Some Remarks About Asymptotic Couples

Matthias Aschenbrenner
Department of Mathematics
University of Illinois at Urbana-Champaign
Urbana, IL 61801, U.S.A.
Current address: Department of Mathematics
University of California at Berkeley
Berkeley, CA 94720, U.S.A.
maschenb@math.uiuc.edu

Abstract. Asymptotic couples (of H-type) try to capture the structure induced by the derivation of a Hardy field K on the value group of the natural valuation on K. In this note we continue the study of algebraic and model-theoretic aspects of asymptotic couples undertaken in [1]. We give a short exposition of some basic facts about asymptotic couples, and address a few topics left out in that paper: the (non-) minimality of the "closure" of an asymptotic couple of H-type, the Vapnik-Chernovenkis property for sets definable in closed asymptotic couples of H-type, and the relation of asymptotic couples of H-type to the "contraction groups" of [6].

Introduction

Let K be a Hardy field, that is (see [3], [17]), an ordered differential field of germs at $+\infty$ of real-valued differentiable functions defined on intervals $(a,+\infty)$, with $a\in\mathbb{R}$. (So two such functions determine the same element of K if they coincide on an interval $(b,+\infty)$ on which they are both defined; we will use the same letter for a function and its germ.) Every element f of K is ultimately monotonic, so $\lim_{x\to\infty} f(x)$ exists as an element of $\mathbb{R}\cup\{\pm\infty\}$. The valuation

$$v\colon K^\times = K\setminus\{0\}\to V = v(K^\times)$$

associated to the place $f\mapsto \lim_{x\to\infty} f(x)$ (where we identify $+\infty$ and $-\infty$) has the crucial property that $v(f')$ only depends on $v(f)$, for $f\in K^\times$ with $v(f)\neq 0$. (This is a consequence of L'Hospital's Rule, see [17].) So we have a well-defined map $\psi\colon V^* = V\setminus\{0\}\to V$ given by

$$\psi\big(v(f)\big) := v(f'/f) \qquad \text{for any } f\in K^\times \text{ such that } v(f)\neq 0.$$

1991 *Mathematics Subject Classification.* Primary 03C10, 06F20; Secondary 26A12, 12H05.
Key words and phrases. Asymptotic couples, Hardy fields.

The pair $\mathcal{V} = (V, \psi)$, called the **asymptotic couple of** K, is of key importance in understanding the interaction of the ordering and the derivation of K. It has the following fundamental properties: For all elements $f, g \in K^\times$ with $a = v(f) \neq 0$, $b = v(g) \neq 0$,

(A1) $\psi(ra) = \psi(a)$ for all $r \in \mathbb{Z}$, $r \neq 0$,
(A2) $\psi(a + b) \geq \min\{\psi(a), \psi(b)\}$, where $\psi(0) := \infty > V$,
(A3) $\psi(a) < \psi(b) + |b|$.

(See [14], Theorem 4.) Property (A2) expresses the fact that ψ is a valuation on the ordered abelian group V (taking values in V itself). In particular, it follows that for a, b as above, $\psi(a + b) = \min\{\psi(a), \psi(b)\}$ if $\psi(a) \neq \psi(b)$. Property (A3) may be seen as a valuation-theoretic formulation of L'Hospital's Rule, see [15].

Moreover, the map ψ is *decreasing* on the set of positive elements of V: For all $a, b \in V$,

(H) $0 < a \leq b \Longrightarrow \psi(a) \geq \psi(b)$.

(Hence by (A1), ψ is increasing on the set of negative elements of V.) Note that if the Hardy field K contains the germ x of the identity function on $\mathbb{R}$, then $\psi(1) = 1$, where we put $1 := v(x^{-1}) > 0$.

By an **asymptotic couple,** we mean a pair $\mathcal{V} = (V, \psi)$ consisting of an ordered abelian group V and a map $\psi \colon V^* \to V$ satisfying (A1)–(A3) above, for all $a, b \in V^*$. As in [2], we say that an asymptotic couple $\mathcal{V} = (V, \psi)$ is **of H-type** if (H) holds for all $a, b \in V$. We will sometimes also say "$\mathcal{V}$ is an **H-asymptotic couple**" instead of "$\mathcal{V}$ is an asymptotic couple of H-type." Rosenlicht, in a series of papers ([14], [15], [16], [18]) studied in detail the asymptotic couples (V, ψ) where the ordered abelian group V has finite rank. The paper [1] contains an investigation of the basic model-theoretic properties of H-asymptotic couples. In this note, we want to supplement it by considering a few issues left open in that paper. Our hope is that insight into algebraic and model-theoretic properties of asymptotic couples will ultimately become useful in the recently initiated project of understanding the model theory of Hardy fields and the field of LE-series. (See [2], [5].)

In section 1, we first review some basic facts about asymptotic couples. We only give a few proofs, referring to [1] and [2] for a more detailed exposition. From results of [1], it follows that the theory of H-asymptotic couples has a model companion (in a natural language), the theory of "closed H-asymptotic couples." (The definition of a closed H-asymptotic couple, as well as the statement of the main theorem from [1], can be found in section 2. See [12] for the notion "model companion".) In particular, each H-asymptotic couple $\mathcal{V}$ can be embedded into a closed one. In section 3, we show that in general there is no closed H-asymptotic couple containing $\mathcal{V}$ *minimally.* Section 4 consists of a few remarks about another model-theoretic property of the class of closed H-asymptotic couples, called the independence property. Finally, in section 5 we discuss a connection to Kuhlmann's "contraction groups" from [6].

In [1], we mainly worked in the setting of "H-couples": these are H-asymptotic couples (V, ψ) of a certain kind, where V has additional structure as an ordered vector space over an ordered field. In Section 6 of that paper, we showed how to adapt the results about H-couples to the case of H-asymptotic couples. Here, we right away restrict our attention to H-asymptotic couples, for convenience. We don't assume familiarity of the reader with [1]. In fact, sections 1–4 of the present note may serve as a quick overview of some of the results from that paper. However, we will freely use basic model-theoretic notions (see e.g. [12]).

Acknowledgements. The author would like to thank Lou van den Dries for numerous discussions around the topics of this paper, and Franz-Viktor Kuhlmann, Salma Kuhlmann, and Murray Marshall for organizing the Valuation Theory Conference 1999, where some of the results of this paper were first presented.

Notations. Throughout, m and n range over the set $\mathbb{N} = \{0, 1, 2, \dots\}$ of natural numbers. Our notations concerning linearly ordered sets and ordered abelian groups are fairly standard. (If in doubt, see [1], §1.) If V is an ordered abelian group, we let $V^* = V \setminus \{0\}$, and we put $S^{>0} = \{v \in S : v > 0\}$, $S^{<0} = \{v \in S : v < 0\}$, for a subset S of V. We define an equivalence relation $\sim$ on V by

$$v \sim w \quad :\Longleftrightarrow \quad |v| \le m|w| \text{ and } |w| \le n|v| \text{ for some } m, n > 0.$$

The equivalence class of an element $v \in V$ is written as $[v]$, and is called its **archimedean class.** By $[V]$ we denote the set of archimedean classes of V, and we set $[V^*] := [V] \setminus \big\{[0]\big\}$. We linearly order $[V]$ by setting

$$\begin{aligned} [v] < [w] \ &:\Longleftrightarrow\ n|v| < |w| \text{ for all } n \\ &\Longleftrightarrow\ [v] \ne [w] \text{ and } |v| < |w|. \end{aligned}$$

Some simple facts about archimedean classes: For $v, w \in V$, $r \in \mathbb{Z} \setminus \{0\}$, we have

1. $[v] = \{0\} \Longleftrightarrow v = 0$,
2. $[v] = [rv]$,
3. $[v + w] \le \max\big\{[v], [w]\big\}$, with $[v + w] = \max\big\{[v], [w]\big\}$, if $[v] \ne [w]$.

If V' is an ordered abelian group containing V as ordered subgroup, the inclusion map $V \hookrightarrow V'$ induces an embedding $[V] \to [V']$ of linearly ordered sets. We identify $[V]$ with its image under this embedding.

We consider V as a subgroup of the divisible abelian group $\mathbb{Q}V = \mathbb{Q} \otimes_{\mathbb{Z}} V$ by means of the embedding $v \mapsto 1 \otimes v$. We equip $\mathbb{Q}V$ with the unique linear ordering extending the one on V and making $\mathbb{Q}V$ into an ordered abelian group. Note that $[\mathbb{Q}V] = [V]$. We consider a divisible ordered abelian group as an ordered vector space over the ordered field $\mathbb{Q}$ as usual.

1 Basic Properties

In this section, $\mathcal{V} = (V, \psi)$ is an asymptotic couple. We set $\Psi := \psi(V^*)$, and with id denoting the identity function on V, we let

$$(\mathrm{id} + \psi)(V^*) = \big\{x + \psi(x) : x \in V^*\big\}.$$

Similarly, we define $(\mathrm{id} + \psi)\big(V^{>0}\big)$ and $(\mathrm{id} + \psi)\big(V^{<0}\big)$. Also, let $V_\infty := V \cup \{\infty\}$, with $\infty > v$ for all $v \in V$ and $v + \infty = \infty + v = -\infty = \infty$ for all $v \in V_\infty$. It is convenient to extend ψ to a map $V_\infty \to V_\infty$ by $\psi(0) := \psi(\infty) := \infty$.

Remark For $a \in V$ we define $\psi + a \colon V^* \to V$ by $(\psi + a)(x) := \psi(x) + a$. Then clearly $(V, \psi + a)$ is also an asymptotic couple, with $(\psi + a)(V^*) = \Psi + a$. Also, (V, ψ) satisfies (H) if and only if $(V, \psi + a)$ does.

For the next proposition, see also [1], §3, and [2], Proposition 2.3. Part (2.) is Theorem 5 in [15]; we give here a much shorter proof.

Proposition 1.1 *Let $v, w \in V$.*

1. *If $v, w \ne 0$, then $n\big(\psi(w) - \psi(v)\big) < |v|$.*
2. *If $v, w, v - w \ne 0$, then $\big[\psi(v) - \psi(w)\big] < [v - w]$.*
3. *The map $x \mapsto x + \psi(x) \colon V^* \to V$ is strictly increasing.*

Proof For (1.), let $v, w \in V^*$, $n \in \mathbb{N}$. We may assume $\psi(w) > \psi(v)$, $v, w > 0$, and $n > 0$. By passing from ψ to $\psi - \psi(v)$, if necessary, we can reduce to the case that $\psi(v) = 0 < \psi(w)$. We then have to show $n\psi(w) < v$. We proceed by induction on n. In the case $n = 1$, $n\psi(w) = \psi(w) < v + \psi(v) = v$ holds by axiom (A3) for asymptotic couples. Now assume that $n\psi(w) < v$. If $(n+1)\psi(w) \leq \psi(u)$ for some $u \in V^*$, we clearly have $(n+1)\psi(w) \leq \psi(u) < v + \psi(v) = v$, by axiom (A3) again. So we can assume that $(n+1)\psi(w) > \Psi$. Note that $\psi^2(w) > 0$, since $\psi(w) > 0$, so $\psi(w) < \psi(w) + \psi\big(\psi(w)\big)$. Hence

$$\psi\big(v - (n+1)\psi(w)\big) = \min\big\{\psi(v), \psi^2(w)\big\} = \psi(v) = 0,$$

by (A2), so we get

$$\Psi < |v - (n+1)\psi(w)| + \psi\big(v - (n+1)\psi(w)\big) = |v - (n+1)\psi(w)|.$$

Suppose $v \leq (n+1)\psi(w)$. Then, in particular, $\psi(w) < (n+1)\psi(w) - v$, hence $v < n\psi(w)$, contradicting the induction hypothesis. Therefore $(n+1)\psi(w) < v$, completing the induction step.— For (2.), let $v, w \neq 0$ with $d := v - w \neq 0$. We have to show $n|\psi(v) - \psi(w)| < |d|$ for all n. If $\psi(d) > \psi(w)$, then $\psi(v) = \psi(w)$, since ψ is a valuation on the ordered abelian group V. Suppose $\psi(d) \leq \psi(w)$. Then we have $\psi(d) \leq \psi(v)$, hence by (1.):

$$n\psi(d) \leq n\psi(w) < n\psi(d) + |d|, \quad n\psi(d) \leq n\psi(v) < n\psi(d) + |d|.$$

Thus $n|\psi(v) - \psi(w)| < |d|$ in all cases.— Property (3.) follows easily from (2.). $\square$

By (A1) and part (1.) of the proposition above, ψ extends uniquely to a map $(\mathbb{Q}V)^* \to V$, also denoted by ψ, such that $(\mathbb{Q}V, \psi)$ is an asymptotic couple. Note that $\psi\big((\mathbb{Q}V)^*\big) = \Psi$.

Some properties of H-asymptotic couples. From now on, we want to concentrate on H-asymptotic couples. So suppose that $\mathcal{V} = (V, \psi)$ is of H-type. Note that by axioms (A1) and (H), we have

$$[v] \leq [w] \quad \Longrightarrow \quad \psi(v) \geq \psi(w) \qquad \text{for all } v, w \in V^*. \tag{1.1}$$

In particular, ψ is constant on archimedean classes of V, i.e., for all $v, w \in V$ with $[v] = [w]$, we have $\psi(v) = \psi(w)$. The argument used for making $\mathbb{Q}V$ into an asymptotic couple extending (V, ψ) may be generalized, using (1.1), to show:

Corollary 1.2 *Let V' be an ordered abelian group containing V as ordered subgroup such that $[V] = [V']$. Then there is a unique extension of ψ to a function $\psi' \colon (V')^* \to V'$ such that (V', ψ') is an H-asymptotic couple.* $\square$

Lemma 1.3 *Let $w \in V^*$. If $\big[\psi(w)\big] \geq [w]$, then $\big[\psi\big(\psi(w)\big)\big] = \big[\psi(w)\big]$.*

Proof By (1.1), we may suppose that $\big[\psi(w)\big] > [w]$. By Proposition 1.1, (2.), and property (3.) of archimedean classes listed in the introduction, we have $\big[\psi(w) - \psi\big(\psi(w)\big)\big] < \big[w - \psi(w)\big] = \big[\psi(w)\big]$ and hence $\big[\psi\big(\psi(w)\big)\big] = \big[\psi(w)\big]$. $\square$

Remark The lemma and (1.1) imply that if $w \in V^*$ satisfies $[w] \leq \big[\psi(w)\big]$, then $y + \psi(y) = 0$ for $y = -\psi(w)$ or $y = -\psi\big(\psi(w)\big)$.

The following facts about $\mathrm{id} + \psi$ are fundamental (see also [1], Section 3):

Corollary 1.4 *The set* $(\mathrm{id}+\psi)\big(V^{>0}\big)$ *is closed upward. The set* $(\mathrm{id}+\psi)\big(V^{<0}\big)$ *is closed downward, and*

$$(-\mathrm{id}+\psi)\big(V^{>0}\big) = (\mathrm{id}+\psi)\big(V^{<0}\big) = \big\{a \in V : a < \psi(x) \text{ for some } x \in V^*\big\}. \quad (1.2)$$

There is at most one element $v \in V$ *such that* $\Psi < v < (\mathrm{id}+\psi)\big(V^{>0}\big)$. *If* Ψ *has a largest element, then there is no* $v \in V$ *with* $\Psi < v < (\mathrm{id}+\psi)\big(V^{>0}\big)$.

Proof Let $a > x + \psi(x)$ for some $x > 0$; we want to show $a \in (\mathrm{id}+\psi)\big(V^{>0}\big)$. Passing from (V,ψ) to $(V, \psi - a)$ if necessary, we reduce to the case $a = 0$. Then $[x] \leq \big[\psi(x)\big]$, hence $a = 0 \in (\mathrm{id}+\psi)\big(V^{>0}\big)$ by the previous remark and Proposition 1.1, (3.). So $(\mathrm{id}+\psi)\big(V^{>0}\big)$ is closed upward, and similarly one shows that $(\mathrm{id}+\psi)\big(V^{<0}\big)$ is closed downward.

The equalities in (1.2) are clear except for the inclusion "$\supseteq$" in the last equation. For this, let $a, x \in V$, $x < 0$, with $a < \psi(x)$; we want to show that $a \in (\mathrm{id}+\psi)\big(V^{<0}\big)$. As above, we may assume that $a = 0$. If $[x] \leq \big[\psi(x)\big]$, it follows as before that $0 \in (\mathrm{id}+\psi)\big(V^{<0}\big)$. If $\big[\psi(x)\big] < [x]$, then $0 < x + \psi(x)$, hence $0 \in (\mathrm{id}+\psi)\big(V^{<0}\big)$, since $(\mathrm{id}+\psi)\big(V^{<0}\big)$ is closed downward.

If $u, v \in V$ satisfy $\psi(w) \leq u < v < w + \psi(w)$ for all $w \in V^{>0}$, then $v < (v-u) + \psi(v-u) \leq (v-u) + u = v$, a contradiction. This shows the rest. □

As a consequence of the last corollary, $V \setminus (\mathrm{id}+\psi)(V^*)$ has at most one element, and $(\mathrm{id}+\psi)(V^*) \neq V$ if and only if Ψ has a supremum in V, and in this case $V \setminus (\mathrm{id}+\psi)(V^*) = \{\sup \Psi\}$. We refer the reader to [1], Figure 1, for a picture of the behavior of the maps ψ and $\mathrm{id}+\psi$ on V^*.

2 Closed H-Asymptotic Couples

A **cut** of an H-asymptotic couple (V,ψ) is a set $P \subseteq V$ which is closed downward, contains Ψ, and is disjoint from $(\mathrm{id}+\psi)\big(V^{>0}\big)$. (So $P < (\mathrm{id}+\psi)\big(V^{>0}\big)$.) By Corollary 1.4, an H-asymptotic couple (V,ψ) has at most two cuts, and it has two cuts if and only if $\Psi < v < (\mathrm{id}+\psi)\big(V^{>0}\big)$ for some $v \in V$. If Ψ has a maximum, then (V,ψ) has exactly one cut $P = \big\{a \in V : a \leq \psi(x) \text{ for some } x \in V^*\big\}$.

Definition 2.1 An H-asymptotic couple $\mathcal{V} = (V,\psi)$ is **closed** if

1. V is divisible (as an abelian group),
2. $(\mathrm{id}+\psi)(V^*) = V$, and
3. $\Psi = (\mathrm{id}+\psi)\big(V^{<0}\big)$.

(In this case, $P = \Psi$ is the only cut of $\mathcal{V}$.)

Example 1 Let K be a Hardy field containing $\mathbb{R}$ and closed under exponentiation (that is, $f \in K \Rightarrow \exp f \in K$) and integration (i.e. $f \in K \Rightarrow \exists g \in K : g' = f$). Then the asymptotic couple of K (as defined in the introduction) is a closed H-asymptotic couple.

In [1], Definition 6.2, we also introduced the following notion, under the somewhat technical name "H_0-triple":

Definition 2.2 An **asymptotic triple of H-type**, or **H-asymptotic triple** for short, is a triple (V,ψ,P), where (V,ψ) is an H-asymptotic couple and P a cut of (V,ψ), such that

1. V is divisible, and

2. there exists a positive element 1 of V with $\psi(1) = 1$. (Equivalently, $0 \in (\mathrm{id} + \psi)\big(V^{<0}\big)$.)

By Proposition 1.1, (2.), the element 1 in (2.) is uniquely determined. If (V, ψ, P) is an H-asymptotic triple such that (V, ψ) is a closed H-asymptotic couple, then $P = \Psi$, and (V, ψ, Ψ) is called a **closed H-asymptotic triple.**

We can naturally consider asymptotic couples (V, ψ) as model-theoretic structures (V_∞, ψ) in the first-order language $\mathcal{L} = \{0, +, -, \psi, \infty\}$. The H-asymptotic couples are then the models of a universal theory in $\mathcal{L}$. Similarly, when dealing with H-asymptotic triples (V, ψ, P) as model-theoretic objects, we construe them as $\mathcal{L}_P$-structures $(V_\infty, \psi, 1, P)$, where $\mathcal{L}_P$ is the extension of $\mathcal{L}$ by

1. a constant symbol 1 for the distinguished element $1 \in V^{>0}$ with $\psi(1) = 1$,
2. a unary predicate symbol for P, and
3. unary function symbols δ_n for each $n > 0$, to be interpreted on V as the scalar multiplication by $1/n$ (and $\delta_n(\infty) := \infty$).

The H-asymptotic triples are models of a universal theory in $\mathcal{L}_P$. Let T be the theory of closed H-asymptotic couples, in the language $\mathcal{L}$, and let T_P be the theory of closed H-asymptotic triples, in the language $\mathcal{L}_P$. One of the main results from [1] (Corollary 6.2) is:

Theorem 2.3 *The theory T_P is complete, decidable, and has elimination of quantifiers. It is the model completion of the theory of H-asymptotic triples.*

From this we get immediately:

Corollary 2.4 *The theory T is the model companion of the theory of H-asymptotic couples.* □

Remark The division symbols δ_n are included in the language $\mathcal{L}_P$ in order to guarantee quantifier elimination for T_P. Here is an instructive example to show that if we omit them, then in the resulting smaller language the theory of closed H-asymptotic triples would not eliminate quantifiers.

Let (W, ψ) be a closed H-asymptotic couple. Choose an element $b \notin W$ in an ordered vector space $W' := W \oplus \mathbb{Q}b$ over $\mathbb{Q}$ extending W, such that $\Psi < \frac{b}{2} < (\mathrm{id} + \psi)\big(W^{>0}\big)$. Then, by Lemma 4.5 in [1], $[W] = [W']$, hence ψ extends uniquely to a map $\psi' \colon (W')^* \to W$ such that $\mathcal{W}' = (W', \psi')$ is an H-asymptotic couple (Corollary 1.2). Note that $[W] = [W']$ implies

$$\Psi' = \Psi < \frac{b}{2} < (\mathrm{id} + \psi')\big((W')^{>0}\big).$$

Hence (W', ψ') has two cuts. Now consider the ordered abelian group $V := W \oplus \mathbb{Z}b \subseteq W'$. Since $\Psi' = \Psi \subseteq W$, $(V, \psi'|V^*)$ is an H-asymptotic couple with $(V, \psi'|V^*) \subseteq (W', \psi')$. One checks easily that the two distinct cuts of (W', ψ') have the same intersection with V, namely $\big\{v \in V : v \leq \psi(w) \text{ for some } w \in W\big\}$.

3 Non-Minimality of Closures

According to Theorem 2.3, every H-asymptotic triple can be embedded into a closed H-asymptotic triple. In fact, in the course of the proof of this theorem, we showed a more precise statement:

Proposition 3.1 *Every H-asymptotic triple $\mathcal{V} = (V, \psi, P)$ has a* **closure,** *that is, a closed H-asymptotic triple $\mathcal{V}^c = (V^c, \psi^c, P^c)$ extending $\mathcal{V}$, such that any embedding $\mathcal{V} \to \mathcal{V}'$ into a closed H-asymptotic triple $\mathcal{V}'$ extends to an embedding $\mathcal{V}^c \to \mathcal{V}'$. Any two closures of $\mathcal{V}$ are isomorphic over $\mathcal{V}$.*

(See [1], Corollaries 5.3 and 6.1.) A natural question is if the closure $\mathcal{V}^c$ of an H-asymptotic triple $\mathcal{V}$ is always *minimal over $\mathcal{V}$,* i.e. there exists no closed H-asymptotic triple $\mathcal{W} \supseteq \mathcal{V}$ strictly contained in $\mathcal{V}^c$ as an $\mathcal{L}_P$-substructure. This turns out to be false, in a very strong way:

Proposition 3.2 *Let $\mathcal{V} = (V, \psi, P)$ be an H-asymptotic triple which is* not *closed. Then the closure $\mathcal{V}^c$ of $\mathcal{V}$ is not minimal over $\mathcal{V}$.*

(This is similar, e.g., to the situation encountered with differential fields and differential closures, [13].)

Before we give a proof of Proposition 3.2, we outline how $\mathcal{V}^c$ is constructed from $\mathcal{V}$. One first shows the following embedding statements (see [1], Lemmas 3.5, 3.6 and 3.7 for a proof):

Lemma 3.3 *Let $\mathcal{V} = (V, \psi, P)$ be an H-asymptotic triple.*

1. *Suppose P has a largest element, and let $V^\varepsilon := V \oplus \mathbb{Q}\varepsilon$ be an extension of the $\mathbb{Q}$-vector space V. Then there exists a unique linear ordering of V^ε, a unique map $\psi^\varepsilon\colon (V^\varepsilon)^* \to V^\varepsilon$, and a unique subset P^ε of V^ε such that $(V^\varepsilon, \psi^\varepsilon, P^\varepsilon)$ is an H-asymptotic triple extending (V, ψ, P) with $\varepsilon > 0$ and $\max P = -\varepsilon + \psi^\varepsilon(\varepsilon)$.*
2. *Suppose there exists $b \in V$ with $P < b < (\mathrm{id} + \psi)\big(V^{>0}\big)$. Let $V^\varepsilon := V \oplus \mathbb{Q}\varepsilon$ be an extension of the $\mathbb{Q}$-vector space V. Then there exists a unique linear ordering of V^ε, a unique map $\psi^\varepsilon\colon (V^\varepsilon)^* \to V^\varepsilon$, and a unique subset $P^\varepsilon \subseteq V^\varepsilon$ such that $(V^\varepsilon, \psi^\varepsilon, P^\varepsilon)$ is an H-asymptotic triple extending (V, ψ, P) with $\varepsilon > 0$ and $b = \varepsilon + \psi^\varepsilon(\varepsilon)$.*
3. *Suppose $b \in P \setminus \Psi$. Let $V^a := V \oplus \mathbb{Q}a$ be an extension of the $\mathbb{Q}$-vector space V. There exists a unique linear ordering of V^a, a unique map $\psi^a\colon (V^a)^* \to V^a$, and a unique $P^a \subseteq V^a$, such that (V^a, ψ^a, P^a) is an H-asymptotic triple extending (V, ψ, P) with $a > 0$ and $\psi^a(a) = b$.*

Note that $(V^\varepsilon, \psi^\varepsilon)$ as in (1.) or (2.) of the lemma has the property that $\psi^\varepsilon\big((V^\varepsilon)^*\big)$ has a maximum. So part (1.) applies to $(V^\varepsilon, \psi^\varepsilon, P^\varepsilon)$ in place of (V, ψ, P). Also, if $(\mathrm{id} + \psi)(V^*) = V$, then $(\mathrm{id} + \psi^a)\big((V^a)^*\big) = V^a$. Therefore, iterating (1.)–(3.), if necessary transfinitely often, we can obtain an increasing chain of H-asymptotic triples extending (V, ψ, P) whose union is a closure of (V, ψ, P).

Proof of Proposition 3.2. Let $\mathcal{V} = (V, \psi, P)$ be an H-asymptotic triple which is *not* closed. We have to find a closed H-asymptotic triple $\mathcal{W}$ with $\mathcal{V} \subseteq \mathcal{W}$ which is strictly contained (as a substructure) in a closure of $\mathcal{V}$. Let us first consider a special case:

Lemma 3.4 *Suppose that P does not have a supremum in V, and $P \setminus \Psi$ contains a strictly increasing sequence $(a_n)_{n \in \mathbb{N}}$. Then the closure of $\mathcal{V}$ is not minimal over $\mathcal{V}$.*

Proof Using Lemma 3.3, (3.), we construct a strictly increasing sequence of H-asymptotic triples $(\mathcal{V}_n)_{n \in \mathbb{N}}$ such that

1. $\mathcal{V}_0 = \mathcal{V} = (V, \psi, P)$,

2. $V_{n+1} = V_n \oplus \mathbb{Q}v_n$, $\psi_{n+1}(v_n) = a_n$, for all n.

We let

$$V_\omega := V \oplus \bigoplus_{n\in\mathbb{N}} \mathbb{Q}v_n = \bigcup_{n<\omega} V_n, \quad P_\omega := \bigcup_{n<\omega} P_n,$$

and ψ_ω the common extension of all ψ_n to V_ω. Now construct another strictly increasing sequence of H-asymptotic triples $(\mathcal{V}'_n)_{n\in\mathbb{N}}$, contained in the H-asymptotic triple $\mathcal{V}_\omega = (V_\omega, \psi_\omega, P_\omega)$, with the following properties:

1. $\mathcal{V}'_0 = \mathcal{V}$,
2. $V'_{n+1} = V'_n \oplus \mathbb{Q}v'_n$ with $v'_n := v_n + v_{n+1}$, $\psi'_{n+1}(v'_n) = a_n$, for all n.

Note that $[V_n] = [V'_n]$ for all $n < \omega$. Again we let

$$V'_\omega := \bigcup_{n<\omega} V'_n \subseteq V_\omega, \quad P'_\omega := P_\omega \cap V'_\omega,$$

and ψ'_ω the common extension of all ψ'_n to V'_ω. Then $\mathcal{V}'_\omega = (V'_\omega, \psi'_\omega, P'_\omega)$ is an asymptotic couple such that $\mathcal{V} \subseteq \mathcal{V}'_\omega \subseteq \mathcal{V}_\omega$, $[V'_\omega] = [V_\omega]$, and one easily verifies that $v_n \notin V'_\omega$ for all n. Fix a closure $\mathcal{V}^{\mathrm{c}} = (V^{\mathrm{c}}, \psi^{\mathrm{c}}, P^{\mathrm{c}})$ of $\mathcal{V}_\omega$. (So $\mathcal{V}^{\mathrm{c}}$ is also a closure of $\mathcal{V}$.) Now using Lemma 3.3, (3.) repeatedly again, starting from $\mathcal{V}'_\omega$, we obtain a strictly increasing sequence $(\mathcal{V}'_\alpha)_{\omega\leq\alpha<\mu}$ of H-asymptotic triples (for some ordinal μ) such that

1. $\mathcal{V}'_\omega \subseteq \mathcal{V}'_\alpha \subseteq \mathcal{V}^{\mathrm{c}}$ for all $\alpha < \mu$,
2. $V'_{\alpha+1} = V'_\alpha \oplus \mathbb{Q}v'_\alpha$ with $\psi'_{\alpha+1}(v'_\alpha) \in P'_\alpha \setminus \psi'_\alpha\big((V'_\alpha)^*\big)$ for all $\alpha < \alpha + 1 < \mu$,
3. $\mathcal{V}'_\lambda = \bigcup_{\alpha<\lambda} \mathcal{V}'_\alpha$ for all limit ordinals $\lambda < \mu$, and
4. $(\mathcal{V}')^{\mathrm{c}} := \bigcup_{\alpha<\lambda} \mathcal{V}'_\alpha$ is closed.

So $(\mathcal{V}')^{\mathrm{c}}$ is a closure of $\mathcal{V}'_\omega$, and hence a closure of $\mathcal{V}$, contained in the closure $\mathcal{V}^{\mathrm{c}}$ of $\mathcal{V}$. One verifies easily that $v_n \notin (V')^{\mathrm{c}}$ for all n. Hence $\mathcal{V}^{\mathrm{c}}$ is not minimal over $\mathcal{V}$, as claimed. □

We now turn to the general case. Since $\mathcal{V}$ is assumed to be non-closed, one of the parts of Lemma 3.3 is applicable. The following three possibilities arise:

Case 1: The cut P has a maximum. Let $(V^\varepsilon, \psi^\varepsilon, P^\varepsilon)$ be as in Lemma 3.3, (1.); then $P^\varepsilon \setminus \Psi^\varepsilon$ is the union of $P \setminus \Psi$ and the set

$$\big\{v + \lambda\varepsilon : v \in V,\ \lambda \in \mathbb{Q}^\times, \text{ and } v < \max P \text{ or } v = \max P \ \&\ \lambda < 1\big\}.$$

Hence $P^\varepsilon \setminus \Psi^\varepsilon$ certainly contains a strictly increasing sequence $(a_n)_{n\in\mathbb{N}}$. If (V', ψ', P') is obtained from $(V^\varepsilon, \psi^\varepsilon, P^\varepsilon)$ by ω many applications of Lemma 3.3, (1.), then $P' \setminus \Psi'$ also contains the sequence (a_n), and P' does not have a supremum in V'.

Case 2: We have $P < b < (\mathrm{id} + \psi)\big(V^{>0}\big)$ for some (uniquely determined) element $b \in V$. Then we let $(V^\varepsilon, \psi^\varepsilon, P^\varepsilon)$ be as in Lemma 3.3, (2.). The set $P^\varepsilon \setminus \Psi^\varepsilon$ is the union of $P \setminus \Psi$ and

$$\big\{v + \lambda\varepsilon : v \in V,\ \lambda \in \mathbb{Q}^\times, \text{ and } v < b \text{ or } v = b \ \&\ \lambda < -1\big\},$$

so contains a strictly increasing sequence (a_n). If $(V', \psi', P') \supseteq (V^\varepsilon, \psi^\varepsilon, P^\varepsilon)$ is obtained from $(V^\varepsilon, \psi^\varepsilon, P^\varepsilon)$ by ω many applications of Lemma 3.3, (1.), then $P' \setminus \Psi'$ also contains the sequence (a_n), and P' does not have a supremum in V'.

Case 3: The cut P does not have a supremum in V, and there exists $b \in P \setminus \Psi$. Let (V^a, ψ^a, P^a) be as in Lemma 3.3, (2.). Then one readily verifies (see proof of Lemma 3.7 in [1]) that $P^a \setminus \Psi^a$ equals the union of $P \setminus (\Psi \cup \{b\})$ and

$$\{v + \lambda a : v \in V,\ \lambda \in \mathbb{Q}^\times,\ \lambda > 0, b - v > a \text{ or } \lambda < 0, v - b < a\}.$$

In particular, $P^a \setminus \Psi^a$ contains a strictly increasing sequence (a_n).

Hence in all three situations, we can reduce to the special case treated in the lemma, and thus finish the proof of the proposition.

4 The Independence Property for Closed H-Asymptotic Couples

Let $\mathcal{L}$ be a language (in the sense of first-order logic) and $\varphi(x, y)$ an $\mathcal{L}$-formula, where $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_n)$. We say that the formula $\varphi(x, y)$ **has the independence property** with respect to an $\mathcal{L}$-structure $\mathbf{A} = (A, \dots)$ if for every $k \in \mathbb{N}$ there is a sequence $(a_1, \dots, a_k)$ of elements of A^m such that for all subsets I of $\{1, \dots, k\}$, there exists $b_I \in A^n$ with

$$\mathbf{A} \models \varphi(a_i, b_I) \qquad \Longleftrightarrow \qquad i \in I,$$

for all $i = 1, \dots, k$. A theory T in the language $\mathcal{L}$ is said to have the independence property if all formulas $\varphi(x, y)$ as above have the independence property with respect to all $\mathbf{A} \models T$. A theory T *not* having the independence property signifies a certain well-behavedness of T, on a model-theoretic level: in this case, T shares many properties with stable theories (see [12]). There is an intriguing connection between the independence property and the notion of a Vapnik-Chernovenkis (VC) class from probability theory: A collection $\mathcal{C}$ of subsets of a set X is called a **VC class** if $f_{\mathcal{C}}(n) < 2^n$ for some n, where

$$f_{\mathcal{C}}(n) := \max\{|\mathcal{C} \cap F| : F \text{ is an } n\text{-element subset of } X\}.$$

(In this case, $f_{\mathcal{C}} \colon \mathbb{N} \to \mathbb{N}$ is in fact of polynomial growth; see [4], Chapter 5, for this and some other properties of VC classes.) Laskowski [9] proved that a formula $\varphi(x, y)$ does not have the independence property with respect to $\mathbf{A}$ if and only if the collection $\mathcal{C}_\varphi = \{\varphi^{\mathbf{A}}(a, y) : a \in A^m\}$, where $\varphi^{\mathbf{A}}(a, y) := \{b \in A^n : \mathbf{A} \models \varphi(a, b)\}$, is a VC class.

Suppose now that $\mathcal{L}$ contains a binary relation symbol $<$, and that T is a complete theory with quantifier elimination, all of whose models $\mathbf{A} = (A, <, \dots)$ are expansions of a dense linear ordering $(A, <)$ without endpoints. A **cut in** $(A, <)$ is a downward closed subset $C \subseteq A$. The following is a special case of a criterion due to Poizat [11]:

Lemma 4.1 *The theory T does not have the independence property if for all models $\mathbf{A}$ and $\mathbf{B}$ of T with $\mathbf{A} \preceq \mathbf{B}$ and all cuts C of A, there exist at most $2^{|A|}$ simple extensions $\mathbf{A} \subseteq \mathbf{A}\langle c\rangle \subseteq \mathbf{B}$ with $C < c < A \setminus C$, up to isomorphism over $\mathbf{A}$.*

In [1], §6, we proved that given closed H-asymptotic triples $(V, \psi, P) \subseteq (V', \psi', P')$ and a cut C in V, there exist at most two simple extensions of (V, ψ, P) inside (V', ψ', P') with generator $c \in V'$ such that $C < c < V \setminus C$, up to isomorphism over V. This implies:

Corollary 4.2 *The theory T_P of closed H-asymptotic triples does not have the independence property.* (*Hence the theory T of closed H-asymptotic couples also does not have the independence property.*) □

In fact, in [1] (Proposition 6.2) we showed something more: the theory T_P is *weakly o-minimal,* that is, for every closed H-asymptotic triple $\mathcal{V} = (V, \psi, P)$, every $\mathcal{L}_P$-formula $\varphi(x, y)$ with $x = (x_1, \ldots, x_m)$ and a single variable y, and every $v \in V^m$, the set $\varphi^{\mathcal{V}}(v, y)$ is a boolean combination of cuts in $(V, <)$. This also implies the corollary above, by Proposition 7.3 in [10]. We want to remark that the argument indicated here also works in the two-sorted setting of "closed H-triples" as defined in [1], thus giving a natural example of a *locally o-minimal* (but not weakly o-minimal) theory without the independence property. (See [1], Proposition 5.1 for the definition of "locally o-minimal" and a proof of the local o-minimality of the theory of closed H-triples.) Unlike in the weakly o-minimal case, it seems not to be known whether every locally o-minimal theory extending the theory of dense linear orders does not have the independence property.

5 Relation to Contraction Groups

Our couples resemble the contraction groups of Kuhlmann [6], [7], and there is indeed a formal connection as indicated below. (A difference is that contraction groups have nothing like our cut P.)

Contraction groups arise as follows: let K be a Hardy field closed under taking logarithms (i.e. $f \in K^{>0} \Rightarrow \log f \in K$), with its valuation $v\colon K^\times \to V = v(K^\times)$. The logarithm map then induces a so-called contraction map $\chi\colon V^{<0} \to V^{<0}$ by

$$\chi\big(v(f)\big) := v(\log f) \quad \text{for all } f \in K^{>0} \text{ with } v(f) < 0,$$

which we extend to a map $V \to V$ by requiring $\chi(-y) = -\chi(y)$. If K is also closed under exponentiation, then V is divisible, and χ is surjective ($\chi(V) = V$). This means that the pair (V, χ) (ordered group with contraction map) is a **divisible centripetal contraction group**, as axiomatized in [6], where it was shown that the elementary theory of non-trivial divisible centripetal contraction groups is complete and has quantifier elimination in its natural language. (See the appendix of [8] for an exposition of these results.)

In the example above, we have for $f \in K^{>0}$, with $y = v(f) < 0$:

$$\psi(y) = v\big((\log f)'\big) = v\big((\log f)'/\log f\big) + v(\log f) = \psi\big(\chi(y)\big) + \chi(y)$$

Let now (V, ψ) be any closed H-asymptotic couple. For $y < 0$ in V, let $\chi(y) = z$ be the unique solution in V^* of the equation

$$z + \psi(z) = \psi(y). \tag{5.1}$$

For $y > 0$, set $\chi(y) := -\chi(-y)$, and $\chi(0) := 0$. It is easily seen that then (V, χ) is a non-trivial divisible centripetal contraction group; clearly χ is definable (without parameters) in (V, ψ). Hence in particular, (V, χ) is weakly o-minimal, by Proposition 6.2 in [1]. We want to point out that the weak o-minimality of the theory of non-trivial divisible centripetal contraction groups (proved in [7]; see also [8], Theorem A.34) is a consequence of its completeness and the preceding observation: any model of this theory can be elementarily embedded into one of the form (V, χ) with χ definable in a closed H-asymptotic couple (V, ψ) (by choosing (V, ψ) sufficiently saturated), and hence is weakly o-minimal. (As the theory of closed H-asymptotic couples, the theory of non-trivial divisible centripetal contraction groups does not have the Steinitz exchange property for the definable closure operation.)

However, we cannot definably reconstruct ψ in (V, χ):

Proposition 5.1 *In* no *divisible centripetal contraction group* (V, χ) *can one define, even allowing parameters, a function* $\psi\colon V^* \to V$ *such that* (V, ψ) *is a closed* H*-asymptotic couple and* $\chi + \psi \circ \chi = \psi$ *on* $V^{<0}$.

Before we can prove this we need some preparations. We let (V, ψ) denote a closed H-asymptotic couple. We also assume that $0 \in \Psi$, so there exists $1 \in V^*$ such that $\psi(1) = 1 > 0$.

Iterates of ψ. For $n > 0$, let $\psi^n\colon V_\infty \to V_\infty$ be the n-fold functional composition $\psi \circ \psi \circ \cdots \circ \psi$. Put

$$D_n := \big\{v \in V : \psi^n(v) \neq \infty\big\}.$$

For example $D_1 = V^*$, $D_2 = V^* \setminus \psi^{-1}(0)$, etc. By induction on n one shows easily that $\psi^n(D_n) = \Psi$.

Lemma 5.2 *Let* $v \in V^*$ *and* $n > 0$ *such that* $\psi^n(v) < 0$. *Then* $\psi^i(v) < 0$ *for all* $i = 1, \ldots, n$*, and*

$$\big[\psi^n(v)\big] < \big[\psi^{n-1}(v)\big] < \cdots < \big[\psi(v)\big] < [v].$$

Proof For $n = 1$, note that $[v] \leq \big[\psi(v)\big]$ and (1.1) imply $\psi(v) \geq \psi\big(\psi(v)\big)$, hence $-\psi(v) + \psi\big(-\psi(v)\big) \leq 0 < (\mathrm{id} + \psi)\big(V^{>0}\big)$. Thus $\psi(v) > 0$, a contradiction.

Assume inductively that the lemma holds for a certain $n > 0$. Let $v \in D_{n+1}$ with $\psi^{n+1}(v) < 0$. Applying the case $n = 1$ to $\psi^n(v)$ instead of v gives $\big[\psi^{n+1}(v)\big] < \big[\psi^n(v)\big]$. By the inductive assumption the remaining inequalities will follow from $\psi^n(v) < 0$. Suppose $\psi^n(v) \geq 0$. Then $\psi^n(v) \in \Psi^{>0}$, thus $\big[\psi^n(v)\big] \leq [1]$ by (A3). Hence $0 > \psi^{n+1}(v) \geq \psi(1) = 1$ by (1.1), a contradiction. □

Let $D_\infty := \bigcap_{n>0} D_n$ and

$$V_{\text{inf}} := \big\{v \in D_\infty : \psi^n(v) < 0 \text{ for all } n > 0\big\},$$
$$V_{\text{fin}} := V \setminus V_{\text{inf}}.$$

Note that $[v_0] < [v]$ for all $v \in V_{\text{inf}}$, and that $V_{\text{inf}} \cap V^{>0}$ is closed upward and $V_{\text{inf}} \cap V^{<0}$ is closed downward.

Remark The previous lemma, together with $\psi^n(D_n) = \Psi$, implies that for all $n > 0$, we can find an element $v \in D_n$ such that all iterates

$$\psi(v), \psi^2(v), \ldots, \psi^n(v)$$

are negative. Hence if (V, ψ) is $\aleph_0$-saturated, then $V_{\text{inf}} \neq \emptyset$.

The proof of the next lemma is easy and left to the reader.

Lemma 5.3 V_{fin} *is a convex subspace of* V*, and* $(V_{\text{fin}}, \psi|V^*_{\text{fin}})$ *is a closed* H*-asymptotic couple. Moreover,* $\psi(V_{\text{inf}}) = V_{\text{inf}} \cap V^{<0}$. □

Let χ be the contraction map defined by $\psi(v) = \chi(v) + \psi\big(\chi(v)\big)$ for all $v < 0$.

Lemma 5.4 *Let* $v \in V^{<0}$ *and* $\psi^3(v) < 0$. *Then* $\chi(v) = \psi(v) - \psi^2(v)$.

Proof We have $[v] > \big[\psi(v)\big]$, so $\psi(v) - \psi^2(v) < 0$. We compute:

$$\big(\psi(v) - \psi^2(v)\big) + \psi\big(\psi(v) - \psi^2(v)\big) = \big(\psi(v) - \psi^2(v)\big) + \psi^2(v) = \psi(v).$$

By the defining equation (5.1) of χ, it follows that $\chi(v) = \psi(v) - \psi^2(v)$. □

Proof of Proposition 5.1. Suppose (V, ψ) is a closed H-asymptotic couple such that we can define ψ in (V, χ). We may assume that (V, ψ) is $\aleph_0$-saturated. For ease of notation we shall also assume that ψ is actually defined without parameters in (V, χ). (In the general case the role of V_{fin} below is taken over by the convex hull in V of a closure inside (V, ψ) of the substructure of (V, ψ) generated by the finitely many parameters used to define ψ.) If $0 \in (\text{id} + \psi)\big(V^{<0}\big)$, then (V, ψ, Ψ) is a closed H-asymptotic triple. Otherwise, we let $1 \in V^{>0}$ be the unique solution to the equation $x + \psi(x) = 0$, and pass from (V, ψ) to (V, ψ_0), where $\psi_0 := \psi + 1 - \psi(1)$, so that $\psi_0(1) = 1 > 0$. We see that we may in fact assume that (V, ψ, Ψ) is a closed H-asymptotic triple, with a distinguished positive element 1.

We now modify ψ to a function $\widetilde{\psi}\colon V^* \to V$ by putting

$$\widetilde{\psi}(v) := \begin{cases} \psi(v), & \text{if } v \in V^*_{\text{fin}} \\ \psi(v) + 1, & \text{if } v \in V_{\text{inf}}. \end{cases}$$

Then $(V, \widetilde{\psi})$ is still an H-asymptotic couple, and $\widetilde{\psi}(V_{\text{inf}}) = \psi(V_{\text{inf}})$, as is easily checked. Thus $\Psi = \widetilde{\psi}(V^*)$, so $(V, \widetilde{\psi})$ is even a *closed* H-asymptotic couple. Let $\widetilde{\chi}$ be the contraction map associated to $(V, \widetilde{\psi})$. By completeness of the theory of closed H-asymptotic triples, the same formula that defines ψ in (V, χ) will define $\widetilde{\psi}$ in $(V, \widetilde{\chi})$. By Lemma 5.4, $\chi = \widetilde{\chi}$, hence $\psi = \widetilde{\psi}$, contradiction. □

References

1. M. Aschenbrenner, L. van den Dries, *Closed asymptotic couples,* J. Algebra **225** (2000), 309–358.
2. ______, *H-fields and their Liouville extensions,* Math. Z. **242** (2002), 543–588.
3. N. Bourbaki, *Fonctions d'une Variable Réelle,* Chapitre V, Appendice *Corps de Hardy. Fonctions (H),* Hermann, Paris (1976).
4. L. van den Dries, *Tame Topology and O-Minimal Structures,* London Mathematical Society Lecture Note Series **248**, Cambridge University Press, Cambridge (1998).
5. L. van den Dries, D. Marker, A. Macintyre, *Logarithmic-exponential series,* Ann. Pure Appl. Logic **111** (2001), 61–113.
6. F.-V. Kuhlmann, *Abelian groups with contractions,* I, Contemp. Math. **171** (1994), 217–241.
7. ______, *Abelian groups with contractions,* II: *Weak o-minimality,* in: A. Facchini, C. Menini (eds.): *Abelian Groups and Modules,* Kluwer, Dordrecht (1995), 323–342.
8. S. Kuhlmann, *Ordered Exponential Fields,* Fields Institute Monographs **12**, American Mathematical Society, Providence, RI (2000).
9. M. C. Laskowski, *Vapnik-Chernovenkis classes of definable sets,* J. London Math. Soc. (2) **45** (1992), 377–384.
10. D. Macpherson, D. Marker, C. Steinhorn, *Weakly o-minimal structures and real closed fields,* Trans. Amer. Math. Soc. **352** (2000), 5435–5483 (electronic).
11. B. Poizat, *Théories instables,* J. Symbolic Logic **46** (1981), 513–522.
12. ______, *Cours de Théorie des Modèles,* Nur al-Mantiq wal-Ma'rifah, Lyon (1985).
13. M. Rosenlicht, *The nonminimality of the differential closure,* Pacific J. Math. **52** (1974), 529–537.
14. ______, *On the value group of a differential valuation,* Amer. J. Math. **101** (1979), 258–266.
15. ______, *Differential valuations,* Pacific J. Math. **86** (1980), 301–319.
16. ______, *On the value group of a differential valuation,* II, Amer. J. Math. **103** (1981), 977–996.
17. ______, *Hardy fields,* J. Math. Analysis and Appl. **93** (1983), 297–311.
18. ______, *The rank of a Hardy field,* Trans. Amer. Math. Soc. **280** (1983), 659–671.