SHORT HARDY FIELDS

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Abstract. Differentially algebraic Hardy field extensions of short Hardy fields are short. This is proved in the more general setting of H-fields. As an application we extend a theorem of Rosenlicht (1981) by showing that each short asymptotic couple of Hardy type with small derivation is isomorphic to the asymptotic couple of an analytic Hardy field.

Introduction

An ordered set—here and below "ordered" means "totally ordered"— is said to be *short* if each ordered subset of it has countable cofinality and countable coinitiality. *Example*: the real line. Shortness is a rather robust property, and [3, Section 5] considers this property for Hardy fields (which are naturally ordered fields) and the ordered differential field $\mathbb{T}$ of transseries. In fact, $\mathbb{T}$ is short [3, Corollary 5.20]. Left open in [3] is whether every differentially algebraic Hardy field extension of a short Hardy field is short. Here we give an affirmative answer. We actually prove something more general for H-fields.

An *H-field* is by definition an ordered field H with a derivation $h \mapsto h'$ on it that interacts with the ordering as follows: for the constant field C of H and the convex subring $\mathcal{O} := \big\{h \in H : |h| \leqslant c \text{ for some } c \in C\big\}$ of H we have:

(H1) for all $h \in H$, if $h > C$, then $h' > 0$;

(H2) $\mathcal{O} = C + \smallo$, where $\smallo$ is the maximal ideal of the valuation ring $\mathcal{O}$ of H.

Hardy fields that contain $\mathbb{R}$ as a subfield are H-fields with constant field $\mathbb{R}$, as is $\mathbb{T}$.

Theorem A. *If E is a differentially algebraic H-field extension of a short H-field and the constant field of E is short, then E is short.*

This answers the above question from [3] for Hardy fields containing $\mathbb{R}$. Let H be any short Hardy field and E a differentially algebraic Hardy field extension of H. Then the Hardy field $H(\mathbb{R})$ is short, by Lemma 3.2 below, and $E(\mathbb{R})$ is a differentially algebraic Hardy field extension of $H(\mathbb{R})$, so $E(\mathbb{R})$ and thus E are short. This answers the question for all Hardy fields.

We use Theorem A to realize short H-fields and short asymptotic couples in the realm of Hardy fields. Some terminology: A Hardy field is said to be *analytic* if each element of it has an analytic representative $(a, +\infty) \to \mathbb{R}$ ($a \in \mathbb{R}$). An analytic Hardy field not contained in any strictly larger analytic Hardy field is called *maximal.* (By Zorn, each analytic Hardy field extends to a maximal one. Every maximal analytic Hardy field contains $\mathbb{R}$.) A valued differential field K is said to have *small derivation* if $\smallo' \subseteq \smallo$, where $\smallo$ is the maximal ideal of the valuation ring of K. Hardy fields have small derivation, as has $\mathbb{T}$. By [3, Corollary 7.9], $\mathbb{T}$ is isomorphic to an analytic Hardy field. We improve this as follows:

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Corollary B. *Every short H-field with small derivation and archimedean constant field embeds into every maximal analytic Hardy field.*

In statements like these, embeddings (and isomorphisms as a special case) are valued field embeddings that respect the ordering and the derivation. As an application of Corollary B and its proof we show that the H-field $\mathbf{No}(\omega_1)$ with the derivation from [9] embeds into every maximal analytic Hardy field.

Let H be an H-field, and let $h \mapsto vh\colon H^\times \to \Gamma = v(H^\times)$ be the valuation on H with valuation ring $\mathcal{O}$ as above. Then the logarithmic derivative map

$$h \mapsto h^\dagger := h'/h \;:\; H^\times \to H$$

descends to Γ: there is a map $\psi\colon \Gamma^{\neq} = \Gamma \setminus \{0\} \to \Gamma$ such that $\psi(vh) = v(h^\dagger)$ for all $h \in H^\times$ with $vh \neq 0$, and such that for all $\alpha, \beta \in \Gamma^{\neq}$:

(A1) $\alpha + \beta \neq 0 \Rightarrow \psi(\alpha+\beta) \geqslant \min\big(\psi(\alpha), \psi(\beta)\big)$;
(A2) $\psi(k\alpha) = \psi(\alpha)$ for $k \in \mathbb{Z} \setminus \{0\}$;
(A3) $\alpha > 0 \Rightarrow \alpha + \psi(\alpha) > \psi(\beta)$.

The pair (Γ, ψ) is called the *asymptotic couple* of H. Any pair (Γ, ψ) where Γ is an ordered abelian group and $\psi\colon \Gamma^{\neq} \to \Gamma$ satisfies (A1)–(A3) for all $\alpha, \beta \in \Gamma^{\neq}$ is called an *asymptotic couple.* Such an asymptotic couple (Γ, ψ) is said to be *of Hardy type* if for all $\alpha, \beta \in \Gamma^{\neq}$: $\psi(\alpha) > \psi(\beta)$ iff $n|\alpha| < |\beta|$ for all n, to have *small derivation* if for all $\alpha > 0$ in Γ we have $\alpha + \psi(\alpha) > 0$, and to be *short* if the ordered set Γ is short. The asymptotic couple of any Hardy field is of Hardy type with small derivation (as is that of $\mathbb{T}$). Rosenlicht [13, Theorem 3 and Remark 3 following it] showed conversely that every asymptotic couple (Γ, ψ) of Hardy type with small derivation and Γ of finite archimedean rank is isomorphic to the asymptotic couple of an analytic Hardy field containing $\mathbb{R}$. Using Corollary B we generalize this result:

Corollary C. *Let M be a maximal analytic Hardy field. Then any short asymptotic couple of Hardy type with small derivation is isomorphic to the asymptotic couple of a spherically complete Hardy subfield of M containing $\mathbb{R}$.*

In the rest of this paper we freely use notation and terminology from [ADH] (and its results!). For a summary of relevant material from [ADH], see *Concepts and Results from* [ADH] in the introduction to [7][1].We also refer to various basic facts on shortness from [3]. As in [3] we call an H-field *closed* if it is ω-free, Liouville closed, and newtonian.

Organization of the paper. Section 1 contains preliminary observations about short ordered fields. The key point in the proof of Theorem A is the discussion in Section 2 to the effect that a certain construction $H \mapsto H^*$ for real closed ω-free H-fields H preserves shortness. This is combined with a more economical way of generating the Newton-Liouville closure of an ω-free H-field than in [ADH]. We also need some more routine lemmas to reduce to the case of a real closed ω-free H-field. These lemmas are in Section 3, where we complete the proof of Theorem A and obtain Corollary B. In Section 4 we prove Corollary C; this requires Corollary B and an extension of a construction from [2, Section 11].

1. Short Ordered Fields

This section we make two observations on short ordered fields, Lemmas 1.3 and 1.4, to be used in the next two sections. First a reminder about composing valuations.

Composing valuations. Let K be a field and $\mathcal{O}_K$ be a valuation ring of K, and let $\pi_K\colon \mathcal{O}_K \to R$ be the residue map onto its residue field R. Let also $\mathcal{O}_R$ be a valuation ring of R. Then $\mathcal{O} := \pi_K^{-1}(\mathcal{O}_R)$ is a subring of $\mathcal{O}_K$.

Claim: *$\mathcal{O}$ is a valuation ring of K.*

First, if $a \in K$ and $a \notin \mathcal{O}_K$, then a^{-1} is in the maximal ideal of $\mathcal{O}_K$, which is the kernel of π_K, and thus $a^{-1} \in \mathcal{O}$. Next, let $a \in \mathcal{O}_K$ and $a \notin \mathcal{O}$. Then a does not lie in the maximal ideal of $\mathcal{O}_K$, so $a^{-1} \in \mathcal{O}_K$. Also $\pi_K(a) \notin \mathcal{O}_R$, so $\pi_K(a)^{-1} = \pi_K(a^{-1}) \in \mathcal{O}_R$, and thus $a^{-1} \in \mathcal{O}$. A useful consequence of the claim is that the maximal ideal of $\mathcal{O}_K$ is a prime ideal of $\mathcal{O}$.

Let $\pi_R\colon \mathcal{O}_R \to \boldsymbol{k}$ be the residue morphism onto the residue field $\boldsymbol{k}$ of $\mathcal{O}_R$, and

$$\pi \ := \ \pi_R \circ (\pi_K|_{\mathcal{O}}) \ : \ \mathcal{O} \to \mathcal{O}_R \to \boldsymbol{k}.$$

We identify the surjective ring morphism $\pi\colon \mathcal{O} \to \boldsymbol{k}$ with the residue map of $\mathcal{O}$ onto its residue field. (The place π is said to be the *composition of the places π_K and π_R.*) Let now $v\colon K^\times \to \Gamma$ be a valuation on K with valuation ring $\mathcal{O}$, let $v_K\colon K^\times \to \Gamma_K$ be a valuation on K with valuation ring $\mathcal{O}_K$, and $v_R\colon R^\times \to \Gamma_R$ a valuation on R with valuation ring $\mathcal{O}_R$.

It is routine to check that we have an order preserving group embedding $\Gamma_R \to \Gamma$ sending $v_R(\pi_K(a))$ to $v(a)$ for $a \in \mathcal{O}_K$ with $\pi_K(a) \neq 0$. We identify Γ_R with its image in Γ via this embedding. The surjective group morphism $\Gamma \to \Gamma_K$ sending $v(a)$ to $v_K(a)$ for $a \in K^\times$ is order preserving with kernel Γ_R. It follows that Γ_R is a convex subgroup of Γ and $\Gamma/\Gamma_R \cong \Gamma_K$ as ordered abelian groups.

Observations on short ordered fields. From [3, Lemma 5.17] we recall that an ordered abelian group Γ is short iff its ordered set $[\Gamma]$ of archimedean classes is short. From [3, Corollary 5.18] we also quote a basic fact about the preservation of shortness under extensions of ordered abelian groups:

Lemma 1.1. *Let $\Delta \subseteq \Gamma$ be an extension of ordered abelian groups.*
(i) *If $\operatorname{rank}_{\mathbb{Q}}(\Gamma/\Delta) \leqslant \aleph_0$, then Γ is short iff Δ is short;*
(ii) *if Δ is convex, then Γ is short iff Δ and Γ/Δ are short.*

The following is [3, Lemma 5.19]:

Lemma 1.2. *Let K be an ordered field equipped with a convex valuation whose ordered residue field is archimedean. Then K is short iff its value group is short.*

The next result extends this to nonarchimedean residue fields.

Lemma 1.3. *Let K be an ordered field equipped with a convex valuation. Then K is short iff its ordered residue field R and value group Γ_K are short.*

Proof. Suppose first that R and Γ_K are short. Let $v_K\colon K^\times \to \Gamma_K$ be the given convex valuation on K, with valuation ring $\mathcal{O}_K$ and residue map $\pi\colon \mathcal{O}_K \to R$. Set $\mathcal{O}_R := \{x \in R : |x| \leqslant n \text{ for some } n\}$, the smallest convex subring of the ordered field R. Then $\mathcal{O}_R$ has archimedean ordered residue field $\boldsymbol{k}$. Let $v_R\colon R^\times \to \Gamma_R$ be a valuation on R with valuation ring $\mathcal{O}_R$.

The subring $\mathcal{O} := \pi^{-1}(\mathcal{O}_R)$ of $\mathcal{O}_K$ is convex. Let $v\colon K^\times \to \Gamma$ be a convex valuation on K with $\mathcal{O}$ as valuation ring. The considerations in the previous subsection show that we may consider $\boldsymbol{k}$ as the ordered residue field of $\mathcal{O}$, and Γ_R as a convex subgroup of Γ with $\Gamma/\Gamma_R \cong \Gamma_K$, as ordered abelian groups.

Now R is short, hence so are Γ_R and $\boldsymbol{k}$ by Lemma 1.2. Thus Γ is short by Lemma 1.1(ii) and so is K, using the (convex) valuation v on K and again Lemma 1.2.

This shows the "if" direction. For the converse, suppose K is short. Then the valuation ring of K and its maximal ideal are short convex ordered additive subgroups of K, so R is short by Lemma 1.1(ii). The ordered subset $K^{>}$ of K is also short, hence so is its image Γ_K under the decreasing surjection v_K. □

Lemma 1.4. *Let K be a short ordered field and L an ordered field extension of countable transcendence degree over K. Then L is short.*

Proof. We equip K and L with their standard convex valuation whose ordered residue fields are archimedean. Then the value group of L has countable rational rank over the value group of K, and the latter being short, so is the former by Lemma 1.1(i). Hence L is short by Lemma 1.2. □

2. Alternative Construction of Newton-Liouville Closures

In this section H is an H-field with asymptotic couple (Γ, ψ). For $s \in H$ we set

$$\Gamma_s \ := \ \{v(s - h^\dagger) : h \in H^\times\} \ \subseteq \ \Gamma_\infty.$$

In particular, $\infty \in \Gamma_s$ iff $s \in H^\dagger := \{h^\dagger : \ h \in H^\times\}$.

Lemma 2.1. *The following are equivalent:*

(i) *H is closed;*
(ii) *H is ω-free, real closed, newtonian, and there is no $s \in H$ with $\Gamma_s \subseteq \Psi^\downarrow$.*

Proof. If H is closed, then $H^\dagger = H$, so (ii) holds. Now assume (ii). To derive (i), it suffices to show that H is Liouville closed. Now H is newtonian, so is closed under integration. Let $s \in H$; it is enough to show that then $s \in H^\dagger$. Now $\Gamma_s \not\subseteq \Psi^\downarrow$, so we have $h \in H^\times$ with $s - h^\dagger \in \mathrm{I}(H)$, so $s - h^\dagger \in (1 + \smallo)^\dagger$ by [ADH, 14.2.5], and thus $s \in H^\dagger$. □

Next a part of [ADH, 10.5.20] (replacing s, f there by $-s$, f^{-1} if necessary):

Lemma 2.2. *Suppose $\Gamma \neq \{0\}$, H is real closed, $s \in H$, and $\Gamma_s \subseteq \Psi^\downarrow$. Then there exists an f in an H-field extension of H such that:*

(i) *f is transcendental over H and $f^\dagger = s$;*
(ii) *the pre-H-field extension $H(f)$ of H is an H-field with the same constant field as H; and*
(iii) *for the asymptotic couple (Γ_f, ψ_f) of $H(f)$, viewed as an extension of (Γ, ψ), we have $vf \in \Gamma_f \setminus \Gamma$, $\Gamma_f = \Gamma \oplus \mathbb{Z}vf$, with Ψ_f cofinal in Ψ.*

Suppose H is real closed with $\Gamma \neq \{0\}$. Transfinitely iterating the extension procedure of the lemma above, alternating it with taking real closures, and taking unions at limit stages, we obtain a real closed d-algebraic H-field extension H^* of H with asymptotic couple (Γ^*, ψ^*) of H^* extending (Γ, ψ), such that:

(1) H^* has the same constant field as H, and Ψ is cofinal in Ψ^*;
(2) for all $s \in H$ we have $\Gamma^*_s \not\subseteq \Psi^{*\downarrow}$; and
(3) $\Gamma^* = \Gamma \oplus \bigoplus_{i \in I} \mathbb{Q}vf_i$ (internal direct sum) where (f_i) is a family of nonzero elements of H^* with $f_i^\dagger \in H$ for all $i \in I$.

Given any archimedean class $[\gamma^*]$ with $0 \neq \gamma^* \in \Gamma^*$ we choose $a \in H^\times$, distinct $i_1, \dots, i_n \in I$, and $k_1, \dots, k_n \in \mathbb{Z}^{\neq}$, such that

$$af_{i_1}^{k_1} \cdots f_{i_n}^{k_n} \succ 1, \qquad v\big(af_{i_1}^{k_1} \cdots f_{i_n}^{k_n}\big) \in [\gamma^*].$$

Associating to $[\gamma^*]$ the element $(af_{i_1}^{k_1} \cdots f_{i_n}^{k_n})^\dagger = a^\dagger + k_1 f_{i_1}^\dagger + \cdots + k_n f_{i_n}^\dagger$ of $H^>$, we obtain a strictly increasing map $[(\Gamma^*)^{\neq}] \to H^>$, by [ADH, 10.5.2]. Recall that the ordered residue field of H is isomorphic to the ordered constant field of H. It follows that if H is short, then so are the ordered residue field of H and the ordered set $[\Gamma^*]$, hence Γ^* is short as well by the remark before Lemma 1.1. Thus if H is short, then so is H^* by Lemma 1.3.

The process leading from H to H^* can now be applied to H^* instead of H, and iterating this process ω times, and taking a union gives us a real closed H-field extension $H^\#$ of H with asymptotic couple $\big(\Gamma^\#, \psi^\#\big)$ extending (Γ, ψ) such that:

(4) $H^\#$ has the same constant field as H, and Ψ is cofinal in $\Psi^\#$;
(5) there is no $s \in H^\#$ with $\Gamma^\#_s \subseteq \Psi^{\#\downarrow}$; and
(6) if H is short, then so is $H^\#$.

Let now H be ω-free and real closed. We build a sequence (H_n) of ω-free real closed d-algebraic H-field extensions of H, with H_{n+1} extending H_n for all n:

- $H_0 := H$,
- for even n, $H_{n+1} := H_n^\#$,
- for odd n, H_{n+1} is a maximal immediate d-algebraic H-field extension of H_n (so H_{n+1} is newtonian by [ADH, 10.5.8, 14.0.1]).

Then $H_\infty := \bigcup_n H_n$ is closed by Lemma 2.1, in view of (5) above. Since H_∞ is also d-algebraic over H and has the same constant field as H, it follows that H_∞ is a Newton-Liouville closure of H, by [ADH, pp. 669, 685].

Now shortness is inherited by immediate extensions of H-fields, so if H is short, then so is H_∞ in view of (6) above. This leads to:

Corollary 2.3. *Suppose H is ω-free, real closed, and short, and E is a differentially algebraic H-field extension of H with the same constant field as H. Then E is short.*

Proof. Take a Newton-Liouville closure of E. This is also a Newton-Liouville closure of H and isomorphic to the H_∞ above, by [ADH, 16.2.1]. So this Newton-Liouville closure of E is short, and thus E is short. □

3. Proof of the Main Theorem

We still need a few generalities about preserving shortness:

From pre-H-fields to ω-free H-fields. By [ADH, p. 445], a pre-H-field K has a "smallest" H-field extension, the *H-field hull* $\mathrm{H}(K)$ of K, whose underlying valued differential field is the pre-d-valued hull $\mathrm{dv}(K)$ of K.

Lemma 3.1. *Let K be a short pre-H-field. Then* $\mathrm{H}(K)$ *is short.*

Proof. By [ADH, 10.3.2] and Lemma 1.1(i), the value group of $\mathrm{H}(K)$ is short, and by [ADH, remarks preceding 10.3.2] its ordered residue field equals that of K, and so is short as well. Hence $\mathrm{H}(K)$ is short by Lemma 1.3. □

The next lemma has almost the same proof as [3, Corollary 2.18]. It is used for Hardy fields that do not contain $\mathbb{R}$ and might therefore not be H-fields.

Lemma 3.2. *If the Hardy field H is short, then so is the Hardy field $H(\mathbb{R})$.*

Proof. This is clear if $H \subseteq \mathbb{R}$. Assume H is a short Hardy field and $H \not\subseteq \mathbb{R}$. Let E be the H-field hull of H, taken as an H-subfield of the Hardy field extension $H(\mathbb{R})$ of H. Then E is short by Lemma 3.1. Now use that $H(\mathbb{R}) = E(\mathbb{R})$ and $\Gamma_{E(\mathbb{R})} = \Gamma_E \neq \{0\}$ by [ADH, 10.5.15 and remark preceding 4.6.16]. □

Elaborating the proof of [ADH, 11.5.15] we now show:

Lemma 3.3. *Let K be a short H-field and L a Schwarz closed H-field extension of K. Then there exists a short ω-free H-subfield of L containing K.*

Proof. If K is grounded, this follows from [ADH, 11.7.17] and Lemma 1.4. Suppose K is ungrounded. Then K has a gap or has asymptotic integration by [ADH, 9.2.16]. If K has a gap vs with $s \in K$, then we take $y \in L$ with $y \not\asymp 1$ and $y' = s$, so that by [ADH, 10.2.1, 10.2.2 and subsequent remarks], $K(y)$ is a grounded H-subfield of L, and we are back to the grounded case. In general we arrange, by passing to the real closure of K in L, that the value group Γ_K of the valued field K is divisible. For such K it remains to consider the case that K has asymptotic integration (and thus rational asymptotic integration) and is not ω-free. Assume we are in this case. We distinguish two subcases:

Subcase 1: *K is not λ-free.*

Then [ADH, 11.5.14] yields $s \in K$ creating a gap over K, with $S := \big\{v(s - a^\dagger) : a \in K^\times\big\}$ a cofinal subset of $\Psi_K^\downarrow$ and $s \neq 0$ by [ADH, 11.5.13]. Taking $f \in L^\times$ with $f^\dagger = s$, $K(f)$ has a gap by [ADH, remark preceding 11.5.15]. Moreover, $K(f)$ is an H-subfield of L by [ADH, 10.4.5(iv)], so $K(f)$ falls under the "gap" case treated earlier.

Subcase 2: *K is λ-free.*

As in [ADH, 11.5, 11.7] we have the pc-sequence (ω_ρ) in K. Now K is not ω-free, so we can take $\omega \in K$ such that $\omega_\rho \rightsquigarrow \omega$. The Schwarz closed H-field L has a unique expansion to a $\Lambda\Omega$-field $\boldsymbol{L}$, and we let $\boldsymbol{K} = (K, I, \Lambda, \Omega)$ be the $\Lambda\Omega$-field expansion of K such that $\boldsymbol{K} \subseteq \boldsymbol{L}$. We are now in the situation of [ADH, 16.4.6], whose proof gives two possibilities:

(a): $\Omega = \omega(K)^\downarrow$. As in the proof of that lemma for Case 1 this yields a pre-H-field extension $K\langle\gamma\rangle$ of K with a gap that embeds over K into L. The residue field of $K\langle\gamma\rangle$ equals that of K, and so $K\langle\gamma\rangle$ is an H-field by [ADH, 9.1.2]. Moreover, $K\langle\gamma\rangle$ is short by Lemma 1.4, so $K\langle\gamma\rangle$ falls under the "gap" case treated earlier.

(b): $\Omega = K \setminus \sigma\big(\Gamma(K)\big)^\uparrow$. As in Case 2 in the proof of [ADH, 16.4.6] this yields an immediate pre-H-field extension $K(\lambda)$ of K that is not λ-free and that embeds over K into L. Then $K(\lambda)$ is an H-field by [ADH, 9.1.2] and is short, so $K(\lambda)$ falls under Subcase 1. □

Finishing the proof. The following is a bit more general than Theorem A:

Theorem 3.4. *Let H be a short pre-H-field and E a pre-H-field extension of H and* d*-algebraic over H with short ordered residue field. Then E is short.*

Proof. Expand E to a pre-$\Lambda\Omega$-field $\boldsymbol{E}$. The proof of [ADH, 16.4.9] yields a Newton-Liouville closure $\boldsymbol{L} = (L, \dots)$ of $\boldsymbol{E}$ such that L is d-algebraic over E and thus over H, and the ordered residue field of L is a real closure of the ordered residue field of E. In particular, the ordered constant field D of L, being isomorphic to the ordered residue field of L, is short. It is enough to prove that L is short. Let K be the H-field hull of H in L. Then K is short by Lemma 3.1, and so Lemma 3.3 yields a short $\upomega$-free H-subfield $F \supseteq K$ of L. Now $F(D)$ is an H-subfield of L and has the same (short) value group as F, by [ADH, 10.5.15], and short constant field D. So $F(D)$ is short by Lemma 1.3. Hence the real closure of $F(D)$ in L is short and $\upomega$-free. Thus L is short by Corollary 2.3. □

Embeddings into closed η_1-ordered H-fields. Maximal Hardy fields, maximal smooth Hardy fields, and maximal analytic Hardy fields are closed H-fields, and are η_1-ordered by [6, Theorem A] and [3, Theorem A and subsequent remark]. More generally, *in this subsection L is a closed η_1-ordered H-field.* Using Theorem A from the introduction we generalize [3, Proposition 7.6]:

Proposition 3.5. *Let E be an $\upomega$-free pre-H-field and K be a short pre-H-field extending E such that* $\operatorname{res}(E) = \operatorname{res}(K)$. *Then any embedding $E \to L$ extends to an embedding $K \to L$.*

Proof. By Lemma 3.1 and [3, remark after Lemma 5.19], the real closure $\mathrm{H}(K)^{\mathrm{rc}}$ of the H-field hull of K is short. Moreover, the H-field $\mathrm{H}(E)^{\mathrm{rc}}$ is $\upomega$-free by [ADH, 13.6.1], and each embedding $E \to L$ extends to an embedding $\mathrm{H}(E)^{\mathrm{rc}} \to L$. The ordered residue field of $\mathrm{H}(E)^{\mathrm{rc}}$ and of $\mathrm{H}(K)^{\mathrm{rc}}$ is the real closure of $\operatorname{res}(E) = \operatorname{res}(K)$, cf. [ADH, remark before 10.3.2]. Hence replacing K by $\mathrm{H}(K)^{\mathrm{rc}}$ and then E by $\mathrm{H}(E)^{\mathrm{rc}}$, taken inside $\mathrm{H}(K)^{\mathrm{rc}}$, we arrange E, K to be H-fields with real closed constant fields $C_E = C_K$.

Next expand K to a $\Lambda\Omega$-field $\boldsymbol{K}$, and let $\boldsymbol{M} = (M, \dots)$ be a Newton-Liouville closure of $\boldsymbol{K}$ such that M is d-algebraic over K and $C_M = C_K$. (See the proof of Theorem 3.4.) Then M is closed, and short by Theorem A, so by [3, Proposition 7.6] any embedding $E \to L$ extends to an embedding $M \to L$. □

Next a generalization of [3, Lemma 7.8], where we recall that a valued differential field is said to have *very small derivation* if $\mathcal{O}' \subseteq o$ (with $\mathcal{O}$ and o as usual).

Lemma 3.6. *Suppose L has small derivation and $C_L = \mathbb{R}$. Then any short pre-H-field with very small derivation and archimedean residue field embeds into L.*

Proof. Let E be a short pre-H-field with very small derivation and archimedean residue field. To embed E into L we pass to $\mathrm{H}(E)^{\mathrm{rc}}$ to arrange that E is a real closed H-field with small derivation. Now Theorem A yields a closed short H-field extension K of E with $C_E = C_K$. Then [3, Lemma 7.8] yields an embedding of K (and thus of E) into L. □

By [8, Proposition 13.11] the universal part of the theory of Hardy fields, viewed as structures in the language (specified there) of ordered valued differential fields, is the theory of pre-H-fields with very small derivation. The following complements this result and includes Corollary B from the introduction:

Corollary 3.7. *Any short pre-H-field with very small derivation and archimedean residue field embeds into every maximal Hardy field. Likewise with "maximal" replaced by "maximal smooth", respectively "maximal analytic".*

This is a consequence of Lemma 3.6. We finish this section with an application to the H-field $\mathbf{No}(\omega_1)$ of surreal numbers of countable length equipped with the

Berarducci-Mantova derivation [9]. (Note, however, that the Continuum Hypothesis gives *isomorphism* results [3, 6] stronger than Corollary 3.8.)

Corollary 3.8. *Let M be a maximal Hardy field. Then $\mathbf{No}(\omega_1)$ embeds into M. Likewise with "maximal smooth" and "maximal analytic" in place of "maximal".*

Proof. The H-field $\mathbf{No}(\omega_1)$ is exhibited in [4, Remark after Corollary 4.5] as the increasing union of grounded and short H-subfields K_ε, with ε ranging over the countable ε-numbers. Let α, β range over countable infinite limit ordinals, and set

$$E_\alpha \;:=\; \bigcup_{\varepsilon<\varepsilon_\alpha} K_\varepsilon.$$

Then E_α is a short H-subfield of $\mathbf{No}(\omega_1)$ with constant field $\mathbb{R}$, and is ω-free by [ADH, 11.7.15]. Thus E_ω embeds into M by Corollary 3.7. Transfinite recursion on α using Proposition 3.5 then yields for all α an embedding $h_\alpha\colon E_\alpha \to M$ such that $h_\alpha = h_\beta|_{E_\alpha}$ for $\alpha \leqslant \beta$. It follows that the common extension of the h_α embeds $\mathbf{No}(\omega_1)$ into M as required. □

4. Constructing Hardy Fields with Given Asymptotic Couple

In this section $\boldsymbol{k}$ is an ordered field, and we consider H-couples (Γ,ψ) over $\boldsymbol{k}$ as defined in [5]. We revisit a construction from [2, Section 11][2] which associates to each H-couple (Γ,ψ) of Hahn type over $\boldsymbol{k}$ satisfying (+) a spherically complete H-field with constant field $\boldsymbol{k}$ and H-couple (Γ,ψ) over $\boldsymbol{k}$, and closed under powers. Here (+) is the condition that $\psi(\gamma)=\gamma$ for some $\gamma>0$ in Γ, and that Γ admits a valuation basis for its $\boldsymbol{k}$-valuation. In the first subsection we carry out this construction without assuming (+): Corollary 4.7. This leads to Corollary C from the introduction. The construction involves equipping suitable Hahn fields with the "right" derivation. Other explorations of derivations on Hahn fields are in [12, 14].

Generalizing [2, Section 11]**.** Let Γ be an ordered vector space over $\boldsymbol{k}$ and suppose it is a spherically complete Hahn space over $\boldsymbol{k}$; see [ADH, 2.4] for the definitions. Using [ADH, 2.3.2, 2.4.23] we identify Γ with the Hahn product $H[I,\boldsymbol{k}]$ where $I := \big([\Gamma^{\neq}]_{\boldsymbol{k}}$ with reversed ordering$\big)$, and we let i, j range over I. The elements of Γ are thus the $\gamma=(\gamma_i)\in\boldsymbol{k}^I$ whose support $\operatorname{supp}\gamma=\{i:\gamma_i\neq 0\}$ is a well-ordered subset of I. Let $e_i=(e_{ij})\in\Gamma$ be given by $e_{ij}=0$ if $i\neq j$ and $e_{ii}=1$. Then $e_i>0$ for each i, and the map $i\mapsto[e_i]_{\boldsymbol{k}}\colon I\to[\Gamma^{\neq}]_{\boldsymbol{k}}$ is decreasing and bijective (so $\{e_i : i\in I\}$ is valuation-independent with respect to the $\boldsymbol{k}$-valuation on the ordered $\boldsymbol{k}$-vector space Γ). We think of each $\gamma=(\gamma_i)\in\Gamma$ as an infinite sum

$$\gamma \;=\; \sum_i \gamma_i e_i.$$

Let α, β, γ range over Γ. We say that e_i **occurs in** γ if $i\in\operatorname{supp}\gamma$. Thus the set of "Hahn basis elements" e_i that occur in γ is reverse well-ordered, and for $\gamma\neq 0$ and $i_0=\min\operatorname{supp}\gamma$ (so e_{i_0} is the largest Hahn basis element occurring in γ), we have $[\gamma]_{\boldsymbol{k}}=[e_{i_0}]_{\boldsymbol{k}}$, and the equivalence $\gamma>0\Leftrightarrow\gamma_{i_0}>0$. Note: $i=[e_i]_{\boldsymbol{k}}$.

Let also $\psi\colon\Gamma^{\neq}\to\Gamma$ be such that (Γ,ψ) is an H-couple over $\boldsymbol{k}$, and assume it is of Hahn type, that is, for all $\alpha,\beta\neq 0$ with $\psi(\alpha)=\psi(\beta)$ there is a $c\in\boldsymbol{k}^\times$ with $\psi(\alpha-c\beta)>\psi(\alpha)$; see [5] or [6, Section 8], also for the fact that then

$$\psi(\alpha)\leqslant\psi(\beta) \iff [\alpha]_{\boldsymbol{k}}\geqslant[\beta]_{\boldsymbol{k}} \qquad (\alpha,\beta\neq 0),$$

and thus $i < j \Leftrightarrow \psi(e_i) < \psi(e_j)$. Let t^Γ be a multiplicative copy of the (additive) ordered abelian group Γ, ordered such that $\gamma \mapsto t^\gamma \colon \Gamma \to t^\Gamma$ is an order-reversing isomorphism, and consider the valued ordered Hahn field $K := \boldsymbol{k}(\!(t^\Gamma)\!)$ over $\boldsymbol{k}$, cf. [ADH, 3.1, 3.5]. Its elements are the formal series

$$f = \sum_\gamma f_\gamma t^\gamma \qquad (f_\gamma \in \boldsymbol{k})$$

whose support $\operatorname{supp} f = \{\gamma : f_\gamma \neq 0\}$ is a well-ordered subset of Γ. Recall that a family $(f_\lambda)_{\lambda\in\Lambda}$ of elements of K is said to be *summable* if $\bigcup_\lambda \operatorname{supp} f_\lambda$ is well-ordered and for each γ there are only finitely many $\lambda \in \Lambda$ such that $f_{\lambda,\gamma} \neq 0$; in this case we define $\sum_\lambda f_\lambda$ to be the series $f \in K$ with $f_\gamma = \sum_\lambda f_{\lambda,\gamma}$ for each γ [ADH, p. 712]. Also recall from [ADH, p. 713] that a map $\Phi\colon K \to K$ is said to be *strongly additive* if for every summable family (f_λ) in K the family $\big(\Phi(f_\lambda)\big)$ is summable and

$$\Phi\left(\sum_\lambda f_\lambda\right) \ = \ \sum_\lambda \Phi(f_\lambda).$$

Lemma 4.1. *Let $S \subseteq \Gamma$ be well-ordered. Then*

(i) *for each γ there are only finitely many $\alpha \in S$ such that $\gamma = \alpha + \psi(e_i)$ for some e_i occurring in α;*

(ii) *the set of all $\alpha + \psi(e_i)$ with $\alpha \in S$ and e_i occurring in α is well-ordered.*

Proof. For (i), suppose $\gamma = \alpha+\psi(e_i) = \beta+\psi(e_j)$ for elements $\alpha < \beta$ in S, with e_i, e_j occurring in α, β, respectively. Then $\psi(e_i) - \psi(e_j) = \beta - \alpha > 0$, so $[e_i]_{\boldsymbol{k}} < [e_j]_{\boldsymbol{k}}$ and $[\beta-\alpha]_{\boldsymbol{k}} = \big[\psi(e_i)-\psi(e_j)\big]_{\boldsymbol{k}} < [e_i - e_j]_{\boldsymbol{k}} = [e_j]_{\boldsymbol{k}}$. Hence e_j occurs in α. Thus if we have a strictly increasing sequence (α_n) in S and a sequence (i_n) in I such that e_{i_n} occurs in α_n and $\alpha_n + \psi(e_{i_n}) = \alpha_{n+1} + \psi(e_{i_{n+1}})$, for all n, then all e_{i_n} occur in α_0 and (e_{i_n}) is strictly increasing, contradicting that the set of e_i occurring in α_0 is reverse well-ordered.

For (ii), suppose towards a contradiction that (i_n) is a sequence in I and (α_n) is a sequence in S such that e_{i_n} occurs in α_n and $\alpha_n + \psi(e_{i_n}) > \alpha_{n+1} + \psi(e_{i_{n+1}})$ for all n. Passing to a subsequence and using the well-orderedness of S we arrange that $\alpha_n \leqslant \alpha_{n+1}$ for all n. Then $0 \leqslant \alpha_{n+1} - \alpha_0 < \psi(e_{i_0}) - \psi(e_{i_{n+1}})$, so

$$[\alpha_{n+1} - \alpha_0]_{\boldsymbol{k}} \ \leqslant \ \big[\psi(e_{i_0}) - \psi(e_{i_{n+1}})\big]_{\boldsymbol{k}} \ < \ [e_{i_0} - e_{i_{n+1}}]_{\boldsymbol{k}} \ = \ [e_{i_{n+1}}]_{\boldsymbol{k}}.$$

Thus all e_{i_n} occur in α_0. Also $\psi(e_{i_{n+1}}) < \psi(e_{i_n}) + (\alpha_n - \alpha_{n+1}) \leqslant \psi(e_{i_n})$ and so $e_{i_{n+1}} > e_{i_n}$ for all n, contradicting that the set of e_i occurring in α_0 is reverse well-ordered. □

For $\alpha = \sum_i \alpha_i e_i$ ($\alpha_i \in \boldsymbol{k}$) the set of e_i with $\alpha_i \neq 0$ is reverse well-ordered, so the subset $\big\{\alpha + \psi(e_i) : e_i \text{ occurs in } \alpha\big\}$ of Γ is well-ordered and is the support of $(t^\alpha)' := -\sum_i \alpha_i\, t^{\alpha+\psi(e_i)} \in K$. Let $f = \sum_\alpha f_\alpha t^\alpha$ range over K. Then the family $\big(f_\alpha (t^\alpha)'\big)$ is summable by Lemma 4.1, and we put

$$f' \ := \ \sum_\alpha f_\alpha (t^\alpha)' \in K \quad (\text{so } f' = 0 \text{ for } f \in \boldsymbol{k}).$$

Using $(t^{\alpha+\beta})' = -\sum_i(\alpha_i+\beta_i)t^{\alpha+\beta+\psi(e_i)} = (t^\alpha)'t^\beta + t^\alpha(t^\beta)'$ and [11, Corollary 3.9] one verifies that the map $f \mapsto f'\colon K \to K$ is a strongly additive $\boldsymbol{k}$-linear derivation on K. For $\alpha \neq 0$ we set $\alpha' := \alpha + \psi(\alpha)$.

Lemma 4.2. *If $0 \neq f \prec 1$, then*

$$f' \sim -\alpha_i f_\alpha t^{\alpha'} \qquad \text{where } \alpha := vf,\ i := [\alpha]_{\boldsymbol{k}}.$$

Hence if $0 \neq f \not\asymp 1$, then $v(f') = (vf)'$.

Proof. For $\alpha \neq 0$ and $i := [\alpha]_{\boldsymbol{k}}$ we have $i = [e_i]_{\boldsymbol{k}}$, hence $(t^\alpha)' \sim -\alpha_i t^{\alpha+\psi(e_i)} = -\alpha_i t^{\alpha'}$. If $0 < \alpha < \beta$, then $\alpha' < \beta'$ and thus $(t^\alpha)' \succ (t^\beta)'$. This yields the first implication. For the second, let $0 \neq f \not\asymp 1$, arrange $f \prec 1$ by replacing f with $1/f$ if necessary, and use the first part of the lemma. □

Lemma 4.3. *The derivation $f \mapsto f'$ makes K into an H-field with constant field $\boldsymbol{k}$ and asymptotic couple (Γ, ψ).*

Proof. Let $f \in K \setminus \boldsymbol{k}$. Then $f' \neq 0$: arranging $0 \neq f \not\asymp 1$ by subtracting $f_0 \in \boldsymbol{k}$ from f in case $f \asymp 1$, we have $v(f') = (vf)'$ by Lemma 4.2, in particular, $f' \neq 0$. Thus the constant field of our derivation is $\boldsymbol{k}$. The valuation ring $\mathcal{O}$ of K is the convex hull of $\boldsymbol{k}$ in K, and $\mathcal{O} = \boldsymbol{k} + o$. Finally, suppose $0 < f \prec 1$. Then $\alpha := vf > 0$, so $\alpha_i > 0$ for $i := [\alpha]_{\boldsymbol{k}}$. Also $f_\alpha > 0$ and $f' \sim -\alpha_i f_\alpha t^{\alpha'}$ and thus $f' < 0$. □

An H-field H with constant field C is said to be *closed under powers* if $H^\dagger$ is a C-linear subspace of H. In that case, the asymptotic couple of H is an H-couple over C of Hahn type in a natural way, by [2, Lemma 7.4, Proposition 7.5]. See [2, Sections 7, 8] for other basic facts about H-fields closed under powers.

Proposition 4.4. *The H-field K is closed under powers, and its associated H-couple over $\boldsymbol{k}$ is (Γ, ψ).*

This follows from Lemma 4.3 using the strong additivity of the derivation $f \mapsto f'$ of K and the fact that $(t^{c\alpha})^\dagger = c(t^\alpha)^\dagger$ for all α and all $c \in \boldsymbol{k}$, just like in [2], Proposition 11.4 followed from Lemmas 11.1, 11.2.

Remark 4.5. Let Γ_0 be a subgroup of Γ with $\Psi = \psi(\Gamma^{\neq}) \subseteq \Gamma_0$. Then $K_0 := \boldsymbol{k}(\!(t^{\Gamma_0})\!)$ is an H-subfield of K with constant field $\boldsymbol{k}$ and asymptotic couple $(\Gamma_0, \psi|_{\Gamma_0^{\neq}})$. If Γ_0 is a $\boldsymbol{k}$-linear subspace of Γ, then K_0 is closed under powers, and its associated H-couple over $\boldsymbol{k}$ is $(\Gamma_0, \psi|_{\Gamma_0^{\neq}})$.

The construction above generalizes that of the derivation defined in [10] on the H-field of logarithmic hyperseries:

Example. With $\boldsymbol{k} = \mathbb{R}$, let I be an ordinal and $\Gamma := H[I, \mathbb{R}] = \mathbb{R}^I$. Let i range over I and take $\psi\colon \Gamma^{\neq} \to \Gamma$ to be constant on archimedean classes with $\psi(e_i) = \sum_{j \leqslant i} e_j$ for all i. Then (Γ, ψ) is an H-couple over $\mathbb{R}$ of Hahn type. Note that $\psi(e_0) = e_0$ is the smallest element of $\Psi = \psi(\Gamma^{\neq})$, and (Γ, ψ) has gap $\sum_i e_i$. The construction above yields the H-field $K := \mathbb{R}(\!(t^\Gamma)\!)$ with constant field $\mathbb{R}$. This K is closed under powers and has associated H-couple (Γ, ψ) over $\mathbb{R}$. Changing notation, let I be the ordinal α and $\mathbb{L}_{<\alpha} = \mathbb{R}[[\mathfrak{L}_{<\alpha}]]$ the H-field of logarithmic hyperseries defined in [10, Sections 1, 3]. Then the unique strongly additive field isomorphism $h\colon K \to \mathbb{L}_{<\alpha}$ over $\mathbb{R}$ with $h(t^\gamma) = \prod_i \ell_i^{-\gamma_i}$ for $\gamma = \sum_i \gamma_i e_i$ is an isomorphism of H-fields.

So far the ordered vector space Γ over $\boldsymbol{k}$ was a spherically complete Hahn space over $\boldsymbol{k}$ and (Γ, ψ) an H-couple over $\boldsymbol{k}$ of Hahn type. We now relax this: *for the rest of this subsection (Γ, ψ) is an arbitrary H-couple over $\boldsymbol{k}$.* We shall need a variant of [1, Lemma 3.2]:

Lemma 4.6. *Let Γ_1 be an ordered vector space over $\boldsymbol{k}$ extending Γ with $[\Gamma]_{\boldsymbol{k}} = [\Gamma_1]_{\boldsymbol{k}}$. Then there is a unique map $\psi_1\colon \Gamma_1^{\neq} \to \Gamma_1$ such that (Γ_1, ψ_1) is an H-couple over $\boldsymbol{k}$ extending (Γ, ψ). If (Γ, ψ) is of Hahn type and Γ_1 is a Hahn space over $\boldsymbol{k}$, then (Γ_1, ψ_1) is of Hahn type.*

Proof. To verify axiom (A3) of asymptotic couples, use that for distinct $\alpha, \beta \neq 0$ we have $\big[\psi(\alpha) - \psi(\beta)\big]_{\boldsymbol{k}} < [\alpha - \beta]_{\boldsymbol{k}}$, by [5, p. 536]. □

Now assume (Γ, ψ) is of Hahn type. By the Hahn Embedding Theorem for Hahn spaces [ADH, 2.4.23] we may view Γ as an ordered $\boldsymbol{k}$-linear subspace of the ordered vector space $\widehat{\Gamma} := H[I, \boldsymbol{k}]$ over $\boldsymbol{k}$, where $I := \big([\Gamma^{\neq}]$ with reversed ordering$\big)$. Lemma 4.6 yields a unique map $\widehat{\psi}\colon \widehat{\Gamma}^{\neq} \to \widehat{\Gamma}$ making $(\widehat{\Gamma}, \widehat{\psi})$ an H-couple over $\boldsymbol{k}$ extending (Γ, ψ). Then $(\widehat{\Gamma}, \widehat{\psi})$ is of Hahn type. Let $\widehat{K} := \boldsymbol{k}(\!(t^{\widehat{\Gamma}})\!)$ be the H-field with constant field $\boldsymbol{k}$ and H-couple $(\widehat{\Gamma}, \widehat{\psi})$ over $\boldsymbol{k}$ and closed under powers that was constructed above, with $\widehat{K}$, $(\widehat{\Gamma}, \widehat{\psi})$ in the roles of K, (Γ, ψ), respectively. Then $\widehat{K}$ has the H-subfield $K := \boldsymbol{k}(\!(t^{\Gamma})\!)$ which is closed under powers and has (Γ, ψ) as its H-couple over $\boldsymbol{k}$, by Remark 4.5. This shows:

Corollary 4.7. *Every H-couple over $\boldsymbol{k}$ of Hahn type is the H-couple of a spherically complete H-field with constant field $\boldsymbol{k}$ and closed under powers.*

Proof of Corollary C. *In this subsection L is a closed η_1-ordered H-field with small derivation and constant field $\mathbb{R}$.* Every ordered vector space over $\mathbb{R}$ is a Hahn space over $\mathbb{R}$, so an H-couple (Γ, ψ) over $\mathbb{R}$ is of Hahn type iff it is of Hardy type. Thus by Lemma 3.6 and Corollary 4.7:

Lemma 4.8. *If (Γ, ψ) is a short H-couple over $\mathbb{R}$ of Hardy type with small derivation, then (Γ, ψ) is isomorphic to the H-couple over $\mathbb{R}$ of a spherically complete H-subfield of L containing $\mathbb{R}$ and closed under powers.*

In the same way that Lemma 3.6 gave rise to Corollary 3.7, Lemma 4.8 yields:

Corollary 4.9. *If M is a maximal Hardy field, then every short H-couple over $\mathbb{R}$ of Hardy type with small derivation is isomorphic to the H-couple over $\mathbb{R}$ of a spherically complete H-subfield of M containing $\mathbb{R}$ and closed under powers. Likewise with "maximal smooth" (respectively, "maximal analytic") in place of "maximal".*

The following is clear from [ADH, 9.8.1].

Lemma 4.10. *Let (Γ, ψ) be an H-asymptotic couple and Γ^* be an ordered vector space over $\mathbb{R}$ containing Γ as an ordered subgroup with $[\Gamma] = [\Gamma^*]$. Then there is a unique map $\psi^*\colon (\Gamma^*)^{\neq} \to \Gamma^*$ extending ψ which makes (Γ^*, ψ^*) an H-asymptotic couple. Moreover, (Γ^*, ψ^*) is an H-couple over $\mathbb{R}$, and (Γ^*, ψ^*) is of Hahn type iff (Γ, ψ) is of Hardy type.*

Lemma 4.11. *Let (Γ, ψ) be a short asymptotic couple of Hardy type with small derivation. Then (Γ, ψ) is isomorphic to the asymptotic couple of a spherically complete H-subfield of L containing $\mathbb{R}$.*

Proof. The Hahn product $\Gamma^* := H[I, \mathbb{R}]$, with $I := \big([\Gamma^{\neq}]$ with reversed ordering$\big)$ is an ordered vector space over $\mathbb{R}$; see [ADH, p. 98]. The Hahn Embedding Theorem [ADH, 2.3.4, 2.4.18, 2.4.19] yields an ordered group embedding $\iota\colon \Gamma \to \Gamma^*$ such that $\big[\iota(\Gamma)\big] = [\Gamma^*]$. Identify Γ with its image in Γ^* via ι. Lemma 4.10 gives a unique extension $\psi^*\colon (\Gamma^*)^{\neq} \to \Gamma^*$ of ψ such that (Γ^*, ψ^*) is an H-asymptotic couple. Then (Γ^*, ψ^*) is an H-couple over $\mathbb{R}$ of Hahn type, and is short by [3, Lemma 5.16]. Let $K^* := \mathbb{R}(\!(t^{\Gamma^*})\!)$ be the H-field closed under powers with constant field $\mathbb{R}$ and H-couple (Γ^*, ψ^*) over $\mathbb{R}$ constructed in the previous subsection. Then by Remark 4.5, $K := \mathbb{R}(\!(t^{\Gamma})\!)$ is an H-subfield of K^* with constant field $\mathbb{R}$ and asymptotic couple (Γ, ψ) and K has small derivation. Since K is short, it embeds into L by Lemma 3.6. □

Lemma 4.11 now yields Corollary C from the introduction:

Corollary 4.12. *Let M be a maximal Hardy field. Then every short asymptotic couple of Hardy type with small derivation is isomorphic to the asymptotic couple of a spherically complete H-subfield of M containing $\mathbb{R}$. Likewise with "maximal smooth"* (*respectively, "maximal analytic"*) *in place of "maximal".*

Notes

1. As a consequence of Corollary C, each asymptotic couple of Hardy type with small derivation and countable rank is isomorphic to the asymptotic couple of a Hardy field extending $\mathbb{R}$.
2. For a list of errata to [ADH], see [7].
3. In [1, 2] an H-couple (Γ, ψ) satisfies additional requirements: $\psi(\gamma) = \gamma$ for some $\gamma > 0$ in Γ, and (Γ, ψ) is of Hahn type.

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The citation [ADH] refers to the book

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