dominance relation on K;

(2) for all $f, g \in K$: $f \preccurlyeq g$ or $g \preccurlyeq f$;
(3) for all $f \in K$: $f \preccurlyeq 1$ or $1 \preccurlyeq f$.

Moreover, the following are equivalent:

(1) $K = H[\mathrm{i}]$ for some Hausdorff field H;
(2) $\mathrm{i} \in K$ and $\overline{f} \in K$ for each $f \in K$;
(3) $\mathrm{i} \in K$ and $\operatorname{Re} f, \operatorname{Im} f \in K$ for each $f \in K$.

Composition. Let $g \in \mathcal{C}$, and suppose that $\lim_{t \to +\infty} g(t) = +\infty$; equivalently, $g \geqslant 0$ and $g \succ 1$. Then the composition operation

$$f \mapsto f \circ g \ : \ \mathcal{C}[\mathrm{i}] \to \mathcal{C}[\mathrm{i}], \qquad (f \circ g)(t) \ := \ f\big(g(t)\big) \ \text{ eventually,}$$

is an injective endomorphism of the ring $\mathcal{C}[\mathrm{i}]$ that is the identity on the subring $\mathbb{C}$. For $f_1, f_2 \in \mathcal{C}[\mathrm{i}]$ we have: $f_1 \preccurlyeq f_2 \Leftrightarrow f_1 \circ g \preccurlyeq f_2 \circ g$, and likewise with $\prec$, $\sim$. This endomorphism of $\mathcal{C}[\mathrm{i}]$ commutes with the automorphism $f \mapsto \overline{f}$ of $\mathcal{C}[\mathrm{i}]$, and maps each subfield K of $\mathcal{C}[\mathrm{i}]$ isomorphically onto the subfield $K \circ g = \{f \circ g : f \in K\}$ of $\mathcal{C}[\mathrm{i}]$. Note that if the subfield K of $\mathcal{C}[\mathrm{i}]$ contains x, then $K \circ g$ contains g. Moreover, $f \mapsto f \circ g$ restricts to an endomorphism of the subring $\mathcal{C}$ of $\mathcal{C}[\mathrm{i}]$ such that if $f_1, f_2 \in \mathcal{C}$ and $f_1 \leqslant f_2$, then $f_1 \circ g \leqslant f_2 \circ g$. This endomorphism of $\mathcal{C}$ maps each Hausdorff field H isomorphically (as an ordered field) onto the Hausdorff field $H \circ g$.

Occasionally it is convenient to extend the composition operation on $\mathcal{C}$ to the ring $\mathcal{G}$ of all (not necessarily continuous) germs. Let $g \in \mathcal{G}$ with $\lim_{t \to +\infty} g(t) = +\infty$. Then for $f \in \mathcal{G}$ we have the germ $f \circ g \in \mathcal{G}$ with

$$(f \circ g)(t) \ := \ f\big(g(t)\big) \ \text{ eventually.}$$

The map $f \mapsto f \circ g$ is an endomorphism of the $\mathbb{R}$-algebra $\mathcal{G}$. Let $f_1, f_2 \in \mathcal{G}$. Then $f_1 \leqslant f_2 \Rightarrow f_1 \circ g \leqslant f_2 \circ g$, and likewise with $\preccurlyeq$ and $\prec$ instead of $\leqslant$, where we extend the binary relations $\preccurlyeq$, $\prec$ from $\mathcal{C}$ to $\mathcal{G}$ in the natural way:

$$f_1 \preccurlyeq f_2 \quad :\Longleftrightarrow \quad \text{there exists } c \in \mathbb{R}^{>} \text{ such that } |f_1(t)| \leqslant c|f_2(t)|, \text{ eventually;}$$
$$f_1 \prec f_2 \quad :\Longleftrightarrow \quad f_2 \in \mathcal{G}^{\times} \text{ and } \lim_{t \to \infty} f_1(t)/f_2(t) = 0.$$

Compositional inversion. Suppose that $g \in \mathcal{C}$ is eventually strictly increasing such that $\lim_{t \to +\infty} g(t) = +\infty$. Then its compositional inverse $g^{\mathrm{inv}} \in \mathcal{C}$ is given by $g^{\mathrm{inv}}\big(g(t)\big) = t$, eventually, and g^{inv} is also eventually strictly increasing with $\lim_{t \to +\infty} g^{\mathrm{inv}}(t) = +\infty$. Then $f \mapsto f \circ g$ is an automorphism of the ring $\mathcal{C}[\mathrm{i}]$, with inverse $f \mapsto f \circ g^{\mathrm{inv}}$. In particular, $g \circ g^{\mathrm{inv}} = g^{\mathrm{inv}} \circ g = x$. Moreover, $f \mapsto f \circ g$ restricts to an automorphism of $\mathcal{C}$, and if $h \in \mathcal{C}$ is eventually strictly increasing with $g \leqslant h$, then $h^{\mathrm{inv}} \leqslant g^{\mathrm{inv}}$.

Let now $f, g \in \mathcal{C}$ with $f, g \geqslant 0$, $f, g \succ 1$. It is not true in general that if f, g are eventually strictly increasing and $f \sim g$, then $f^{\mathrm{inv}} \sim g^{\mathrm{inv}}$. (Counterexample: $f = \log x$, $g = \log 2x$.) Corollary 2.6 below gives a useful condition on f, g under which this implication does hold. In addition, let $h \in \mathcal{C}^{\times}$ be eventually monotone and continuously differentiable with $h'/h \preccurlyeq 1/x$.

Lemma 2.5 (Entringer [24])**.** *Suppose $f \sim g$. Then $h \circ f \sim h \circ g$.*

Proof. Replacing h by $-h$ if necessary we arrange that $h \geqslant 0$, so $h(t) > 0$ eventually. Set $p := \min(f, g) \in \mathcal{C}$ and $q := \max(f, g) \in \mathcal{C}$. Then $0 \leqslant p \succ 1$ and $f - g \prec p$. The Mean Value Theorem gives $\xi \in \mathcal{G}$ such that $p \leqslant \xi \leqslant q$ (so $0 \leqslant \xi \succ 1$) and

$$h \circ f - h \circ g \;=\; (h' \circ \xi) \cdot (f - g).$$

From $h'/h \preccurlyeq 1/x$ we obtain $h' \circ \xi \preccurlyeq (h \circ \xi)/\xi \preccurlyeq (h \circ \xi)/p$, hence $h \circ f - h \circ g \prec h \circ \xi$. Set $u := \max(h \circ p, h \circ q)$. Then $0 \leqslant h \circ \xi \leqslant u$, hence $h \circ f - h \circ g \prec u$. Also $(u - h \circ f)(u - h \circ g) = 0$, so Lemma 2.3 yields $h \circ f \sim h \circ g$. □

Corollary 2.6. *Suppose f, g are eventually strictly increasing such that $f \sim g$ and $f^{\text{inv}} \sim h$. Then $g^{\text{inv}} \sim h$.*

Proof. By the lemma above we have $h \circ f \sim h \circ g$, and from $f^{\text{inv}} \sim h$ we obtain $x = f^{\text{inv}} \circ f \sim h \circ f$. Therefore $g^{\text{inv}} \circ g = x \sim h \circ g$ and thus $g^{\text{inv}} \sim h$. □

Extending ordered fields inside an ambient partially ordered ring. Let R be a commutative ring with $1 \neq 0$, equipped with a translation-invariant partial ordering $\leqslant$ such that $r^2 \geqslant 0$ for all $r \in R$, and $rs \geqslant 0$ for all $r, s \in R$ with $r, s \geqslant 0$. It follows that for $a, b, r \in R$ we have:

(1) if $a \leqslant b$ and $r \geqslant 0$, then $ar \leqslant br$;
(2) if a is a unit and $a > 0$, then $a^{-1} = a \cdot (a^{-1})^2 > 0$;
(3) if a, b are units and $0 < a \leqslant b$, then $0 < b^{-1} \leqslant a^{-1}$.

Relevant cases: $R = \mathcal{G}$ and $R = \mathcal{C}$, with partial ordering given by (2.1).

An *ordered subring of R* is a subring of R that is totally ordered by the partial ordering of R. An *ordered subfield of R* is an ordered subring H of R which happens to be a field; then H equipped with the induced ordering is indeed an ordered field, in the usual sense of that term. (Thus any Hausdorff field is an ordered subfield of the partially ordered ring $\mathcal{C}$.) We identify $\mathbb{Z}$ with its image in R via the unique ring embedding $\mathbb{Z} \to R$, and this makes $\mathbb{Z}$ with its usual ordering into an ordered subring of R.

Lemma 2.7. *Assume D is an ordered subring of R and every nonzero element of D is a unit of R. Then D generates an ordered subfield* $\operatorname{Frac} D$ *of R.*

Proof. It is clear that D generates a subfield $\operatorname{Frac} D$ of R. For $a \in D$, $a > 0$, we have $a^{-1} > 0$. It follows that $\operatorname{Frac} D$ is totally ordered. □

Thus if every $n \geqslant 1$ is a unit of R, then we may identify $\mathbb{Q}$ with its image in R via the unique ring embedding $\mathbb{Q} \to R$, making $\mathbb{Q}$ into an ordered subfield of R.

Lemma 2.8. *Suppose H is an ordered subfield of R, all $g \in R$ with $g > H$ are units of R, and $H < f \in R$. Then we have an ordered subfield $H(f)$ of R.*

Proof. For $P \in H[Y]$ of degree $d \geqslant 1$ with leading coefficient $a > 0$ we have $P(f) = af^d(1 + \varepsilon)$ with $-1/n < \varepsilon < 1/n$ for all $n \geqslant 1$, in particular, $P(f) > H$ is a unit of R. It remains to appeal to Lemma 2.7. □

Lemma 2.9. *Let H be a real closed ordered subfield of R. Let A be a nonempty downward closed subset of H such that A has no largest element and $B := H \setminus A$ is nonempty and has no least element. Let $f \in R$ be such that $A < f < B$. Then the subring $H[f]$ of R has the following properties:*

(i) *$H[f]$ is a domain;*

(ii) *$H[f]$ is an ordered subring of R;*
(iii) *H is cofinal in $H[f]$;*
(iv) *for all $g \in H[f] \setminus H$ and $a \in H$, if $a < g$, then $a < b < g$ for some $b \in H$, and if $g < a$, then $g < b < a$ for some $b \in H$.*

Proof. Let $P \in H[Y]^{\neq}$; to obtain (i) and (ii) it suffices to show that then $P(f) < 0$ or $P(f) > 0$. We have

$$P \;=\; c\,Q\,(Y - a_1)\cdots(Y - a_n)$$

where $c \in H^{\neq}$, Q is a product of monic quadratic irreducibles in $H[Y]$, and $a_1, \dots, a_n \in H$. This gives $\delta \in H^{>}$ such that $Q(r) \geqslant \delta$ for all $r \in R$. Assume $c > 0$. (The case $c < 0$ is handled similarly.) We can arrange that $m \leqslant n$ is such that $a_i \in A$ for $1 \leqslant i \leqslant m$ and $a_j \in B$ for $m < j \leqslant n$. Take $\varepsilon > 0$ in H such that $a_i + \varepsilon \leqslant f$ for $1 \leqslant i \leqslant m$ and $f \leqslant a_j - \varepsilon$ for $m < j \leqslant n$. Then

$$P(f) \;=\; c\,Q(f)\,(f - a_1)\cdots(f - a_m)(f - a_{m+1})\cdots(f - a_n),$$

and $(f-a_1)\cdots(f-a_m) \geqslant \varepsilon^m$. If $n-m$ is even, then $(f-a_{m+1})\cdots(f-a_n) \geqslant \varepsilon^{n-m}$, so $P(f) \geqslant c\delta\varepsilon^n > 0$. If $n-m$ is odd, then $(f-a_{m+1})\cdots(f-a_n) \leqslant -\varepsilon^{n-m}$, so $P(f) \leqslant -c\delta\varepsilon^n < 0$. These estimates also yield (iii) and (iv). □

Lemma 2.10. *With H, A, f as in Lemma 2.9, suppose all $g \in R$ with $g \geqslant 1$ are units of R. Then we have an ordered subfield $H(f)$ of R such that* (iii) *and* (iv) *of Lemma 2.9 go through for $H(f)$ in place of $H[f]$.*

Proof. Note that if $g \in R$ and $g \geqslant \delta \in H^{>}$, then $g\delta^{-1} \geqslant 1$, so g is a unit of R and $0 < g^{-1} \leqslant \delta^{-1}$. For $Q \in H[Y]^{\neq}$ with $Q(f) > 0$ we can take $\delta \in H^{>}$ such that $Q(f) \geqslant \delta$, so $Q(f) \in R^{\times}$ and $0 < Q(f)^{-1} \leqslant \delta^{-1}$. Thus we have an ordered subfield $H(f)$ of R by Lemma 2.7, and the rest now follows easily. □

Adjoining pseudolimits and increasing the value group. Let H be a real closed Hausdorff field and view H as an ordered valued field as before. Let (a_ρ) be a strictly increasing divergent pc-sequence in H. Set

$$A \;:=\; \{a \in H :\; a < a_\rho \text{ for some } \rho\}, \qquad B \;:=\; \{b \in H :\; b > a_\rho \text{ for all } \rho\},$$

so A is nonempty and downward closed without a largest element. Moreover, $B = H \setminus A$ is nonempty and has no least element, since a least element of B would be a limit and thus a pseudolimit of (a_ρ). Let $f \in \mathcal{C}$ satisfy $A < f < B$. Then by Lemma 2.10 for $R = \mathcal{C}$ we have an ordered subfield $H(f)$ of $\mathcal{C}$, and:

Lemma 2.11. *$H(f)$ is an immediate valued field extension of H with $a_\rho \rightsquigarrow f$.*

Proof. We can assume that $v(a_\tau - a_\sigma) > v(a_\sigma - a_\rho)$ for all indices $\tau > \sigma > \rho$. Set $d_\rho := a_{s(\rho)} - a_\rho$ ($s(\rho) :=$ successor of ρ). Then $a_\rho + 2d_\rho \in B$ for all indices ρ; see the discussion preceding [ADH, 2.4.2]. It then follows from that lemma that $a_\rho \rightsquigarrow f$. Now (a_ρ) is a divergent pc-sequence in the henselian valued field H, so it is of transcendental type over H, and thus $H(f)$ is an immediate extension of H. □

Lemma 2.12. *Let H be a Hausdorff field with divisible value group $\Gamma := v(H^{\times})$. Let P be a nonempty upward closed subset of Γ, and let $f \in \mathcal{C}$ be such that $a < f$ for all $a \in H^{>}$ with $va \in P$, and $f < b$ for all $b \in H^{>}$ with $vb < P$. Then f generates a Hausdorff field $H(f)$ with $P > vf > Q$, $Q := \Gamma \setminus P$.*

Proof. For any positive $a \in H^{\mathrm{rc}}$ there is $b \in H^{>}$ with $a \asymp b$ and $a < b$, and also an element $b \in H^{>}$ with $a \asymp b$ and $a > b$. Thus by Proposition 2.4 we can replace H by H^{rc} and arrange in this way that H is real closed. Set

$$A \ := \ \{a \in H : \ a \leqslant 0 \text{ or } va \in P\}, \qquad B := H \setminus A.$$

Then we are in the situation of Lemma 2.9 for $R = \mathcal{C}$, so by that lemma and Lemma 2.10 we have a Hausdorff field $H(f)$. Clearly then $P > vf > Q$. □

Non-oscillation. A germ $f \in \mathcal{C}$ is said to **oscillate** if $f(t) = 0$ for arbitrarily large t and $f(t) \neq 0$ for arbitrarily large t. Thus for $f, g \in \mathcal{C}$,

$$f - g \text{ is non-oscillating} \quad \Longleftrightarrow \quad \begin{cases} \text{either } f(t) < g(t) \text{ eventually, or } f = g, \\ \text{or } f(t) > g(t) \text{ eventually.} \end{cases}$$

In particular, $f \in \mathcal{C}$ does not oscillate iff $f = 0$ or $f \in \mathcal{C}^{\times}$. If $g \in \mathcal{C}$ and $g(t) \to +\infty$ as $t \to +\infty$, then $f \in \mathcal{C}$ oscillates iff $f \circ g$ oscillates. The following two lemmas are included for use in [13]:

Lemma 2.13. *Let $f \in \mathcal{C}$ be such that for every $q \in \mathbb{Q}$ the germ $f - q$ is non-oscillating. Then $\lim_{t\to\infty} f(t)$ exists in $\mathbb{R} \cup \{-\infty, +\infty\}$.*

Proof. Set $S := \{s \in \mathbb{Q} : f(t) > s \text{ eventually}\}$. If $S = \emptyset$, then $\lim_{t\to\infty} f(t) = -\infty$, whereas if $S = \mathbb{Q}$, then $\lim_{t\to\infty} f(t) = +\infty$. If $S \neq \emptyset, \mathbb{Q}$, then for $\ell := \sup S \in \mathbb{R}$ we have $\lim_{t\to\infty} f(t) = \ell$. □

Lemma 2.14. *Let H be a real closed Hausdorff field and $f \in \mathcal{C}$. Then f lies in a Hausdorff field extension of H iff $f - h$ is non-oscillating for all $h \in H$.*

Proof. The forward direction is clear. For the converse, suppose $f - h$ is non-oscillating for all $h \in H$. We assume $f \notin H$, so $h < f$ or $h > f$ for all $h \in H$. Set $A := \{h \in H : h < f\}$, a downward closed subset of H. If $A = H$, then we are done by Lemma 2.8 applied to $R = \mathcal{C}$; if $A = \emptyset$ then we apply the same lemma to $R = \mathcal{C}$ and $-f$ in place of f. Suppose $A \neq \emptyset, H$. If A has a largest element a, then we replace f by $f - a$ to arrange $0 < f(t) < h(t)$ eventually, for all $h \in H^{>}$, and then Lemma 2.8 applied to $R = \mathcal{C}$, f^{-1} in place of f yields that f^{-1}, and hence also f, lies in a Hausdorff field extension of H. The case that $B := H \setminus A$ has a least element is handled in the same way. If A has no largest element and B has no least element, then we are done by Lemma 2.10. □

3. Germs of Differentiable Functions

In this section we fix notations and conventions concerning differentiable functions and summarize well-known results on second-order linear differential equations as needed later. (Basic facts about linear differential equations can be found in [23, Ch. X], [33, Ch. XI], and [62, Ch. IV].)

Differentiable functions. Let r range over $\mathbb{N} \cup \{\infty\}$, and let U be a nonempty open subset of $\mathbb{R}$. Then $\mathcal{C}^r(U)$ denotes the $\mathbb{R}$-algebra of r-times continuously differentiable functions $U \to \mathbb{R}$, with the usual pointwise defined algebra operations. (We use "$\mathcal{C}$" instead of "C" since C will often denote the constant field of a differential field.) For $r = 0$ this is the $\mathbb{R}$-algebra $\mathcal{C}(U)$ of continuous real-valued functions on U, so

$$\mathcal{C}(U) \ = \ \mathcal{C}^0(U) \ \supseteq \ \mathcal{C}^1(U) \ \supseteq \ \mathcal{C}^2(U) \ \supseteq \ \cdots \ \supseteq \ \mathcal{C}^{\infty}(U).$$

For $r \geqslant 1$ we have the derivation $f \mapsto f' \colon \mathcal{C}^r(U) \to \mathcal{C}^{r-1}(U)$ (with $\infty - 1 := \infty$). This makes $\mathcal{C}^\infty(U)$ a differential ring, with its subalgebra $\mathcal{C}^\omega(U)$ of real-analytic functions $U \to \mathbb{R}$ as a differential subring. The algebra operations on the algebras below are also defined pointwise. Note that

$$\mathcal{C}^r(U)^\times \ = \ \big\{ f \in \mathcal{C}^r(U) : f(t) \neq 0 \text{ for all } t \in U \big\},$$

also for ω in place of r [23, (9.2), ex. 4].

Let a range over $\mathbb{R}$. Then $\mathcal{C}^r_a$ denotes the $\mathbb{R}$-algebra of functions $[a, +\infty) \to \mathbb{R}$ that extend to a function in $\mathcal{C}^r(U)$ for some open $U \supseteq [a, +\infty)$. Thus $\mathcal{C}^0_a$ (also denoted by $\mathcal{C}_a$) is the $\mathbb{R}$-algebra of real-valued continuous functions on $[a, +\infty)$, and

$$\mathcal{C}^0_a \ \supseteq \ \mathcal{C}^1_a \ \supseteq \ \mathcal{C}^2_a \ \supseteq \ \cdots \ \supseteq \mathcal{C}^\infty_a.$$

We have the subalgebra $\mathcal{C}^\omega_a$ of $\mathcal{C}^\infty_a$, consisting of the functions $[a, +\infty) \to \mathbb{R}$ that extend to a real-analytic function $U \to \mathbb{R}$ for some open $U \supseteq [a, +\infty)$. For $f \in \mathcal{C}^1_a$ and $g \in \mathcal{C}^1(U)$ extending f with open $U \subseteq \mathbb{R}$ containing $[a, +\infty)$, the restriction of g' to $[a, +\infty) \to \mathbb{R}$ depends only on f, not on g, and so we may define $f' := g'|_{[a,+\infty)} \in \mathcal{C}_a$. For $r \geqslant 1$ this gives the derivation $f \mapsto f' \colon \mathcal{C}^r_a \to \mathcal{C}^{r-1}_a$. This makes $\mathcal{C}^\infty_a$ a differential ring with $\mathcal{C}^\omega_a$ as a differential subring.

For each of the algebras A above we also consider its complexification $A[\mathrm{i}]$ which consists by definition of the $\mathbb{C}$-valued functions $f = g + h i$ with $g, h \in A$, so $g = \operatorname{Re} f$ and $h = \operatorname{Im} f$ for such f. We consider $A[\mathrm{i}]$ as a $\mathbb{C}$-algebra with respect to the natural pointwise defined algebra operations. We identify each complex number with the corresponding constant function to make $\mathbb{C}$ a subfield of $A[\mathrm{i}]$ and $\mathbb{R}$ a subfield of A. (This justifies the notation $A[\mathrm{i}]$.) We have $\mathcal{C}^r_a[\mathrm{i}]^\times = \mathcal{C}_a[\mathrm{i}]^\times \cap \mathcal{C}^r_a[\mathrm{i}]$ and $(\mathcal{C}^r_a)^\times = \mathcal{C}^\times_a \cap \mathcal{C}^r_a$, and likewise with r replaced by ω.

For $r \geqslant 1$ we extend $g \mapsto g' \colon \mathcal{C}^r_a \to \mathcal{C}^{r-1}_a$ to the derivation

$$g + h i \mapsto g' + h' \mathrm{i} \ : \ \mathcal{C}^r_a[\mathrm{i}] \to \mathcal{C}^{r-1}_a[\mathrm{i}] \qquad (g, h \in \mathcal{C}^r_a[\mathrm{i}]),$$

which for $r = \infty$ makes $\mathcal{C}^\infty_a$ a differential subring of $\mathcal{C}^\infty_a[\mathrm{i}]$. We shall use the map

$$f \mapsto f^\dagger := f'/f \ : \ \mathcal{C}^1_a[\mathrm{i}]^\times = \big(\mathcal{C}^1_a[\mathrm{i}]\big)^\times \to \mathcal{C}^0_a[\mathrm{i}],$$

with

$$(fg)^\dagger = f^\dagger + g^\dagger \qquad \text{for } f, g \in \mathcal{C}^1_a[\mathrm{i}]^\times,$$

in particular the fact that $f \in \mathcal{C}^1_a[\mathrm{i}]^\times$ and $f^\dagger \in \mathcal{C}^0_a[\mathrm{i}]$ are related by

$$f(t) \ = \ f(a) \exp\left[\int_a^t f^\dagger(s)\, ds\right] \qquad (t \geqslant a).$$

For $g \in \mathcal{C}^0_a[\mathrm{i}]$, let $\exp \int g$ denote the function $t \mapsto \exp\left[\int_a^t g(s)\, ds\right]$ in $\mathcal{C}^1_a[\mathrm{i}]^\times$. Then

$$(\exp {\textstyle\int} g)^\dagger = g \quad \text{ and } \quad \exp {\textstyle\int} (g + h) = (\exp {\textstyle\int} g) \cdot (\exp {\textstyle\int} h) \qquad \text{for } g, h \in \mathcal{C}^0_a[\mathrm{i}].$$

Therefore $f \mapsto f^\dagger \colon \mathcal{C}^1_a[\mathrm{i}]^\times \to \mathcal{C}^0_a[\mathrm{i}]$ is surjective.

Notation. For $b \geqslant a$ and $f \in \mathcal{C}_a[\mathrm{i}]$ we set $f|_b := f|_{[b,+\infty)} \in \mathcal{C}_a[\mathrm{i}]$.

Differentiable germs. Let $r \in \mathbb{N} \cup \{\infty\}$ and let a range over $\mathbb{R}$. Let $\mathcal{C}^r$ be the partially ordered subring of $\mathcal{C}$ consisting of the germs at $+\infty$ of the functions in $\bigcup_a \mathcal{C}_a^r$; thus $\mathcal{C}^0 = \mathcal{C}$ consists of the germs at $+\infty$ of the continuous real valued functions on intervals $[a, +\infty)$, $a \in \mathbb{R}$. Note that $\mathcal{C}^r$ with its partial ordering satisfies the conditions on R from Section 2. Also, every $g \geqslant 1$ in $\mathcal{C}^r$ is a unit of $\mathcal{C}^r$, so Lemmas 2.8 and 2.10 apply to ordered subfields of $\mathcal{C}^r$. We have

$$\mathcal{C}^0 \supseteq \mathcal{C}^1 \supseteq \mathcal{C}^2 \supseteq \cdots \supseteq \mathcal{C}^\infty.$$

Each subring $\mathcal{C}^r$ of $\mathcal{C}$ yields the subring $\mathcal{C}^r[i] = \mathcal{C}^r + \mathcal{C}^r i$ of $\mathcal{C}^0[i] = \mathcal{C}[i]$, with

$$\mathcal{C}^0[i] \supseteq \mathcal{C}^1[i] \supseteq \mathcal{C}^2[i] \supseteq \cdots \supseteq \mathcal{C}^\infty[i].$$

Suppose $r \geqslant 1$; then for $f \in \mathcal{C}_a^r[i]$ the germ of $f' \in \mathcal{C}_a^{r-1}[i]$ only depends on the germ of f, and we thus obtain a derivation $g \mapsto g' \colon \mathcal{C}^r[i] \to \mathcal{C}^{r-1}[i]$ with (germ of $f)' =$ (germ of f') for $f \in \bigcup_a \mathcal{C}_a^r[i]$. This derivation restricts to a derivation $\mathcal{C}^r \to \mathcal{C}^{r-1}$. Note that $\mathcal{C}[i]^\times \cap \mathcal{C}^r[i] = \mathcal{C}^r[i]^\times$, and hence $\mathcal{C}^\times \cap \mathcal{C}^r = (\mathcal{C}^r)^\times$. Given $g \in \mathcal{C}^r$ with $g(t) \to +\infty$ as $t \to +\infty$ and $f \in \mathcal{C}^r[i]$, the germ $f \circ g$ (as defined in Section 2) also lies in $\mathcal{C}^r[i]$, with $f \circ g \in \mathcal{C}^r$ if $f \in \mathcal{C}^r$.

We set

$$\mathcal{C}^{<\infty}[i] := \bigcap_n \mathcal{C}^n[i].$$

Thus $\mathcal{C}^{<\infty}[i]$ is naturally a differential ring with $\mathbb{C}$ as its ring of constants. We also have the differential subring

$$\mathcal{C}^{<\infty} := \bigcap_n \mathcal{C}^n$$

of $\mathcal{C}^{<\infty}[i]$, with $\mathbb{R}$ as its ring of constants and $\mathcal{C}^{<\infty}[i] = \mathcal{C}^{<\infty} + \mathcal{C}^{<\infty} i$. Note that $\mathcal{C}^{<\infty}[i]$ has $\mathcal{C}^\infty[i]$ as a differential subring. Similarly, $\mathcal{C}^{<\infty}$ has $\mathcal{C}^\infty$ as a differential subring, and the differential ring $\mathcal{C}^\infty$ has in turn the differential subring $\mathcal{C}^\omega$, whose elements are the germs at $+\infty$ of the functions in $\bigcup_a \mathcal{C}_a^\omega$. We have $\mathcal{C}[i]^\times \cap \mathcal{C}^{<\infty}[i] = (\mathcal{C}^{<\infty}[i])^\times$ and $\mathcal{C}^\times \cap \mathcal{C}^{<\infty} = (\mathcal{C}^{<\infty})^\times$, and likewise with $\mathcal{C}^\omega$ in place of $\mathcal{C}^{<\infty}$. If R is a subring of $\mathcal{C}^1$ such that $f' \in R$ for all $f \in R$, then $R \subseteq \mathcal{C}^{<\infty}$ is a differential subring of $\mathcal{C}^{<\infty}$.

Compositional conjugation of differentiable germs. Let $\ell \in \mathcal{C}^1$, $\ell'(t) > 0$ eventually (so ℓ is eventually strictly increasing) and $\ell(t) \to +\infty$ as $t \to +\infty$. Then $\phi := \ell' \in \mathcal{C}^\times$, and the compositional inverse $\ell^{\text{inv}} \in \mathcal{C}^1$ of ℓ satisfies

$$\ell^{\text{inv}} > \mathbb{R}, \qquad (\ell^{\text{inv}})' = (1/\phi) \circ \ell^{\text{inv}} \in \mathcal{C}.$$

The $\mathbb{C}$-algebra automorphism $f \mapsto f^\circ := f \circ \ell^{\text{inv}}$ of $\mathcal{C}[i]$ (with inverse $g \mapsto g \circ \ell$) maps $\mathcal{C}^1[i]$ onto itself and satisfies for $f \in \mathcal{C}^1[i]$ a useful identity:

$$(f^\circ)' = (f \circ \ell^{\text{inv}})' = (f' \circ \ell^{\text{inv}}) \cdot (\ell^{\text{inv}})' = (f'/\ell') \circ \ell^{\text{inv}} = (\phi^{-1} f')^\circ.$$

Hence if $n \geqslant 1$ and $\ell \in \mathcal{C}^n$, then $\ell^{\text{inv}} \in \mathcal{C}^n$ and $f \mapsto f^\circ$ maps $\mathcal{C}^n[i]$ and $\mathcal{C}^n$ onto themselves, for each n. Therefore, if $\ell \in \mathcal{C}^{<\infty}$, then $\ell^{\text{inv}} \in \mathcal{C}^{<\infty}$ and $f \mapsto f^\circ$ maps $\mathcal{C}^{<\infty}[i]$ and $\mathcal{C}^{<\infty}$ onto themselves; likewise with $\mathcal{C}^\infty$ or $\mathcal{C}^\omega$ in place of $\mathcal{C}^{<\infty}$. *In the rest of this subsection we assume $\ell \in \mathcal{C}^{<\infty}$.* Denote the differential ring $\mathcal{C}^{<\infty}[i]$ by R, and as usual let R^ϕ be R with its derivation $f \mapsto \partial(f) = f'$ replaced by the derivation $f \mapsto \delta(f) = \phi^{-1} f'$ [ADH, 5.7]. Then $f \mapsto f^\circ \colon R^\phi \to R$ is an isomorphism of differential rings by the identity above. We extend it to the isomorphism

$$Q \mapsto Q^\circ : R^\phi\{Y\} \to R\{Y\}$$

of differential rings given by $Y^\circ = Y$. Let $y \in R$. Then

$$Q(y)^\circ \;=\; Q^\circ(y^\circ) \qquad \text{for } Q \in R^\phi\{Y\}$$

and thus

$$P(y)^\circ \;=\; P^\phi(y)^\circ \;=\; (P^\phi)^\circ(y^\circ) \qquad \text{for } P \in R\{Y\}.$$

Second-order differential equations. Let $f \in \mathcal{C}_a$, that is, $f\colon [a,\infty) \to \mathbb{R}$ is continuous. We consider the differential equation

$$\text{(L)} \qquad Y'' + fY \;=\; 0.$$

The solutions $y \in \mathcal{C}^2_a$ of (L) form an $\mathbb{R}$-linear subspace $\operatorname{Sol}(f)$ of $\mathcal{C}^2_a$. The solutions $y \in \mathcal{C}^2_a[i]$ of (L) are the $y_1 + y_2 i$ with $y_1, y_2 \in \operatorname{Sol}(f)$ and form a $\mathbb{C}$-linear subspace $\operatorname{Sol}_{\mathbb{C}}(f)$ of $\mathcal{C}^2_a[i]$. For any complex numbers c, d there is a unique solution $y \in \mathcal{C}^2_a[i]$ of (L) with $y(a) = c$ and $y'(a) = d$, and the map that assigns to (c,d) in $\mathbb{C}^2$ this unique solution is an isomorphism $\mathbb{C}^2 \to \operatorname{Sol}_{\mathbb{C}}(f)$ of $\mathbb{C}$-linear spaces; it restricts to an $\mathbb{R}$-linear bijection $\mathbb{R}^2 \to \operatorname{Sol}(f)$. We have $f \in \mathcal{C}^n_a \Rightarrow \operatorname{Sol}(f) \subseteq \mathcal{C}^{n+2}_a$ (hence $f \in \mathcal{C}^\infty_a \Rightarrow \operatorname{Sol}(f) \subseteq \mathcal{C}^\infty_a$) and $f \in \mathcal{C}^\omega_a \Rightarrow \operatorname{Sol}(f) \subseteq \mathcal{C}^\omega_a$. (See [23, (10.5.3)].)

Let $y_1, y_2 \in \operatorname{Sol}(f)$, with Wronskian $w = y_1 y_2' - y_1' y_2$. Then $w \in \mathbb{R}$, and

$$w \neq 0 \iff y_1,\ y_2 \text{ are } \mathbb{R}\text{-linearly independent.}$$

By [14, Chapter 6, Lemmas 2 and 3] we have:

Lemma 3.1. *Let $y_1, y_2 \in \operatorname{Sol}(f)$ be $\mathbb{R}$-linearly independent and $g \in \mathcal{C}_a$. Then*

$$t \mapsto y(t) \;:=\; -y_1(t)\int_a^t \frac{y_2(s)}{w} g(s)\,ds + y_2(t)\int_a^t \frac{y_1(s)}{w} g(s)\,ds \;:\; [a,+\infty) \to \mathbb{R}$$

lies in $\mathcal{C}^2_a$ and satisfies $y'' + fy = g$, $y(a) = y'(a) = 0$.

Lemma 3.2. *Let $y_1 \in \operatorname{Sol}(f)$ with $y_1(t) \neq 0$ for $t \geqslant a$. Then the function*

$$t \mapsto y_2(t) := y_1(t)\int_a^t \frac{1}{y_1(s)^2}\,ds\colon [a,+\infty) \to \mathbb{R}$$

also lies in $\operatorname{Sol}(f)$, and y_1, y_2 are $\mathbb{R}$-linearly independent.

From [14, Chapter 2, Lemma 1] we also recall:

Lemma 3.3 (Gronwall's Lemma)**.** *Let $C \in \mathbb{R}^{\geqslant}$, $v, y \in \mathcal{C}_a$ satisfy $v(t), y(t) \geqslant 0$ for all $t \geqslant a$ and*

$$y(t) \;\leqslant\; C + \int_a^t v(s)y(s)\,ds \quad \text{for all } t \geqslant a.$$

Then

$$y(t) \;\leqslant\; C\exp\left[\int_a^t v(s)\,ds\right] \quad \text{for all } t \geqslant a.$$

In the rest of this subsection we assume that $a \geqslant 1$ and that $c \in \mathbb{R}^{>}$ is such that $|f(t)| \leqslant c/t^2$ for all $t \geqslant a$. Under this hypothesis, Lemma 3.3 yields the following bound on the growth of the solutions $y \in \operatorname{Sol}(f)$; the proof we give is similar to that of [14, Chapter 6, Theorem 5].

Proposition 3.4. *Let $y \in \operatorname{Sol}(f)$. Then there is $C \in \mathbb{R}^{\geqslant}$ such that $|y(t)| \leqslant Ct^{c+1}$ and $|y'(t)| \leqslant Ct^c$ for all $t \geqslant a$.*

Proof. Let t range over $[a,+\infty)$. Integrating $y''=-fy$ twice between a and t, we obtain constants c_1, c_2 such that for all t,

$$y(t) \;=\; c_1+c_2t-\int_a^t\int_a^{t_1} f(t_2)y(t_2)\,dt_2\,dt_1 \;=\; c_1+c_2t-\int_a^t (t-s)f(s)y(s)\,ds$$

and hence, with $C:=|c_1|+|c_2|$,

$$|y(t)| \;\leqslant\; Ct+t\int_a^t |f(s)|\cdot|y(s)|\,ds, \quad\text{so}\quad \frac{|y(t)|}{t} \;\leqslant\; C+\int_a^t s|f(s)|\cdot\frac{|y(s)|}{s}\,ds.$$

Then by Lemma 3.3,

$$\frac{|y(t)|}{t} \;\leqslant\; C\exp\left[\int_a^t s|f(s)|\,ds\right] \;\leqslant\; C\exp\left[\int_1^t c/s\,ds\right] \;=\; Ct^c$$

and thus $|y(t)|\leqslant Ct^{c+1}$. Now

$$\begin{aligned} y'(t) \;&=\; c_2-\int_a^t f(s)y(s)\,ds, \text{ so}\\ |y'(t)| \;&\leqslant\; |c_2|+\int_a^t |f(s)y(s)|\,ds \;\leqslant\; C+Cc\int_1^t s^{c-1}\,ds\\ &=\; C+Cc\left[\frac{t^c}{c}-\frac{1}{c}\right] \;=\; Ct^c. \end{aligned}$$

□

Let $y_1,y_2\in\mathrm{Sol}(f)$ be $\mathbb{R}$-linearly independent. Recall that $w=y_1y_2'-y_1'y_2\in\mathbb{R}^\times$. It follows that y_1 and y_2 cannot be simultaneously very small:

Lemma 3.5. *There is a positive constant d such that*

$$\max\bigl(|y_1(t)|,|y_2(t)|\bigr) \;\geqslant\; dt^{-c} \qquad \textit{for all } t\geqslant a.$$

Proof. Proposition 3.4 yields $C\in\mathbb{R}^>$ such that $|y_i'(t)|\leqslant Ct^c$ for $i=1,2$ and all $t\geqslant a$. Hence $|w|\leqslant 2\max\bigl(|y_1(t)|,|y_2(t)|\bigr)Ct^c$ for $t\geqslant a$, so

$$\max\bigl(|y_1(t)|,|y_2(t)|\bigr) \;\geqslant\; \frac{|w|}{2C}t^{-c} \qquad (t\geqslant a).$$

□

Corollary 3.6. *Set $y:=y_1+y_2\mathrm{i}$ and $z:=y^\dagger$. Then for some $D\in\mathbb{R}^>$,*

$$|z(t)| \;\leqslant\; Dt^{2c} \qquad \textit{for all } t\geqslant a.$$

Proof. Take C as in the proof of Lemma 3.5, and d as in that lemma. Then

$$|z(t)| \;=\; \frac{|y_1'(t)+y_2'(t)\mathrm{i}|}{|y_1(t)+y_2(t)\mathrm{i}|} \;\leqslant\; \frac{|y_1'(t)|+|y_2'(t)|}{\max\bigl(|y_1(t)|,|y_2(t)|\bigr)} \;\leqslant\; \left(\frac{2C}{d}\right)t^{2c}$$

for $t\geqslant a$. □

Oscillation. Let $y\in\mathcal{C}_a$. We say that y **oscillates** if its germ in $\mathcal{C}$ oscillates. So y does not oscillate iff $\operatorname{sign} y(t)$ is constant, eventually. If y oscillates, then so does cy for $c\in\mathbb{R}^\times$. If $y\in\mathcal{C}_a^1$ oscillates, then so does $y'\in\mathcal{C}_a$, by Rolle's Theorem.

We now continue with the study of (L). Let $y\in\mathrm{Sol}(f)^{\neq}$, and let $Z:=y^{-1}(0)$ be the set of zeros of y, so $Z\subseteq[a,+\infty)$ is closed in $\mathbb{R}$. Moreover, Z has no limit points, hence $Z\cap[a,b]$ is finite for every $b\geqslant a$. Thus

$$y \text{ does not oscillate} \quad\Longleftrightarrow\quad Z \text{ is finite} \quad\Longleftrightarrow\quad Z \text{ is bounded.}$$

If $t_0\in Z$ is not the largest element of Z, then $Z\cap(t_0,t_1)=\emptyset$ for some $t_1>t_0$ in Z. We say that a pair of zeros $t_0<t_1$ of y is **consecutive** if $Z\cap(t_0,t_1)=\emptyset$.

Sturm's Separation Theorem says that if $y, z \in \mathrm{Sol}(f)$ are $\mathbb{R}$-linearly independent and $t_0 < t_1$ are consecutive zeros of z, then (t_0, t_1) contains a unique zero of y [62, §27, VI]. Thus:

Lemma 3.7. *Some $y \in \mathrm{Sol}(f)^{\neq}$ oscillates $\iff$ every $y \in \mathrm{Sol}(f)^{\neq}$ oscillates.*

We say that f **generates oscillation** if some element of $\mathrm{Sol}(f)^{\neq}$ oscillates.

Lemma 3.8. *Let $b \in \mathbb{R}^{\geqslant a}$. Then*

$$f \text{ generates oscillation} \iff f|_b \in \mathcal{C}_b \text{ generates oscillation.}$$

Proof. The forward direction is obvious. For the backward direction, use that every $y \in \mathcal{C}_b^2$ with $y'' + gy = 0$ for $g := f|_b$ extends uniquely to a solution of (L). □

By this lemma, whether f generates oscillation depends only on its germ in $\mathcal{C}$. So this induces the notion of an element of $\mathcal{C}$ generating oscillation. Here is another result of Sturm [62, loc. cit.] that we use below:

Theorem 3.9 (Sturm's Comparison Theorem)**.** *Let $g \in \mathcal{C}_a$ with $f(t) \geqslant g(t)$ for all $t \geqslant a$. Let $y \in \mathrm{Sol}(f)^{\neq}$ and $z \in \mathrm{Sol}(g)^{\neq}$, and let $t_0 < t_1$ be consecutive zeros of z. Then either (t_0, t_1) contains a zero of y, or on $[t_0, t_1]$ we have $f = g$ and $y = cz$ for some constant $c \in \mathbb{R}^{\times}$.*

Here is an immediate consequence:

Corollary 3.10. *If $g \in \mathcal{C}_a$ generates oscillation and $f(t) \geqslant g(t)$, eventually, then f also generates oscillation.*

Example. For $k \in \mathbb{R}^{\times}$ we have the differential equation of the harmonic oscillator,

$$y'' + k^2 y = 0.$$

A function $y \in \mathcal{C}_a^2$ is a solution iff for some real constants c, t_0 and all $t \geqslant a$,

$$y(t) = c \sin k(t - t_0).$$

For $c \neq 0$, any function $y \in \mathcal{C}_a^2$ as displayed oscillates. Thus if $f(t) \geqslant \varepsilon$, eventually, for some constant $\varepsilon > 0$, then f generates oscillation.

To (L) we associate the corresponding **Riccati equation**

$$\text{(R)} \qquad z' + z^2 + f = 0.$$

Let $y \in \mathrm{Sol}(f)^{\neq}$ be a non-oscillating solution to (L), and take $b \geqslant a$ with $y(t) \neq 0$ for $t \geqslant b$. Then the function

$$t \mapsto z(t) := y'(t)/y(t) : [b, +\infty) \to \mathbb{R}$$

in $\mathcal{C}_b^1$ satisfies (R). Conversely, if $z \in \mathcal{C}_b^1$ $(b \geqslant a)$ is a solution to (R), then

$$t \mapsto y(t) := \exp\left(\int_b^t z(s)\, ds\right) : [b, +\infty) \to \mathbb{R}$$

is a non-oscillating solution to (L) with $y \in (\mathcal{C}_b^1)^{\times}$ and $z = y^{\dagger}$.

Let $g \in \mathcal{C}_a^1$, $h \in \mathcal{C}_a^0$ and consider the second-order linear differential equation

$$(\widetilde{\mathrm{L}}) \qquad y'' + gy' + hy = 0.$$

Corollary 3.11. *Set $f := -\frac{1}{2}g' - \frac{1}{4}g^2 + h \in \mathcal{C}_a$. Then the following are equivalent:*

(i) *some nonzero solution of $(\widetilde{\mathrm{L}})$ oscillates;*

(ii) *all nonzero solutions of* $(\widetilde{\mathrm{L}})$ *oscillate;*
(iii) f *generates oscillation.*

Proof. Let $G \in (\mathcal{C}_a^2)^\times$ be given by $G(t) := \exp\Big(-\frac{1}{2}\int_a^t g(s)\,ds\Big)$. Then $y \in \mathcal{C}_a^2$ is a solution to (L) iff Gy is a solution to $(\widetilde{\mathrm{L}})$; cf. [ADH, 5.1.13]. □

Non-oscillation. We continue with (L). Let y_1, y_2 range over elements of $\mathrm{Sol}(f)$, and recall that its Wronskian $w = y_1y_2' - y_1'y_2$ lies in $\mathbb{R}$.

Lemma 3.12. *Suppose $b \geqslant a$ is such that $y_2(t) \neq 0$ for $t \geqslant b$. Then for $q := y_1/y_2 \in \mathcal{C}_b^2$ we have $q'(t) = -w/y_2(t)^2$ for $t \geqslant b$, so q is monotone and $\lim_{t\to\infty} q(t)$ exists in $\mathbb{R} \cup \{-\infty, +\infty\}$.*

This leads to:

Corollary 3.13. *Suppose $b \geqslant a$ and $y_1(t), y_2(t) \neq 0$ for $t \geqslant b$. For $i = 1, 2$, set*

$$h_i(t) \ := \ \int_b^t \frac{1}{y_i(s)^2}\,ds \qquad \text{for } t \geqslant b, \text{ so } h_i \in \mathcal{C}_b^3.$$

Then: $y_1 \prec y_2 \iff h_1 \succ 1 \succcurlyeq h_2$.

Proof. Suppose $y_1 \prec y_2$. Then y_1, y_2 are $\mathbb{R}$-linearly independent, so $w \neq 0$. Moreover, $q \prec 1$ with q as in in Lemma 3.12, and $q' = -wh_2'$ by that lemma, so $q + wh_2$ is constant, and thus $h_2 \preccurlyeq 1$. Note that h_1 is strictly increasing. If $h_1(t) \to r \in \mathbb{R}$ as $t \to +\infty$, then $z := (r - h_1)y_1 \in \mathrm{Sol}(f)$ by Lemma 3.12 with y_1 and y_2 interchanged, and $z \prec y_1$, so $z = 0$, hence $h_1 = r$, a contradiction. Thus $h_1 \succ 1$.

For the converse, suppose $h_1 \succ 1 \succcurlyeq h_2$. Then y_1, y_2 are $\mathbb{R}$-linearly independent, so $w \neq 0$. From $h_2 \preccurlyeq 1$ and $q + wh_2$ being constant we obtain $q \preccurlyeq 1$. If $q(t) \to r \neq 0$ as $t \to +\infty$, then $y_1 = qy_2 \asymp y_2$, and thus $h_1 \asymp h_2$, a contradiction. Hence $q \prec 1$, and thus $y_1 \prec y_2$. □

The pair (y_1, y_2) is said to be a **principal system** of solutions of (L) if

(1) $y_1(t), y_2(t) > 0$ eventually, and
(2) $y_1 \prec y_2$.

Then y_1, y_2 form a basis of the $\mathbb{R}$-linear space $\mathrm{Sol}(f)$, and f does not generate oscillation, by Lemma 3.7. Moreover, for $y = c_1y_1 + c_2y_2$ with $c_1, c_2 \in \mathbb{R}$, $c_2 \neq 0$ we have $y \sim c_2y_2$. Here are some facts about this notion:

Lemma 3.14. *If (y_1, y_2), (z_1, z_2) are principal systems of solutions of* (L)*, then there are $c_1, d_1, d_2 \in \mathbb{R}$ such that $z_1 = c_1y_1$, $z_2 = d_1y_1 + d_2y_2$, and $c_1, d_2 > 0$.*

Lemma 3.15. *Suppose f does not generate oscillation. Then* (L) *has a principal system of solutions.*

Proof. It suffices to find a basis y_1, y_2 of $\mathrm{Sol}(f)$ with $y_1 \prec y_2$. Suppose y_1, y_2 is any basis of $\mathrm{Sol}(f)$, and set $c := \lim_{t\to\infty} y_1(t)/y_2(t) \in \mathbb{R} \cup \{-\infty, +\infty\}$. If $c = \pm\infty$, then interchange y_1, y_2, otherwise replace y_1 by $y_1 - cy_2$. Then $c = 0$, so $y_1 \prec y_2$. □

One calls y_1 a **principal** solution of (L) if (y_1, y_2) is a principal system of solutions of (L) for some y_2. (See [33, Theorem XI.6.4] and [43, 45].) By the previous two lemmas, (L) has a principal solution iff f does not generate oscillation, and any two principal solutions differ by a multiplicative factor in $\mathbb{R}^>$. If $y_1 \in (\mathcal{C}_a)^\times$ and y_2 is given as in Lemma 3.2, then y_2 is a non-principal solution of (L) and $y_1 \notin \mathbb{R}y_2$.

Remark 3.16 (Hardy-type inequality associated to (L)). Suppose $f(t) > 0$ for all $t > a$ and (L) has a solution y such that $y(t), y'(t) > 0$ for all $t > a$. Then for some $C = C_f \in \mathbb{R}^{\geqslant}$, every $u \in \mathcal{C}^1_a$ with $u(a) = 0$ satisfies

$$\int_a^\infty |u(t)|^2 f(t)\,dt \;\leqslant\; C \int_a^\infty |u'(t)|^2\,dt. \tag{3.1}$$

For $a > 0$ and $f(t) := \frac{1}{4t^2}$ for $t \geqslant a$ this was shown by Hardy [27, 29]; here one can take $C = 1$, and this is optimal [41]. For the general case, see [52, Theorem 4.1].

4. Hardy Fields

In this brief section we introduce Hardy fields and review some classical extension theorems for Hardy fields.

Hardy fields. A *Hardy field* is a subfield of $\mathcal{C}^{<\infty}$ that is closed under the derivation of $\mathcal{C}^{<\infty}$; see also [ADH, 9.1]. Let H be a Hardy field. Then H is considered as an ordered valued differential field in the obvious way; see Section 2 for the ordering and valuation on H. The field of constants of H is $\mathbb{R} \cap H$. Hardy fields are pre-H-fields, and H-fields if they contain $\mathbb{R}$; see [ADH, 9.1.9(i), (iii)]. As in Section 2 we equip the differential subfield $H[i]$ of $\mathcal{C}^{<\infty}[i]$ with the unique valuation ring whose intersection with H is the valuation ring of H. Then $H[i]$ is a pre-d-valued field of H-type with small derivation and constant field $\mathbb{C} \cap H[i]$; if $H \supseteq \mathbb{R}$, then $H[i]$ is d-valued with constant field $\mathbb{C}$. (Section 5 has an example of a differential subfield of $\mathcal{C}^{<\infty}[i]$ that is not contained in $H[i]$ for any Hardy field H.)

We also consider variants: a $\mathcal{C}^\infty$-*Hardy field* is a Hardy field $H \subseteq \mathcal{C}^\infty$, and a $\mathcal{C}^\omega$-*Hardy field* (also called an *analytic Hardy field*) is a Hardy field $H \subseteq \mathcal{C}^\omega$. Most Hardy fields arising in practice are actually $\mathcal{C}^\omega$-Hardy fields.

Hardian germs. Let $y \in \mathcal{G}$. Following [57] we call y **hardian** if it lies in a Hardy field (and thus $y \in \mathcal{C}^{<\infty}$). We also say that y is $\mathcal{C}^\infty$**-hardian** if y lies in a $\mathcal{C}^\infty$-Hardy field, equivalently, $y \in \mathcal{C}^\infty$ and y is hardian; likewise with $\mathcal{C}^\omega$ in place of $\mathcal{C}^\infty$. Let H be a Hardy field. Call $y \in \mathcal{G}$ H**-hardian** (or **hardian over** H) if y lies in a Hardy field extension of H. (Thus y is hardian iff y is $\mathbb{Q}$-hardian.) If H is a $\mathcal{C}^\infty$-Hardy field and $y \in \mathcal{C}^\infty$ is hardian over H, then y generates a $\mathcal{C}^\infty$-Hardy field extension $H\langle y\rangle$ of H; likewise with $\mathcal{C}^\omega$ in place of $\mathcal{C}^\infty$.

Maximal and perfect Hardy fields. Let H be a Hardy field. Call H *maximal* if no Hardy field properly contains H. Following Boshernitzan [18] we denote by $\operatorname{E}(H)$ the intersection of all maximal Hardy fields containing H; thus $\operatorname{E}(H)$ is a Hardy field extension of H, and a maximal Hardy field contains H iff it contains $\operatorname{E}(H)$, so $\operatorname{E}(\operatorname{E}(H)) = \operatorname{E}(H)$. If H^* is a Hardy field extension of H, then $\operatorname{E}(H) \subseteq \operatorname{E}(H^*)$; hence if H^* is a Hardy field with $H \subseteq H^* \subseteq \operatorname{E}(H)$, then $\operatorname{E}(H^*) = \operatorname{E}(H)$. Note that $\operatorname{E}(H)$ consists of the $f \in \mathcal{G}$ that are hardian over each Hardy field $E \supseteq H$. Hence $\operatorname{E}(\mathbb{Q})$ consists of the germs in $\mathcal{G}$ that are hardian over each Hardy field. As in [18] we also say that H is **perfect** if $\operatorname{E}(H) = H$. (This terminology is slightly unfortunate, since Hardy fields, being of characteristic zero, are perfect as fields.) Thus $\operatorname{E}(H)$ is the smallest perfect Hardy field extension of H. Maximal Hardy fields are perfect.

Differentially maximal Hardy fields. Let H be a Hardy field. We now define differentially-algebraic variants of the above: call H **differentially maximal**, or d**-maximal** for short, if H has no proper d-algebraic Hardy field extension. Every maximal Hardy field is d-maximal, so each Hardy field is contained in a d-maximal one; in fact, by Zorn, each Hardy field H has a d-maximal Hardy field extension which is d-algebraic over H. Let $\mathrm{D}(H)$ be the intersection of all d-maximal Hardy fields containing H. Then $\mathrm{D}(H)$ is a d-algebraic Hardy field extension of H with $\mathrm{D}(H) \subseteq \mathrm{E}(H)$. We have $\mathrm{D}(H) = \mathrm{E}(H)$ iff $\mathrm{E}(H)$ is d-algebraic over H:

Lemma 4.1. $\mathrm{D}(H) = \big\{f \in \mathrm{E}(H) : f$ *is* d*-algebraic over* $H\big\}$.

Proof. We only need to show the inclusion "$\supseteq$". For this let $f \in \mathrm{E}(H)$ be d-algebraic over H, and let E be a d-maximal Hardy field extension of H; we need to show $f \in E$. To see this extend E to a maximal Hardy field M; then $f \in M$, hence f generates a Hardy field extension $E\langle f\rangle$ of E. Since f is d-algebraic over H and thus over E, this yields $f \in E$ by d-maximality of E, as required. □

A d-maximal Hardy field contains H iff it contains $\mathrm{D}(H)$, hence $\mathrm{D}(\mathrm{D}(H)) = \mathrm{D}(H)$. For any Hardy field $H^* \supseteq H$ we have $\mathrm{D}(H^*) \supseteq \mathrm{D}(H)$, hence if also $H^* \subseteq \mathrm{D}(H)$, then $\mathrm{D}(H^*) = \mathrm{D}(H)$. We say that H is d**-perfect** if $\mathrm{D}(H) = H$. Thus $\mathrm{D}(H)$ is the smallest d-perfect Hardy field extension of H. Every perfect Hardy field is d-perfect, as is every d-maximal Hardy field. The following diagram summarizes the various implications among these properties of Hardy fields:

$$\begin{array}{ccc} \text{maximal} & \Longrightarrow & \text{perfect} \\ \Downarrow & & \Downarrow \\ \text{d-maximal} & \Longrightarrow & \text{d-perfect} \end{array}$$

We call $\mathrm{D}(H)$ the d**-perfect hull** of H, and $\mathrm{E}(H)$ the **perfect hull** of H.

Variants of the perfect hull. Let H be a $\mathcal{C}^r$-Hardy field where $r \in \{\infty, \omega\}$. We say that H is $\mathcal{C}^r$**-maximal** if no $\mathcal{C}^r$-Hardy field properly contains it. By Zorn, H has a $\mathcal{C}^r$-maximal extension. In analogy with $\mathrm{E}(H)$, define the $\mathcal{C}^r$**-perfect hull** $\mathrm{E}^r(H)$ of H to be the intersection of all $\mathcal{C}^r$-maximal Hardy fields containing H. We say that H is $\mathcal{C}^r$**-perfect** if $\mathrm{E}^r(H) = H$. The penultimate subsection goes through with *Hardy field*, *maximal*, *hardian*, $\mathrm{E}(\,\cdot\,)$, and *perfect* replaced by $\mathcal{C}^r$*-Hardy field*, $\mathcal{C}^r$*-maximal*, $\mathcal{C}^r$*-hardian*, $\mathrm{E}^r(\,\cdot\,)$, and $\mathcal{C}^r$*-perfect*, respectively. (In [12] we show that no analogue of $\mathrm{D}(H)$ is needed for the $\mathcal{C}^r$-category.)

Some basic extension theorems. We summarize some well-known extension results for Hardy fields, cf. [20], [57], [55]:

Proposition 4.2. *Any Hardy field H has the following Hardy field extensions:*

- (i) $H(\mathbb{R})$*, the subfield of* $\mathcal{C}^{<\infty}$ *generated by* H *and* $\mathbb{R}$*;*
- (ii) H^{rc}*, the real closure of* H *as defined in Proposition 2.4;*
- (iii) $H(\mathrm{e}^f)$ *for any* $f \in H$*;*
- (iv) $H(f)$ *for any* $f \in \mathcal{C}^1$ *with* $f' \in H$*;*
- (v) $H(\log f)$ *for any* $f \in H^{>}$*.*

If H is contained in $\mathcal{C}^\infty$, then so are the Hardy fields in (i), (ii), (iii), (iv), (v)*; likewise with $\mathcal{C}^\omega$ instead of $\mathcal{C}^\infty$.*

Note that (v) is a special case of (iv), since $(\log f)' = f^\dagger \in H$ for $f \in H^>$. Another special case of (iv) is that a Hardy field H yields a Hardy field $H(x)$. It also yields the Hardy field H_{LE} from the introduction as the smallest (under inclusion) Hardy field extension of $\mathbb{R}(x)$ that is log-closed and exp-closed.

A consequence of the proposition is that any Hardy field H has a smallest real closed Hardy field extension L with $\mathbb{R} \subseteq L$ such that for all $f \in L$ we have $e^f \in L$ and $g' = f$ for some $g \in L$. Note that then L is a Liouville closed H-field as defined in Section 1. Let H be a Hardy field with $H \supseteq \mathbb{R}$. As in [3] and [ADH, p. 460] we then denote the above L by $\text{Li}(H)$; so $\text{Li}(H)$ is the smallest Liouville closed Hardy field containing H, called the *Hardy-Liouville closure* of H in [8]. We have $\text{Li}(H) \subseteq \text{D}(H)$, hence if H is d-perfect, then H is a Liouville closed H-field. Moreover, if $H \subseteq \mathcal{C}^\infty$ then $\text{Li}(H) \subseteq \mathcal{C}^\infty$, and similarly with $\mathcal{C}^\omega$ in place of $\mathcal{C}^\infty$.

The next more general result in Rosenlicht [55] is attributed there to M. Singer:

Proposition 4.3. *Let H be a Hardy field and $p, q \in H[Y]$, $y \in \mathcal{C}^1$, such that $y'q(y) = p(y)$ with $q(y) \in \mathcal{C}^\times$. Then y generates a Hardy field $H(y)$ over H.*

Note that for H, p, q, y as in the proposition we have $y \in \text{D}(H)$.

Compositional conjugation in Hardy fields. Let now H be a Hardy field, and let $\ell \in \mathcal{C}^1$ be such that $\ell > \mathbb{R}$ and $\ell' \in H$. Then $\ell \in \mathcal{C}^{<\infty}$, $\phi := \ell'$ is active in H, $\phi > 0$, and we have a Hardy field $H(\ell)$. The $\mathbb{C}$-algebra automorphism $f \mapsto f^\circ := f \circ \ell^{\text{inv}}$ of $\mathcal{C}[i]$ restricts to an ordered field isomorphism

$$h \mapsto h^\circ \ : \ H \to H^\circ := H \circ \ell^{\text{inv}}.$$

The identity $(f^\circ)' = (\phi^{-1} f')^\circ$, valid for each $f \in \mathcal{C}^1[i]$, shows that H° is again a Hardy field. Conversely, if E is a subfield of $\mathcal{C}^{<\infty}$ with $\phi \in E$ and $E^\circ := E \circ \ell^{\text{inv}}$ is a Hardy field, then E is a Hardy field. If $H \subseteq \mathcal{C}^\infty$ and $\ell \in \mathcal{C}^\infty$, then $H^\circ \subseteq \mathcal{C}^\infty$; likewise with $\mathcal{C}^\omega$ instead of $\mathcal{C}^\infty$. If E is a Hardy field extension of H, then E° is a Hardy field extension of H°, and E is d-algebraic over H iff E° is d-algebraic over H°. Hence H is maximal iff H° is maximal, and likewise with "d-maximal" in place of "maximal". So $\text{E}(H^\circ) = \text{E}(H)^\circ$ and $\text{D}(H^\circ) = \text{D}(H)^\circ$, and thus H is perfect iff H° is perfect, and likewise with "d-perfect" in place of "perfect". The next lemma is [17, Corollary 6.5]; see also [5, Theorem 1.7].

Lemma 4.4. *The germ ℓ^{inv} is hardian. Moreover, if ℓ is $\mathcal{C}^\infty$-hardian, then ℓ^{inv} is also $\mathcal{C}^\infty$-hardian, and likewise with $\mathcal{C}^\omega$ in place of $\mathcal{C}^\infty$.*

Proof. By Proposition 4.2(iv) we can arrange that our Hardy field H contains both ℓ and x. Then $\ell^{\text{inv}} = x \circ \ell^{\text{inv}}$ is an element of the Hardy field $H \circ \ell^{\text{inv}}$. □

Next we consider the pre-d-valued field $K := H[i]$ of H-type, which gives rise to

$$K^\circ := K \circ \ell^{\text{inv}} = H^\circ[i],$$

also a pre-d-valued field of H-type, and we have the valued field isomorphism

$$h \mapsto h^\circ \ : \ K \to K^\circ.$$

Note: $h \mapsto h^\circ : H^\phi \to H^\circ$ is an isomorphism of pre-H-fields, and $h \mapsto h^\circ : K^\phi \to K^\circ$ is an isomorphism of valued differential fields. Recall that K and K^ϕ have the same underlying field.

Lemma 4.5. *From the isomorphisms $H^\phi \cong H^\circ$ and $K^\phi \cong K^\circ$ we obtain: If H is Liouville closed, then so is H°. If $\text{I}(K) \subseteq K^\dagger$, then $\text{I}(K^\circ) \subseteq (K^\circ)^\dagger$.*

5. Upper and Lower Bounds on the Growth of Hardian Germs

In this section we use logarithmic decompositions to simplify arguments in [18, 19, 56]. It is not used for proving our main theorem, but some of it is needed for its applications, in the proofs of Corollary 7.10, Proposition 8.1, and Theorem 8.6.

Generalizing logarithmic decomposition. In this subsection K is a differential ring and $y \in K$. In [ADH, p. 213] we defined the nth iterated logarithmic derivative of $y^{\langle n\rangle}$ when K is a differential field. (See also Section 1.) Generalizing this, set $y^{\langle 0\rangle} := y$, and recursively, if $y^{\langle n\rangle} \in K$ is defined and a unit in K, then $y^{\langle n+1\rangle} := (y^{\langle n\rangle})^\dagger$, while otherwise $y^{\langle n+1\rangle}$ is not defined. (Thus if $y^{\langle n\rangle}$ is defined, then so are $y^{\langle 0\rangle}, \dots, y^{\langle n-1\rangle}$.) With L_n in $\mathbb{Z}[X_1, \dots, X_n]$ as in [ADH, p. 213], if $y^{\langle n\rangle}$ is defined, then

$$y^{(n)} \;=\; y^{\langle 0\rangle} \cdot L_n(y^{\langle 1\rangle}, \dots, y^{\langle n\rangle}).$$

If $y^{\langle n\rangle}$ is defined and $\boldsymbol{i} = (i_0, \dots, i_n) \in \mathbb{N}^{1+n}$, we set

$$y^{\langle \boldsymbol{i}\rangle} \;:=\; (y^{\langle 0\rangle})^{i_0} (y^{\langle 1\rangle})^{i_1} \cdots (y^{\langle n\rangle})^{i_n} \in K.$$

Hence if H is a differential subfield of K, $P \in H\{Y\}$ has order at most n and logarithmic decomposition $P = \sum_{\boldsymbol{i}} P_{\langle \boldsymbol{i}\rangle} Y^{\langle \boldsymbol{i}\rangle}$ ($\boldsymbol{i}$ ranging over $\mathbb{N}^{1+n}$, all $P_{\langle \boldsymbol{i}\rangle} \in H$, and $P_{\langle \boldsymbol{i}\rangle} = 0$ for all but finitely many $\boldsymbol{i}$), and $y^{\langle n\rangle}$ is defined, then $P(y) = \sum_{\boldsymbol{i}} P_{\langle \boldsymbol{i}\rangle} y^{\langle \boldsymbol{i}\rangle}$. Below we apply these remarks to $K = \mathcal{C}^{<\infty}$, where for $y \in K^\times$ we have $y^\dagger = (\log|y|)'$, hence $y^{\langle n+1\rangle} = (\log|y^{\langle n\rangle}|)'$ if $y^{\langle n+1\rangle}$ is defined.

Transexponential germs. For $f \in \mathcal{C}$ we recursively define the germs $\exp_n f$ in $\mathcal{C}$ by $\exp_0 f := f$ and $\exp_{n+1} f := \exp(\exp_n f)$. Following [18] we say that a germ $y \in \mathcal{C}$ is **transexponential** if $y \geqslant \exp_n x$ for all n. *In the rest of this subsection H is a Hardy field.* By Corollary 1.9 and Proposition 4.2:

Lemma 5.1. *If the H-hardian germ y is* d*-algebraic over H, then $y \leqslant \exp_n h$ for some n and some $h \in H(x)$.*

Thus each transexponential hardian germ is d-transcendental (over $\mathbb{R}$). *In the rest of this subsection: $y \in \mathcal{C}^{<\infty}$ is transexponential and hardian, and $z \in \mathcal{C}^{<\infty}[i]$.* Then $y^{\langle n\rangle}$ is defined, and $y^{\langle n\rangle}$ is also transexponential and hardian, for all n. Next some variants of results from Section 1. For this, let n be given and let $f \in \mathcal{C}^{<\infty}$, not necessarily hardian, be such that $f \succ 1$, $f \geqslant 0$, and $y \succcurlyeq \exp_{n+1} f$.

Lemma 5.2. *We have $y^\dagger \succcurlyeq \exp_n f$ and $y^{\langle n\rangle} \succcurlyeq \exp f$.*

Proof. Since $y \succcurlyeq \exp_2 x$, we have $\log y \succcurlyeq \exp x$ by Lemma 2.2, and thus $y^\dagger = (\log y)' \succcurlyeq \log y$. Since $y \succcurlyeq \exp_{n+1} f$, the same lemma gives $\log y \succcurlyeq \exp_n f$. Thus $y^\dagger \succcurlyeq \exp_n f$. Now the second statement follows by an easy induction. □

Corollary 5.3. *Let $\boldsymbol{i} \in \mathbb{Z}^{1+n}$ and suppose $\boldsymbol{i} > 0$ lexicographically. Then $y^{\langle \boldsymbol{i}\rangle} \succ f$.*

Proof. Let $m \in \{0, \dots, n\}$ be minimal such that $i_m \neq 0$; so $i_m \geqslant 1$. The remarks after Corollary 1.2 then give $y^{\langle \boldsymbol{i}\rangle} \succ 1$ and $[v(y^{\langle \boldsymbol{i}\rangle})] = [v(y^{\langle m\rangle})]$, so we have $k \in \mathbb{N}$, $k \geqslant 1$, such that $y^{\langle \boldsymbol{i}\rangle} \succcurlyeq (y^{\langle m\rangle})^{1/k}$. Then Lemma 5.2 gives $y^{\langle \boldsymbol{i}\rangle} \succcurlyeq (y^{\langle m\rangle})^{1/k} \succcurlyeq (\exp f)^{1/k} \succ f$ as required. □

In the next proposition and lemma $P \in H\{Y\}^{\neq}$ has order at most n, and $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ range over $\mathbb{N}^{1+n}$. Let $\boldsymbol{j}$ be lexicographically maximal such that $P_{\langle \boldsymbol{j}\rangle} \neq 0$, and choose $\boldsymbol{k}$ so that $P_{\langle \boldsymbol{k}\rangle}$ has minimal valuation. If $P_{\langle \boldsymbol{k}\rangle}/P_{\langle \boldsymbol{j}\rangle} \succ x$, set $f := |P_{\langle \boldsymbol{k}\rangle}/P_{\langle \boldsymbol{j}\rangle}|$; otherwise set $f := x$. Then $f \in H(x)$, $f > 0$, $f \succ 1$, and $f \succcurlyeq P_{\langle \boldsymbol{i}\rangle}/P_{\langle \boldsymbol{j}\rangle}$ for all $\boldsymbol{i}$.

Proposition 5.4. *We have $P(y) \sim P_{\langle \boldsymbol{j}\rangle} y^{\langle \boldsymbol{j}\rangle}$ and thus*

$$P(y) \in \left(\mathcal{C}^{<\infty}\right)^{\times}, \qquad \operatorname{sign} P(y) \ = \ \operatorname{sign} P_{\langle \boldsymbol{j}\rangle} \neq 0.$$

Proof. For $\boldsymbol{i} < \boldsymbol{j}$ we have $y^{\langle \boldsymbol{j}-\boldsymbol{i}\rangle} \succ f \succcurlyeq P_{\langle \boldsymbol{i}\rangle}/P_{\langle \boldsymbol{j}\rangle}$ by Corollary 5.3, hence $P_{\langle \boldsymbol{j}\rangle} y^{\langle \boldsymbol{j}\rangle} \succ P_{\langle \boldsymbol{i}\rangle} y^{\langle \boldsymbol{i}\rangle}$. Thus $P(y) \sim P_{\langle \boldsymbol{j}\rangle} y^{\langle \boldsymbol{j}\rangle}$. □

Lemma 5.5. *Suppose that $z^{\langle n\rangle}$ is defined and $y^{\langle i\rangle} \sim z^{\langle i\rangle}$ for $i = 0,\dots,n$. Then $P(y) \sim P(z)$.*

Proof. For all $\boldsymbol{i}$ with $P_{\langle \boldsymbol{i}\rangle} \neq 0$ we have $P_{\langle \boldsymbol{i}\rangle} y^{\langle \boldsymbol{i}\rangle} \sim P_{\langle \boldsymbol{i}\rangle} z^{\langle \boldsymbol{i}\rangle}$, by Lemma 2.1. Now use that for $\boldsymbol{i} \neq \boldsymbol{j}$ we have $P_{\langle \boldsymbol{i}\rangle} y^{\langle \boldsymbol{i}\rangle} \prec P_{\langle \boldsymbol{j}\rangle} y^{\langle \boldsymbol{j}\rangle}$ by the proof of Proposition 5.4. □

From here on n is no longer fixed.

Corollary 5.6 (Boshernitzan [18, Theorem 12.23])**.** *If $y \geqslant \exp_n h$ for all $h \in H(x)$ and all n, then y is H-hardian.*

This is an immediate consequence of Proposition 5.4. (In [18], the proof of this fact is only indicated.) From Lemma 5.5 we also obtain:

Corollary 5.7. *Suppose that y is as in Corollary 5.6 and $z \in \mathcal{C}^{<\infty}$, and $z^{\langle n\rangle}$ is defined and $y^{\langle n\rangle} \sim z^{\langle n\rangle}$, for all n. Then z is H-hardian, and there is a unique ordered differential field isomorphism $H\langle y\rangle \to H\langle z\rangle$ over H which sends y to z.*

Lemma 5.13 below contains another criterion for z to be H-hardian. This involves a certain binary relation $\sim_\infty$ on germs defined in the next subsection. Lemma 5.5 also yields a complex version of Corollary 5.7:

Corollary 5.8. *Suppose that y is as in Corollary 5.6 and that $z^{\langle n\rangle}$ is defined and $y^{\langle n\rangle} \sim z^{\langle n\rangle}$, for all n. Then z generates a differential subfield $H\langle z\rangle$ of $\mathcal{C}^{<\infty}[\mathrm{i}]$, and there is a unique differential field isomorphism $H\langle y\rangle \to H\langle z\rangle$ over H which sends y to z. Moreover, the binary relation $\preccurlyeq$ on $\mathcal{C}[\mathrm{i}]$ restricts to a dominance relation on $H\langle z\rangle$ which makes this an isomorphism of valued differential fields.*

A useful equivalence relation. We set

$$\mathcal{C}^{<\infty}[\mathrm{i}]^{\preccurlyeq} \ := \ \left\{f \in \mathcal{C}^{<\infty}[\mathrm{i}] : f^{(n)} \preccurlyeq 1 \text{ for all } n\right\} \ \subseteq \ \mathcal{C}[\mathrm{i}]^{\preccurlyeq},$$

a differential $\mathbb{C}$-subalgebra of $\mathcal{C}^{<\infty}[\mathrm{i}]$, and

$$\mathcal{I} \ := \ \left\{f \in \mathcal{C}^{<\infty}[\mathrm{i}] : f^{(n)} \prec 1 \text{ for all } n\right\} \ \subseteq \ \mathcal{C}^{<\infty}[\mathrm{i}]^{\preccurlyeq},$$

a differential ideal of $\mathcal{C}^{<\infty}[\mathrm{i}]^{\preccurlyeq}$ (thanks to the Product Rule). Recall from the remarks preceding Lemma 2.1 that $(\mathcal{C}[\mathrm{i}]^{\preccurlyeq})^{\times} = \mathcal{C}[\mathrm{i}]^{\asymp}$.

Lemma 5.9. *The group of units of $\mathcal{C}^{<\infty}[\mathrm{i}]^{\preccurlyeq}$ is*

$$\mathcal{C}^{<\infty}[\mathrm{i}]^{\asymp} \ := \ \mathcal{C}^{<\infty}[\mathrm{i}]^{\preccurlyeq} \cap \mathcal{C}[\mathrm{i}]^{\asymp} \ = \ \left\{f \in \mathcal{C}^{<\infty}[\mathrm{i}] : f \asymp 1,\ f^{(n)} \preccurlyeq 1 \textit{ for all } n\right\}.$$

Moreover, $1 + \mathcal{I}$ is a subgroup of $\mathcal{C}^{<\infty}[\mathrm{i}]^{\asymp}$.

Proof. It is clear that

$$(\mathcal{C}^{<\infty}[i]^{\preccurlyeq})^{\times} \ \subseteq\ \mathcal{C}^{<\infty}[i]^{\preccurlyeq} \cap (\mathcal{C}[i]^{\preccurlyeq})^{\times} \ =\ \mathcal{C}^{<\infty}[i]^{\preccurlyeq} \cap \mathcal{C}[i]^{\asymp} \ =\ \mathcal{C}^{<\infty}[i]^{\asymp}.$$

Conversely, suppose $f \in \mathcal{C}^{<\infty}[i]$ satisfies $f \asymp 1$ and $f^{(n)} \preccurlyeq 1$ for all n. For each n we have $Q_n \in \mathbb{Q}\{X\}$ such that $(1/f)^{(n)} = Q_n(f)/f^{n+1}$, hence $(1/f)^{(n)} \preccurlyeq 1$. Thus $f \in (\mathcal{C}^{<\infty}[i]^{\preccurlyeq})^{\times}$. This shows the first statement. Clearly $1 + \mathcal{I} \subseteq \mathcal{C}^{<\infty}[i]^{\asymp}$, and $1+\mathcal{I}$ is closed under multiplication. If $\delta \in \mathcal{I}$, then $1+\delta$ is a unit of $\mathcal{C}^{<\infty}[i]^{\preccurlyeq}$ and $(1+\delta)^{-1} = 1 + \varepsilon$ where $\varepsilon = -\delta(1+\delta)^{-1} \in \mathcal{I}$. □

For $y, z \in \mathcal{C}[i]^{\times}$ we define

$$y \sim_\infty z \quad :\Longleftrightarrow \quad y \in z \cdot (1 + \mathcal{I});$$

hence $y \sim_\infty z \Rightarrow y \sim z$. Lemma 5.9 yields that $\sim_\infty$ is an equivalence relation on $\mathcal{C}[i]^{\times}$, and for $y_i, z_i \in \mathcal{C}[i]^{\times}$ $(i = 1, 2)$ we have

$$y_1 \sim_\infty y_2 \quad \& \quad z_1 \sim_\infty z_2 \qquad \Longrightarrow \qquad y_1 z_1 \sim_\infty y_2 z_2, \quad y_1^{-1} \sim_\infty y_2^{-1}.$$

Lemma 5.10. *Let $y, z \in \mathcal{C}^1[i]^{\times}$ with $y \sim_\infty z$ and $z \in z'\,\mathcal{C}^{<\infty}[i]^{\preccurlyeq}$. Then*

$$y', z' \in \mathcal{C}[i]^{\times}, \qquad y' \sim_\infty z'.$$

Proof. Let $\delta \in \mathcal{I}$ and $f \in \mathcal{C}^{<\infty}[i]^{\preccurlyeq}$ with $y = z(1+\delta)$ and $z = z' f$. Then $z' \in \mathcal{C}[i]^{\times}$ and $y' = z'(1+\delta) + z\delta' = z'(1 + \delta + f\delta')$ where $\delta + f\delta' \in \mathcal{I}$, so $y' \sim_\infty z'$. □

If $\ell \in \mathcal{C}^n[i]$ and $f \in \mathcal{C}^n$ with $f \geqslant 0$, $f \succ 1$, then $\ell \circ f \in \mathcal{C}^n[i]$. In fact, for $n \geqslant 1$ and $1 \leqslant k \leqslant n$ we have a differential polynomial $Q^n_k \in \mathbb{Q}\{X'\} \subseteq \mathbb{Q}\{X\}$ of order $\leqslant n$, isobaric of weight n, and homogeneous of degree k, such that for all such ℓ, f,

$$(\ell \circ f)^{(n)} \ =\ (\ell^{(n)} \circ f)\, Q^n_n(f) + \cdots + (\ell' \circ f)\, Q^n_1(f).$$

For example,

$$Q^1_1 = X', \quad Q^2_2 = (X')^2,\ Q^2_1 = X'', \quad Q^3_3 = (X')^3,\ Q^3_2 = 3X'X'',\ Q^3_1 = X'''.$$

The following lemma is only used in the proof of Theorem 8.6 below.

Lemma 5.11. *Let $f, g \in \mathcal{C}^{<\infty}$ be such that $f, g \geqslant 0$ and $f, g \succ 1$, and set $r := g - f$. Suppose $P(f) \cdot Q(r) \prec 1$ for all $P, Q \in \mathbb{Q}\{Y\}$ with $Q(0) = 0$, and let $\ell \in \mathcal{C}^{<\infty}[i]$ be such that $\ell' \in \mathcal{I}$. Then $\ell \circ g - \ell \circ f \in \mathcal{I}$.*

Proof. Treating real and imaginary parts separately we arrange $\ell \in \mathcal{C}^{<\infty}$. Note that $r \prec 1$. Taylor expansion [ADH, 4.2] for $P \in \mathbb{Q}\{X\}$ of order $\leqslant n$ gives

$$P(g) - P(f) \ =\ \sum_{|\boldsymbol{i}| \geqslant 1} \frac{1}{\boldsymbol{i}!} P^{(\boldsymbol{i})}(f) \cdot r^{\boldsymbol{i}} \qquad (\boldsymbol{i} \in \mathbb{N}^{1+n}),$$

and thus $P(g) - P(f) \prec 1$ and $rP(g) \prec 1$. The Mean Value Theorem yields a germ $r_n \in \mathcal{G}$ such that

$$\ell^{(n)} \circ g - \ell^{(n)} \circ f \ =\ \big(\ell^{(n+1)} \circ (f + r_n)\big) \cdot r \quad \text{and} \quad |r_n| \leqslant |r|.$$

Now $r_0 \prec 1$, so $\ell' \circ (f + r_0) \prec 1$, hence $\ell \circ g - \ell \circ f \prec 1$. For $1 \leqslant k \leqslant n$,

$$(\ell^{(k)} \circ g)\, Q^n_k(g) - (\ell^{(k)} \circ f)\, Q^n_k(f) = \\ (\ell^{(k)} \circ f)\, \big(Q^n_k(g) - Q^n_k(f)\big) + \big(\ell^{(k+1)} \circ (f + r_k)\big) \cdot r Q^n_k(g),$$

so $(\ell^{(k)} \circ g)\, Q^n_k(g) - (\ell^{(k)} \circ f)\, Q^n_k(f) \prec 1$, and thus $\big(\ell \circ g - \ell \circ f\big)^{(n)} \prec 1$. □

We consider next the differential $\mathbb{R}$-subalgebra

$$(\mathcal{C}^{<\infty})^{\preccurlyeq} \ := \ \mathcal{C}^{<\infty}[i]^{\preccurlyeq} \cap \mathcal{C}^{<\infty} \ \subseteq \ \mathcal{C}^{\preccurlyeq}$$

of $\mathcal{C}^{<\infty}$. In the rest of this subsection H is a Hardy field and $y, z \in \mathcal{C}^{<\infty}$, $y, z \succ 1$. Note that $(\mathcal{C}^{<\infty})^{\preccurlyeq} \cap H = \mathcal{O}$ is the valuation ring of H and $\mathcal{I} \cap H = \scriptstyle\mathcal{O}$ is the maximal ideal of $\mathcal{O}$. This yields:

Lemma 5.12. *Suppose $y - z \in (\mathcal{C}^{<\infty})^{\preccurlyeq}$ and z is hardian. Then $y \sim_\infty z$.*

Proof. From $y = z + f$ with $f \in (\mathcal{C}^{<\infty})^{\preccurlyeq}$ we obtain $y = z(1 + fz^{-1})$. Now $z^{-1} \in \mathcal{I}$, so $fz^{-1} \in \mathcal{I}$, and thus $y \sim_\infty z$. $\square$

We now formulate a sufficient condition involving $\sim_\infty$ for y to be H-hardian.

Lemma 5.13. *Suppose z is H-hardian with $z \geqslant \exp_n h$ for all $h \in H(x)$ and all n, and $y \sim_\infty z$. Then y is H-hardian, and there is a unique ordered differential field isomorphism $H\langle y\rangle \to H\langle z\rangle$ which is the identity on H and sends y to z.*

Proof. By Lemma 5.1 we may replace H by the Hardy subfield $\operatorname{Li}\big(H(\mathbb{R})\big)$ of $\operatorname{E}(H)$ to arrange that $H \supseteq \mathbb{R}$ is Liouville closed. By Corollary 5.7 (with the roles of y, z reversed) it is enough to show that for each n, $y^{\langle n\rangle}$ is defined, $y^{\langle n\rangle} \succ 1$, and $y^{\langle n\rangle} \sim_\infty z^{\langle n\rangle}$. This holds by hypothesis for $n = 0$. By Lemma 1.3, $z > H$ gives $z^\dagger > H$, so $z = z'f$ with $f \prec 1$ in the Hardy field $H\langle z\rangle$, hence $f^{(n)} \prec 1$ for all n. So by Lemma 5.10, $y^{\langle 1\rangle} = y^\dagger$ is defined, $y^{\langle 1\rangle} \in (\mathcal{C}^{<\infty})^\times$, $y^{\langle 1\rangle} \sim_\infty z^{\langle 1\rangle}$, and thus $y^{\langle 1\rangle} \succ 1$. Assume for a certain $n \geqslant 1$ that $y^{\langle n\rangle}$ is defined, $y^{\langle n\rangle} \succ 1$, and $y^{\langle n\rangle} \sim_\infty z^{\langle n\rangle}$. Then $z^{\langle n\rangle}$ is H-hardian and $H < z^{\langle n\rangle}$ by Lemma 1.5. Hence by the case $n = 1$ applied to $y^{\langle n\rangle}$, $z^{\langle n\rangle}$ in place of y, z, respectively, $y^{\langle n+1\rangle} = (y^{\langle n\rangle})^\dagger$ is defined, $y^{\langle n+1\rangle} \succ 1$, and $y^{\langle n+1\rangle} \sim_\infty z^{\langle n+1\rangle}$. $\square$

The next two corollaries are Theorems 13.6 and 13.10, respectively, in [18].

Corollary 5.14. *Suppose z is transexponential and hardian, and $y - z \in (\mathcal{C}^{<\infty})^{\preccurlyeq}$. Then y is hardian, and there is a unique isomorphism $\mathbb{R}\langle y\rangle \to \mathbb{R}\langle z\rangle$ of ordered differential fields that is the identity on $\mathbb{R}$ and sends y to z.*

Proof. Take $H := \operatorname{Li}(\mathbb{R})$. Then z lies in a Hardy field extension of H, namely $\operatorname{Li}\big(\mathbb{R}\langle z\rangle\big)$, and $H < z$. So $y \sim_\infty z$ by Lemma 5.12. Now use Lemma 5.13. $\square$

Corollary 5.15. *If $z \in \operatorname{E}(H)^{>\mathbb{R}}$, then $z \leqslant \exp_n h$ for some $h \in H(x)$ and some n. (Thus if $x \in H$ and $\exp H \subseteq H$, then $H^{>\mathbb{R}}$ is cofinal in $\operatorname{E}(H)^{>\mathbb{R}}$.)*

Proof. Towards a contradiction, suppose $z \in \operatorname{E}(H)^{>\mathbb{R}}$ and $z > \exp_n h$ in $\operatorname{E}(H)$ for all $h \in H(x)$ and all n. Set $y := z + \sin x$. Then y is H-hardian by Lemmas 5.12 and 5.13, so y, z lie in a common Hardy field extension of H, a contradiction. $\square$

Remark. The same proof shows that Corollary 5.15 remains true if H is a $\mathcal{C}^\infty$-Hardy field and $\operatorname{E}(H)$ is replaced by $\operatorname{E}^\infty(H)$; likewise for ω in place of ∞.

Next a lemma similar to Lemma 5.13, but obtained using Corollary 5.8 instead of Corollary 5.7:

Lemma 5.16. *Let H be a Hardy field, let $z \in \mathcal{C}^{<\infty}$ be H-hardian with $z \geqslant \exp_n h$ for all $h \in H(x)$ and all n, and $y \in \mathcal{C}^{<\infty}[i]$ with $y \sim_\infty z$. Then y generates a differential subfield $H\langle y\rangle$ of $\mathcal{C}^{<\infty}[i]$, and there is a unique differential field isomorphism $H\langle y\rangle \to H\langle z\rangle$ which is the identity on H and sends y to z. The binary* *relation $\preccurlyeq$ on $\mathcal{C}[i]$ restricts to a dominance relation on $H\langle y\rangle$ which makes this an isomorphism of valued differential fields.*

We use the above at the end of the next subsection to produce a differential subfield of $\mathcal{C}^{<\infty}[i]$ that is not contained in $H[i]$ for any Hardy field H.

Boundedness. Let $H \subseteq \mathcal{C}$. We say that $b \in \mathcal{C}$ **bounds** H if $h \leqslant b$ for each $h \in H$. We call H **bounded** if some $b \in \mathcal{C}$ bounds H, and we call H **unbounded** if H is not bounded. If $H_1, H_2 \subseteq \mathcal{C}$ and for each $h_2 \in H_2$ there is an $h_1 \in H_1$ with $h_2 \leqslant h_1$, then any $b \in \mathcal{C}$ bounding H_1 also bounds H_2. Every bounded subset of $\mathcal{C}$ is bounded by a germ in $\mathcal{C}^{\omega}$; this follows from [18, Lemma 14.3]:

Lemma 5.17. *For every $b \geqslant 0$ in $\mathcal{C}^{\times}$ there is a $\phi \geqslant 0$ in $(\mathcal{C}^{\omega})^{\times}$ such that $\phi^{(n)} \prec b$ for all n.*

Every countable subset of $\mathcal{C}$ is bounded, by du Bois-Reymond [15]; see also [28, Chapter II] or [20, Chapitre V, p. 53, ex. 8]. Thus $H \subseteq \mathcal{C}$ is bounded if it is totally ordered by the partial ordering $\leqslant$ of $\mathcal{C}$ and has countable cofinality. If H is a Hausdorff field and $b \in \mathcal{C}$ bounds H, then b also bounds the real closure $H^{\mathrm{rc}} \subseteq \mathcal{C}$ of H [ADH, 5.3.2]. *In the rest of this subsection H is a Hardy field.*

Lemma 5.18. *Let H^* be a* d*-algebraic Hardy field extension of H and suppose H is bounded. Then H^* is also bounded.*

Proof. By [ADH, 3.1.11] we have $f \in H(x)^{>}$ such that for all $g \in H(x)^{\times}$ there are $h \in H^{\times}$ and $q \in \mathbb{Q}$ with $g \asymp hf^q$. Hence $H(x)$ is bounded. Replacing H, H^* by $H(x)^{\mathrm{rc}}$, $\mathrm{Li}\big(H^*(\mathbb{R})\big)$, respectively, we arrange that H is real closed with $x \in H$, and $H^* \supseteq \mathbb{R}$ is Liouville closed. Let $b \in \mathcal{C}$ bound H. Then any $b^* \in \mathcal{C}$ such that $\exp_n b \leqslant b^*$ for all n bounds H^*, by Lemma 5.1. □

In particular, if H is bounded, then so is $\mathrm{Li}\big(H(\mathbb{R})\big)$. We use this to show:

Lemma 5.19. *Suppose that H is bounded and $f \in \mathcal{C}^{<\infty}$ is hardian over H. Then $H\langle f\rangle$ is bounded.*

Proof. Using that f remains hardian over the bounded Hardy field $\mathrm{Li}\big(H(\mathbb{R})\big)$, we arrange that H is Liouville closed. The case that $H\langle f\rangle$ has no element $> H$ is trivial, so assume we have $y \in H\langle f\rangle$ with $y > H$. Then y is d-transcendental over H and the sequence $y, y^2, y^3, \dots$ is cofinal in $H\langle y\rangle$, by Corollary 1.8, so $H\langle y\rangle$ is bounded. Now use that f is d-algebraic over $H\langle y\rangle$. □

Theorem 5.20 (Boshernitzan [18, Theorem 14.4])**.** *Suppose H is bounded. Then the perfect hull $\mathrm{E}(H)$ of H is* d*-algebraic over H and hence bounded. If $H \subseteq \mathcal{C}^{\infty}$, then $\mathrm{E}^{\infty}(H)$ is* d*-algebraic over H; likewise with ω in place of ∞.*

Using the results above the proof is not difficult. It is omitted in [18], but we include it here for the sake of completeness. First, a lemma also needed for the proof of Theorem 8.6:

Lemma 5.21. *Let $b \in \mathcal{C}^{\times}$ bound H, let $\phi \geqslant 0$ in $\mathcal{C}^{<\infty}$ satisfy $\phi^{(n)} \prec b^{-1}$ for all n, and let $r \in \phi \cdot (\mathcal{C}^{<\infty})^{\preccurlyeq}$. Then $Q(r) \prec 1$ for all $Q \in H\{Y\}$ with $Q(0) = 0$.*

Proof. From $\phi \in \mathcal{I}$ we obtain $r \in \mathcal{I}$, so it is enough that $hr^{(n)} \prec 1$ for all $h \in H$ and all n. Now use the Product Rule and $h\phi^{(n)} \prec hb^{-1} \preccurlyeq 1$ for $h \in H^{\times}$. □

Proof of Theorem 5.20. Using Lemma 5.18, replace H by $\mathrm{Li}\big(H(\mathbb{R})\big)$ to arrange that $H \supseteq \mathbb{R}$ is Liouville closed. Let $b \in \mathcal{C}$ bound H. Then b also bounds $\mathrm{E}(H)$, by Corollary 5.15. Lemma 5.17 yields $\phi \geqslant 0$ in $(\mathcal{C}^\omega)^\times$ such that $\phi^{(n)} \prec b^{-1}$ for all n; set $r := \phi \cdot \sin x \in \mathcal{C}^\omega$. Then $Q(r) \prec f$ for all $f \in \mathrm{E}(H)^\times$ and $Q \in \mathrm{E}(H)\{Z\}$ with $Q(0) = 0$, by Lemma 5.21.

Suppose towards a contradiction that $f \in \mathrm{E}(H)$ is d-transcendental over H, and set $g := f + r \in \mathcal{C}^{<\infty}$. Then f, g are not in a common Hardy field, so g is not hardian over H. On the other hand, let $P \in H\{Y\}^{\neq}$. Then $P(f) \in \mathrm{E}(H)^\times$, and by Taylor expansion,

$$P(f+Z) \;=\; P(f) + Q(Z) \quad \text{ where } Q \in \mathrm{E}(H)\{Z\} \text{ with } Q(0) = 0,$$

so $P(g) = P(f + r) \sim P(f)$. Hence g is hardian over H, a contradiction.

The proof in the case where $H \subseteq \mathcal{C}^\infty$ is similar, using the version of Corollary 5.15 for $\mathrm{E}^\infty(H)$; similarly for ω in place of ∞. □

As to the existence of transexponential hardian germs, we have:

Theorem 5.22. *For every $b \in \mathcal{C}$ there is a $\mathcal{C}^\omega$-hardian germ $y \geqslant b$.*

This is Boshernitzan [19, Theorem 1.2], and leads to [19, Theorem 1.1]:

Corollary 5.23. *No maximal Hardy field is bounded.*

Proof. Suppose $x \in H$, and $b \in \mathcal{C}$ bounds H. Take $b^* \in \mathcal{C}$ such that $b^* \geqslant \exp_n b$ for all n. Now Theorem 5.22 yields a $\mathcal{C}^\omega$-hardian germ $y \geqslant b^*$. By Corollary 5.6, y is H-hardian, so $H\langle y\rangle$ is a proper Hardy field extension of H. □

The same proof shows also that no $\mathcal{C}^\infty$-maximal Hardy field and no $\mathcal{C}^\omega$-maximal Hardy field is bounded. In particular (Boshernitzan [19, Theorem 1.3]):

Corollary 5.24. *Every maximal Hardy field contains a transexponential germ. Likewise with "$\mathcal{C}^\infty$-maximal" or "$\mathcal{C}^\omega$-maximal" in place of "maximal".*

Remark. For $\mathcal{C}^\infty$-Hardy fields, some of the above is in Sjödin's [57], predating [18, 19]: if H is a bounded $\mathcal{C}^\infty$-Hardy field, then so is $\mathrm{Li}\big(H(\mathbb{R})\big)$ [57, Theorem 2]; no maximal $\mathcal{C}^\infty$-Hardy field is bounded [57, Theorem 6]; and $E := \mathrm{E}^\infty(\mathbb{Q})$ is bounded [57, Theorem 10].

We can now produce a differential subfield K of $\mathcal{C}^\omega[\mathrm{i}]$ containing i such that $\preccurlyeq$ restricts to a dominance relation on K making K a d-valued field of H-type with constant field $\mathbb{C}$, yet $K \nsubseteq H[\mathrm{i}]$ for every H: Take a transexponential $\mathcal{C}^\omega$-hardian germ z, and $h \in \mathbb{R}(x)$ with $0 \neq h \prec 1$. Then $\varepsilon := h\,\mathrm{e}^{x\mathrm{i}} \in \mathcal{I}$, so $y := z(1+\varepsilon) \in \mathcal{C}^\omega[\mathrm{i}]$ with $y \sim_\infty z$. Lemma 5.16 applied with $H = \mathbb{R}$ shows that y generates a differential subfield $K_0 := \mathbb{R}\langle y\rangle$ of $\mathcal{C}^\omega[\mathrm{i}]$, and $\preccurlyeq$ restricts to a dominance relation on K_0 making K_0 a d-valued field of H-type with constant field $\mathbb{R}$. Then $K := K_0[\mathrm{i}]$ is a differential subfield of $\mathcal{C}^\omega[\mathrm{i}]$ with constant field $\mathbb{C}$. Moreover, $\preccurlyeq$ also restricts to a dominance relation on K, and this dominance relation makes K a d-valued field of H-type [ADH, 10.5.15]. We cannot have $K \subseteq H[\mathrm{i}]$ for any H, since $\operatorname{Im} y = zh\sin x \notin H$.

Lower bounds on d-algebraic hardian germs. In this subsection H is a Hardy field. Let $f \in \mathcal{C}$ and $f \succ 1$, $f \geqslant 0$. Then the germ $\log f \in \mathcal{C}$ also satisfies $\log f \succ 1$, $\log f \geqslant 0$. So we may inductively define the germs $\log_n f$ in $\mathcal{C}$ by $\log_0 f := f$, $\log_{n+1} f := \log\log_n f$. (So $\ell_n = \log_n x$ for each n.) Lemma 5.1 gives exponential upper bounds on d-algebraic H-hardian germs. The next result leads to logarithmic lower bounds on such germs when H is grounded.

Theorem 5.25 (Rosenlicht [56, Theorem 3])**.** *Suppose H is grounded, and let E be a Hardy field extension of H such that $|\Psi_E \setminus \Psi_H| \leqslant n$ (so E is also grounded). Then there are $r, s \in \mathbb{N}$ with $r + s \leqslant n$ such that*

(i) *for any $h \in H^>$ with $h \succ 1$ and $\max \Psi_H = v(h^\dagger)$, there exists $g \in E^>$ such that $g \asymp \log_r h$ and $\max \Psi_E = v(g^\dagger)$;*
(ii) *for any $g \in E$ there exists $h \in H$ such that $g < \exp_s h$.*

This theorem is most useful in combination with the following lemma, which is [56, Proposition 5] (and also [4, Lemma 2.1] in the context of pre-H-fields).

Lemma 5.26. *Let E be a Hardy field extension of H such that* $\operatorname{trdeg}(E|H) \leqslant n$. *Then $|\Psi_E \setminus \Psi_H| \leqslant n$.*

From [ADH, 9.1.11] we recall that for $f, g \succ 1$ in a Hardy field we have $f^\dagger \preccurlyeq g^\dagger$ iff $|f| \leqslant |g|^n$ for some $n \geqslant 1$. Thus by Lemma 5.26 and Theorem 5.25:

Corollary 5.27. *Let $h \in H^>$, $h \succ 1$, and $\max \Psi_H = v(h^\dagger)$. Then for any Hardy field extension E of H with* $\operatorname{trdeg}(E|H) \leqslant n$*: E is grounded, and for all $g \in E$ with $g \succ 1$ there is an $m \geqslant 1$ such that $\log_n h \preccurlyeq g^m$ (and so $\log_{n+1} h \prec g$).*

Hence for h as in Corollary 5.27 and H-hardian $y \in \mathcal{C}$, if y is d-algebraic over H, then the Hardy field $E = H\langle y\rangle$ is grounded, and there is an n such that $\log_n h \prec g$ for all $g \in E$ with $g \succ 1$. Applying this to $H = \mathbb{R}(x)$, $h = x$ yields:

Corollary 5.28 (Boshernitzan [18, Proposition 14.11])**.** *If $y \in \mathcal{C}$ is hardian and* d*-algebraic over $\mathbb{R}$, then the Hardy field $E = \mathbb{R}(x)\langle y\rangle$ is grounded, and there is an n such that $\ell_n \prec g$ for all $g \in E$ with $g \succ 1$.*

Following [19] we say that $y \in \mathcal{C}$ is **translogarithmic** if $r \leqslant y \leqslant \ell_n$ for all n and all $r \in \mathbb{R}$. Thus for eventually strictly increasing $y \succ 1$ in $\mathcal{C}$, y is translogarithmic iff its compositional inverse y^{inv} is transexponential. By Lemma 4.4 and Corollary 5.24 there exist $\mathcal{C}^\omega$-hardian translogarithmic germs; see also [ADH, 13.9]. Translogarithmic hardian germs are d-transcendental, by Corollary 5.28.

6. Second-Order Linear Differential Equations over Hardy Fields

In this section we review Boshernitzan's work [18, §16] on adjoining non-oscillating solutions of second-order linear differential equations to Hardy fields in the light of results from [ADH], and deduce some consequences about complex exponentials over Hardy fields for use in [12]. *Throughout this section H is a Hardy field.*

Oscillation over Hardy fields. In this subsection we assume $f \in H$ and consider the linear differential equation

$$4Y'' + fY = 0 \qquad \text{(4L)}$$

over H. The factor 4 is to simplify certain expressions, in conformity with [ADH, 5.2]. There we also defined for any differential field K the function $\omega\colon K \to K$ given by $\omega(z) = -2z' - z^2$, and the function $\sigma\colon K^\times \to K$ given by $\sigma(y) = \omega(z) + y^2$ for $z := -y^\dagger$. We define likewise

$$\omega \;:\; \mathcal{C}^1[i] \to \mathcal{C}^0[i], \qquad \sigma \;:\; \mathcal{C}^2[i]^\times \to \mathcal{C}^0[i]$$

by

$$\omega(z) \;=\; -2z' - z^2 \quad \text{and} \quad \sigma(y) \;=\; \omega(z) + y^2 \text{ for } z := -y^\dagger.$$

Note that $\omega(\mathcal{C}^1) \subseteq \mathcal{C}^0$ and $\sigma\big((\mathcal{C}^2)^\times\big) \subseteq \mathcal{C}^0$, and $\sigma(y) = \omega(z + y\mathrm{i})$ for $z := -y^\dagger$. To clarify the role of ω and σ in connection with second-order linear differential equations, suppose $y \in \mathcal{C}^2$ is a non-oscillating solution to (4L) with $y \neq 0$. Then $z := 2y^\dagger \in \mathcal{C}^1$ satisfies $-2z' - z^2 = f$, so z generates a Hardy field $H(z)$ with $\omega(z) = f$, by Proposition 4.3, which in turn yields a Hardy field $H(z,y)$ with $2y^\dagger = z$. Thus $y_1 := y$ lies in a Hardy field extension of H. From Lemma 3.2 and Proposition 4.2(iv) we also obtain a solution y_2 of (4L) in a Hardy field extension of $H\langle y_1\rangle = H(y,z)$ such that y_1, y_2 are $\mathbb{R}$-linearly independent; see also [55, Theorem 2, Corollary 2]. This shows:

Proposition 6.1. *If $f/4$ does not generate oscillation, then* $\mathrm{D}(H)$ *contains $\mathbb{R}$-linearly independent solutions y_1, y_2 to* (4L).

Indeed, if $f/4$ does not generate oscillation, then $\mathrm{D}(H)$ contains solutions y_1, y_2 of (4L) with $y_1, y_2 > 0$ and $y_1 \prec y_2$. Here y_1 is determined up to multiplication by a factor in $\mathbb{R}^>$; we call such y_1 a **principal solution** of (4L). (Lemmas 3.14, 3.15.)

Notation. Let K be a differential field. Then $K[\partial]$ denotes the ring of linear differential operators over K; see [ADH, 5.1]. Let $A \in K[\partial]$. Then $A_{\ltimes b} := b^{-1}Ab \in K[\partial]$ is the *twist* of A by b. We say that A *splits over* K if $A = a(\partial - b_1)\cdots(\partial - b_n)$ for some $a \in K^\times$, $b_1, \dots, b_n \in K$. If A splits over K, then so does $A_{\ltimes b}$ for each $b \in K^\times$.

By [ADH, p. 259], with $A := 4\partial^2 + f \in H[\partial]$ we have

$$4y'' + fy = 0 \text{ for some } y \in H^\times \;\Rightarrow\; A \text{ splits over } H \;\Longleftrightarrow\; f \in \omega(H).$$

To simplify the discussion we now also introduce the subset

$$\overline{\omega}(H) := \big\{f \in H : f/4 \text{ does not generate oscillation}\big\}$$

of H. If E is a Hardy field extension of H, then $\overline{\omega}(E) \cap H = \overline{\omega}(H)$. By Corollary 3.10, $\overline{\omega}(H)$ is downward closed, and $\omega(H) \subseteq \overline{\omega}(H)$ by the discussion following (R) in Section 3.

Corollary 6.2. *If H is* d*-perfect, then*

$$\omega(H) \;=\; \overline{\omega}(H) \;=\; \big\{f \in H :\; 4y'' + fy = 0 \textit{ for some } y \in H^\times\big\},$$

and $\omega(H)$ is downward closed in H.

Lemma 3.1 and Proposition 6.1 also yield:

Corollary 6.3. *If $f \in \overline{\omega}(H)$, then each $y \in \mathcal{C}^2$ such that $4y'' + fy \in H$ is in* $\mathrm{D}(H)$.

Next some consequences of Proposition 6.1 for more general linear differential equations of order 2: Let $g, h \in H$, and consider the linear differential equation

$$(\widetilde{\mathrm{L}}) \qquad Y'' + gY' + hY \;=\; 0$$

over H. An easy induction on n shows that for a solution $y \in \mathcal{C}^2$ of $(\widetilde{\mathrm{L}})$ we have $y \in \mathcal{C}^n$ with $y^{(n)} \in Hy + Hy'$ for all n, so $y \in \mathcal{C}^{<\infty}$. To reduce $(\widetilde{\mathrm{L}})$ to an equation (4L) we take

$$f \;:=\; \omega(g) + 4h \;=\; -2g' - g^2 + 4h \in H,$$

take $a \in \mathbb{R}$, and take a representative of g in $\mathcal{C}^1_a$, also denoted by g, and let $G \in (\mathcal{C}^2)^\times$ be the germ of

$$t \mapsto \exp\left(-\frac{1}{2}\int_a^t g(s)\,ds\right) \qquad (t \geqslant a).$$

This gives an isomorphism $y \mapsto Gy$ from the $\mathbb{R}$-linear space of solutions of (4L) in $\mathcal{C}^2$ onto the $\mathbb{R}$-linear space of solutions of $(\widetilde{\mathrm{L}})$ in $\mathcal{C}^2$, and $y \in \mathcal{C}^2$ oscillates iff Gy oscillates. By Proposition 4.2, $G \in \mathrm{D}(H)$. Using $\frac{f}{4} = -\frac{1}{2}g' - \frac{1}{4}g^2 + h$ we now obtain the following germ version of Corollary 3.11:

Corollary 6.4. *The following are equivalent:*

(i) *some solution in $\mathcal{C}^2$ of $(\widetilde{\mathrm{L}})$ oscillates;*
(ii) *all nonzero solutions in $\mathcal{C}^2$ of $(\widetilde{\mathrm{L}})$ oscillate;*
(iii) *$-\frac{1}{2}g' - \frac{1}{4}g^2 + h$ generates oscillation.*

Moreover, if $-\frac{1}{2}g' - \frac{1}{4}g^2 + h$ does not generate oscillation, then all solutions of $(\widetilde{\mathrm{L}})$ in $\mathcal{C}^2$ belong to $\mathrm{D}(H)$.

Set $A := \partial^2 + g\partial + h$, and let $f = \omega(g) + 4h$, G be as above. Then $A_{\ltimes G} = \partial^2 + \frac{f}{4}$. Thus by combining Corollary 6.3 and Corollary 6.4 we obtain:

Corollary 6.5. *If $(\widetilde{\mathrm{L}})$ has no oscillating solution in $\mathcal{C}^2$, and $y \in \mathcal{C}^2$ is such that $y'' + gy' + hy \in H$, then $y \in \mathrm{D}(H)$.*

The next corollary follows from Proposition 6.1 and [ADH, 5.1.21]:

Corollary 6.6. *The following are equivalent, for $A \in H[\partial]$ and f as above:*

(i) *$f/4$ does not generate oscillation;*
(ii) *A splits over some Hardy field extension of H;*
(iii) *A splits over $\mathrm{D}(H)$.*

For $A \in H[\partial]$ and f as before we have $A_{\ltimes G} = \partial^2 + \frac{f}{4}$ and $G^\dagger = -\frac{1}{2}g \in H$, so:

Corollary 6.7. *A splits over $H[\mathrm{i}] \iff \partial^2 + \frac{f}{4}$ splits over $H[\mathrm{i}]$.*

Proposition 6.1 and its corollaries 6.3–6.5 are from [18, Theorems 16.17, 16.18, 16.19], and Corollary 6.2 is essentially [18, Lemma 17.1].

Proposition 6.1 applies only when (4L) has a solution in $(\mathcal{C}^2)^\times$. Such a solution might not exist, but (4L) does have $\mathbb{R}$-linearly independent solutions $y_1, y_2 \in \mathcal{C}^2$, so $w := y_1y_2' - y_1'y_2 \in \mathbb{R}^\times$. Set $y := y_1 + y_2 i$. Then $4y'' + fy = 0$ and $y \in \mathcal{C}^2[\mathrm{i}]^\times$, and for $z = 2y^\dagger \in \mathcal{C}^1[\mathrm{i}]$ we have $-2z' - z^2 = f$. Now

$$z \;=\; \frac{2y_1' + 2\mathrm{i}y_2'}{y_1 + \mathrm{i}y_2} \;=\; \frac{2y_1'y_1 + 2y_2'y_2 - 2\mathrm{i}(y_1'y_2 - y_1y_2')}{y_1^2 + y_2^2} \;=\; \frac{2(y_1'y_1 + y_2'y_2) + 2\mathrm{i}w}{y_1^2 + y_2^2},$$

$$\text{so} \quad \operatorname{Re} z \;=\; \frac{2(y_1'y_1 + y_2'y_2)}{y_1^2 + y_2^2} \in \mathcal{C}^1, \qquad \operatorname{Im} z \;=\; \frac{2w}{y_1^2 + y_2^2} \in \mathcal{C}^2.$$

Thus $\operatorname{Im} z \in (\mathcal{C}^2)^\times$ and $(\operatorname{Im} z)^\dagger = -\operatorname{Re} z$, and so

$$\sigma(\operatorname{Im} z) \;=\; \omega\big(-(\operatorname{Im} z)^\dagger + (\operatorname{Im} z)\mathrm{i}\big) \;=\; \omega(z) \;=\; f \qquad \text{in } \mathcal{C}^1.$$

Replacing y_1 by $-y_1$ changes w to $-w$; this way we can arrange $w > 0$, so $\operatorname{Im} z > 0$.

Conversely, every $u \in (\mathcal{C}^2)^\times$ such that $u > 0$ and $\sigma(u) = f$ arises in this way. To see this, suppose we are given such u, take $\phi \in \mathcal{C}^3$ with $\phi' = \frac{1}{2}u$, and set

$$y_1 \ := \ \frac{1}{\sqrt{u}}\cos\phi, \qquad y_2 \ := \ \frac{1}{\sqrt{u}}\sin\phi \qquad (\text{elements of } \mathcal{C}^2).$$

Then $\mathrm{wr}(y_1, y_2) = 1/2$, and y_1, y_2 solve (4L). To see the latter, consider

$$y \ := \ y_1 + y_2\mathrm{i} \ = \ \frac{1}{\sqrt{u}}\,\mathrm{e}^{\phi\mathrm{i}} \in \mathcal{C}^2[\mathrm{i}]^\times$$

and note that $z := 2y^\dagger$ satisfies

$$\omega(z) \ = \ \omega(-u^\dagger + u\mathrm{i}) \ = \ \sigma(u) \ = \ f,$$

hence $4y'' + fy = 0$. The computation above shows $\operatorname{Im} z = 1/(y_1^2 + y_2^2) = u$. We have $\phi' > 0$, so either $\phi > \mathbb{R}$ or $\phi - c \prec 1$ for some $c \in \mathbb{R}$, with $\phi > \mathbb{R}$ iff $f/4$ generates oscillation. As to uniqueness of the above pair (y_1, y_2), we have:

Lemma 6.8. *Suppose $f \notin \overline{\omega}(H)$. Let $\widetilde{y}_1, \widetilde{y}_2 \in \mathcal{C}^2$ be $\mathbb{R}$-linearly independent solutions of* (4L) *with $\mathrm{wr}(\widetilde{y}_1, \widetilde{y}_2) = 1/2$. Set $\widetilde{y} := \widetilde{y}_1 + \widetilde{y}_2\mathrm{i}$, $\widetilde{z} := 2\widetilde{y}^\dagger$. Then*

$$\operatorname{Im}\widetilde{z} = u \quad \Longleftrightarrow \quad \widetilde{y} = \mathrm{e}^{\theta\mathrm{i}}\,y \text{ for some } \theta \in \mathbb{R}.$$

Proof. If $\widetilde{y} = \mathrm{e}^{\theta\mathrm{i}}\,y$ ($\theta \in \mathbb{R}$), then clearly $\widetilde{z} = 2\widetilde{y}^\dagger = 2y^\dagger = z$, hence $\operatorname{Im} z = \operatorname{Im}\widetilde{z}$. For the converse, let A be the invertible 2×2 matrix with real entries and $Ay = \widetilde{y}$; here $y = (y_1, y_2)^t$ and $\widetilde{y} = (\widetilde{y}_1, \widetilde{y}_2)^t$, column vectors with entries in $\mathcal{C}^2$. As in the proof of [ADH, 4.1.18], $\mathrm{wr}(y_1, y_2) = \mathrm{wr}(\widetilde{y}_1, \widetilde{y}_2)$ yields $\det A = 1$.

Suppose $\operatorname{Im}\widetilde{z} = u$, so $y_1^2 + y_2^2 = \widetilde{y}_1^2 + \widetilde{y}_2^2$. Choose $a \in \mathbb{R}$ and representatives for u, y_1, y_2, $\widetilde{y}_1$, $\widetilde{y}_2$ in $\mathcal{C}_a$, denoted by the same symbols, such that in $\mathcal{C}_a$ we have $Ay = \widetilde{y}$ and $y_1^2 + y_2^2 = \widetilde{y}_1^2 + \widetilde{y}_2^2$, and $u(t)\cdot\big(y_1(t)^2 + y_2(t)^2\big) = 1$ for all $t \geqslant a$. With $\|\cdot\|$ the usual euclidean norm on $\mathbb{R}^2$, we then have $\|Ay(t)\| = \|y(t)\| = 1/\sqrt{u(t)}$ for $t \geqslant a$. Since $f/4$ generates oscillation, we have $\phi > \mathbb{R}$, and we conclude that $\|Av\| = 1$ for all $v \in \mathbb{R}^2$ with $\|v\| = 1$. It is well-known that then $A = \left(\begin{smallmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{smallmatrix}\right)$ with $\theta \in \mathbb{R}$ (see, e.g., [42, Chapter XV, Exercise 2]), so $\widetilde{y} = \mathrm{e}^{\theta\mathrm{i}}\,y$. □

The observations above will be used in the proof of Theorems 7.14 below. We finish this subsection with miscellaneous historical remarks (not used later):

Remarks. The connection between the second-order linear differential equation (4L) and the third-order non-linear differential equation $\sigma(u) = f$ was first investigated by Kummer [40] in 1834. Appell [1] noted that the linear differential equation

$$Y''' + fY' + (f'/2)Y \ = \ 0$$

has $\mathbb{R}$-linearly independent solutions $y_1^2, y_1y_2, y_2^2 \in \mathcal{C}^{<\infty}$, though some cases were known earlier [22, 49]; in particular, $1/u = y_1^2 + y_2^2$ is a solution. Hartman [32, 34] investigates monotonicity properties of $y_1^2 + y_2^2$. Steen [59] in 1874, and independently Pinney [53], remarked that $r := 1/\sqrt{u} = \sqrt{y_1^2 + y_2^2} \in \mathcal{C}^{<\infty}$ satisfies $4r'' + fr = 4w^2/r^3$ where $w := y_1y_2' - y_1'y_2 \in \mathbb{R}^\times$. (See also [54].)

Complex exponentials over Hardy fields. We now use some of the above to prove an extension theorem for Hardy fields (cf. [18, Lemma 11.6(6)]). Recall that the H-asymptotic field extension $K := H[\mathrm{i}]$ of H is a differential subring of $\mathcal{C}^{<\infty}[\mathrm{i}]$.

Proposition 6.9. *If $\phi \in H$ and $\phi \preccurlyeq 1$, then $\cos\phi, \sin\phi \in \mathrm{D}(H)$.*

Proof. Replacing H by $\mathrm{D}(H)$ we arrange $\mathrm{D}(H)=H$. Then by Proposition 4.2, $H\supseteq\mathbb{R}$ is a Liouville closed H-field, and by Corollary 6.2, $\omega(H)$ is downward closed. Hence by [11, Lemma 1.2.17], there is for all $\phi\preccurlyeq 1$ in H a (necessarily unique) $y\in K$ with $y\sim 1$ and $y^\dagger=\phi'\mathrm{i}$. Let now $\phi\in H$ and $\phi\preccurlyeq 1$. Then $(\mathrm{e}^{\phi\mathrm{i}})^\dagger=\phi'\mathrm{i}\in K^\dagger$, so $\cos\phi+\mathrm{i}\sin\phi=\mathrm{e}^{\phi\mathrm{i}}\in K$ using $K\supseteq\mathbb{C}$. Thus $\cos\phi,\sin\phi\in H$. □

Corollary 6.10. *Let $\phi\in H$ and $\phi\preccurlyeq 1$. Then $\cos\phi$, $\sin\phi$ generate a* d*-algebraic Hardy field extension $E:=H(\cos\phi,\sin\phi)$ of H. If H is a $\mathcal{C}^\infty$-Hardy field, then so is E, and likewise with $\mathcal{C}^\omega$ in place of $\mathcal{C}^\infty$.*

In [11, Part 4] we sometimes assume $\mathrm{I}(K)\subseteq K^\dagger$, a condition that we consider more closely in the next proposition:

Proposition 6.11. *Suppose $H\supseteq\mathbb{R}$ is closed under integration, that is, $\partial(H)=H$. Then the following conditions are equivalent:*

(i) $\mathrm{I}(K)\subseteq K^\dagger$;
(ii) $\mathrm{e}^f\in K$ *for all* $f\in K$ *with* $f\prec 1$;
(iii) $\mathrm{e}^\phi,\cos\phi,\sin\phi\in H$ *for all* $\phi\in H$ *with* $\phi\prec 1$.

Proof. Assume (i), and let $f\in K$, $f\prec 1$. Then $f'\in\mathrm{I}(K)$, so we have $g\in K^\times$ with $f'=g^\dagger$ and thus $\mathrm{e}^f=cg$ for some $c\in\mathbb{C}^\times$. Therefore $\mathrm{e}^f\in K$. This shows (i) ⇒ (ii), and (ii) ⇒ (iii) is clear. Assume (iii), and let $f\in\mathrm{I}(K)$. Then $f=g+h\mathrm{i}$, $g,h\in\mathrm{I}(H)$. Taking $\phi,\theta\prec 1$ in H with $\phi'=g$ and $\theta'=h$,

$$\exp(\phi+\theta\mathrm{i})\ =\ \exp(\phi)\big(\cos(\theta)+\sin(\theta)\mathrm{i}\big)\in H[\mathrm{i}]\ =\ K$$

has the property that $f=\big(\exp(\phi+\theta\mathrm{i})\big)^\dagger\in K^\dagger$. This shows (iii) ⇒ (i). □

From Propositions 6.9 and 6.11 we obtain:

Corollary 6.12. *If H is* d*-perfect, then* $\mathrm{I}(K)\subseteq K^\dagger$.

Some special subsets of H. Let $f\in H$. Principal solutions of (4L) arise from certain distinguished solutions of the non-linear (Riccati) equation $\omega(z)=f$. To explain this and for later use we recall from [ADH, 11.8] some special subsets of H:

$$\begin{aligned}\Gamma(H)\ &:=\ \big\{h^\dagger:\ h\in H,\ h\succ 1\big\}\ \subseteq\ H^>,\\ \Lambda(H)\ &:=\ \big\{-(h^{\dagger\dagger}):\ h\in H,\ h\succ 1\big\},\\ \Delta(H)\ &:=\ \big\{-(h'^{\dagger}):\ h\in H,\ 0\neq h\prec 1\big\}.\end{aligned}$$

Here $\Lambda(H)=-\Gamma(H)^\dagger$ and $\Delta(H)$ are disjoint, and $\omega\colon H\to H$ and $\sigma\colon H^\times\to H$ are strictly increasing on $\Lambda(H)$ and $\Gamma(H)$, respectively. If H is Liouville closed, then $\Gamma(H)$ is upward closed, $\Lambda(H)$ is downward closed, $H=\Lambda(H)\cup\Delta(H)$, and $\omega\big(\Lambda(H)\big)=\omega\big(\Delta(H)\big)=\omega(H)$; see [ADH, 11.8.13, 11.8.19, 11.8.20, 11.8.29].

Lemma 6.13. *Suppose H is* d*-perfect and $f/4$ does not generate oscillation, and let $y\in H$ be a principal solution of* (4L). *Then $2y^\dagger$ is the unique $z\in\Lambda(H)$ such that $\omega(z)=f$.*

Proof. We already know $\omega(2y^\dagger)=f$, and as ω is strictly increasing on $\Lambda(H)$, it remains to show that $2y^\dagger\in\Lambda(H)$. For this take $h\in H$ with $h'=1/y^2$. Then $h\succ 1$ by Corollary 3.13, hence $1/y^2\in\Gamma(H)$, and thus $2y^\dagger=-(1/y^2)^\dagger\in\Lambda(H)$. □

Combining Lemma 6.13 with the remarks after Proposition 6.1 yields:

Corollary 6.14. *If H is* d*-perfect and $f/4$ does not generate oscillation, then* (4L) *has solutions $y_1, y_2 \in H$ such that $y_1, y_2 > 0$, $y_1 \prec y_2$, $2y_1^\dagger \in \Lambda(H)$, and $2y_2^\dagger \in \Delta(H)$ (and thus $y_2' > 0$ in view of $-(\mathrm{e}^x)^{\dagger\dagger} = 0 \in \Lambda(H)$).*

Remark 6.15. Suppose $f/4 > 0$ does not generate oscillation. Remark 3.16 and Corollary 6.14 yield $a \in \mathbb{R}$, a representative of f in $\mathcal{C}_a$, also denoted by f, and a constant $C \in \mathbb{R}^{\geqslant}$ such that the inequality (3.1) holds for all $u \in \mathcal{C}_a^1$ with $u(a) = 0$. (This is not used later.)

We have $\omega(H) < \sigma\big(\Gamma(H)\big)$ by [ADH, remark before 11.8.29]. Recall that $\overline{\omega}(H)$ is downward closed and $\omega(H) \subseteq \overline{\omega}(H)$, with equality for d-perfect H. (Corollary 6.2.) This yields a property of $\overline{\omega}(H)$ used in the proof of Corollary 6.24:

Lemma 6.16. $\overline{\omega}(H) < \Gamma(H)$.

Proof. We arrange that H is d-perfect, hence $H \supseteq \mathbb{R}$ is Liouville closed and $\overline{\omega}(H) = \omega(H)$. From $x^{-1} = x^\dagger \in \Gamma(H)$ and $\sigma(x^{-1}) = 2x^{-2} \asymp (x^{-1})' \prec \ell^\dagger$ for all $\ell \succ 1$ in H we obtain $\Gamma(H) \subseteq \sigma\big(\Gamma(H)\big)^\uparrow$, so $\omega(H) < \Gamma(H)$. □

The concepts of λ-freeness and ω-freeness for Hardy fields. Suppose H has asymptotic integration. Then by [ADH, 11.8.16, 11.8.30]: H is λ-free iff there is no $\lambda \in H$ such that $\Lambda(H) < \lambda < \Delta(H)$, and H is ω-free iff there is no $\omega \in H$ such that $\omega\big(\Lambda(H)\big) < \omega < \sigma\big(\Gamma(H)\big)$. By [ADH, 11.7.3], if H is ω-free, then H is λ-free. If $H \supseteq \mathbb{R}$ is Liouville closed, then H is λ-free, and

$$H \text{ is } \omega\text{-free} \iff \omega(H)^\downarrow = H \setminus \sigma\big(\Gamma(H)\big)^\uparrow.$$

In [ADH, 16.3] we also introduced the concept of a *ΛΩ-cut* in a pre-H-field; every pre-H-field has exactly one or exactly two ΛΩ-cuts [ADH, remark before 16.3.19]. By [ADH, 16.3.14, 16.3.16]:

Lemma 6.17. *Suppose H is* d*-perfect. Then $\big(\mathrm{I}(H), \Lambda(H), \overline{\omega}(H)\big)$ is a ΛΩ-cut in H, and this is the unique ΛΩ-cut in H iff H is ω-free.*

Thus in general,

$$\big(\mathrm{I}\big(\mathrm{D}(H)\big) \cap H,\ \Lambda\big(\mathrm{D}(H)\big) \cap H,\ \overline{\omega}(H)\big)$$

is a ΛΩ-cut in H, and hence $\overline{\omega}(H) < \sigma\big(\Gamma(H)\big)^\uparrow$ (see [ADH, p. 692]). The classification of ΛΩ-cuts in H from [ADH, 16.3] can be used to narrow down the possibilities for $\overline{\omega}(H)$. With $\mathcal{O}$ = valuation ring of H and $\smallo$ = maximal ideal of $\mathcal{O}$, we have:

Lemma 6.18. *Let $\phi \in H^>$ be such that $v\phi \notin (\Gamma_H^{\neq})'$. Then*

$$\overline{\omega}(H) \ = \ \omega(-\phi^\dagger) + \phi^2 \smallo^\downarrow \quad \text{or} \quad \overline{\omega}(H) \ = \ \omega(-\phi^\dagger) + \phi^2 \mathcal{O}^\downarrow.$$

The first alternative holds if H is grounded, and the second alternative holds if $v\phi$ is a gap in H with $\phi \asymp b'$ for some $b \asymp 1$ in H.

Proof. Either $v\phi = \max \Psi_H$ or $v\phi$ is a gap in H, by [ADH, 9.2]. The remark before the lemma yields an ΛΩ-cut (I, Λ, Ω) in H where $\Omega = \overline{\omega}(H)$. Now use the proofs of [ADH, 16.3.11, 16.3.12, 16.3.13] together with the transformation formulas [ADH, (16.3.1)] for ΛΩ-cuts. □

By [ADH, 16.3.15] we have:

Lemma 6.19. *If H has asymptotic integration and the set $2\Psi_H$ does not have a supremum in Γ_H, then*

$$\overline{\omega}(H) \ = \ \omega\big(\Lambda(H)\big)^\downarrow \ = \ \omega(H)^\downarrow \quad \text{or} \quad \overline{\omega}(H) \ = \ H \setminus \sigma\big(\Gamma(H)\big)^\uparrow.$$

Corollary 6.20. *Suppose H is ω-free. Then*

$$\overline{\omega}(H) \;=\; \omega\big(\Lambda(H)\big)^{\downarrow} \;=\; \omega(H)^{\downarrow} \;=\; H\setminus\sigma\big(\Gamma(H)\big)^{\uparrow}.$$

Proof. By [ADH, 11.8.30] we have $\omega\big(\Lambda(H)\big)^{\downarrow} = \omega(H)^{\downarrow} = H\setminus\sigma\big(\Gamma(H)\big)^{\uparrow}$. It follows from [ADH, 9.2.19] that $2\Psi_H$ has no supremum in Γ_H. Now use Lemma 6.19. □

Non-oscillation and compositional conjugation. Which "changes of variable" preserve the general form of the linear differential equation (4L)? The next lemma and Corollary 6.22 below give an answer.

Lemma 6.21. *Let K be a differential ring, $f\in K$, and $P := 4Y'' + fY \in K\{Y\}$. Then for $g\in K^{\times}$ and $\phi := g^{-2}$ we have*

$$g^3 P^{\phi}_{\times g} \;=\; 4Y'' + g^3P(g)Y.$$

Proof. Let $g,\phi\in K^{\times}$. Then

$$\begin{aligned} P_{\times g} &= 4gY'' + 8g'Y' + (4g''+fg)Y \;=\; 4gY''+8g'Y'+P(g)Y, \quad\text{so}\\ P^{\phi}_{\times g} &= 4g(\phi^2Y''+\phi'Y') + 8g'\phi Y' + P(g)Y\\ &= 4g\phi^2Y'' + (4g\phi'+8g'\phi)Y' + P(g)Y. \end{aligned}$$

Now $4g\phi'+8g'\phi=0$ is equivalent to $\phi^{\dagger}=-2g^{\dagger}$, which holds for $\phi=g^{-2}$. For this ϕ we get $P^{\phi}_{\times g}(Y) = g^{-3}\big(4Y''+g^3P(g)Y\big)$, that is, $g^3P^{\phi}_{\times g}(Y) = 4Y''+g^3P(g)Y$. □

Now let $\ell\in\mathcal{C}^1$ be such that $\ell>\mathbb{R}$ and $\phi:=\ell'\in H$. We use the superscript $\circ$ as in the subsection on compositional conjugation in Hardy fields of Section 4. Let $P := 4Y''+fY$ where $f\in H$. Recall that if $y\in\mathcal{C}^2[\mathrm{i}]$ and $4y''+fy=0$, then $y\in\mathcal{C}^{<\infty}[\mathrm{i}]$. Towards using Lemma 6.21, suppose $\phi=g^{-2}$, $g\in H^{\times}$, and set $h := \big(g^3P(g)\big)^{\circ}\in H^{\circ}$. We then obtain the following reduction of solving the differential equation (4L) to solving a similar equation over H°:

Corollary 6.22. *Let $y\in\mathcal{C}^2[\mathrm{i}]$. Then $z := (y/g)^{\circ}\in\mathcal{C}^2[\mathrm{i}]$, and*

$$4y''+fy \;=\; 0 \quad\Longleftrightarrow\quad 4z''+hz \;=\; 0.$$

In particular, $f/4$ generates oscillation iff $h/4$ does.

Remark. Corollary 6.22 is a special case of a result of Kummer [40]; cf. [16, §11] and [64, §2]. Lie [46] and Stäckel [58] proved uniqueness results about the transformation $y\mapsto (y/h)^{\circ}$; see [16, §22].

Consider the increasing bijection

$$f\mapsto \Phi(f) \;:=\; \Big(\big(f-\omega(-\phi^{\dagger})\big)/\phi^2\Big)^{\circ} \;:\; H\to H^{\circ},$$

and note that

$$g^3P(g) \;=\; g^3(4g''+fg) \;=\; \big(f-\omega(-\phi^{\dagger})\big)/\phi^2,$$

so $h=\Phi(f)$.

Lemma 6.23. $\Phi\big(\overline{\omega}(H)\big) = \overline{\omega}(H^{\circ})$.

Proof. First replace H by its real closure to arrange that H is real closed, then take $g\in H^{\times}$ with $g^{-2}=\phi$, and use the remarks above. □

The bijection

$$y \mapsto (y/\phi)^{\circ} \;:\; H \to H^{\circ}$$

restricts to bijections $\mathrm{I}(H) \to \mathrm{I}(H^{\circ})$ and $\Gamma(H) \to \Gamma(H^{\circ})$, and the bijection

$$z \mapsto \big((z+\phi^{\dagger})/\phi\big)^{\circ} \;:\; H \to H^{\circ}$$

restricts to bijections $\Lambda(H) \to \Lambda(H^{\circ})$ and $\Delta(H) \to \Delta(H^{\circ})$. (See the transformation formulas in [ADH, p. 520].) Then for $y \in H^{\times}$, $z \in H$ we have

$$\sigma\big((y/\phi)^{\circ}\big) \;=\; \Phi\big(\sigma(y)\big), \qquad \omega\big(((z+\phi^{\dagger})/\phi)^{\circ}\big) \;=\; \Phi\big(\omega(z)\big).$$

(See the formulas in [ADH, pp. 518–519].) Hence Φ also restricts to bijections

$$\sigma(H^{\times}) \to \sigma\big((H^{\circ})^{\times}\big), \quad \sigma\big(\mathrm{I}(H)^{\neq}\big) \to \sigma\big(\mathrm{I}(H^{\circ})^{\neq}\big), \quad \sigma\big(\Gamma(H)\big) \to \sigma\big(\Gamma(H^{\circ})\big),$$

and

$$\omega(H) \to \omega(H^{\circ}), \quad \omega\big(\Lambda(H)\big) \to \omega\big(\Lambda(H^{\circ})\big), \quad \omega\big(\Delta(H)\big) \to \omega\big(\Delta(H^{\circ})\big).$$

To illustrate the above, we use it to prove the Fite-Leighton-Wintner oscillation criterion for self-adjoint second-order linear differential equations over H [25, 44, 65]. (See also [37, §2] and [61, p. 45].) For this, let $A = f\partial^2 + f'\partial + g$ where $f \in H^{\times}$ and $g \in H$. For each $h \in \mathcal{C}$ we choose a germ $\int h$ in $\mathcal{C}^1$ such that $(\int h)' = h$.

Corollary 6.24. *Suppose $\int f^{-1} > \mathbb{R}$ and $\int g > \mathbb{R}$. Then $A(y) = 0$ for some oscillating $y \in \mathcal{C}^{<\infty}$.*

Proof. We arrange that $H \supseteq \mathbb{R}$ is Liouville closed. Then $f^{-1}, g \in \Gamma(H)$ by [ADH, 11.8.19]. Note that $\phi := f^{-1}$ is active in H. Put $B := 4\phi A_{\ltimes \phi^{1/2}}$, so $B = 4\partial^2 + h$ with $h := \omega(-\phi^{\dagger}) + 4g\phi$. Then

$$\begin{aligned} A(y) = 0 \text{ for some oscillating } y \in \mathcal{C}^2 \;&\Longleftrightarrow\; B(z) = 0 \text{ for some oscillating } z \in \mathcal{C}^2 \\ &\Longleftrightarrow\; h \notin \overline{\omega}(H), \end{aligned}$$

by Corollary 6.4. The latter is equivalent to $(4g/\phi)^{\circ} \notin \overline{\omega}(H^{\circ})$, by Lemma 6.23 applied to h in place of f. Now $\Gamma(H^{\circ}) \cap \overline{\omega}(H^{\circ}) = \emptyset$ by Lemma 6.16, so it remains to note that $4g \in \Gamma(H)$ yields $(4g/\phi)^{\circ} \in \Gamma(H^{\circ})$. ☐

7. Extending Hardy Fields to ω-free Hardy Fields

In this section H is a Hardy field. We first discuss in more detail the fundamental property of ω-freeness and then prove a conjecture from [18, §17]. Next we establish the main result of the paper, Theorem 7.14, to the effect hat every maximal Hardy field is ω-free. We finish with some complements to the theorem.

Iterated logarithms and ω-freeness. *In this subsection $H \supseteq \mathbb{R}(x)$ is log-closed.* As in the introduction this yields a log-sequence (ℓ_ρ) in $H^{>\mathbb{R}}$ from which we obtain sequences (γ_ρ), (λ_ρ), (ω_ρ) in H as follows:

$$\gamma_\rho \;:=\; \ell_\rho^{\dagger} \in \Gamma(H), \qquad \lambda_\rho \;:=\; -\gamma_\rho^{\dagger} \in \Lambda(H), \qquad \omega_\rho \;:=\; \omega(\lambda_\rho) \in \omega\big(\Lambda(H)\big).$$

Also $\lambda_{\rho+1} = \lambda_\rho + \gamma_{\rho+1}$ and $\omega_{\rho+1} = \omega_\rho + \gamma_{\rho+1}^2$. The sequence $\big(v(\gamma_\rho)\big)$ is strictly increasing and cofinal in Ψ_H by [ADH, beginning of 11.5], so the sequence (γ_ρ) is strictly decreasing and coinitial in $\Gamma(H)$. Using in addition that for all $f, g \in H^{\times}$ we have $f \prec g \Rightarrow f^{\dagger} < g^{\dagger}$, it follows that the pc-sequence (λ_ρ) is strictly increasing and cofinal in $\Lambda(H)$. Recall that the function $\omega\colon H \to H$ is strictly increasing on $\Lambda(H)$; so the pc-sequence (ω_ρ) is strictly increasing and cofinal in $\omega\big(\Lambda(H)\big)$. Hence using also Corollary 6.2, we obtain:

Corollary 7.1. *If H is* d*-perfect, then* (ω_ρ) *is cofinal in* $\overline{\omega}(H)$.

Note that Corollary 7.1 applies in particular to the d-perfect hull of any Hardy field. We have $\sigma(\gamma_\rho) = \omega_\rho + \gamma_\rho^2$, and by [ADH, 11.8.29] the sequence $\big(\sigma(\gamma_\rho)\big)$ is strictly decreasing and coinitial in $\sigma\big(\Gamma(H)\big)$. Thus by Corollary 6.20:

if H is ω-free, then (ω_ρ) is cofinal in $\overline{\omega}(H)$ and $\big(\sigma(\gamma_\rho)\big)$ is coinitial in $H \setminus \overline{\omega}(H)$.

Lemma 7.2. *Suppose H is ω-free and $f \in H$. Then the following are equivalent:*

(i) $f \in \overline{\omega}(H)$;
(ii) $f < \omega_\rho$ *for some* ρ;
(iii) $f < \omega_\rho + c\gamma_\rho^2$ *for all* $c \in \mathbb{R}^>$ *and all* ρ;
(iv) *there exists* $c \in \mathbb{R}^>$ *such that for all* ρ *we have* $f < \omega_\rho + c\gamma_\rho^2$.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) holds by the sentence preceding the lemma. The implication (ii) $\Rightarrow$ (iii) follows from [ADH, 11.8.22], and (iii) $\Rightarrow$ (iv) is obvious. Since $0 < \gamma_{\rho+1} \prec \gamma_\rho$ we obtain for $c \in \mathbb{R}^>$:

$$\omega_{\rho+1} + c\gamma_{\rho+1}^2 \ = \ \omega_\rho + \gamma_{\rho+1}^2 + c\gamma_{\rho+1}^2 \ < \ \omega_\rho + \gamma_\rho^2 \ = \ \sigma(\gamma_\rho).$$

This yields (iv) $\Rightarrow$ (i) by the sentence before the lemma. □

Corollary 7.3. *The following are equivalent:*

(i) H *is* ω*-free;*
(ii) (ω_ρ) *is cofinal in* $\overline{\omega}(H)$ *and* $\big(\sigma(\gamma_\rho)\big)$ *is coinitial in* $H \setminus \overline{\omega}(H)$.

Proof. The sentence preceding Lemma 7.2 yields (i) $\Rightarrow$ (ii). For the converse, assume (ii). Now (ω_ρ) is cofinal in $\omega\big(\Lambda(H)\big)$ and $\big(\sigma(\gamma_\rho)\big)$ is coinitial in $\sigma\big(\Gamma(H)\big)$, so there is no $\omega \in H$ with $\omega\big(\Lambda(H)\big) < \omega < \sigma\big(\Gamma(H)\big)$. Thus if H has asymptotic integration, then H is ω-free by the remarks preceding Lemma 6.16. Suppose H does not have asymptotic integration. Then we have a gap $v\gamma$ in H, $\gamma \in H^\times$, so $\lambda_\rho \rightsquigarrow \lambda := -\gamma^\dagger$ by [ADH, remark after 11.5.9], hence $\omega_\rho \rightsquigarrow \omega := \omega(\lambda)$ by [ADH, 11.7.3] and so $\omega_\rho < \omega \in \overline{\omega}(H)$ for all ρ, contradicting (ii). □

Lemma 7.2 also yields a generalization of Hartman's [31, Lemma 1]:

Corollary 7.4. *Suppose H is ω-free, and $f \in H$. Then there exists ρ such that $f - \omega_{\rho'} \sim f - \omega_\rho$ for all $\rho' \geqslant \rho$; for such ρ we have $f < \omega_\rho$ iff $f \in \overline{\omega}(H)$.*

Proof. If $f \notin \overline{\omega}(H)$, then (i)$\Rightarrow$(iv) in Lemma 7.2 yields ρ such that $f - \omega_\rho \geqslant \gamma_\rho^2$ and so $f - \omega_\rho \succ \gamma_{\rho+1}^2$, hence $f - \omega_\rho \succ \omega_{\rho'} - \omega_\rho$ for $\rho' > \rho$, and thus $f - \omega_{\rho'} \sim f - \omega_\rho$ for $\rho' > \rho$, as required. Suppose $f \in \overline{\omega}(H)$, and take an index σ such that $f \leqslant \omega_\sigma$, and put $\rho := \sigma + 1$. Then $\omega_\rho - f \geqslant \omega_\rho - \omega_\sigma > 0$ and thus for all $\rho' > \rho$ we have $\omega_\rho - f \succcurlyeq \gamma_\rho^2 \succ \gamma_{\rho+1}^2 \asymp \omega_{\rho'} - \omega_\rho$ and so $\omega_\rho - f \sim \omega_{\rho'} - f$. □

In the proof of Theorem 7.14 we shall use:

Lemma 7.5. *Let $\gamma \in (\mathcal{C}^1)^\times$, $\gamma > 0$, and $\lambda := -\gamma^\dagger$ with $\lambda_\rho < \lambda < \lambda_\rho + \gamma_\rho$ in $\mathcal{C}$, for all ρ. Then $\gamma_\rho > \gamma > \gamma_\rho/\ell_\rho = (-1/\ell_\rho)'$ in $\mathcal{C}$, for all ρ.*

Proof. Pick $a \in \mathbb{R}$ (independent of ρ) and functions in $\mathcal{C}_a$ whose germs at $+\infty$ are the elements $\ell_\rho, \gamma_\rho, \lambda_\rho$ of H; denote these functions also by $\ell_\rho, \lambda_\rho, \gamma_\rho$. From $\ell_\rho^\dagger = \gamma_\rho$ and $\gamma_\rho^\dagger = -\lambda_\rho$ in H we obtain $c_\rho, d_\rho \in \mathbb{R}^>$ such that for all sufficiently large $t \geqslant a$,

$$\ell_\rho(t) \ = \ c_\rho \exp\left[\int_a^t \gamma_\rho(s)\,ds\right], \quad \gamma_\rho(t) \ = \ d_\rho \exp\left[-\int_a^t \lambda_\rho(s)\,ds\right].$$

(How large is "sufficiently large" depends on ρ.) Likewise we pick functions in $\mathcal{C}_a$ whose germ at $+\infty$ are $\upgamma$, $\uplambda$, and also denote these functions by $\upgamma$, $\uplambda$. From $\upgamma^\dagger = -\uplambda$ in H we obtain a real constant $d > 0$ such that for all sufficiently large $t \geqslant a$,

$$\upgamma(t) \;=\; d \exp\left[-\int_a^t \uplambda(s)\,ds\right].$$

Also, $\uplambda_\rho < \uplambda < \uplambda_\rho + \upgamma_\rho$ yields constants $a_\rho, b_\rho \in \mathbb{R}$ such that for all $t \geqslant a$

$$\int_a^t \uplambda_\rho(s)\,ds \;<\; a_\rho + \int_a^t \uplambda(s)\,ds \;<\; b_\rho + \int_a^t \uplambda_\rho(s)\,ds + \int_a^t \upgamma_\rho(s)\,ds,$$

which by applying $\exp(-*)$ yields that for all sufficiently large $t \geqslant a$,

$$\frac{1}{d_\rho}\upgamma_\rho(t) \;>\; \frac{1}{\mathrm{e}^{a_\rho}\,d}\upgamma(t) \;>\; \frac{c_\rho}{\mathrm{e}^{b_\rho}\,d_\rho}\upgamma_\rho(t)/\ell_\rho(t).$$

Here the positive constant factors don't matter, since the valuation of $\upgamma_\rho$ is strictly increasing and that of $\upgamma_\rho/\ell_\rho = (-1/\ell_\rho)'$ is strictly decreasing with ρ. Thus for all ρ we have $\upgamma_\rho > \upgamma > \upgamma_\rho/\ell_\rho = (-1/\ell_\rho)'$, in $\mathcal{C}$. □

Constructing $\upomega$-free Hardy field extensions: special cases. If H is $\upomega$-free, then so is every d-algebraic Hardy field extension of H, by [ADH, 13.6.1]. Thus if H is $\upomega$-free, then the Hardy-Liouville closure $\operatorname{Li}\big(H(\mathbb{R})\big)$ of $H(\mathbb{R})$ is $\upomega$-free. Moreover, by [11, Lemma 1.3.20]:

Proposition 7.6. *If H is not $\uplambda$-free, then $\operatorname{Li}\big(H(\mathbb{R})\big)$ is $\upomega$-free.*

Together with Corollary 5.27, this yields:

Corollary 7.7. *Suppose H is grounded. Then $L := \operatorname{Li}\big(H(\mathbb{R})\big)$ is $\upomega$-free. Moreover, if for some m we have $h \succ \ell_m$ for all $h \in H$ with $h \succ 1$, then for all $g \in L$ with $g \succ 1$ there is an n with $g \succ \ell_n$.*

In particular, $\operatorname{Li}(\mathbb{R}) = \operatorname{Li}\big(\mathbb{R}(x)\big)$ is $\upomega$-free, and (ℓ_n) is coinitial in $\operatorname{Li}(\mathbb{R})^{>\mathbb{R}}$. We now use these observations to establish [18, Conjecture 17.11]: Corollary 7.10. We first note that for $c \in \mathbb{R}$, the germ $(\upomega_n + c\upgamma_n^2)/4$ generates oscillation iff $c > 0$. (See also the introduction.) This follows from the next corollary applied to $f = \upomega_n + c\upgamma_n^2$ and the grounded Hardy subfield $H := \mathbb{R}\langle \ell_n \rangle = \mathbb{R}(\ell_0, \dots, \ell_n)$ of $\operatorname{Li}(\mathbb{R})$:

Corollary 7.8. *Suppose H is grounded and for some m we have $h \succ \ell_m$ for all $h \in H$ with $h \succ 1$. Then for $f \in H$, the following are equivalent:*

(i) *$f \in \overline{\omega}(H)$;*
(ii) *$f < \upomega_n$ for some n;*
(iii) *$f < \upomega_n + c\upgamma_n^2$ for all n and all $c \in \mathbb{R}^>$;*
(iv) *there exists $c \in \mathbb{R}^>$ such that for all n we have $f < \upomega_n + c\upgamma_n^2$.*

Proof. By Corollary 7.7, $L := \operatorname{Li}\big(H(\mathbb{R})\big)$ is $\upomega$-free, and for all $g \in L$ with $g \succ 1$ there is an n such that $\ell_n \prec g$. Hence the corollary follows from Lemma 7.2 applied to L in place of H, using $\overline{\omega}(H) = H \cap \overline{\omega}(L)$. □

From the above equivalence (i) $\Leftrightarrow$ (ii) we recover [18, Theorem 17.7]:

Corollary 7.9. *Suppose $f \in \mathcal{C}$ is hardian and* d*-algebraic over $\mathbb{R}$. Then*

$$f \text{ generates oscillation} \iff f > \upomega_n/4 \text{ for all } n.$$

Proof. By Corollary 5.28 the Hardy field $H := \mathbb{R}\langle f\rangle$ satisfies the hypotheses of Corollary 7.8. Also, f generates oscillation iff $4f \notin \overline{\omega}(H)$. □

Using the above implication (iv) ⇒ (i) we obtain in the same way:

Corollary 7.10. *Let $f \in \mathcal{C}$ be hardian and* d*-algebraic over $\mathbb{R}$, and suppose for some $c \in \mathbb{R}^{>}$ we have $f < \omega_n + c\gamma_n^2$ for all n. Then $f/4$ does not generate oscillation.*

Next a variant of Proposition 7.6. Let $L \supseteq \mathbb{R}$ be a Liouville closed d-algebraic Hardy field extension of H such that $\omega(L) = \overline{\omega}(L)$. (By Corollary 6.2 this holds for $L = \mathrm{D}(H)$.) Note that then $\overline{\omega}(L) = \omega\big(\Lambda(L)\big)$ by [ADH, 11.8.20].

Lemma 7.11. *If H is not λ-free or $\overline{\omega}(H) = H \setminus \sigma\big(\Gamma(H)\big)^{\uparrow}$, then L is ω-free.*

Proof. If H is ω-free, then L is ω-free by [ADH, 13.6.1]. Hence, if H is not λ-free, then L is ω-free by Proposition 7.6. Suppose H is λ-free but not ω-free, and $\overline{\omega}(H) = H \setminus \sigma\big(\Gamma(H)\big)^{\uparrow}$. Then [ADH, 11.8.30] gives $\omega \in H$ with $\omega\big(\Lambda(H)\big) < \omega < \sigma\big(\Gamma(H)\big)$. Hence $\omega \in \overline{\omega}(H) \subseteq \overline{\omega}(L) = \omega\big(\Lambda(L)\big)$. Thus L is ω-free by [11, Corollary 1.3.21]. □

In [13] we show that for $L = \mathrm{D}(H)$ the converse of Lemma 7.11 holds. Here are examples of (1) a non-ω-free Liouville closed $\mathcal{C}^{\omega}$-Hardy field $L \supseteq \mathbb{R}(x)$, and (2) a non-$\lambda$-free log-closed $\mathcal{C}^{\omega}$-Hardy field $M \supseteq \mathbb{R}(x)$ with asymptotic integration:

Example 7.12. Remarks after Corollary 5.28 yield a translogarithmic and hardian germ $\ell \in \mathcal{C}^{\omega}$. Then $E := \mathbb{R}(\ell_0, \ell_1, \ell_2, \dots)$ is a Hardy subfield of $\mathrm{Li}\big(\mathbb{R}\langle \ell\rangle\big)$. Now E is ω-free by [ADH, 11.7.15], ℓ is E-hardian, and $\mathbb{R} < \ell < E^{>\mathbb{R}}$ by Corollary 7.7. Set

$$\gamma := \ell^{\dagger}, \qquad \lambda := -\gamma^{\dagger}, \qquad \omega := \omega(\lambda), \qquad H := E\langle\omega\rangle, \qquad L := \mathrm{Li}(H).$$

We have $\lambda_n \rightsquigarrow \lambda$ by [ADH, 11.5.7], hence $\omega_n \rightsquigarrow \omega$ by [ADH, 11.7.3]. Then H is an immediate λ-free extension of E by [ADH, 13.6.3, 13.6.4], and $H^{>\mathbb{R}}$ is coinitial in $L^{>\mathbb{R}}$ by [11, Proposition 1.3.15]. Now (ℓ_n) is coinitial in $E^{>\mathbb{R}}$, hence in $H^{>\mathbb{R}}$, and thus in $L^{>\mathbb{R}}$. It follows that L is not ω-free. Also $L \subseteq \mathrm{Li}\big(\mathbb{R}\langle\ell\rangle\big) \subseteq \mathcal{C}^{\omega}$.

Example 7.13. Let E, etc. be as in the previous example. Then E has asymptotic integration, and $E\langle\lambda\rangle$ is an immediate extension of E by [ADH, 13.6.3], so $E\langle\lambda\rangle$ has asymptotic integration. Let M be the smallest Hardy field extension of $E\langle\lambda\rangle$ that is henselian as a valued field and closed under integration (hence log-closed). Then $M \subseteq \mathcal{C}^{\omega}$, M is an immediate extension of $E\langle\lambda\rangle$ by [ADH, 10.2.7], so has asymptotic integration, and is not λ-free in view of $\lambda_n \rightsquigarrow \lambda$.

Proof of the main theorem. Here now is our main result:

Theorem 7.14. *Every Hardy field has a* d*-algebraic ω-free Hardy field extension.*

Proof. It is enough to show that every d-maximal Hardy field is ω-free. That reduces to showing that every non-ω-free Liouville closed Hardy field containing $\mathbb{R}$ has a proper d-algebraic Hardy field extension. So assume $H \supseteq \mathbb{R}$ is Liouville closed and not ω-free. We shall construct a proper d-algebraic Hardy field extension of H. We have $\omega \in H$ such that

$$\omega(H) \ < \ \omega \ < \ \sigma\big(\Gamma(H)\big).$$

With ω in the role of f in the discussion following Corollary 6.7, we have $\mathbb{R}$-linearly independent solutions $y_1, y_2 \in \mathcal{C}^2$ of the differential equation $4Y'' + \omega Y = 0$; in fact, $y_1, y_2 \in \mathcal{C}^{<\infty}$. Then the complex solution $y = y_1 + y_2 i$ is a unit of $\mathcal{C}^{<\infty}[i]$, and so we have $z := 2y^\dagger \in \mathcal{C}^{<\infty}[i]$. We shall prove that the elements $\lambda := \operatorname{Re} z$ and $\gamma := \operatorname{Im} z$ of $\mathcal{C}^{<\infty}$ generate a Hardy field extension $K = H(\lambda, \gamma)$ of H with $\omega = \sigma(\gamma) \in \sigma(K^\times)$. We can assume that $w := y_1 y_2' - y_1' y_2 \in \mathbb{R}^>$, so $\gamma = 2w/|y|^2 \in (\mathcal{C}^{<\infty})^\times$ and $\gamma > 0$.

Choose a log-sequence (ℓ_ρ) in H and define (γ_ρ), (λ_ρ), (ω_ρ) as indicated at the beginning of this section. Then $\omega_\rho \rightsquigarrow \omega$, with $\omega - \omega_\rho \sim \gamma_{\rho+1}^2$ by [ADH, 11.7.1]. For now we fix ρ and set $g_\rho := \gamma_\rho^{-1/2}$, so $2g_\rho^\dagger = \lambda_\rho = -\gamma_\rho^\dagger$. For $h \in H^\times$ we also have $\omega(2h^\dagger) = -4h''/h$, hence $P := 4Y'' + \omega Y \in H\{Y\}$ gives

$$P(g_\rho) \;=\; g_\rho(\omega - \omega_\rho) \;\sim\; g_\rho \gamma_{\rho+1}^2,$$

and so with an eye towards using Lemma 6.21:

$$g_\rho^3 P(g_\rho) \;\sim\; g_\rho^4 \gamma_{\rho+1}^2 \;\sim\; \gamma_{\rho+1}^2/\gamma_\rho^2 \;\asymp\; 1/\ell_{\rho+1}^2.$$

Thus with $g := g_\rho = \gamma_\rho^{-1/2}$, $\phi := g^{-2} = \gamma_\rho$ we have $A_\rho \in \mathbb{R}^>$ such that

$$g^3 P^\phi_{\times g}(Y) \;=\; 4Y'' + g^3 P(g) Y, \quad |g^3 P(g)| \;\leqslant\; A_\rho/\ell_{\rho+1}^2. \tag{7.1}$$

From $P(y) = 0$ we get $P^\phi_{\times g}(y/g) = 0$, that is, $y/g \in \mathcal{C}^{<\infty}[i]^\phi$ is a solution of

$$4Y'' + g^3 P(g) Y \;=\; 0, \quad \text{with } g^3 P(g) \in H \subseteq \mathcal{C}^{<\infty}.$$

Set $\ell := \ell_{\rho+1}$, so $\ell' = \ell_\rho^\dagger = \phi$. The subsection on compositional conjugation in Section 4 yields the isomorphism

$$h \mapsto h^\circ = h \circ \ell^{\mathrm{inv}} \colon H^\phi \to H^\circ$$

of H-fields, where ℓ^{inv} is the compositional inverse of ℓ. Under this isomorphism the equation $4Y'' + g^3 P(g) Y = 0$ corresponds to the equation

$$4Y'' + f_\rho Y \;=\; 0, \qquad f_\rho \;:=\; \big(g^3 P(g)\big)^\circ \in H^\circ \;\subseteq\; \mathcal{C}^{<\infty}.$$

By Corollary 6.22, the equation $4Y'' + f_\rho Y = 0$ has the "real" solutions

$$y_{j,\rho} \;:=\; (y_j/g)^\circ \in (\mathcal{C}^{<\infty})^\circ \;=\; \mathcal{C}^{<\infty} \qquad (j = 1, 2),$$

and the "complex" solution

$$y_\rho \;:=\; y_{1,\rho} + y_{2,\rho} i \;=\; (y/g)^\circ,$$

which is a unit of the ring $\mathcal{C}^{<\infty}[i]$. Set $z_\rho := 2y_\rho^\dagger \in \mathcal{C}^{<\infty}[i]$. The bound in (7.1) gives $|f_\rho| \leqslant A_\rho/x^2$, which by Corollary 3.6 yields positive constants B_ρ, c_ρ such that $|z_\rho| \leqslant B_\rho x^{c_\rho}$. Using $(f^\circ)' = (\phi^{-1} f')^\circ$ for $f \in \mathcal{C}^{<\infty}[i]$ we obtain

$$z_\rho \;=\; 2\big((y/g)^\circ\big)^\dagger \;=\; 2\big(\phi^{-1}(y/g)^\dagger\big)^\circ \;=\; \big((z - 2g^\dagger)/\phi\big)^\circ$$

In combination with the bound on $|z_\rho|$ this yields

$$\begin{aligned} \left|\frac{z - 2g^\dagger}{\phi}\right| \;&\leqslant\; B_\rho\, \ell_{\rho+1}^{c_\rho}, \quad \text{hence} \\ |z - \lambda_\rho| \;&\leqslant\; B_\rho\, \ell_{\rho+1}^{c_\rho}\, \phi \;=\; B_\rho\, \ell_{\rho+1}^{c_\rho}\, \gamma_\rho, \quad \text{and so} \\ z \;&=\; \lambda_\rho + R_\rho \quad \text{where} \quad |R_\rho| \leqslant B_\rho\, \ell_{\rho+1}^{c_\rho}\, \gamma_\rho. \end{aligned}$$

We now use this last estimate with $\rho + 1$ instead of ρ, together with

$$\lambda_{\rho+1} \;=\; \lambda_\rho + \gamma_{\rho+1}, \quad \ell_{\rho+1}\gamma_{\rho+1} \;=\; \gamma_\rho.$$

This yields

$$
\begin{aligned}
z &= \lambda_\rho + \gamma_{\rho+1} + R_{\rho+1} \\
&\qquad \text{with } |R_{\rho+1}| \leqslant B_{\rho+1}\, \ell_{\rho+2}^{c_{\rho+1}}\, \gamma_{\rho+1} \;=\; B_{\rho+1}\big(\ell_{\rho+2}^{c_{\rho+1}}/\ell_{\rho+1}\big)\,\gamma_\rho, \\
\text{so} \quad z &= \lambda_\rho + o(\gamma_\rho) \ \text{ that is, } z - \lambda_\rho \prec \gamma_\rho, \\
\text{and thus} \quad \lambda &= \operatorname{Re} z \;=\; \lambda_\rho + o(\gamma_\rho), \quad \gamma \;=\; \operatorname{Im} z \;\prec\; \gamma_\rho.
\end{aligned}
$$

Now varying ρ again, (λ_ρ) is a strictly increasing divergent pc-sequence in H which is cofinal in $\Lambda(H)$. By the above, $\lambda = \operatorname{Re} z$ satisfies $\Lambda(H) < \lambda < \Delta(H)$. This yields an ordered subfield $H(\lambda)$ of $\mathcal{C}^{<\infty}$, which by Lemma 2.11 is an immediate valued field extension of H with $\lambda_\rho \rightsquigarrow \lambda$. Now $\lambda = -\gamma^\dagger$ (see discussion before Lemma 6.8), so Lemma 7.5 gives $\gamma_\rho > \gamma > (-1/\ell_\rho)'$ in $\mathcal{C}^{<\infty}$, for all ρ. In view of Lemma 2.12 applied to $H(\lambda)$, γ in the role of K, f this yields an ordered subfield $H(\lambda,\gamma)$ of $\mathcal{C}^{<\infty}$. Moreover, γ is transcendental over $H(\lambda)$ and γ satisfies the second-order differential equation $2yy'' - 3(y')^2 + y^4 - \omega y^2 = 0$ over H (obtained from the relation $\sigma(\gamma) = \omega$ by multiplication with γ^2). It follows that $H(\lambda,\gamma)$ is closed under the derivation of $\mathcal{C}^{<\infty}$, and hence $H(\lambda,\gamma) = H\langle\gamma\rangle$ is a Hardy field that is d-algebraic over H. □

The proof also shows that every $\mathcal{C}^\infty$-Hardy field has an ω-free d-algebraic $\mathcal{C}^\infty$-Hardy field extension, and the same with $\mathcal{C}^\omega$ instead of $\mathcal{C}^{<\infty}$. In [12, 13] we prove that the perfect hull of an ω-free Hardy field is ω-free, but that not every perfect Hardy field is ω-free.

Improving Theorem 7.14. *In this subsection, assume $H \supseteq \mathbb{R}$ is Liouville closed and $\omega \in H$, $\gamma \in (\mathcal{C}^2)^\times$ satisfy $\omega(H) < \omega < \sigma\big(\Gamma(H)\big)$ and $\sigma(\gamma) = \omega$.* Proposition 7.6 and results from [ADH, 13.7] lead to a more explicit version of Theorem 7.14:

Corollary 7.15. *The germ γ generates a Hardy field extension $H\langle\gamma\rangle$ of H with a gap $v\gamma$, and so $\operatorname{Li}\big(H\langle\gamma\rangle\big)$ is an ω-free Hardy field extension of H.*

Proof. Since $\sigma(-\gamma) = \sigma(\gamma)$, we may arrange $\gamma > 0$. The discussion before Lemma 6.8 with ω, γ in the roles of f, g, respectively, yields $\mathbb{R}$-linearly independent solutions $y_1, y_2 \in \mathcal{C}^{<\infty}$ of the differential equation $4Y'' + \omega Y = 0$ with Wronskian $1/2$ such that $\gamma = 1/(y_1^2 + y_2^2)$. The proof of Theorem 7.14 shows that γ generates a Hardy field extension $H\langle\gamma\rangle = H(\lambda,\gamma)$ of H. Recall that $v(\gamma_\rho)$ is strictly increasing as a function of ρ and cofinal in Ψ_H; as $\gamma \prec \gamma_\rho$ for all ρ, this gives $\Psi_H < v\gamma$. Also $\gamma > (-1/\ell_\rho)' > 0$ for all ρ and $v(1/\ell_\rho)'$ is strictly decreasing as a function of ρ and coinitial in $(\Gamma_H^>)'$, and so $v\gamma < (\Gamma_H^>)'$. Then by [ADH, 13.7.1 and subsequent remark (2) on p. 626], $v\gamma$ is a gap in $H\langle\gamma\rangle$, so $\operatorname{Li}\big(H\langle\gamma\rangle\big)$ is ω-free by Proposition 7.6. □

Corollary 7.16. *Suppose $\gamma > 0$. Then with $L := \operatorname{Li}\big(H\langle\gamma\rangle\big)$,*

$$\omega \notin \overline{\omega}(H) \iff \gamma \in \Gamma(L), \qquad \omega \in \overline{\omega}(H) \iff \gamma \in \mathrm{I}(L).$$

Proof. If $\gamma \notin \Gamma(L)$, then $\omega \in \omega(L)^\downarrow$ by [ADH, 11.8.31], hence $\omega \in \overline{\omega}(H)$. If $\gamma \in \Gamma(L)$, then we can use Corollary 6.20 for L to conclude $\omega \notin \overline{\omega}(H)$. The equivalence on the right now follows from that on the left and [ADH, 11.8.19]. □

We also note that if $\omega/4$ generates oscillation, then we have many choices for γ:

Corollary 7.17. *Suppose $\omega/4$ generates oscillation. Then there are continuum many $\widetilde{\gamma} \in (\mathcal{C}^{<\infty})^\times$ with $\widetilde{\gamma} > 0$ and $\sigma(\widetilde{\gamma}) = \omega$; no Hardy field extension of H contains* *more than one such germ $\widetilde{\gamma}$. (Thus H has at least continuum many maximal Hardy field extensions.)*

Proof. As before we arrange $\gamma > 0$ and set $L := \operatorname{Li}\big(H\langle\gamma\rangle\big)$. Take $\phi \in L$ with $\phi' = \frac{1}{2}\gamma$ and consider the germs

$$y_1 := \frac{1}{\sqrt{\gamma}}\cos\phi, \quad y_2 := \frac{1}{\sqrt{\gamma}}\sin\phi \qquad \text{in } \mathcal{C}^{<\infty}.$$

The remarks preceding Lemma 6.8 show that y_1, y_2 solve the differential equation $4Y'' + \omega Y = 0$, their Wronskian equals $1/2$, and $\phi \succ 1$ (since $\omega/4$ generates oscillation). We now dilate y_1, y_2: let $r \in \mathbb{R}^>$ be arbitrary and set

$$y_{1r} \ := \ r y_1, \qquad y_{2r} \ := \ r^{-1} y_2.$$

Then y_{1r}, y_{2r} still solve the equation $4Y'' + \omega Y = 0$, and their Wronskian is $1/2$. Put $\gamma_r := 1/(y_{1r}^2 + y_{2r}^2) \in \mathcal{C}^{<\infty}$. Then $\sigma(\gamma_r) = \omega$. Let $r, s \in \mathbb{R}^>$. Then

$$\gamma_r = \gamma_s \quad \Longleftrightarrow \quad y_{1r}^2 + y_{2r}^2 = y_{1s}^2 + y_{2s}^2 \quad \Longleftrightarrow \quad (r^2 - s^2)\cos^2\phi + \big(\tfrac{1}{r^2} - \tfrac{1}{s^2}\big)\sin^2\phi = 0,$$

so $\gamma_r = \gamma_s$ iff $r = s$. Next, suppose $M \supseteq H$ is a d-perfect Hardy field containing both γ and $\widetilde{\gamma} \in (\mathcal{C}^{<\infty})^\times$ with $\widetilde{\gamma} > 0$ and $\sigma(\widetilde{\gamma}) = \omega$. Corollary 6.2 gives $\omega \notin \omega(M)$, so $\gamma, \widetilde{\gamma} \in \Gamma(M)$ by [ADH, 11.8.31], hence $\gamma = \widetilde{\gamma}$ by [ADH, 11.8.29]. □

8. A Question of Boshernitzan

In this final section we apply Theorem 7.14 to answer a question from [19] and to generalize a theorem from [18].

Translogarithmic germs in maximal Hardy fields. The following analogue of Corollary 5.24 for translogarithmic germs gives a positive answer to Question 4 in [19, §7]:

Proposition 8.1. *Every maximal Hardy field contains a translogarithmic germ.*

Let $H \supseteq \mathbb{R}$ be a Liouville closed Hardy field; then H has no translogarithmic element iff (ℓ_n) is a logarithmic sequence for H in the sense of [ADH, 11.5]. If in this case H is also ω-free, then for each translogarithmic H-hardian germ y the isomorphism type of the ordered differential field $H\langle y\rangle$ over H is uniquely determined. This is part of the next lemma, which follows from [ADH, 13.6.7, 13.6.8]. We need to assume familiarity with the *Newton degree* $\operatorname{ndeg}(P)$ and *Newton weight* $\operatorname{nwt}(P)$ of $P \in H\{Y\}^{\neq}$ for suitable H; see [ADH, 11.1].

Lemma 8.2. *Let H be an ω-free H-field, with asymptotic couple (Γ, ψ), and let $L = H\langle y\rangle$ be a pre-H-field extension of H with $\Gamma^< < vy < 0$. Then for all $P \in H\{Y\}^{\neq}$ we have*

$$v\big(P(y)\big) \ = \ \gamma + \operatorname{ndeg}(P)vy + \operatorname{nwt}(P)\psi_L(vy) \qquad \textit{where } \gamma = v^{\mathrm{e}}(P) \in \Gamma,$$

and thus

$$\Gamma_L \ = \ \Gamma \oplus \mathbb{Z}vy \oplus \mathbb{Z}\psi_L(vy) \qquad (\textit{internal direct sum}).$$

Moreover, if $L^* = H\langle y^*\rangle$ is a pre-H-field extension of H with $\Gamma^< < vy^* < 0$ and $\operatorname{sign} y = \operatorname{sign} y^*$, then there is a unique pre-H-field isomorphism $L \to L^*$ which is the identity on H and sends y to y^*.

This lemma suggests how to obtain Proposition 8.1: follow the arguments in the proof of [ADH, 13.6.7]. In the rest of this subsection we carry out this plan. In the next lemma and corollary $H \supseteq \mathbb{R}$ is a Liouville closed Hardy field and $y \in \mathcal{C}^{<\infty}$. The proof of this lemma assumes familiarity with the binary relation $\preccurlyeq^\flat$ on K and $K\{Y\}$ for suitable H-asymptotic fields K, and their variants; see [ADH, 9.4].

Lemma 8.3. *Suppose H is ω-free and for all $\ell \in H^{>\mathbb{R}}$ we have, in $\mathcal{C}$:*

(i) $1 \prec y \prec \ell$;
(ii) $\delta^n(y) \preccurlyeq 1$ *for all* $n \geqslant 1$, *where* $\delta := \phi^{-1}\partial$, $\phi := \ell'$;
(iii) $y' \in \mathcal{C}^\times$ *and* $(1/\ell)' \preccurlyeq y^\dagger$.

Let $P \in H\{Y\}^{\neq}$. Then in $\mathcal{C}$ we have

$$P(y) \sim a\, y^d\, (y^\dagger)^w \qquad \text{where } a \in H^\times,\ d = \operatorname{ndeg}(P),\ w = \operatorname{nwt}(P).$$

(*Hence y is hardian over H and* d*-transcendental over H.*)

Proof. Since H is real closed, it has a monomial group by [ADH, 3.3.32], so the material of [ADH, 13.3] applies. Then [ADH, 13.3.3] gives a monic $D \in \mathbb{R}[Y]^{\neq}$, $b \in H^\times$, $w \in \mathbb{N}$, and an active element ϕ of H with $0 < \phi \prec 1$ such that:

$$P^\phi \;=\; b \cdot D \cdot (Y')^w + R, \qquad R \in H^\phi\{Y\},\ R \prec^\flat_\phi b.$$

Set $d := \operatorname{ndeg} P$, and note that by [ADH, 13.1.9] we have $d = \deg D + w$ and $w = \operatorname{nwt} P$. Replace P, b, R by $b^{-1}P, 1, b^{-1}R$, respectively, to arrange $b = 1$. Take $\ell \in H$ with $\ell' = \phi$, so $\ell > \mathbb{R}$; we use the superscript $\circ$ as in the subsection on compositional conjugation of Section 4; in particular, $y^\circ = y \circ \ell^{\text{inv}}$ with $(y^\circ)' = (\phi^{-1}y')^\circ$, so $(y^\circ)^\dagger \succcurlyeq 1/x^2$ by hypothesis (iii) of our lemma. For $f, g \in H$ we have

$$f \prec^\flat_\phi g \text{ (in } H) \iff f \prec^\flat g \text{ (in } H^\phi) \iff f^\circ \prec^\flat g^\circ \text{ (in } H^\circ).$$

Hence in $H^\circ\{Y\}$ we have

$$(P^\phi)^\circ \;=\; D \cdot (Y')^w + R^\circ \qquad \text{where} \qquad R^\circ \prec^\flat 1.$$

Evaluating at y° we have $D(y^\circ)\big((y^\circ)'\big)^w \sim (y^\circ)^d\big((y^\circ)^\dagger\big)^w$ and so $D(y^\circ)\big((y^\circ)'\big)^w \succcurlyeq x^{-2w} \asymp^\flat 1$. By (i) we have $(y^\circ)^m \prec x$ for $m \geqslant 1$, and by (ii) we have $(y^\circ)^{(n)} \preccurlyeq 1$ for $n \geqslant 1$. Hence $R^\circ(y^\circ) \preccurlyeq h^\circ$ for some $h \in H$ with $h^\circ \prec^\flat 1$. Thus in $\mathcal{C}$ we have

$$(P^\phi)^\circ(y^\circ) \sim (y^\circ)^d\big((y^\circ)^\dagger\big)^w.$$

Since $P(y)^\circ = (P^\phi)^\circ(y^\circ)$, this yields $P(y) \sim a \cdot y^d \cdot (y^\dagger)^w$ for $a = \phi^{-w}$. □

Corollary 8.4. *Suppose H is ω-free and $1 \prec y \prec \ell$ for all $\ell \in H^{>\mathbb{R}}$. Then the following are equivalent:*

(i) *y is hardian over H;*
(ii) *for all $h \in H^{>\mathbb{R}}$ there is an $\ell \in H^{>\mathbb{R}}$ such that $\ell \preccurlyeq h$ and y, ℓ lie in a common Hardy field;*
(iii) *for all $h \in H^{>\mathbb{R}}$ there is an $\ell \in H^{>\mathbb{R}}$ such that $\ell \preccurlyeq h$ and $y \circ \ell^{\text{inv}}$ is hardian.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is clear. Let $\ell \in H^{>\mathbb{R}}$ be such that $y^\circ := y \circ \ell^{\text{inv}}$ lies in a Hardy field H_0; we arrange $x \in H_0$. For $\phi := \ell'$ we have

$$(\phi^{-1}y^\dagger)^\circ \;=\; (y^\circ)^\dagger \;\succ\; (1/x)' \;=\; -1/x^2$$

and thus $y^\dagger \succ -\phi/\ell^2 = (1/\ell)'$. Also $y^\circ \prec x$, hence $z := (y^\circ)' \prec x' = 1$ and so $z^{(n)} \prec 1$ for all n. With $\delta := \phi^{-1}\partial$ and $n \geqslant 1$ we have $\delta^n(y)^\circ = z^{(n-1)}$, and thus $\delta^n(y) \prec 1$. Moreover, for $h \in H^{>\mathbb{R}}$ with $\ell \preccurlyeq h$ and $\theta := h'$ we have $\theta^{-1}\partial = f\delta$ where $f := \phi/\theta \in H$, $f \preccurlyeq 1$. Let $n \geqslant 1$. Then

$$(\theta^{-1}\partial)^n = (f\delta)^n = G_n^n(f)\delta^n + \cdots + G_1^n(f)\delta \quad \text{on } \mathcal{C}^{<\infty}$$
$$\text{where } G_j^n \in \mathbb{Q}\{X\} \subseteq H^\phi\{X\} \text{ for } j = 1, \ldots, n.$$

As δ is small as a derivation on H, we have $G_j^n(f) \preccurlyeq 1$ for $j = 1, \ldots, n$, and so $(\theta^{-1}\partial)^n(y) \prec 1$. Thus (iii) $\Rightarrow$ (i) by Lemma 8.3. □

Proof of Proposition 8.1. Let $H \supseteq \mathbb{R}$ be any ω-free Liouville closed Hardy field not containing any translogarithmic element; in view of Theorem 7.14 it suffices to show that then some Hardy field extension of H contains a translogarithmic element. The remark before Proposition 8.1 yields a translogarithmic germ y in a $\mathcal{C}^\omega$-Hardy field $H_0 \supseteq \mathbb{R}$. Then for each n, the germs y, ℓ_n are contained in a common Hardy field, namely $\operatorname{Li}(H_0)$. Hence y generates a proper Hardy field extension of H by (ii) $\Rightarrow$ (i) in Corollary 8.4. □

Proposition 8.1 goes through when "maximal" is replaced by "$\mathcal{C}^\infty$-maximal" or "$\mathcal{C}^\omega$-maximal". This follows from its proof, using also remarks after the proof of Theorem 7.14. Here is a conjecture that is much stronger than Proposition 8.1; it postulates an analogue of Corollary 5.23 for infinite "lower bounds":

Conjecture. If H is maximal, then there is no $y \in \mathcal{C}^1$ such that $1 \prec y \prec h$ for all $h \in H^{>\mathbb{R}}$, and $y' \in \mathcal{C}^\times$.

We observe that in this conjecture we may restrict attention to $\mathcal{C}^\omega$-hardian germs y:

Lemma 8.5. *Suppose there exists $y \in \mathcal{C}^1$ such that $1 \prec y \prec h$ for all $h \in H^{>\mathbb{R}}$ and $y' \in \mathcal{C}^\times$. Then there exists such a germ y which is $\mathcal{C}^\omega$-hardian.*

Proof. Take y as in the hypothesis. Replace y by $-y$ if necessary to arrange $y > \mathbb{R}$. Now Theorem 5.22 yields a $\mathcal{C}^\omega$-hardian germ $z \geqslant y^{\text{inv}}$. By Lemma 4.4, the germ z^{inv} is also $\mathcal{C}^\omega$-hardian, and $\mathbb{R} < z^{\text{inv}} \leqslant y \prec h$ for all $h \in H^{>\mathbb{R}}$. □

Lower bounds on the growth of germs in $\operatorname{E}(H)$. *In this subsection H is a Hardy field.* Recall from Corollary 5.15 that for all $f \in \operatorname{E}(H)$ there are $h \in H(x)$ and n such that $f \leqslant \exp_n h$. In particular, the sequence $(\exp_n x)$ is cofinal in $\operatorname{E}(\mathbb{Q})$. By Theorem 5.20 and Corollary 5.27, the sequence $(\ell_n) = (\log_n x)$ is coinitial in $\operatorname{E}(\mathbb{Q})^{>\mathbb{R}}$; see also [18, Theorem 13.2]. Thus for the Hardy field $H = \operatorname{Li}(\mathbb{R})$, the subset $H^{>\mathbb{R}}$ is coinitial in $\operatorname{E}(\mathbb{Q})^{>\mathbb{R}} = \operatorname{E}(H)^{>\mathbb{R}}$, equivalently, $\Gamma_H^<$ is cofinal in $\Gamma_{\operatorname{E}(H)}^<$. We now generalize this fact, also recalling from the remarks after Corollary 7.7 that $\operatorname{Li}(\mathbb{R})$ is ω-free:

Theorem 8.6. *Suppose H is ω-free. Then $\Gamma_H^<$ is cofinal in $\Gamma_{\operatorname{E}(H)}^<$.*

Proof. Replacing H by $\operatorname{Li}\big(H(\mathbb{R})\big)$ and using [ADH, 13.6.1] we arrange that H is Liouville closed and $H \supseteq \mathbb{R}$. Let $y \in \operatorname{E}(H)$ and suppose towards a contradiction that $\mathbb{R} < y < H^{>\mathbb{R}}$. Then $f := y^{\text{inv}}$ is transexponential and hardian (Lemma 4.4). Lemma 5.19 gives a bound $b \in \mathcal{C}^\times$ for $\mathbb{R}\langle f\rangle$. Lemma 5.17 gives $\phi \in (\mathcal{C}^\omega)^\times$ such that $\phi^{(n)} \prec 1/b$ for all n; set $r := \phi \cdot \sin x \in \mathcal{C}^\omega$. Then by Lemma 5.21 (with $\mathbb{R}\langle f\rangle$ in place of H) we have $Q(r) \prec 1$ for all $Q \in \mathbb{R}\langle f\rangle\{Y\}$ with $Q(0) = 0$. Hence $g := f + r$ is eventually strictly increasing with $g \succ 1$, and $y = f^{\text{inv}}$ and $z := g^{\text{inv}} \in \mathcal{C}^{<\infty}$ do not lie in a common Hardy field. Thus in order to achieve the desired contradiction it suffices to show that z is H-hardian. For this we use Corollary 8.4. It is clear that $f \sim g$, so $y \sim z$ by Corollary 2.6, and thus $1 \prec z \prec \ell$ for all $\ell \in H^{>\mathbb{R}}$. Let $\ell \in H^{>\mathbb{R}}$ and $\ell \prec x$; we claim that $z \circ \ell^{\text{inv}}$ is hardian, equivalently, by Lemma 4.4, that $\ell \circ g = (z \circ \ell^{\text{inv}})^{\text{inv}}$ is hardian. Now $\ell \circ f = (y \circ \ell^{\text{inv}})^{\text{inv}}$ is hardian and $\ell \circ f \succ 1$, and Lemma 5.11 gives $\ell \circ f - \ell \circ g \in (\mathcal{C}^{<\infty})^{\preccurlyeq}$. Hence $\ell \circ f \sim_\infty \ell \circ g$ by Lemma 5.12. For all n we have $\ell_n \circ \ell = \log_n \ell \in H^{>\mathbb{R}}$, so $y \leqslant \ell_n \circ \ell$, hence $y \circ \ell^{\text{inv}} \leqslant \ell_n$, which by compositional inversion gives $\ell \circ f \geqslant \exp_n x$. So $\ell \circ g$ is hardian by Corollary 5.14. Thus z is H-hardian by (iii) $\Rightarrow$ (i) of Corollary 8.4. $\square$

If $H \subseteq \mathcal{C}^\infty$ is $\upomega$-free, then $\Gamma_H^<$ is also cofinal in $\Gamma_{\mathrm{E}^\infty(H)}^<$, and similarly with ω in place of ∞. (Same proof as that of Theorem 8.6.) If H is bounded, then $\mathrm{D}(H) = \mathrm{E}(H)$ by Theorem 5.20, in which case Theorem 8.6 already follows from [ADH, 13.6.1]. Boshernitzan [18, p. 144] asked whether $\mathrm{D}(H) = \mathrm{E}(H)$ in general, and he gave Theorem 5.20 as support for a positive answer. Our Theorem 8.6 can be seen as further evidence.

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