LOGARITHMS OF ITERATION MATRICES, AND PROOF OF A CONJECTURE BY SHADRIN AND ZVONKINE

MATTHIAS ASCHENBRENNER

Abstract. A proof of a conjecture by Shadrin and Zvonkine, relating the entries of a matrix arising in the study of Hurwitz numbers to a certain sequence of rational numbers, is given. The main tools used are iteration matrices of formal power series and their (matrix) logarithms.

This note is devoted to the study of the somewhat mysterious-looking sequence

$$0, 1, -\frac{1}{2}, \frac{1}{2}, -\frac{2}{3}, \frac{11}{12}, -\frac{3}{4}, -\frac{11}{6}, \frac{29}{4}, \frac{493}{12}, -\frac{2711}{6}, -\frac{12406}{15}, \frac{2636317}{60}, \dots \tag{S}$$

of rational numbers. I first encountered this sequence in ongoing joint work with van den Dries and van der Hoeven on asymptotic differential algebra [4]. It also appears in a conjecture made in a paper by Shadrin and Zvonkine [31] in connection with a generating series for Hurwitz numbers (which count the number of ramified coverings of the sphere by a surface, depending on certain parameters like the degree of the covering and the genus of the surface). I came across [31] by entering the numerators and denominators of the first few terms of (S) into Sloane's *On-Line Encyclopedia of Integer Sequences* [1]. (The numerator sequence is A134242, the denominator sequence is A134243.) In this note we prove the conjecture from [31]. In the course of doing so, we identify a formula for the sequence (S): denoting its nth term by c_n (so $c_1 = 0$, $c_2 = 1$, $c_3 = -\frac{1}{2}$ etc.), we have

$$c_n = \sum_{\substack{1\leqslant k<n \\ 1<n_1<\cdots<n_{k-1}<n_k=n}} \frac{(-1)^{k+1}}{k} \left\{ {n_2 \atop n_1} \right\} \left\{ {n_3 \atop n_2} \right\} \cdots \left\{ {n_k \atop n_{k-1}} \right\}.$$

Here and below, we denote by $\left\{ {j \atop i} \right\}$ the Stirling numbers of the second kind: $\left\{ {j \atop i} \right\}$ is the number of equivalence relations on a j-element set with i equivalence classes. They obey the recurrence relation

$$\left\{ {j \atop i} \right\} = \left\{ {j-1 \atop i-1} \right\} + i \left\{ {j-1 \atop i} \right\} \qquad (i, j > 0)$$

with initial conditions

$$\left\{ {0 \atop 0} \right\} = 1, \ \left\{ {0 \atop i} \right\} = \left\{ {j \atop 0} \right\} = 0 \qquad (i, j > 0).$$

Partially supported by a grant from the National Science Foundation.
November 2011

For example, we have

$$\begin{aligned}
1-\frac{1}{2}\left\{{3 \atop 2}\right\} &= 1-\frac{1}{2}\cdot 3 &&= -\frac{1}{2} = c_3\\
1-\frac{1}{2}\left(\left\{{4 \atop 2}\right\}+\left\{{4 \atop 3}\right\}\right)+\frac{1}{3}\left\{{3 \atop 2}\right\}\left\{{4 \atop 3}\right\} &= 1-\frac{1}{2}(7+6)+\frac{1}{3}\cdot 3\cdot 6 &&= \frac{1}{2} = c_4\\
\left.\begin{matrix} 1-\tfrac{1}{2}\left(\left\{{5 \atop 2}\right\}+\left\{{5 \atop 3}\right\}+\left\{{5 \atop 4}\right\}\right)+ \\ \tfrac{1}{3}\left(\left\{{3 \atop 2}\right\}\left\{{5 \atop 3}\right\}+\left\{{4 \atop 2}\right\}\left\{{5 \atop 4}\right\}+\left\{{4 \atop 3}\right\}\left\{{5 \atop 4}\right\}\right)- \\ \tfrac{1}{4}\left\{{3 \atop 2}\right\}\left\{{4 \atop 3}\right\}\left\{{5 \atop 4}\right\}\end{matrix}\right\} &= \left\{\begin{matrix} 1-\tfrac{1}{2}(15+25+10)+ \\ \tfrac{1}{3}(3\cdot 25+7\cdot 10+6\cdot 10)- \\ \tfrac{1}{4}\cdot 3\cdot 6\cdot 10\end{matrix}\right\} &&= -\frac{2}{3} = c_5
\end{aligned}$$

A key concept for our study of (S) is the *iteration matrix* of a formal power series; these matrices are well-known in the iteration theory of analytic functions [20, 21] and in combinatorics [11]. The iteration matrix of a power series $f \in \mathbb{Q}[[z]]$ of the form $f = z + z^2 g$ ($g \in \mathbb{Q}[[z]]$) is a certain bi-infinite upper triangular matrix with rational entries associated to f. After stating the conjecture of Shadrin and Zvonkine in Section 1 and making some preliminary reductions, we summarize some general definitions and basic facts about triangular matrices in Section 2 and introduce the group of iteration matrices in Section 3. In Section 4 we determine its Lie algebra of infinitesimal generators, by slightly generalizing results of Schippers [30]. These results tie in with a notion from classical iteration theory: the infinitesimal generator of the iteration matrix of a formal power series f as above is uniquely determined by another power series $\operatorname{itlog}(f) \in z^2\mathbb{Q}[[z]]$, introduced by Jabotinsky [21] and called the *iterative logarithm* of f by Écalle [13]. Some of the properties of iterative logarithms are discussed in Section 5, before we return to the proof of the conjecture of Shadrin-Zvonkine in Section 7. The exponential generating function (egf) of the sequence (c_n), that is, the formal power series

$$\sum_{n\geqslant 1} c_n \frac{z^n}{n!} = \frac{1}{2}z^2 - \frac{1}{12}z^3 + \frac{1}{48}z^4 - \frac{1}{180}z^5 + \cdots,$$

turns out to be nothing else than the iterative logarithm of the power series $e^z - 1$.

The iterative logarithm $\operatorname{itlog}(f)$ of any formal power series f satisfies a certain functional equation found by Jabotinsky [20]. In the case of $f = e^z - 1$, this equation leads to a convolution formula for Stirling numbers (and another formula for the terms of the sequence (c_n)):

$$c_n = \sum_{\substack{1\leqslant k<n\\ 1<n_1<\cdots<n_{k-1}<n_k=n}} \frac{(-1)^{k+1}}{k}\left\{{n_2 \atop n_1}\right\}\left\{{n_3 \atop n_2}\right\}\cdots\left\{{n_k \atop n_{k-1}}\right\} = \sum_{\substack{1\leqslant k<n-1\\ 1<n_1<\cdots<n_{k-1}<n_k=n-1}} \frac{(-1)^{k}}{k+1}\left\{{n_2 \atop n_1}\right\}\left\{{n_3 \atop n_2}\right\}\cdots\left\{{n_k \atop n_{k-1}}\right\} \tag{C}$$

To our knowledge, this formula does not seem to have been noticed before. (For instance, it does not appear in Gould's collection of combinatorial identities [17].) We give a proof of (C) in Section 7.

Shadrin and Zvonkine write that the sequence (S) *seems to be quite irregular* [31, p. 224]. This impression can be substantiated as follows. A formal power series $f \in \mathbb{C}[[z]]$ is said to be *differentially algebraic* if it satisfies an algebraic differential equation, i.e., an equation

$$P(z, f, f', \dots, f^{(n)}) = 0$$

where P is a non-zero polynomial in $n+2$ indeterminates with constant complex coefficients. The coefficient sequence (f_n) of every differentially algebraic power series $f = \sum_{n \geqslant 0} f_n z^n \in \mathbb{Q}[[z]]$ is regular in the sense that it satisfies a certain kind of (generally non-linear) recurrence relation [28, pp. 186–194]. A class of differentially algebraic power series which is of particular importance in combinatorial enumeration is the class of *D-finite* (also called *holonomic*) power series [32, Chapter 6]. These are the series whose coefficient sequence satisfies a homogeneous *linear* recurrence relation of finite degree with polynomial coefficients. Equivalently [32, Proposition 6.4.3] a formal power series $f \in \mathbb{C}[[z]]$ is D-finite if and only if f satisfies a non-trivial *linear* differential equation

$$a_0 f + a_1 f' + \cdots + a_n f^{(n)} = 0 \qquad (a_i \in \mathbb{C}[z],\ a_n \neq 0).$$

(This class includes, e.g., all hypergeometric series.) In Section 7 we will see that the egf of (c_n) is not differentially algebraic. This is a consequence of a result of Boshernitzan and Rubel, stated without proof in [10], which characterizes when the iterative logarithm of a power series satisfies an ADE; in Section 6 below we give a complete proof of this fact. It is also known [8, 25] that the egf of (c_n) has radius of convergence 0. Indeed, a common generalization of these results holds true: *the egf of* (c_n) *does not satisfy an algebraic differential equation over the ring of convergent power series.* The proof of this fact will be given elsewhere [3]. It seems likely (though we have not investigated this further) that the *ordinary* generating function (ogf)

$$\sum_{n \geqslant 1} c_n z^n = z^2 - \frac{1}{2} z^3 + \frac{1}{2} z^4 - \frac{2}{3} z^5 + \cdots$$

of the sequence (S) is also differentially transcendental. (Note, however, that there are examples of sequences of rationals whose egf is differentially transcendental yet whose ogf is differentially algebraic; see [26, Proposition 6.3 (i)].)

Notations and conventions. We let d, m, n, k, possibly with decorations, range over $\mathbb{N} = \{0, 1, 2, \dots\}$. All rings below are assumed to have a unit 1. Given a ring R we denote by $R^\times$ the group of units of R.

Acknowledgements. We thank the anonymous referees whose corrections and suggestions improved the paper.

1. The Conjecture of Shadrin and Zvonkine

Before we can formulate this conjecture, we need to fix some notation. Let K be a commutative ring and let $R = K[[t_0, t_1, \dots]]$ be the ring of powers series in the pairwise distinct indeterminates $t_0, t_1, \dots$, with coefficients from K. We equip R with the $\mathfrak{m}$-adic topology, where $\mathfrak{m}$ is the ideal $(t_0, t_1, \dots)$ of R. In this subsection we let $\boldsymbol{i}$, $\boldsymbol{j}$ range over the set of sequences $\boldsymbol{i} = (i_0, i_1, \dots) \in \mathbb{N}^{\mathbb{N}}$ such that $i_n = 0$ for all but finitely many n. For each $\boldsymbol{i}$ we set

$$t^{\boldsymbol{i}} := t_0^{i_0} t_1^{i_1} \cdots t_n^{i_n} \cdots \in R.$$

Hence every element f of R can be uniquely written in the form

$$f = \sum_{\boldsymbol{i}} f_{\boldsymbol{i}}\, t^{\boldsymbol{i}} \qquad \text{where } f_{\boldsymbol{i}} \in K \text{ for all } \boldsymbol{i}.$$

We call an element of R of the form $at^{\boldsymbol{i}}$, where $0 \neq a \in K$, a monomial. We put

$$\|\boldsymbol{i}\| := 1i_0 + 2i_1 + 3i_2 + \cdots + (n+1)i_n + \cdots \in \mathbb{N},$$

and we define a valuation v on R by setting

$$v(f) := \min_{f_{\boldsymbol{i}} \neq 0} \|\boldsymbol{i}\| \in \mathbb{N} \text{ for } 0 \neq f \in R, \qquad v(0) := \infty > \mathbb{N}.$$

Suppose from now on that $K = \mathbb{Q}[z]$ where z is a new indeterminate over $\mathbb{Q}$. Shadrin and Zvonkine first introduce rational numbers $a_{d,d+k}$ by the equation

$$\sum_{b=1}^{d+1} \binom{d}{b-1} \frac{(-1)^{d-b+1}}{d!} \cdot \frac{1}{1-b\psi} = \sum_{k \geqslant 0} a_{d,d+k} \psi^{d+k} \tag{1.1}$$

in the formal power series ring $\mathbb{Q}[[\psi]]$:

$$\frac{1}{1-\psi} = 1 + \psi + \psi^2 + \cdots \qquad (d=0)$$

$$-\frac{1}{1-\psi} + \frac{1}{1-2\psi} = \psi + 3\psi^2 + 7\psi^3 + \cdots \qquad (d=1)$$

$$\frac{1/2}{1-\psi} - \frac{1}{1-2\psi} + \frac{1/2}{1-3\psi} = \psi^2 + 6\psi^3 + 25\psi^4 + \cdots \qquad (d=2)$$

$$\vdots$$

Using the numbers $a_{d,d+k}$ (which turn out to be positive integers, see Lemma 1.2 below) they then define a sequence $(L_k)_{k>0}$ of differential operators on R: abbreviating the K-derivation $\frac{\partial}{\partial t_n}$ of R by ∂_n, set

$$L_k = \sum_{\substack{0 \leqslant r \leqslant k \\ k_1 + \cdots + k_r = k \\ k_1, \ldots, k_r > 0 \\ n_1, \ldots, n_r \geqslant 0}} \frac{1}{r!}\, a_{n_1, n_1+k_1} \cdots a_{n_r, n_r+k_r}\, t_{n_1+k_1} \cdots t_{n_r+k_r}\, \partial_{n_1} \cdots \partial_{n_r} \qquad (k > 0).$$

Note that the definition of L_k (as a K-linear map $R \to R$) makes sense, since for every $\boldsymbol{i}$, either

$$t_{n_1+k_1} \cdots t_{n_l+k_r}\, \partial_{n_1} \cdots \partial_{n_r}(t^{\boldsymbol{i}})$$

is zero or is a monomial which has valuation $\|\boldsymbol{i}\| + k_1 + \cdots + k_r$ and which is divisible by $t_{n_1+k_1} \cdots t_{n_r+k_r}$; moreover, given $\boldsymbol{j}$ there are only finitely many $\boldsymbol{i}$ with $\|\boldsymbol{i}\| < \|\boldsymbol{j}\|$, and only finitely many $k_1, \ldots, k_r > 0$ and $n_1, \ldots, n_r \geqslant 0$ such that

$j_{n_1+k_1}, \dots, j_{n_r+k_r} > 0$. The first few terms of the sequence (L_k) are

$$\begin{aligned}
L_1 &= \sum_{n_1} a_{n_1,n_1+1} t_{n_1+1}\,\partial_{n_1},\\
L_2 &= \sum_{n_1} a_{n_1,n_1+2} t_{n_1+2}\,\partial_{n_1} + \frac{1}{2!}\sum_{n_1,n_2} a_{n_1,n_1+1}a_{n_2,n_2+1}\,t_{n_1+1}t_{n_2+1}\partial_{n_1}\partial_{n_2}\\
L_3 &= \sum_{n_1} a_{n_1,n_1+3} t_{n_1+3}\,\partial_{n_1} + \frac{1}{2!}\sum_{n_1,n_2} a_{n_1,n_1+1}a_{n_2,n_2+2}\,t_{n_1+1}t_{n_2+2}\partial_{n_1}\partial_{n_2} +\\
&\quad \frac{1}{2!}\sum_{n_1,n_2} a_{n_1,n_1+2}a_{n_2,n_2+1}\,t_{n_1+2}t_{n_2+1}\partial_{n_1}\partial_{n_2} +\\
&\quad \frac{1}{3!}\sum_{n_1,n_2,n_3} a_{n_1,n_1+1}a_{n_2,n_2+1}a_{n_3,n_3+1}\,t_{n_1+1}t_{n_2+1}t_{n_3+1}\partial_{n_1}\partial_{n_2}\partial_{n_3},
\end{aligned}$$

and in general we have

$$L_k = \sum_{n_1} a_{n_1,n_1+k} t_{n_1+k}\,\partial_{n_1} + \text{higher-order operators} \qquad (k > 0). \tag{1.2}$$

To streamline the notation we set $L_0 := \mathrm{id}_R$. The argument above shows that for every $f \in R$ we have $v(L_k(f)) \geqslant k + v(f)$, hence the sequence $(z^k L_k(f))_k$ is summable in R. Thus one may combine the L_k to a K-linear map $\boldsymbol{L}\colon R \to R$ with

$$\boldsymbol{L}(f) = \sum_k z^k L_k(f) = f + zL_1(f) + z^2 L_2(f) + \cdots \qquad \text{for all } f \in R.$$

The operator $\boldsymbol{L}$ is used in [31] to perform a change of variables in a certain formula for Hurwitz numbers coming from [15]. The following proposition is established in [31, Proposition A.8]. (The formula for l_k given in [31] mistakenly omits the summation over n.)

Proposition 1.1. *There are rational numbers $\alpha_{n,n+k}$ such that, setting*

$$l_k = \sum_n \alpha_{n,n+k} t_{n+k}\,\partial_n \qquad (k > 0)$$

and

$$\boldsymbol{l} = z l_1 + z^2 l_2 + \cdots,$$

we have $\boldsymbol{L} = \exp(\boldsymbol{l})$, i.e.,

$$\boldsymbol{L}(f) = \sum_n \frac{1}{n!}\boldsymbol{l}^n(f) \qquad \textit{for every } f \in R. \tag{1.3}$$

(To see that the definition of l_k and $\boldsymbol{l}$ makes sense argue as for L_k and $\boldsymbol{L}$ above; since $v(\boldsymbol{l}(f)) \geqslant v(f) + 1$ we have $v(\boldsymbol{l}^n(f)) \geqslant v(f) + n$ for all n, hence the sum on the right-hand side of the equation in (1.3) exists in R.)

After proving this proposition, Shadrin and Zvonkine make the following conjecture about the form of the $\alpha_{n,n+k}$. (Again, we correct a typo in [31]: in Conjecture A.9 replace $t_n \frac{\partial}{\partial t_{n+k}}$ by $t_{n+k}\frac{\partial}{\partial t_n}$.)

Conjecture. For all $k > 0$ and all n,

$$\alpha_{n,n+k} = c_{k+1}\binom{n+k+1}{k+1}$$

where $(c_k)_{k\geqslant 1}$ is a sequence of rational numbers, with the first terms given by (S).

The first step in our proof of this conjecture is to realize is that the $a_{d,d+k}$ are essentially the Stirling numbers of the second kind. We extend the definition of $a_{d,d+k}$ by setting $a_{dd} := 1$ for every d.

Lemma 1.2. *For every d and k,*

$$a_{d,d+k} = \left\{ {d+k+1 \atop d+1} \right\}.$$

Proof. We expand the left-hand side of (1.1) in powers of ψ:

$$\sum_{b=1}^{d+1} \binom{d}{b-1} \frac{(-1)^{d-b+1}}{d!} \cdot \frac{1}{1-b\psi} = \sum_{i \geqslant 0} \left(\frac{1}{d!} \sum_{b=1}^{d+1} (-1)^{d-b+1} \binom{d}{b-1} b^i \right) \psi^i.$$

Now we focus on the coefficient of ψ^i in the last sum. By the Binomial Theorem, this coefficient can be written as

$$\frac{1}{d!} \sum_{b=0}^{d} (-1)^{d-b} \binom{d}{b} (b+1)^i = \sum_{j=0}^{i} \binom{i}{j} \left(\frac{1}{d!} \sum_{b=0}^{d} (-1)^{d-b} \binom{d}{b} b^j \right).$$

It is well-known that

$$\left\{ {j \atop d} \right\} = \frac{1}{d!} \sum_{b=0}^{d} (-1)^{d-b} \binom{d}{b} b^j$$

and

$$\sum_{j=0}^{i} \binom{i}{j} \left\{ {j \atop d} \right\} = \left\{ {i+1 \atop d+1} \right\}.$$

(See, e.g., identities (6.19) respectively (6.15) in [18].) The lemma follows. $\square$

By (1.2) and the above lemma we therefore have

$$L_k(t_d) = a_{d,d+k} t_{d+k} = \left\{ {d+k+1 \atop d+1} \right\} t_{d+k}$$

and hence

$$\boldsymbol{L}(t_d) = \sum_k \left\{ {d+k+1 \atop d+1} \right\} z^k t_{d+k}. \tag{1.4}$$

Moreover, by definition of l_k we have $l_k(t_d) = \alpha_{d,d+k} t_{d+k}$ for all d and $k > 0$, hence

$$l(t_d) = \sum_{k>0} \alpha_{d,d+k} z^k t_{d+k}$$

and thus for every $n > 0$:

$$l^n(t_d) = \sum_{k_1,\dots,k_n>0} \alpha_{d,d+k_1} \cdots \alpha_{d+k_1+\cdots+k_{n-1},d+k_1+\cdots+k_n} z^{k_1+\cdots+k_n} t_{d+k_1+\cdots+k_n}.$$

This yields

$$\exp(\boldsymbol{l})(t_d) = \sum_k \left(\sum_{\substack{k_1+\cdots+k_n=k \\ n>0,\ k_1,\dots,k_n>0}} \frac{1}{n!} \alpha_{d,d+k_1} \cdots \alpha_{d+k_1+\cdots+k_{n-1},d+k} \right) z^k t_{d+k}$$

and therefore, by (1.4) and Proposition 1.1:

$$\left\{ {d+k+1 \atop d+1} \right\} = \sum_{\substack{k_1+\cdots+k_n=k\\ n>0, k_1,\ldots,k_n>0}} \frac{1}{n!} \alpha_{d,d+k_1} \alpha_{d+k_1,d+k_1+k_2} \cdots \alpha_{d+k_1+\cdots+k_{n-1},d+k}. \quad (1.5)$$

It is suggestive to express this equation as an identity between matrices. We define $\left\{ {j \atop i} \right\} := 0$ for $i > j$, and combine the Stirling numbers of the second kind into a bi-infinite upper triangular matrix:

$$S = (S_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ & 1 & 1 & 1 & 1 & 1 & \cdots \\ & & 1 & 3 & 7 & 15 & \cdots \\ & & & 1 & 6 & 25 & \cdots \\ & & & & 1 & 10 & \cdots \\ & & & & & 1 & \cdots \\ & & & & & & \ddots \end{pmatrix} \qquad \text{where } S_{ij} = \left\{ {j \atop i} \right\}. \quad (1.6)$$

We also introduce the upper triangular matrix

$$A = (\alpha_{ij}) = \begin{pmatrix} 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{2}{3} & \frac{11}{12} & \cdots \\ & 0 & 3 & -2 & \frac{5}{2} & -4 & \cdots \\ & & 0 & 6 & -5 & \frac{15}{2} & \cdots \\ & & & 0 & 10 & 10 & \cdots \\ & & & & 0 & -15 & \cdots \\ & & & & & 0 & \cdots \\ & & & & & & \ddots \end{pmatrix} \qquad \text{where } \alpha_{ij} := 0 \text{ for } i \geqslant j.$$

Then (1.5) may be written as

$$\begin{aligned} S_{i+1,j+1} &= \sum_{\substack{k_1+\cdots+k_n=j-i\\ n>0,\ k_1,\ldots,k_n>0}} \frac{1}{n!}\, \alpha_{i,i+k_1} \alpha_{i+k_1,i+k_1+k_2} \cdots \alpha_{i+k_1+\cdots+k_{n-1},j} \\ &= \sum_{n=1}^{j-i} \frac{1}{n!}\, (A^n)_{ij} \qquad\qquad (i \leqslant j) \end{aligned}$$

or equivalently, writing $S^+ := (S_{i+1,j+1})_{i,j}$ and employing the matrix exponential:

$$S^+ = \sum_{n\geqslant 0} \frac{1}{n!} A^n = \exp(A)$$

Therefore, in order to prove the conjecture from [31], we need to be able to express the matrix logarithm of S^+ in some explicit manner. We show how this can be done (and finish the proof of the conjecture) in Section 7 below; before that, we need to step back and first embark on a systematic study of a class of matrices (iteration matrices) which encompasses S and many other matrices of combinatorial significance (Sections 2 and 3), and of their matrix logarithms (Sections 4 and 5).

2. Triangular Matrices

In this section we let K be a commutative ring.

The K-algebra of triangular matrices. We construe $K^{\mathbb{N}\times\mathbb{N}}$ as a K-module with the componentwise addition and scalar multiplication. The elements $M = (M_{ij})_{i,j\in\mathbb{N}}$ of $K^{\mathbb{N}\times\mathbb{N}}$ may be visualized as bi-infinite matrices with entries in K:

$$M = \begin{pmatrix} M_{00} & M_{01} & M_{02} & \cdots \\ M_{10} & M_{11} & M_{12} & \cdots \\ M_{20} & M_{21} & M_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We say that $M = (M_{ij}) \in K^{\mathbb{N}\times\mathbb{N}}$ is (upper) **triangular** if $M_{ij} = 0$ for all $i, j \in \mathbb{N}$ with $i > j$. We usually write a triangular matrix M in the form

$$M = \begin{pmatrix} M_{00} & M_{01} & M_{02} & M_{03} & \cdots \\ & M_{11} & M_{12} & M_{13} & \cdots \\ & & M_{22} & M_{23} & \cdots \\ & & & M_{33} & \cdots \\ & & & & \ddots \end{pmatrix}.$$

Given triangular matrices $M = (M_{ij})$ and $\widetilde{M} = (\widetilde{M}_{ij})$, the product

$$M \cdot \widetilde{M} := \left(\textstyle\sum_k M_{ik}\widetilde{M}_{kj}\right)_{i,j\in\mathbb{N}}$$

makes sense and is again a triangular matrix. Equipped with this operation, the K-submodule of $K^{\mathbb{N}\times\mathbb{N}}$ consisting of all triangular matrices becomes an associative K-algebra $\mathfrak{tr}_K$ with unit 1 given by the identity matrix. If K is a subring of a commutative ring L, then $\mathfrak{tr}_K$ is a K-subalgebra of the K-algebra $\mathfrak{tr}_L$. We also define

$$[M, N] := MN - NM \qquad \text{for } M, N \in \mathfrak{tr}_K.$$

Then the K-module $\mathfrak{tr}_K$ equipped with the binary operation $[\ ,\ ]$ is a Lie K-algebra.

For every n we set

$$\mathfrak{tr}_K^n := \big\{M = (M_{ij}) \in \mathfrak{tr}_K : M_{ij} = 0 \text{ for all } i, j \in \mathbb{N} \text{ with } i - j + n \geqslant 1\big\}.$$

We call the elements of $\mathfrak{tr}_K^1$ **strictly triangular.** It is easy to verify that the sequence $(\mathfrak{tr}_K^n)$ of K-submodules of $\mathfrak{tr}_K$ is a filtration of the K-algebra $\mathfrak{tr}_K$, i.e.,

(1) $\mathfrak{tr}_K^0 = \mathfrak{tr}_K$;
(2) $\mathfrak{tr}_K^n \supseteq \mathfrak{tr}_K^{n+1}$ for all n;
(3) $\mathfrak{tr}_K^m\,\mathfrak{tr}_K^n \subseteq \mathfrak{tr}_K^{m+n}$ for all m, n; and
(4) $\bigcap_n \mathfrak{tr}_K^n = \{0\}$.

Clearly $\mathfrak{tr}_K$ is complete in the topology making $\mathfrak{tr}_K$ into a topological ring with fundamental system of neighborhoods of 0 given by the $\mathfrak{tr}_K^n$.

The group $\mathfrak{tr}_K^\times$ of units of $\mathfrak{tr}_K$ has the form

$$\mathfrak{tr}_K^\times = D_K \ltimes (1 + \mathfrak{tr}_K^1) \qquad \text{(internal semidirect product of subgroups of } \mathfrak{tr}_K^\times)$$

where D_K is the group of diagonal invertible matrices:

$$D_K := \big\{M = (M_{ij}) \in \mathfrak{tr}_K : M_{ii} \in K^\times \text{ and } M_{ij} = 0 \text{ for } i \neq j\big\}.$$

Diagonals. We say that a matrix $M = (M_{ij}) \in \mathfrak{tr}_K$ is n**-diagonal** if $M_{ij} = 0$ for $j \neq i + n$. We simply call M **diagonal** if M is 0-diagonal. Given a sequence $a = (a_i)_{i\geqslant 0} \in K^{\mathbb{N}}$, we denote by $\operatorname{diag}_n a$ the n-diagonal matrix $M = (M_{ij}) \in K^{\mathbb{N}\times\mathbb{N}}$ with $M_{i,i+n} = a_i$ for every i. The sum of two n-diagonal matrices is n-diagonal. As for products, we have:

Lemma 2.1. *Let $M = \operatorname{diag}_m a$ be m-diagonal and $N = \operatorname{diag}_n b$ be n-diagonal, where $a = (a_i), b = (b_i) \in K^{\mathbb{N}}$. Then $M \cdot N$ is $(m+n)$-diagonal, in fact*

$$M \cdot N = \operatorname{diag}_{m+n}(a_i \cdot b_{i+m})_{i\geqslant 0}.$$

Therefore $[M,N]$ is $(m+n)$-diagonal, with

$$[M,N] = \operatorname{diag}_{m+n}(a_i \cdot b_{i+m} - b_i \cdot a_{i+n})_{i\geqslant 0},$$

and for each k, the matrix M^k is km-diagonal, with

$$M^k = \operatorname{diag}_{km}(a_i \cdot a_{i+m} \cdots a_{i+(k-1)m})_{i\geqslant 0}.$$

Exponential and logarithm of triangular matrices. In this subsection we assume that K contains $\mathbb{Q}$ as a subring. Then for each strictly triangular matrix M, the sequences $\left(\frac{M^n}{n!}\right)_{n\geqslant 0}$ and $\left((-1)^{n+1}\frac{M^n}{n}\right)_{n\geqslant 1}$ are summable, and the maps

$$\mathfrak{tr}_K^1 \to 1 + \mathfrak{tr}_K^1 \colon M \mapsto \exp(M) := \sum_{n\geqslant 0} \frac{M^n}{n!}$$

and

$$1 + \mathfrak{tr}_K^1 \to \mathfrak{tr}_K^1 \colon M \mapsto \log(M) := \sum_{n\geqslant 1} (-1)^{n+1} \frac{(M-1)^n}{n}$$

are mutual inverse; in particular, they are bijective. If $M \in \mathfrak{tr}_K^n$, $n > 0$, then $\exp(M) \in 1 + \mathfrak{tr}_K^n$ and $\log(1+M) \in \mathfrak{tr}_K^n$. It is easy to see that

$$\exp(M)\exp(N) = \exp(M+N) \qquad \text{for all } M, N \in \mathfrak{tr}_K^1 \text{ with } MN = NM. \tag{2.1}$$

In particular

$$\exp(M)^k = \exp(kM) \qquad \text{for all } M \in \mathfrak{tr}_K^1,\ k \in \mathbb{Z}.$$

We also note that given a unit U of $\mathfrak{tr}_K$, we have

$$\exp(UMU^{-1}) = U\exp(M)U^{-1} \qquad \text{for all } M \in \mathfrak{tr}_K^1$$

and

$$\log\left(UMU^{-1}\right) = U\log(M)U^{-1} \qquad \text{for all } M \in 1 + \mathfrak{tr}_K^1. \tag{2.2}$$

Given $M = (M_{ij})_{i,j} \in \mathfrak{tr}_K$ we define $M^+ := (M_{i+1,j+1})_{i,j} \in \mathfrak{tr}_K$. It is easy to see that $M \mapsto M^+$ is a K-algebra morphism $\mathfrak{tr}_K \to \mathfrak{tr}_K$ with $M \in \mathfrak{tr}_K^n \Rightarrow M^+ \in \mathfrak{tr}_K^n$. Thus, for $M \in \mathfrak{tr}_K^1$:

$$\exp(M^+) = \exp(M)^+, \qquad \log(1+M^+) = \log(1+M)^+. \tag{2.3}$$

From Lemma 2.1 we immediately obtain, for all $M = \operatorname{diag}_1 a$ where $a = (a_i) \in K^{\mathbb{N}}$:

$$(\exp M)_{ij} = \frac{1}{(j-i)!}\, a_i \cdot a_{i+1} \cdots a_{j-1} \qquad \text{for all } i, j \in \mathbb{N} \text{ with } i \leqslant j. \tag{2.4}$$

Derivations on the K-algebra of triangular matrices. Let ∂ be a derivation of K, i.e., a map $\partial\colon K \to K$ such that

$$\partial(a+b) = \partial(a) + \partial(b), \quad \partial(ab) = \partial(a)b + a\partial(b) \qquad \text{for all } a, b \in K.$$

Given $M = (M_{ij}) \in \mathfrak{tr}_K$ we let

$$\partial(M) := (\partial(M_{ij})) \in \mathfrak{tr}_K\,.$$

Then $M \mapsto \partial(M)\colon\ \mathfrak{tr}_K \to \mathfrak{tr}_K$ is a derivation of $\mathfrak{tr}_K$, i.e.,

$$\partial(M+N) = \partial(M) + \partial(N), \quad \partial(MN) = \partial(M)N + M\partial(N) \qquad \text{for all } M, N \in \mathfrak{tr}_K.$$

Note that $\partial(\mathfrak{tr}_K^n) \subseteq \mathfrak{tr}_K^n$ for every n.

We now let t be an indeterminate over K, and we work in the polynomial ring $K^* = K[t]$ and in the K^*-algebra $\mathfrak{tr}_{K^*}$ (which contains $\mathfrak{tr}_K$ as a K-subalgebra). We equip K^* with the derivation $\frac{d}{dt}$. The following two elementary observations are used in Section 4. Until the end of this subsection we assume that K contains $\mathbb{Q}$ as a subring.

Lemma 2.2. *Let $M \in \mathfrak{tr}_K^1$. Then*

$$\frac{d}{dt}\exp(tM) = \exp(tM)M.$$

Proof. We have $(tM)^n = t^n M^n$ for every n, hence

$$\exp(tM) = \sum_{n\geqslant 0} \frac{(tM)^n}{n!} = \sum_{n\geqslant 0} \frac{t^n M^n}{n!}$$

and thus

$$\frac{d}{dt}\exp(tM) = \sum_{n\geqslant 0} \frac{d}{dt}\left(\frac{t^n M^n}{n!}\right) = \sum_{n>0} \frac{t^{n-1}M^n}{(n-1)!} = \exp(tM)M.$$

(Similarly, of course, one also sees $\frac{d}{dt}\exp(tM) = M\exp(tM)$, but we won't need this fact.) □

The following lemma is a familiar fact about homogeneous systems of linear differential equations with constant coefficients:

Lemma 2.3. *Let $M, Y_0 \in \mathfrak{tr}_K^1$ and $Y \in \mathfrak{tr}_{K^*}^1$. Then*

$$\frac{dY}{dt} = YM \text{ and } Y\big|_{t=0} = Y_0 \qquad \Longleftrightarrow \qquad Y = Y_0\exp(tM).$$

Proof. Lemma 2.2 shows that if $Y = Y_0\exp(tM)$ then $\frac{dY}{dt} = YM$, and clearly $Y\big|_{t=0} = Y_0\exp(0) = Y_0$. Conversely, suppose $\frac{dY}{dt} = YM$ and $Y\big|_{t=0} = Y_0$. Then $Y_1 := Y - Y_0\exp(tM) \in \mathfrak{tr}_{K^*}^1$ satisfies $\frac{dY_1}{dt} = Y_1 M$ and $Y_1\big|_{t=0} = 0$; hence after replacing Y by Y_1 we may assume that $\frac{dY}{dt} = YM$ and $Y\big|_{t=0} = 0$, and need to show that then $Y = 0$. For a contradiction suppose $Y \neq 0$, and write $Y = (Y_{ij})$ where $Y_{ij} \in K^*$ and $M = (M_{ij})$ where $M_{ij} \in K$. Since $Y\big|_{t=0} = 0$, for each i, j such that $Y_{ij} \neq 0$ we can write $Y_{ij} = t^{n_{ij}} Z_{ij}$ with $n_{ij} \in \mathbb{N}$, $n_{ij} > 0$, and $Z_{ij} \in K^*$, $Z_{ij}(0) \neq 0$. Choose i, j so that n_{ij} is minimal. Then by $\frac{dY}{dt} = YM$ we have

$$n_{ij}t^{n_{ij}-1}Z_{ij} + t^{n_{ij}}\frac{dZ_{ij}}{dt} = \frac{dY_{ij}}{dt} = \sum_k Y_{ik}M_{kj} = \sum_{Y_{ik}\neq 0} t^{n_{ik}} Z_{ik} M_{kj},$$

thus

$$Z_{ij} = \frac{1}{n_{ij}} \left(-t \frac{dZ_{ij}}{dt} + \sum_{Y_{ik} \neq 0} t^{n_{ik} - n_{ij} + 1} Z_{ik} M_{kj} \right)$$

and hence $Z_{ij}(0) = 0$, a contradiction. So $Y = 0$ as desired. $\square$

3. Iteration Matrices

Let K be a commutative ring containing $\mathbb{Q}$ as a subring. Let $A = \mathbb{Q}[y_1, y_2, \dots]$ where $(y_n)_{n \geqslant 1}$ is a sequence of pairwise distinct indeterminates, let z be an indeterminate distinct from each y_n, and let

$$y = \sum_{n \geqslant 1} y_n \frac{z^n}{n!} \in A[[z]].$$

Then, with x another new indeterminate, we have in the power series ring $A[[x, z]]$:

$$\exp(x \cdot y) = \sum_{n \geqslant 0} \frac{(x \cdot y)^n}{n!} = \sum_{i,j \in \mathbb{N}} B_{ij} x^i \frac{z^j}{j!} \tag{3.1}$$

where $B_{ij} = B_{ij}(y_1, y_2, \dots)$ are polynomials in $\mathbb{Q}[y_1, y_2, \dots]$, known as the **Bell polynomials.** A general reference for properties of the B_{ij} is Comtet's book [11]. (Our notation slightly differs from the one used in [11]: $B_{ij} = \mathbf{B}_{ji}$.) We can obtain B_{ij} by differentiating (3.1) appropriately and setting $x = z = 0$:

$$B_{ij} = \frac{1}{i!} \frac{\partial^i \partial^j}{\partial x^i \partial z^j} \exp(x \cdot y) \bigg|_{x=z=0} = \frac{1}{i!} \frac{d^j}{dz^j} y^i \bigg|_{z=0},$$

hence

$$\frac{1}{i!} y^i = \sum_{j \geqslant 0} B_{ij} \frac{z^j}{j!}.$$

In particular, we immediately see that $B_{0j} = 0$ and $B_{1j} = y_j$ for $j \geqslant 1$. Since

$$\frac{1}{i!} y^i = y_1^i \frac{z^i}{i!} + \text{terms of higher degree (in } z)$$

we also see that $B_{ij} = 0$ whenever $i > j$ and $B_{jj} = y_1^j$ for all j. It may also be shown (see [11, Section 3.3, Theorem A]) that $B_{ij} \in \mathbb{Z}[y_1, \dots, y_{j-i+1}]$, and B_{ij} is homogeneous of degree i and isobaric of weight j. (Here each y_j is assigned weight j.) Given a power series $f \in zK[[z]]$, written in the form

$$f = \sum_{n \geqslant 1} f_n \frac{z^n}{n!} \qquad (f_n \in K \text{ for each } n \geqslant 1),$$

we now define the triangular matrix

$$[f] := \big([f]_{ij}\big)_{i,j\in\mathbb{N}} = \big(B_{ij}(f_1,f_2,\dots,f_{j-i+1})\big)_{i,j\in\mathbb{N}} =$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ & f_1 & f_2 & f_3 & f_4 & f_5 & \cdots \\ & & f_1^2 & 3f_1f_2 & 4f_1f_3+3f_2^4 & 5f_1f_4+10f_2f_3 & \cdots \\ & & & f_1^3 & 6f_1^2f_2 & 10f_1^2f_3+15f_1f_2^2 & \cdots \\ & & & & f_1^4 & 10f_1^3f_2 & \cdots \\ & & & & & f_1^5 & \cdots \\ & & & & & & \ddots \end{pmatrix} \in \mathfrak{tr}_K\,.$$

More generally, suppose $\Omega = (\Omega_n)$ is a **reference sequence,** i.e., a sequence of non-zero rational numbers with $\Omega_0 = \Omega_1 = 1$. Then we define the Bell polynomials with respect to Ω by setting

$$y = \sum_{n\geqslant 1} y_n\Omega_n\, z^n \in A[[z]]$$

and expanding

$$\Omega_i y^i = \sum_{j\geqslant 0} B_{ij}^{\Omega}\Omega_j\, z^j \tag{3.2}$$

where $B_{ij}^{\Omega} = B_{ij}^{\Omega}(y_1,y_2,\dots) \in \mathbb{Q}[y_1,y_2,\dots]$. As above, one sees that $B_{0j}^{\Omega}=0$ and $B_{1j}^{\Omega}=y_j$ for $j\geqslant 1$, as well as $B_{ij}^{\Omega}=0$ whenever $i>j$ and $B_{jj}^{\Omega}=y_1^j$ for all j. For

$$f = \sum_{n\geqslant 1} f_n\,\Omega_n z^n \in zK[[z]] \qquad (f_n\in K \text{ for each } n\geqslant 1),$$

we define

$$[f]^{\Omega} := \big([f]_{ij}^{\Omega}\big)_{i,j\in\mathbb{N}} \in \mathfrak{tr}_K \qquad \text{where } [f]_{ij}^{\Omega} := B_{ij}^{\Omega}(f_1,f_2,\dots,f_{j-i+1}).$$

Thus, denoting the reference sequence $(1/n!)$ by Φ, we have $B_{ij}^{\Phi}=B_{ij}$ for each i, j and $[f]^{\Phi}=[f]$ for each $f\in zK[[z]]$. Note that by (3.2) we have, for all reference sequences Ω, $\widetilde{\Omega}$:

$$\frac{\Omega_j}{\Omega_i}\,[f]_{ij}^{\Omega} = \frac{\widetilde{\Omega}_j}{\widetilde{\Omega}_i}\,[f]_{ij}^{\widetilde{\Omega}} \qquad \text{for all } i,\ j, \tag{3.3}$$

that is,

$$(D^{\Omega})^{-1}\,[f]^{\Omega}\,D^{\Omega} = (D^{\widetilde{\Omega}})^{-1}\,[f]^{\widetilde{\Omega}}\,D^{\widetilde{\Omega}} \tag{3.4}$$

where D^{Ω} is the diagonal matrix

$$D^{\Omega} = \begin{pmatrix} \Omega_0 & & & \\ & \Omega_1 & & \\ & & \Omega_2 & \\ & & & \ddots \end{pmatrix} \in \mathfrak{tr}_{\mathbb{Q}}^{\times}\,.$$

In particular, for every reference sequence Ω we have, with $\mathbf{1}$ denoting the constant sequence $(1,1,1,\dots)$:

$$[f]^{\Omega} = D^{\Omega}(D^{\Phi})^{-1}\,[f]\,D^{\Phi}(D^{\Omega})^{-1} = D^{\Omega}\,[f]^{\mathbf{1}}\,(D^{\Omega})^{-1}. \tag{3.5}$$

As first noticed by Jabotinsky [20, 21], a crucial property of $[\ \ ]^\Omega$ is that it converts composition of power series into matrix multiplication [11, Section 3.7, Theorem A]:

$$[f\circ g]^\Omega = [f]^\Omega\cdot[g]^\Omega \qquad \text{for all } f,g\in zK[[z]]. \tag{3.6}$$

To see this, repeatedly use (3.2) to obtain

$$\begin{aligned}\sum_{j\geqslant 0}[f\circ g]^\Omega_{ij}\,\Omega_j z^j = \Omega_i(f\circ g)^i = \Omega_i f^i\circ g &= \sum_{k\geqslant 0}[f]^\Omega_{ik}\,\Omega_k g^k \\ &= \sum_{k\geqslant 0}[f]^\Omega_{ik}\sum_{j\geqslant 0}[g]^\Omega_{kj}\,\Omega_j z^j = \sum_{j\geqslant 0}\left(\sum_{k\geqslant 0}[f]^\Omega_{ik}[g]^\Omega_{kj}\right)\Omega_j z^j\end{aligned}$$

and compare the coefficients of z^j. The matrix $[f]^\Omega$ is called the **iteration matrix of** f with respect to Ω in [11]. (To be precise, [11] uses the transpose of our $[f]^\Omega$.) For $[f]$, the term *convolution matrix* of f is also in use (cf. [22]), and $[f]^{\mathbf{1}}$ is called the *power matrix* of f in [30].

The subset $zK^\times + z^2K[[z]]$ of $zK[[z]]$ forms a group under composition (with identity element z), and $f\mapsto [f]^\Omega$ restricts to an embedding of this group into the group $\mathfrak{tr}_K^\times$ of units of $\mathfrak{tr}_K$. (In particular, $[z]^\Omega = 1$ for each Ω.) As in [11], we say that $f\in zK[[z]]$ is **unitary** if $f_1 = 1$. The set of unitary power series in $K[[z]]$ is a subgroup of $zK^\times + z^2K[[z]]$ under composition, whose image under $f\mapsto[f]^\Omega$ is a subgroup of $1+\mathfrak{tr}_K^1$ which we denote by $\mathcal{M}_K^\Omega$. If Ω is clear from the context, we simply write $\mathcal{M}_K = \mathcal{M}_K^\Omega$. By (3.5), the matrix groups $\mathcal{M}_K^\Omega$, for varying Ω, are all conjugate to each other. We call $\mathcal{M}_K^\Omega$ the **group of iteration matrices over** K with respect to Ω.

Given $f\in K[[z]]$ of the form $f = z + z^{n+1}g$ with $n>0$ and $g\in K[[z]]$ such that $g(0)\neq 0$, we say that the **iterative valuation** of f is n; in symbols: $n = \operatorname{itval}(f)$. (See [13].) It is easy to see that for $f\in zK[[z]]$ and $n>0$, we have $f\in z+z^{n+1}K[[z]]$ if and only if $[f]^\Omega\in 1+\mathfrak{tr}_K^n$. For each $n>0$ we define the subgroup

$$\mathcal{M}_K^{\Omega,n} := \mathcal{M}_K^\Omega\cap(1+\mathfrak{tr}_K^n) = \big\{[f]^\Omega : f\in z+z^{n+1}K[[z]]\big\}$$

of $\mathcal{M}_K^\Omega$. Then

$$\mathcal{M}_K^\Omega = \mathcal{M}_K^{\Omega,1}\supseteq\mathcal{M}_K^{\Omega,2}\supseteq\cdots\supseteq\mathcal{M}_K^{\Omega,n}\supseteq\cdots \quad\text{and}\quad \bigcap_{n>0}\mathcal{M}_K^{\Omega,n} = \{1\},$$

and if $f\in zK[[z]]$ is unitary with $f\neq z$, then $n = \operatorname{itval}(f)$ is the unique $n>0$ such that $[f]^\Omega\in\mathcal{M}_K^{\Omega,n}\setminus\mathcal{M}_K^{\Omega,n+1}$.

As shown by Erdős and Jabotinsky [16], iteration matrices can be used to define "fractional" iterates of formal power series. Let t be a new indeterminate and $K^* = K[t]$.

Proposition 3.1 (Erdős and Jabotinsky). *Suppose K is an integral domain, and let $f\in zK[[z]]$ be unitary. Then there exists a unique power series $f^{[t]}\in zK^*[[z]]$ such that, writing $f^{[a]} := f^{[t]}\big|_{t=a}\in zK[[z]]$ for $a\in K$:*

(1) $f^{[0]} = z$;
(2) $f^{[a+1]} = f^{[a]}\circ f$ *for all* $a,b\in K$.

The power series $f^{[t]}$ is given by

$$f^{[t]} = \sum_{j\geqslant 1} M_{1j}\frac{z^j}{j!} \quad \textit{where } M := \sum_{n\geqslant 0}\binom{t}{n}\big([f]-1\big)^n\in\mathfrak{tr}_{K^*}.$$

Here for every n as usual $\binom{t}{n} = \frac{1}{n!} t(t-1)\cdots(t-n+1) \in \mathbb{Q}[t]$.

Proof. Since $[f] - 1 \in \mathfrak{tr}_K^1$, the sum defining M exists in $\mathfrak{tr}_{K^*}$, and $M\big|_{t=n} = [f]^n$ for every n, by the binomial formula. Let $f^{\circ t} := \sum_{j \geqslant 1} M_{1j} \frac{z^j}{j!}$, and for an element a in a ring extension of K^* write $f^{\circ a} := f^{\circ t}\big|_{t=a}$. Then $[f^{\circ n}]_{1j} = M_{1j}\big|_{t=n} = ([f]^n)_{1j}$ for every $j \geqslant 1$ and thus $f^{\circ n}$ is the nth iterate of f: $f^{\circ n} = f \circ f \circ \cdots \circ f$ (n times). In particular $f^{\circ 1} = f$ and $f^{\circ(m+n)} = f^{\circ m} \circ f^{\circ n}$ for all m, n. Hence if s is another indeterminate, then $f^{\circ(s+t)} = f^{\circ s} \circ f^{\circ t}$ (in $K[s,t][[z]]$), since the coefficients (of equal powers of z) of both sides of this equation are polynomials in s and t with coefficients in K which agree for all integral values of (s,t). This shows that $f^{\circ t}$ satisfies conditions (1) and (2) (with $f^{\circ \cdot}$ replacing $f^{[\,\cdot\,]}$ everywhere). If $f^{[t]} \in K^*[[z]]$ is any power series satisfying (1) and (2), then $f^{[n]} = f^{\circ n}$ is the nth iterate of f, for every n, and as before we deduce $f^{[t]} = f^{\circ t}$. □

The power series $f^{[a]}$ ($a \in K$) in this proposition form a subgroup of $zK[[z]]$ under composition which contains f; they may be thought of as "fractional iterates" of f. (This explains the choice of the term "iteration matrix.")

Some examples of iteration matrices are collected below. Many more (in the case where $\Omega = \Phi$) are given in [22].

Example. Suppose $f = \frac{z}{1-z}$. Then

$$[f]_{ij} = \binom{j-1}{i-1} \frac{j!}{i!} \in \mathbb{N} \qquad (i > 0)$$

are the **Lah numbers**; here and below we set $\binom{j}{i} := 0$ for $i > j$. (See [11, Section 3.3, Theorem B].) Thus if $\Omega_n = \frac{1}{n}$ for each $n > 0$, then by (3.3)

$$[f]^{\Omega}_{ij} = \frac{\Omega_i \Phi_j}{\Omega_j \Phi_i} [f]_{ij} = \binom{j}{i} \qquad \text{for } i > 0,$$

hence

$$[f]^{\Omega} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ & 1 & 2 & 3 & 4 & 5 & \cdots \\ & & 1 & 3 & 6 & 10 & \cdots \\ & & & 1 & 4 & 10 & \cdots \\ & & & & 1 & 5 & \cdots \\ & & & & & 1 & \cdots \\ & & & & & & \ddots \end{pmatrix} \in \mathfrak{tr}_{\mathbb{Z}} \tag{3.7}$$

is Pascal's triangle of binomial coefficients (except for the first row).

Example. The Stirling numbers of the second kind have the egf

$$e^{x(e^z - 1)} = \sum_{i,j} \left\{ {j \atop i} \right\} x^i \frac{z^j}{j!},$$

cf. [11, Section 1.14, (III)] or [18, (7.54)]. Hence by (3.1) we have

$$[e^z - 1] = S, \tag{3.8}$$

where S is as in (1.6). The matrix S is a unit in $\mathfrak{tr}_{\mathbb{Z}}$, and it is well-known (see [11, Section 3.6 (II)]) that the entries of its inverse

$$S^{-1} = (S^{-1}_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ & 1 & -1 & 2 & -6 & 24 & \cdots \\ & & 1 & -3 & 11 & -50 & \cdots \\ & & & 1 & -6 & 35 & \cdots \\ & & & & 1 & -10 & \cdots \\ & & & & & 1 & \cdots \\ & & & & & & \ddots \end{pmatrix} \tag{3.9}$$

are the signed Stirling numbers of the first kind: $S^{-1}_{ij} = (-1)^{j-i}\left[{j \atop i}\right]$, where $\left[{j \atop i}\right]$ denotes the number of permutations of a j-element set having i disjoint cycles. Thus (3.6) and (3.8) yields $\big[\log(1+z)\big] = S^{-1}$.

4. The Lie Algebra of the Group of Iteration Matrices

Throughout this section we let K be a commutative ring which contains $\mathbb{Q}$ as a subring. We let Ω denote a reference sequence. We need a description of the Lie algebra of the matrix group $\mathcal{M}_K = \mathcal{M}_K^{\Omega}$, generalizing the one of the Lie algebra of $\mathcal{M}_{\mathbb{C}}^{\mathbf{1}}$ from [30]. The arguments follow [30], except that we replace the complex-analytic ones used there by algebraic ones.

Definition 4.1. Let $h = \sum_n h_n z^n \in zK[[z]]$. The **infinitesimal iteration matrix** of h with respect to Ω is the triangular matrix

$$\langle h\rangle^{\Omega} = \big(\langle h\rangle^{\Omega}_{ij}\big) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ & h_1 & \frac{\Omega_1}{\Omega_2}\,h_2 & \frac{\Omega_1}{\Omega_3}\,h_3 & \frac{\Omega_1}{\Omega_4}\,h_4 & \cdots \\ & & 2h_1 & \frac{\Omega_2}{\Omega_3}2h_2 & \frac{\Omega_2}{\Omega_4}2h_3 & \cdots \\ & & & 3h_1 & \frac{\Omega_3}{\Omega_4}3h_2 & \cdots \\ & & & & 4h_1 & \cdots \\ & & & & & \ddots \end{pmatrix} \in \mathfrak{tr}_K$$

$$\text{where } \langle h\rangle^{\Omega}_{ij} = \frac{\Omega_i}{\Omega_j}\, i h_{j-i+1}.$$

Note that if Ω, $\widetilde{\Omega}$ are reference sequences, then

$$(D^{\Omega})^{-1}\langle h\rangle^{\Omega} D^{\Omega} = (D^{\widetilde{\Omega}})^{-1}\langle h\rangle^{\widetilde{\Omega}} D^{\widetilde{\Omega}}, \tag{4.1}$$

in particular

$$\langle h\rangle^{\Omega} = D^{\Omega}(D^{\Phi})^{-1}\,\langle h\rangle\, D^{\Phi}(D^{\Omega})^{-1} = D^{\Omega}\,\langle h\rangle^{\mathbf{1}}\,(D^{\Omega})^{-1}.$$

Example 4.2. For $h = \sum_n h_n z^n \in zK[[z]]$ we have

$$\langle h \rangle := \langle h \rangle^{\Phi} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ & h_1 & \frac{2!}{1!}\,h_2 & \frac{3!}{1!}\,h_3 & \frac{4!}{1!}\,h_4 & \cdots \\ & & 2h_1 & \frac{3!}{2!}2h_2 & \frac{4!}{2!}2h_3 & \cdots \\ & & & 3h_1 & \frac{4!}{3!}3h_2 & \cdots \\ & & & & 4h_1 & \cdots \\ & & & & & \ddots \end{pmatrix}$$

$$\text{where } \langle h \rangle_{ij} = \frac{j!}{(i-1)!} h_{j-i+1} \text{ for } i > 0.$$

For each n we have $h \in z^{n+1}K[[z]]$ if and only if $\langle h \rangle^{\Omega} \in \mathfrak{tr}_K^n$. We define the K-submodule

$$\mathfrak{m}_K^{\Omega,n} := \left\{ \langle h \rangle^{\Omega} : h \in z^{n+1}K[[z]] \right\}$$

of $\mathfrak{tr}_K^n$, and we set $\mathfrak{m}_K^{\Omega} := \mathfrak{m}_K^{\Omega,1}$; so

$$\mathfrak{m}_K^{\Omega} = \mathfrak{m}_K^{\Omega,1} \supseteq \mathfrak{m}_K^{\Omega,2} \supseteq \cdots \supseteq \mathfrak{m}_K^{\Omega,n} \supseteq \cdots \quad \text{and} \quad \bigcap_{n>0} \mathfrak{m}_K^{\Omega,n} = \{0\}.$$

If Ω is clear from the context, we abbreviate $\mathfrak{m}_K = \mathfrak{m}_K^{\Omega}$ and $\mathfrak{m}_K^n = \mathfrak{m}_K^{\Omega,n}$. We set

$$e_n^{\Omega} := \langle z^{n+1} \rangle^{\Omega},$$

and we write e_n if the reference sequence Ω is clear from the context. The matrix $e_n = e_n^{\Omega}$ is n-diagonal; in fact

$$e_n = \operatorname{diag}_n\left(\tfrac{\Omega_i}{\Omega_{i+n}} i \right) \in \mathfrak{m}_K^n.$$

Clearly the infinitesimal iteration matrix with respect to Ω of a power series from $zK[[z]]$ can be uniquely written as an infinite sum

$$h_1 e_0 + h_2 e_1 + \cdots \qquad \text{where } h_n \in K \text{ for every } n > 0.$$

Using Lemma 2.1 one verifies easily that

$$[e_m, e_n] = (m-n)e_{m+n} \qquad \text{for all } m,\, n.$$

This implies that

$$\mathfrak{m}_K^n = Ke_n + Ke_{n+1} + \cdots \qquad (n > 0)$$

is an ideal of the Lie K-algebra $\mathfrak{tr}_K^1$. The main goal of this section is to show the following generalization of a result of Schippers [30]:

Theorem 4.3. *Let* $n > 0$. *Then* $\exp(\mathfrak{m}_K^n) = \mathcal{M}_K^n$ (*and hence* $\log(\mathcal{M}_K^n) = \mathfrak{m}_K^n$).

Example 4.4. Let $f = \frac{z}{1-z} \in z\mathbb{Q}[[z]]$, and suppose $\Omega_n = \frac{1}{n}$ for every $n > 0$. Then by (2.4) and (3.7) one sees easily that

$$\log\,[f]^{\Omega} = \operatorname{diag}_1(0,2,3,4,\dots) = \begin{pmatrix} 0 & 0 & 0 & & & \cdots \\ & 0 & 2 & 0 & & \cdots \\ & & 0 & 3 & 0 & \cdots \\ & & & 0 & 4 & \cdots \\ & & & & 0 & \cdots \\ & & & & & \ddots \end{pmatrix} = \langle z^2 \rangle^{\Omega} \in \mathfrak{m}_{\mathbb{Q}}^1.$$

We give the proof of this theorem after some preparatory results. Below we let t be a new indeterminate and $K^* = K[t]$.

Lemma 4.5. *Let $f \in zK^*[[z]]$ and $h \in zK[[z]]$ satisfy*

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial z}\, h.$$

Then

$$\frac{d}{dt}[f]^\Omega = [f]^\Omega \langle h\rangle^\Omega.$$

Proof. We need to show that for all i we have

$$\frac{d}{dt}[f]^\Omega_{ij} = \big([f]^\Omega \langle h\rangle^\Omega\big)_{ij} \qquad \text{for each } j.$$

For $i = 0$ this is an easy computation, so suppose $i > 0$. We have

$$\frac{\partial f^i}{\partial z} = \sum_{j\geqslant 1} j[f]^\Omega_{ij}\Omega_j\, z^{j-1}$$

and hence

$$\frac{\partial f^i}{\partial z}\, h = \sum_{j\geqslant 0}\left(\sum_{k=1}^{j} k[f_{ik}]^\Omega \Omega_k h_{j-k+1}\right) z^j.$$

Moreover

$$\frac{\partial f^i}{\partial t} = \sum_{j\geqslant 0} \frac{d}{dt}[f]^\Omega_{ij}\Omega_j\, z^j.$$

By the hypothesis of the lemma

$$\frac{\partial f^i}{\partial t} = if^{i-1}\frac{\partial f}{\partial t} = if^{i-1}\frac{\partial f}{\partial z}\, h = \frac{\partial f^i}{\partial z}\, h,$$

hence

$$\frac{d}{dt}[f]^\Omega_{ij} = \sum_{k=1}^{j} k[f_{ik}]^\Omega \frac{\Omega_k}{\Omega_j} h_{j-k+1} = \big([f]^\Omega \langle h\rangle^\Omega\big)_{ij}$$

for each j as required. ∎

This lemma is used in the proof of the following important proposition:

Proposition 4.6. *Let $h \in z^{n+1}K[[z]]$, where $n > 0$, and set*

$$f_t := \sum_{j\geqslant 1} (\exp\, t\langle h\rangle^\Omega)_{1j}\Omega_j\, z^j \in z + z^{n+1}K^*[[z]].$$

Then

$$\frac{\partial f_t}{\partial t} = \frac{\partial f_t}{\partial z}\, h \tag{4.2}$$

and hence

$$[f_t]^\Omega = \exp\, t\langle h\rangle^\Omega. \tag{4.3}$$

Proof. By Lemma 2.2 we have

$$\frac{d}{dt}\exp\, t\langle h\rangle^\Omega = (\exp\, t\langle h\rangle^\Omega)\, \langle h\rangle^\Omega.$$

Hence

$$\begin{aligned}\frac{\partial f_t}{\partial t} &= \sum_{j\geqslant 1}\left(\frac{d}{dt}\exp t\langle h\rangle^\Omega\right)_{1j}\Omega_j\, z^j\\ &= \sum_{j\geqslant 1}\big((\exp t\langle h\rangle^\Omega)\langle h\rangle^\Omega\big)_{1j}\Omega_j\, z^j\\ &= \sum_{j\geqslant 1}\left(\sum_{i=1}^{j}(\exp t\langle h\rangle^\Omega)_{1i}\langle h\rangle^\Omega_{ij}\Omega_j\right)z^j\\ &= \sum_{j\geqslant 1}\left(\sum_{i=1}^{j}(\exp t\langle h\rangle^\Omega)_{1i}ih_{j-i+1}\Omega_i\right)z^j = \frac{\partial f_t}{\partial z}\,h.\end{aligned}$$

By Lemma 4.5 this yields $\frac{d}{dt}[f_t]^\Omega = [f_t]^\Omega\langle h\rangle^\Omega$. This shows that both $Y = [f_t]^\Omega$ and $Y = \exp t\langle h\rangle^\Omega$ satisfy $\frac{dY}{dt} = Y\,\langle h\rangle^\Omega$ and $Y\big|_{t=0} = 1$. Hence $[f_t]^\Omega = \exp t\langle h\rangle^\Omega$ by Lemma 2.3. □

The equation (4.2) is called the *formal Loewner partial differential equation* in [30]. The following corollary, obtained by setting $t = 1$ in (4.3) above, shows in particular that $\exp(\mathfrak{m}_K^n) \subseteq \mathcal{M}_K^n$ for each $n > 0$:

Corollary 4.7. *Let $h \in z^{n+1}K[[z]]$, where $n > 0$, and set*

$$f := \sum_{j\geqslant 1}(\exp\langle h\rangle^\Omega)_{1j}\Omega_j\, z^j \in z + z^{n+1}K[[z]].$$

Then $[f]^\Omega = \exp\langle h\rangle^\Omega$.

As above we write $e_k = e_k^\Omega$. Given $k_1,\dots,k_n$ and $k = k_1+\cdots+k_n$, we have

$$e_{k_1}\cdots e_{k_n} = \operatorname{diag}_k\left(\tfrac{\Omega_i}{\Omega_{i+k}}\, i(i+k_1)(i+k_1+k_2)\cdots(i+k_1+\cdots+k_{n-1})\right)_{i\geqslant 0}$$

by Lemma 2.1. Now let $M := \langle h\rangle^\Omega$ where $h \in zK[[z]]$. So

$$M = \langle h\rangle^\Omega = h_1e_0 + h_2e_1 + \cdots$$

and hence

$$M^n = \sum_{k_1,\dots,k_n} h_{k_1+1}\cdots h_{k_n+1}\, e_{k_1}\cdots e_{k_n},$$

that is,

$$(M^n)_{ij} = \sum_{k_1+\cdots+k_n=j-i} h_{k_1+1}\cdots h_{k_n+1}\,\tfrac{\Omega_i}{\Omega_j}\, i(i+k_1)\cdots(i+k_1+\cdots+k_{n-1}) \quad (4.4)$$

for all i, j. This observation leads to:

Lemma 4.8. *Suppose $n > 0$. Then*

$$(M^n)_{11} = h_1^n, \qquad (M^n)_{1j} = \frac{j^n-1}{\Omega_j(j-1)}h_1^{n-1}h_j + P_{nj}^\Omega(h_1,\dots,h_{j-1}) \quad \textit{for } j \geqslant 2,$$

where $P_{nj}^\Omega(Y_0,\dots,Y_{j-2}) \in \mathbb{Q}[Y_0,\dots,Y_{j-2}]$ is homogeneous of degree n and isobaric of weight $j-1$, and independent of h. (Here each Y_i is assigned weight i.)

Proof. Set $i = 1$ in (4.4). Then the only terms involving h_j in this sum are those of the form $h_1^{n-1}h_j\,\frac{1}{\Omega_j}\,j^{n-m}$ where $m \in \{1,\dots,n\}$. This yields the lemma. □

An analogue of the preceding lemma (for $K = \mathbb{C}$ and $\Omega = \mathbf{1}$) is Lemma 3.10 of [30]; however, the formula given there is wrong:

Example. Suppose $h = h_1 z + h_2 z^2$ and $\Omega = \mathbf{1}$. Then

$$M = \langle h \rangle^{\mathbf{1}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \\ & h_1 & h_2 & 0 & 0 & \ddots \\ & & 2h_1 & 2h_2 & 0 & \ddots \\ & & & 3h_1 & 3h_2 & \ddots \\ & & & & 4h_1 & \ddots \\ & & & & & \ddots \end{pmatrix}$$

and hence

$$M^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \\ & h_1^2 & 3h_1h_2 & 2h_2^2 & 0 & \ddots \\ & & 4h_1^2 & 10h_1h_2 & 6h_2^2 & \ddots \\ & & & 9h_1^2 & 21h_1h_2 & \ddots \\ & & & & 16h_1^2 & \ddots \\ & & & & & \ddots \end{pmatrix}.$$

According to [30, Lemma 3.10] we should have, for $j \geqslant 2$:

$$(M^2)_{1j} = 2h_1 h_j + \text{polynomial in } h_1, \dots, h_{j-1}.$$

However $(M^2)_{12} = 3h_1h_2$ is not of this form.

In the proof of Theorem 4.3 we are concerned with the case where $h \in z^2 K[[z]]$, for which we need a refinement of Lemma 4.8:

Lemma 4.9. *Suppose $h \in z^2K[[z]]$ and $n > 0$. Then*

$$(M^n)_{1j} = \begin{cases} \frac{1}{\Omega_j} h_j & \text{if } n = 1, \\ P^{\Omega}_{nj}(h_1, \dots, h_{j-1}) & \text{if } 1 < n < j, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We have $h_1 = 0$, hence if $n > 1$ then $(M^n)_{1j} = P^{\Omega}_{nj}(h_1, \dots, h_{j-1})$ by the previous lemma. We have $M \in \mathfrak{tr}^1_K$ and hence $M^n \in \mathfrak{tr}^n_K$, so $(M^n)_{1j} = 0$ if $j-1 < n$, that is, if $j \leqslant n$. The lemma follows. □

Corollary 4.10. *Suppose $h \in z^2K[[z]]$. Then for $j \geqslant 2$:*

$$(\exp M)_{1j} = \frac{1}{\Omega_j} h_j + P^{\Omega}_j(h_2, \dots, h_{j-1})$$

where $P^{\Omega}_j(Y_1, \dots, Y_{j-2}) \in \mathbb{Q}[Y_1, \dots, Y_{j-2}]$ is independent of h. (In particular, $(\exp M)_{1j}$ is polynomial in $h_2, \dots, h_j$.) Moreover, $P^{\Omega}_2 = 0$, and for $j > 2$, P^{Ω}_j has degree $j-1$ and is isobaric of weight $j-1$.

Proof. By the previous lemma we have

$$(\exp M)_{1j} = \sum_{n=1}^{j-1} \frac{1}{n!}(M^n)_{1j} = \frac{1}{\Omega_j} h_j + \sum_{n=2}^{j-1} \frac{1}{n!} P^{\Omega}_{nj}(h_1, \dots, h_{j-1}).$$

Hence

$$P^{\Omega}_j(Y_1, \dots, Y_{j-2}) := \sum_{n=2}^{j-1} \frac{1}{n!} P^{\Omega}_{nj}(0, Y_1, \dots, Y_{j-2})$$

has the right properties. □

Theorem 4.3 now follows immediately from Corollary 4.7 and the following:

Proposition 4.11. *Let $f \in zK[[z]]$ be unitary, $n = \operatorname{itval}(f)$. Then $\log [f]^{\Omega} \in \mathfrak{m}^n_K$.*

Proof. We define a sequence $(h_j)_{j\geqslant 1}$ recursively as follows: set $h_1 := 0$, and assuming inductively that $h_2, \dots, h_j$ have been defined already, where $j > 0$, let $h_{j+1} := (f_{j+1} - P^{\Omega}_{j+1}(h_2, \dots, h_j))\Omega_{j+1}$. Let $h := \sum_{j\geqslant 1} h_j z^j \in z^{n+1}K[[z]]$ and $M := \langle h\rangle^{\Omega}$. Then by the corollary above, we have $(\exp M)_{1j} = f_j$ for every j. Corollary 4.7 now yields $\exp M = [f]^{\Omega}$ and hence $\log [f]^{\Omega} = M = \langle h\rangle^{\Omega} \in \mathfrak{m}^n_K$. □

Remark. The mistake in [30, Lemma 3.10] pointed out in the example following the proof of Lemma 4.8 affects the statements of items 3.14 and 3.15 and the proofs of 3.13–3.17 in loc. cit. (which concern the shape of $\log [f]$ for non-unitary $f \in z\mathbb{C}[[z]]$); however, based on the correct formula in Lemma 4.8 above, it is routine to make the necessary changes. For example, the corrected version of [30, Corollary 3.14] states that (using our notation) for $h \in z\mathbb{C}[[z]]$ and $j \geqslant 2$ we have

$$[\exp\langle h\rangle^{\mathbf{1}}]_{1j} = \frac{h_j}{j-1}\left(\frac{e^{jh_1} - e^{h_1}}{h_1}\right) + \Phi_j(h_1, \dots, h_{j-1})$$

where Φ_j is an entire function $\mathbb{C}^{j-1} \to \mathbb{C}$.

5. The Iterative Logarithm

In this section we let K be an integral domain which contains $\mathbb{Q}$ as a subring, and Ω be a reference sequence. Let $f \in zK[[z]]$ be unitary. By Theorem 4.3 there exists a (unique) power series $h \in z^2K[[z]]$ such that $\log [f]^{\Omega} = \langle h\rangle^{\Omega}$. The identities (2.2), (3.4) and (4.1) show that h does not depend on Ω. Indeed, we have

$$h = \sum_{n\geqslant 1} \frac{(-1)^{n-1}}{n} h[n]$$

where $h[0] = z$ and $h[n+1] = h[n] \circ f - h[n] \in z^{n+1}K[[z]]$ for every n.

As in [13], we call the power series h the **iterative logarithm of** f, and we denote it by $h = \operatorname{itlog}(f)$ or $h = f_*$. In the following we let s, t be new distinct indeterminates, and we write

$$f^{[t]} = \sum_{j\geqslant 1} (\exp\, t\langle f_*\rangle^{\Omega})_{1j}\Omega_j\, z^j \in z + z^{n+1}K[t][[z]], \qquad n = \operatorname{itval}(f).$$

Note that $f^{[t]}$ does not depend on the choice of reference sequence Ω. For an element a of a ring extension K^* of K let

$$f^{[a]} := f^{[t]}\big|_{t=a} \in z + z^{n+1}K^*[[z]],$$

so $f^{[0]} = z$ and $f^{[1]} = f$. The notations $f^{[t]}$ and $f^{[a]}$ do not conflict with the ones introduced in Proposition 3.1: by (2.1) and (4.3) (in Proposition 4.6) we have

$$[f^{[s+t]}]^\Omega = \exp{(s+t)}\langle h\rangle^\Omega = \exp\, s\langle h\rangle^\Omega \cdot \exp\, t\langle h\rangle^\Omega = [f^{[s]}]^\Omega \cdot [f^{[t]}]^\Omega = [f^{[s]} \circ f^{[t]}]^\Omega$$

and hence

$$f^{[s+t]} = f^{[s]} \circ f^{[t]} \tag{5.1}$$

in $K[s,t][[z]]$. Equation (4.2) also yields

$$\operatorname{itlog}(f) = \left.\frac{\partial f^{[t]}}{\partial t}\right|_{t=0}.$$

If $a \in K$ then $(f^{[a]})^{[t]} = f^{[at]}$ by the uniqueness statement in Proposition 3.1 and hence

$$\operatorname{itlog}(f^{[a]}) = a \operatorname{itlog}(f) \qquad \text{for all } a \in K. \tag{5.2}$$

Aczél [2] and Jabotinsky [20] also showed that the iterative logarithm satisfies a functional equation (although [19] suggests that Frege had already been aware of this equation much earlier):

Proposition 5.1 (Aczél and Jabotinsky)**.**

$$f_* \cdot \frac{\partial f^{[t]}}{\partial z} = \frac{\partial f^{[t]}}{\partial t} = f_* \circ f^{[t]} \tag{5.3}$$

and hence

$$f_* \cdot \frac{df}{dz} = f_* \circ f. \tag{5.4}$$

The equation (5.4) is known as *Julia's equation* in iteration theory. (See [24, Section 8.5A].) The first equation in (5.3) is simply (4.2). To show the second equation $\frac{\partial f^{[t]}}{\partial t} = f_* \circ f^{[t]}$, simply differentiate (5.1) with respect to s:

$$\left.\frac{\partial f^{[u]}}{\partial u}\right|_{u=s+t} = \left.\frac{\partial f^{[u]}}{\partial u}\right|_{u=s+t} \cdot \frac{\partial (s+t)}{\partial s} = \frac{\partial f^{[s+t]}}{\partial s} = \frac{\partial (f^{[s]} \circ f^{[t]})}{\partial s} = \frac{\partial f^{[s]}}{\partial s} \circ f^{[t]}.$$

Setting $s = 0$ yields the desired result.

Suppose now that $K = \mathbb{C}$. Even if f is convergent, for given $a \in \mathbb{C}$ the formal power series $f^{[a]}$ is not necessarily convergent. In fact, by remarkable results of Baker [7], Écalle [14] and Liverpool [27], there are only three possibilities:

(1) $f^{[a]}$ has radius of convergence 0 for all $a \in \mathbb{C}$, $a \neq 0$;
(2) there is some non-zero $a_1 \in \mathbb{C}$ such that $f^{[a]}$ has positive radius of convergence if and only if a is an integer multiple of a_1; or
(3) $f^{[a]}$ has positive radius of convergence for all $a \in \mathbb{C}$.

If (3) holds, then one calls f **embeddable** (in a continuous group of analytic iterates of f). This is a very rare circumstance; for example, Baker [6] and Szekeres [33] showed that if f is the Taylor series at 0 of a meromorphic function on the whole complex plane which is regular at 0, then f is not embeddable except in the case where

$$f = \frac{z}{1-cz} \qquad (c \in \mathbb{C}).$$

In this case, $\operatorname{itlog}(f) = cz^2$ by Example 4.4 and (5.2). Erdős and Jabotinsky [16] showed that in general, f is embeddable if and only if $f_* = \operatorname{itlog}(f)$ has a positive radius of convergence. (See also [23, Theorem 9.15] or [29] for an exposition.) As a consequence, very rarely does f_* have a positive radius of convergence. (However, Écalle [12] has shown that f_* is always Borel summable.) In particular, we obtain a negative answer to the question posed in [30, Question 4.3]: *if f is convergent, is f_* convergent?* Contrary to what is conjectured in [30], the converse question (Question 4.1 in [30]), however, is seen to have a positive answer: *if f_* is convergent, then f is convergent.*

In the next section we discuss when iterative logarithms satisfy algebraic differential equations.

6. Differential Transcendence of Iterative Logarithms

Before we state the main result of this section, we introduce basic terminology concerning differential rings and differential polynomials.

Differential rings. Let R be a **differential ring**, that is, a commutative ring R equipped with a derivation ∂ of R. We also write y' instead of $\partial(y)$ and similarly $y^{(n)}$ instead of $\partial^n(y)$, where ∂^n is the nth iterate of ∂. The set $C_R := \{y \in R : y' = 0\}$ is a subring of R, called the ring of constants of R. A subring of R which is closed under ∂ is called a **differential subring** of R. If R is a differential subring of a differential ring $\widetilde{R}$ and $y \in \widetilde{R}$, the smallest differential subring of $\widetilde{R}$ containing $R \cup \{y\}$ is the subring $R\{y\} := R[y, y', y'', \dots]$ of $\widetilde{R}$ generated by R and all the derivatives $y^{(n)}$ of y. A differential field is a differential ring whose underlying ring happens to be a field. The ring of constants of a differential field F is a subfield of F. The derivation of a differential ring whose underlying ring is an integral domain extends uniquely to a derivation of its fraction field, and we always consider the derivation extended in this way. If R is a differential subring of a differential field F and $y \in F^\times$, then $R_y := \{a/y^n : a \in R,\ n \geqslant 0\}$ is a differential subring of F.

Differential polynomials. Let Y be a differential indeterminate over the differential ring R. Then $R\{Y\}$ denotes the ring of differential polynomials in Y over R. As ring, $R\{Y\}$ is just the polynomial ring $R[Y, Y', Y'', \dots]$ in the distinct indeterminates $Y^{(n)}$ over R, where as usual we write $Y = Y^{(0)}$, $Y' = Y^{(1)}$, $Y'' = Y^{(2)}$. We consider $R\{Y\}$ as the differential ring whose derivation, extending the derivation of R and also denoted by ∂, is given by $\partial(Y^{(n)}) = Y^{(n+1)}$ for every n. For $P(Y) \in R\{Y\}$ and y an element of a differential ring containing R as a differential subring, we let $P(y)$ be the element of that extension obtained by substituting $y, y', \dots$ for $Y, Y', \dots$ in P, respectively. We call an equation of the form

$$P(Y) = 0 \qquad (\text{where } P \in R\{Y\},\ P \neq 0)$$

an **algebraic differential equation** (ADE) over R, and a solution of such an ADE is an element y of a differential ring extension of R with $P(y) = 0$. We say that an element y of a differential ring extension of R is **differentially algebraic over** R if y is the solution of an ADE over R, and if y is not differentially algebraic over R, then y is said to be **differentially transcendental over** R**.** Clearly to be algebraic over R means in particular to be differentially algebraic over R.

Being differentially algebraic is transitive; this well-known fact follows from basic properties of transcendence degree of field extensions:

Lemma 6.1. *Let F be a differential field and let R be a differential subring of F. If $f \in F$ is differentially algebraic over R and $g \in F$ is differentially algebraic over $R\{f\}$, then g is differentially algebraic over R.*

Differential transcendence of iterative logarithms. Let now K be an integral domain containing $\mathbb{Q}$ as a subring, and let z be an indeterminate over K. We view $K[[z]]$ as a differential ring with the derivation $\frac{d}{dz}$. The ring of constants of $K[[z]]$ is K. We simply say that $f \in K[[z]]$ is **differentially algebraic** or **differentially transcendental** if f is differentially algebraic respectively differentially transcendental over $K[z]$. If $f \in K[[z]]$ is differentially algebraic, then f is actually differentially algebraic over K, by Lemma 6.1.

As above, we let t be a new indeterminate over K, and $K^* = K[t]$. The goal of this section is to show:

Theorem 6.2. *Let $f \in zK[[z]]$ be unitary. Then $f_* \in z^2K[[z]]$ is differentially algebraic if and only if $f^{[t]} \in zK^*[[z]]$ is differentially algebraic, if and only if $f^{[t]}$ is differentially algebraic over K^*.*

Before we give the proof, we introduce some more terminology concerning differential polynomials, and we make a few observations about how the derivation $\frac{d}{dz}$ of $K[[z]]$ and composition in $K[[z]]$ interact with each other, in particular in connection with solutions of Julia's equation.

More terminology about differential polynomials. Let R be a differential ring and $P \in R\{Y\}$. The smallest $r \in \mathbb{N}$ such that $P \in R[Y, Y', \dots, Y^{(r)}]$ is called the **order** of the differential polynomial P. Given a non-zero $P \in R\{Y\}$ we define its **rank** to be the pair $(r, d) \in \mathbb{N}^2$ where $r = \operatorname{order}(P)$ and d is the degree of P in the indeterminate $Y^{(r)}$. In this context we order $\mathbb{N}^2$ lexicographically.

For any $(r+1)$-tuple $\boldsymbol{i} = (i_0, \dots, i_r)$ of natural numbers and $Q \in R\{Y\}$, put

$$Q^{\boldsymbol{i}} := Q^{i_0}(Q')^{i_1} \cdots (Q^{(r)})^{i_r}.$$

In particular, $Y^{\boldsymbol{i}} = Y^{i_0}(Y')^{i_1} \cdots (Y^{(r)})^{i_r}$, and $y^{\boldsymbol{i}} = y^{i_0}(y')^{i_1} \cdots (y^{(r)})^{i_r}$ for $y \in R$.

Let $P \in R\{Y\}$ have order r, and let $\boldsymbol{i} = (i_0, \dots, i_r)$ range over $\mathbb{N}^{1+r}$. We denote by $P_{\boldsymbol{i}} \in R$ the coefficient of $Y^{\boldsymbol{i}}$ in P; then

$$P(Y) = \sum_{\boldsymbol{i}} P_{\boldsymbol{i}}\, Y^{\boldsymbol{i}}.$$

We also define the support of P as

$$\operatorname{supp} P := \big\{\boldsymbol{i} : P_{\boldsymbol{i}} \neq 0\big\}.$$

We set

$$|\boldsymbol{i}| := i_0 + \cdots + i_r, \qquad \|\boldsymbol{i}\| := i_1 + 2i_2 + \cdots + ri_r.$$

For non-zero $P \in R\{Y\}$ we call

$$\deg(P) = \max_{\boldsymbol{i} \in \operatorname{supp} P} |\boldsymbol{i}|, \qquad \operatorname{wt}(P) = \max_{\boldsymbol{i} \in \operatorname{supp} P} \|\boldsymbol{i}\|$$

the **degree** of P respectively **weight** of P. We say that P is **homogeneous** if $|\boldsymbol{i}| = \deg(P)$ for every $\boldsymbol{i} \in \operatorname{supp} P$ and **isobaric** if $\|\boldsymbol{i}\| = \operatorname{wt}(P)$ for every $\boldsymbol{i} \in \operatorname{supp} P$.

Transformation formulas. Let X be a differential indeterminate over $K[[z]]$. An easy induction on n shows that for each $n > 0$ there are differential polynomials $G_{mn} \in \mathbb{Z}\{X\}$ $(1 \leqslant m \leqslant n)$ such that for all $f \in zK[[z]]$ and $h \in K[[z]]$ we have

$$(h^{(n)} \circ f) \cdot (f')^{2n-1} = G_{1n}(f)\,(h \circ f)' + G_{2n}(f)\,(h \circ f)'' + \cdots + G_{nn}(f)\,(h \circ f)^{(n)}.$$

Moreover, G_{mn} has order $n-m+1$, and is homogeneous of degree $n-1$ and isobaric of weight $2n-m-1$. Set $G_{mn} := 0$ if $m > n$ or $m = 0 < n$, and $G_{00} := (X')^{-1} \in \mathbb{Z}\{X\}_{X'}$. Then the G_{mn} satisfy the recurrence relation

$$G_{m,n+1} = (1-2n)G_{mn}X'' + (G'_{mn} + G_{m-1,n})X' \qquad (m > 0).$$

Organizing the G_{mn} into a triangular matrix we obtain:

$$G := (G_{mn})_{m,n} = \begin{pmatrix} (X')^{-1} & 0 & 0 & 0 & \cdots \\ & 1 & -X'' & 3(X'')^2 - X'X^{(3)} & \cdots \\ & & X' & -3X'X'' & \cdots \\ & & & (X')^2 & \cdots \\ & & & & \ddots \end{pmatrix}. \tag{6.1}$$

Note that $G_{nn} = (X')^{n-1}$ for every n. Now set

$$H_{kn} = \sum_{m=k}^{n} \binom{m}{k} X^{(m-k+1)} G_{mn} \in \mathbb{Z}\{X\} \qquad \text{for } k = 0, \dots, n.$$

So if we define the triangular matrix

$$B := (B_{km}) = \begin{pmatrix} X' & X'' & X^{(3)} & X^{(4)} & \cdots \\ & X' & 2X'' & 3X^{(3)} & \cdots \\ & & X' & 3X'' & \cdots \\ & & & X' & \cdots \\ & & & & \ddots \end{pmatrix}$$

$$\text{where } B_{km} = \binom{m}{k} X^{(m-k+1)} \text{ for } m \geqslant k,$$

then

$$H := (H_{kn}) = B \cdot G =$$

$$\begin{pmatrix} 1 & X'' & X'X^{(3)} - (X'')^2 & (X')^2X^{(4)} - 4X'X''X^{(3)} + 3(X'')^3 & \cdots \\ & X' & X'X'' & -3X'(X'')^2 + 2(X')^2X^{(3)} & \cdots \\ & & (X')^2 & 0 & \cdots \\ & & & (X')^3 & \cdots \\ & & & & \ddots \end{pmatrix}.$$

Each differential polynomial H_{kn} has order at most $n-k+1$, and if non-zero, is homogeneous of degree n and isobaric of weight $2n-k$. Note that for $n > 0$, H_{0n} has the form

$$H_{0n} = \sum_{m=1}^{n} X^{(m+1)} G_{mn} = (X')^{n-1} X^{(n+1)} + H_n \qquad \text{where } H_n \in \mathbb{Z}[X', \dots, X^{(n)}];$$

in particular $\operatorname{order}(H_{0n}) = n+1 > \operatorname{order}(H_{kn})$ for $k = 1, \dots, n$.

Let now $f \in zK[[z]]$ and $h \in K[[z]]$ satisfy Julia's equation

$$h \cdot f' = h \circ f.$$

We assume $f \neq 0$ (and hence $f' \neq 0$). Then for every n:

$$(h^{(n)} \circ f) \cdot (f')^{2n-1} = H_{0n}(f)\, h + H_{1n}(f)\, h' + \cdots + H_{nn}(f)\, h^{(n)}.$$

Let $R := K\{X\}_{X'}$, and denote the R-algebra automorphism of $R\{Y\}$ with

$$Y^{(n)} \mapsto (X')^{1-2n}\big(H_{0n}\,Y + H_{1n}\,Y' + \cdots + H_{nn}\,Y^{(n)}\big) \qquad \text{for every } n$$

also by H. Then for every $P \in K\{Y\}$ we have

$$P(h) \circ f = H(P)\big|_{X=f,Y=h}.$$

Note that for every $i \in \mathbb{N}$ and n we can write

$$(X')^{(2n-1)i} \cdot H\big((Y^{(n)})^i\big) = (X')^{i(n-1)} Y^i (X^{(n+1)})^i + a_i$$
$$\text{where } a_i \in \mathbb{Z}[X',\dots,X^{(n+1)},Y,Y',\dots,Y^{(n)}] \text{ with } \deg_{X^{(n+1)}} a_i < i.$$

Hence given $\boldsymbol{i} = (i_0,\dots,i_r) \in \mathbb{N}^{r+1}$, setting $d = |\boldsymbol{i}|$ and $w = \|\boldsymbol{i}\|$, we may write

$$(X')^{2w-d} \cdot H(Y^{\boldsymbol{i}}) = (X')^{w-d} (X')^{\boldsymbol{i}} Y^d + a_{\boldsymbol{i}}$$
$$\text{where } a_{\boldsymbol{i}} \in \mathbb{Z}[X',\dots,X^{(r+1)},Y,Y',\dots,Y^{(r)}] \text{ with } \deg_{X^{(r+1)}} a_{\boldsymbol{i}} < i_r.$$

Proof of Theorem 6.2. Let $f \in zK[[z]]$ be unitary. Suppose first that $f^{[t]}$ is differentially algebraic over K^*. Let $P \in K^*\{Y\}$ be non-zero of lowest rank such that $P(f^{[t]}) = 0$. Differentiating with respect to t on both sides of this equation yields

$$P^*(f^{[t]}) + \sum_{i=0}^{r} \frac{\partial P}{\partial Y^{(i)}}(f^{[t]}) \cdot \frac{\partial (f^{[t]})^{(i)}}{\partial t} = 0.$$

Here $r = \operatorname{order}(P)$ and $P^*(Y) \in K^*\{Y\}$ is the differential polynomial obtained by applying $\frac{d}{dt}$ to each coefficient of the differential polynomial P. Now by Proposition 5.1 we further have

$$\frac{\partial (f^{[t]})^{(i)}}{\partial t} = \left(\frac{\partial f^{[t]}}{\partial t}\right)^{(i)} = \big(f_* \cdot (f^{[t]})'\big)^{(i)} = \sum_{j=0}^{i} \binom{j}{i} (f^{[t]})^{(i-j+1)} f_*^{(j)}.$$

Since $\frac{\partial P}{\partial Y^{(r)}}$ has lower rank than P, by choice of P we have $\frac{\partial P}{\partial Y^{(r)}}(f^{[t]}) \neq 0$. Hence f_* satisfies a non-trivial (inhomogeneous) linear differential equation with coefficients from $K^*\{f^{[t]}\}$, and so by Lemma 6.1, is differentially algebraic over K^*. Specializing t to a suitable rational number in an ADE over K^* satisfied by f_* shows that then f_* also satisfies an ADE over K, that is, f_* is differentially algebraic over K.

Conversely, suppose that f_* is differentially algebraic. Let $P \in K\{Y\}$ be non-zero, of some order r, such that $P(f_*) = 0$. Then

$$H(P)(f^{[t]}, f_*) = P(f_*) \circ f^{[t]} = 0.$$

Let $d = \deg_{Y^{(r)}} P$. By the remarks in the previous subsection, for sufficiently large $N \in \mathbb{N}$ we have

$$(X')^N\, H(P) = \sum_{\boldsymbol{i}: i_r = d} P_{\boldsymbol{i}}\, (X')^{N - \|\boldsymbol{i}\|} Y^{|\boldsymbol{i}|} + A$$
$$\text{where } A \in K[X',\dots,X^{(r+1)},Y,Y',\dots,Y^{(r)}] \text{ with } \deg_{X^{(r+1)}} A < d.$$

For such N, the differential polynomial

$$Q(X) := (X')^N\, H(P)\big|_{Y=f_*} \in R\{X\}$$

is non-zero, where $R = K\{f_*\}$, and satisfies $Q(f^{[t]}) = 0$. Thus $f^{[t]}$ is differentially algebraic over R and hence (by Lemma 6.1) over K, as required. □

Let $\mathcal{F}$ be a family of elements of $K[[z]]$. Following [10] we say that $\mathcal{F}$ is **coherent** if there is a non-zero differential polynomial $P \in K[z]\{Y\}$ such that $P(f) = 0$ for every $f \in \mathcal{F}$. If $\mathcal{F}$ is coherent, then P with these properties may actually be chosen to have coefficients in K; see [10, Lemma 2.1]. If $\mathcal{F}$ is not coherent, then we say that $\mathcal{F}$ is **incoherent**; we also say that $\mathcal{F}$ is **totally incoherent** if every infinite subset of $\mathcal{F}$ is incoherent. From the previous theorem we immediately obtain a result stated without proof in [10]:

Corollary 6.3 (Boshernitzan and Rubel [10])**.** *Let $f \in zK[[z]]$ be unitary and let $\mathcal{F} := \{f^{[0]}, f^{[1]}, f^{[2]}, \dots\}$ be the family of iterates of f. Then exactly one of the following holds:*

(1) *f_* is differentially algebraic and $\mathcal{F}$ is coherent;*
(2) *f_* is differentially transcendental and $\mathcal{F}$ is totally incoherent.*

Proof. By the theorem above, it suffices to show: if $f^{[t]}$ is differentially algebraic, then $\mathcal{F}$ is coherent, and if $f^{[t]}$ is differentially transcendental, then $\mathcal{F}$ is totally incoherent. The first implication is obvious (specialize t to n in a given ADE for $f^{[t]}$). For the second implication, suppose $\mathcal{F}$ is not totally incoherent. Then there exists an infinite sequence (n_i) of pairwise distinct natural numbers such that $\{f^{[n_i]}\}$ is coherent. Let $P \in K\{Y\}$, $P \neq 0$, be such that $P(f^{[n_i]}) = 0$ for every i. With $g := P(f^{[t]}) \in K^*[[z]]$ we then have $g\big|_{t=n_i} = 0$ for every i; thus $g = 0$ (since the coefficients of g are polynomials in t with coefficients from the integral domain K^* of characteristic 0). This shows that $f^{[t]}$ is differentially algebraic. $\square$

7. The Iterative Logarithm of $e^z - 1$

In this section we apply the results obtained in Sections 4 and 5 to the unitary power series $f = e^z - 1 \in z\mathbb{Q}[[z]]$. Recall that the iteration matrix $[e^z - 1]$ of this power series is the matrix $S = (S_{ij}) \in 1 + \mathfrak{tr}^1_{\mathbb{Q}}$ consisting of the Stirling numbers $S_{ij} = \left\{ {j \atop i} \right\}$ of the second kind (cf. (3.8)).

Proof of the conjecture. We first finish the proof of the conjecture stated in Section 1. The matrix S is related to $A = (\alpha_{ij}) \in \mathfrak{tr}^1_{\mathbb{Q}}$ via the equation

$$S^+ = \exp(A),$$

or equivalently (cf. (2.3)):

$$A = \log(S)^+.$$

(Recall: for a given matrix $M = (M_{ij}) \in \mathfrak{tr}_{\mathbb{Q}}$ we defined $M^+ = (M_{i+1,j+1})_{i,j} \in \mathfrak{tr}_{\mathbb{Q}}$.) The conjecture postulates the existence of a sequence $(c_n)_{n\geqslant 1}$ of rational numbers such that

$$\alpha_{ij} = c_{j-i+1}\binom{j+1}{i} \qquad \text{for } i < j. \tag{7.1}$$

This now follows easily from the results of Section 4:

Proposition 7.1. *Let $h = \operatorname{itlog}(e^z - 1) \in z^2\mathbb{Q}[[z]]$, write $h = \sum_{n\geqslant 1} h_n z^n$ where $h_n \in \mathbb{Q}$, and define $c_n := n!\, h_n$ for $n \geqslant 1$. Then (7.1) holds, and*

$$c_n = \sum_{\substack{1\leqslant k<n \\ 1<n_1<\cdots<n_{k-1}<n_k=n}} \frac{(-1)^{k+1}}{k} \left\{ {n_2 \atop n_1} \right\} \left\{ {n_3 \atop n_2} \right\} \cdots \left\{ {n_k \atop n_{k-1}} \right\}$$

for every $n \geqslant 1$.

Proof. We have $\log(S) = \langle h \rangle$ by Theorem 4.3. Hence, using the formula for $\langle h \rangle_{ij}$ from Example 4.2 we obtain for $i < j$, as required:

$$\alpha_{ij} = \langle h \rangle_{i+1,j+1} = \frac{(j+1)!}{i!} h_{j-i+1} = \frac{(j+1)!}{i!(j-i+1)!} c_{j-i+1} = c_{j-i+1} \binom{j+1}{i}$$

The displayed identity for c_n follows from $c_n = \langle h \rangle_{1n} = \log(S)_{1n}$. $\square$

We note that the c_n may also be expressed using the Stirling numbers of the first kind, using $\langle h \rangle = -\log(S^{-1})$:

$$c_n = \sum_{\substack{1 \leqslant k < n \\ 1 < n_1 < \cdots < n_{k-1} < n_k = n}} \frac{(-1)^{k+n-n_1}}{k} \begin{bmatrix} n_2 \\ n_1 \end{bmatrix} \begin{bmatrix} n_3 \\ n_2 \end{bmatrix} \cdots \begin{bmatrix} n_k \\ n_{k-1} \end{bmatrix} \qquad (n \geqslant 1).$$

Proof of the convolution identity. We now turn to the convolution identity (C) for Stirling numbers stated in the introduction. Jabotinsky's functional equation (5.4) for $f = e^z - 1$, writing again $h = f_*$, reads as follows:

$$h \circ (e^z - 1) = e^z\, h.$$

Taking derivatives on both sides of this equation and dividing by e^z we obtain:

$$h' \circ (e^z - 1) = h + h'. \tag{7.2}$$

Now define, for $M \in 1 + \mathfrak{tr}^1_{\mathbb{Q}}$:

$$\Lambda(M) := \sum_n \frac{(-1)^n}{n+1} (M-1)^n \in 1 + \mathfrak{tr}^1_{\mathbb{Q}},$$

so

$$\Lambda(M) \cdot (M-1) = \log(M). \tag{7.3}$$

For later use we note that then for every $j \geqslant 1$:

$$\sum_{k=1}^{j} \Lambda(M)_{1k} M_{k,j+1} = \sum_{k=1}^{j+1} \Lambda(M)_{1k} (M-1)_{k,j+1} = \\ (\Lambda(M) \cdot (M-1))_{1,j+1} = \log(M)_{1,j+1}, \tag{7.4}$$

where in the last equation we used (7.3).

Taking $M = S$ we compute

$$\log(S) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{2}{3} & \frac{11}{12} & \cdots \\ & & 0 & 3 & -2 & \frac{5}{2} & -4 & \cdots \\ & & & 0 & 6 & -5 & \frac{15}{2} & \cdots \\ & & & & 0 & 10 & 10 & \cdots \\ & & & & & 0 & -15 & \cdots \\ & & & & & & 0 & \cdots \\ & & & & & & & \ddots \end{pmatrix}$$

and

$$\Lambda(S) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ & 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{2}{3} & \frac{11}{12} & \cdots \\ & & 1 & -\frac{3}{2} & \frac{5}{2} & -\frac{25}{6} & \cdots \\ & & & 1 & -3 & \frac{15}{2} & \cdots \\ & & & & 1 & -5 & \cdots \\ & & & & & 1 & \cdots \\ & & & & & & \ddots \end{pmatrix}$$

We observe that the first row of $\Lambda(S)$ agrees with the first row of $\log(S)$ shifted by one place to the left. (This is simply a reformulation of the formula (C).)

Proposition 7.2. *For every $j \geqslant 1$,*

$$\Lambda(S)_{1j} = \log(S)_{1,j+1}.$$

Proof. As observed in (7.4),

$$\sum_{k=1}^{j} \Lambda(S)_{1k} \left\{ {j+1 \atop k} \right\} = c_{j+1} \qquad \text{for } j \geqslant 1. \tag{7.5}$$

On the other hand, by (7.2) we have $[h'] \cdot S = [h+h']$; thus

$$\sum_{k=1}^{j+1} c_{k+1} \left\{ {j+1 \atop k} \right\} = \sum_{k=1}^{j+1} [h']_{1k} S_{k,j+1} = ([h'] \cdot S)_{1,j+1} = [h+h']_{1,j+1} = c_{j+1} + c_{j+2}$$

and hence

$$\sum_{k=1}^{j} c_{k+1} \left\{ {j+1 \atop k} \right\} = c_{j+1} \qquad \text{for } j \geqslant 1. \tag{7.6}$$

An easy induction on j using (7.5) and (7.6) now yields $\Lambda(S)_{1j} = c_{j+1} = \log(S)_{1,j+1}$ for each $j \geqslant 1$, as claimed. □

Differential transcendence of the egf of (c_n). It is easy to see that for $n > 0$, the nth iterate $\phi^{[n]}$ of $\phi = e^z - 1$ is a solution of an ADE over $\mathbb{Q}$ of order n. However, it is well-known that $\phi^{[n]}$ does not satisfy an ADE over $\mathbb{C}[z]$ of order $< n$. (See, e.g., [5, Corollary 3.7].) The egf of the sequence (c_n) is $\operatorname{itlog}\phi$, hence from Corollary 6.3 we obtain the fact (mentioned in the introduction) that this egf is differentially transcendental. In fact, Bergweiler [9] showed the more general result that if f is (the Taylor series at 0 of) any transcendental entire function, then $\operatorname{itlog}(f)$ is differentially transcendental (equivalently, by Corollary 6.3, the family of iterates of f is totally incoherent). Moreover, by the results quoted at the end of the previous section, $\operatorname{itlog}\phi$ is not convergent. (This can also be shown directly; cf. [25].) See [3] for a proof of a common generalization of these two facts.

References

1. *The On-Line Encyclopedia of Integer Sequences,* published online at `http://oeis.org`, 2010.
2. J. Aczél, *Einige aus Funktionalgleichungen zweier Veränderlichen ableitbare Differentialgleichungen,* Acta Univ. Szeged. Sect. Sci. Math. **13** (1950), 179–189.
3. M. Aschenbrenner and W. Bergweiler, *Julia's equation and differential transcendence,* manuscript (2010).
4. M. Aschenbrenner and L. van den Dries, *Asymptotic differential algebra,* in: O. Costin, M. D. Kruskal, and A. Macintyre (eds.), *Analyzable Functions and Applications,* 49–85, Contemp. Math. vol. 373, Amer. Math. Soc., Providence, RI (2005).

5. ———, *Liouville closed H-fields,* J. Pure Appl. Algebra **197** (2005), 83–139.
6. I. N. Baker, *Fractional iteration near a fixpoint of multiplier* 1*,* J. Austral. Math. Soc. **4** (1964), 143–148.
7. ———, *Permutable power series and regular iteration,* J. Austral. Math. Soc. **2** (1961/1962), 265–294.
8. ———, *Zusammensetzungen ganzer Funktionen,* Math. Z. **69** (1958), 121–163.
9. W. Bergweiler, *Solution of a problem of Rubel concerning iteration and algebraic differential equations,* Indiana Univ. Math. J. **44** (1995), no. 1, 257–268.
10. M. Boshernitzan and L. Rubel, *Coherent families of polynomials,* Analysis **6** (1986), no. 4, 339–389.
11. L. Comtet, *Advanced Combinatorics,* D. Reidel Publishing Co., Dordrecht, 1974.
12. J. Écalle, *Sommations de séries divergentes en théorie de l'itération des applications holomorphes,* C. R. Acad. Sci. Paris Sér. A-B **282** (1976), no. 4, Aii, A203–A206.
13. ———, *Théorie itérative: introduction à la théorie des invariants holomorphes,* J. Math. Pures Appl. (9) **54** (1975), 183–258.
14. ———, *Nature du groupe des ordres d'itération complexes d'une transformation holomorphe au voisinage d'un point fixe de multiplicateur* 1*,* C. R. Acad. Sci. Paris Sér. A-B **276** (1973), A261–A263.
15. T. Ekedahl, S. Lando, M. Shapiro, and A. Vainshtein, *Hurwitz numbers and intersections on moduli spaces of curves,* Invent. Math. **146** (2001), 297–327.
16. P. Erdős and E. Jabotinsky, *On analytic iteration,* J. Analyse Math. **8** (1960/1961), 361–376.
17. H. Gould, *Tables of Combinatorial Identities,* available online at `http://www.math.wvu.edu/~gould`, 2010.
18. R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics,* 2nd ed., Addison-Wesley Publishing Company, Reading, MA, 1994.
19. D. Gronau, *Gottlob Frege, a pioneer in iteration theory,* in: L. Reich, J. Smítal, and G. Targonski, *Iteration Theory (ECIT 94),* pp. 105–119, Grazer Math. Ber., vol. 334, Karl-Franzens-Univ. Graz, Graz, 1997.
20. E. Jabotinsky, *Analytic iteration,* Trans. Amer. Math. Soc. **108** (1963), 457–477.
21. ———, *Sur la reprsentation de la composition de fonctions par un produit de matrices. Application à l'itération de e^z et de $e^z - 1$,* C. R. Acad. Sci. Paris **224** (1947), 323–324.
22. D. E. Knuth, *Convolution polynomials,* Mathematica J. **2** (1992), no. 4, 67–78.
23. M. Kuczma, *Functional Equations in a Single Variable,* Monografie Matematyczne, vol. 46, Państwowe Wydawnictwo Naukowe, Warsaw, 1968.
24. M. Kuczma, B. Choczewski, and G. Roman, *Iterative Functional Equations,* Encyclopedia of Mathematics and its Applications, vol. 32, Cambridge University Press, Cambridge, 1990.
25. M. Lewin, *An example of a function with non-analytic iterates,* J. Austral. Math. Soc. **5** (1965), 388–392.
26. L. Lipshitz and L. Rubel, *A gap theorem for power series solutions of algebraic differential equations,* Amer. J. Math. **108** (1986), no. 5, 1193–1213.
27. L. S. O. Liverpool, *Fractional iteration near a fix point of multiplier* 1*,* J. London Math. Soc. (2) **9** (1974/75), 599–609.
28. K. Mahler, *Lectures on Transcendental Numbers,* Lecture Notes in Mathematics, vol. 546, Springer-Verlag, Berlin-New York, 1976.
29. S. Scheinberg, *Power series in one variable,* J. Math. Anal. Appl. **31** (1970), 321–333.
30. E. Schippers, *A power matrix approach to the Witt algebra and Loewner equations,* Comput. Methods Funct. Theory **10** (2010), no. 1, 399–420.
31. S. Shadrin and D. Zvonkine, *Changes of variables in ELSV-type formulas,* Michigan Math. J. **55** (2007), no. 1, 209–228.
32. R. Stanley, *Enumerative Combinatorics,* vol. 2, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999.
33. G. Szekeres, *Fractional iteration of entire and rational functions,* J. Austral. Math. Soc. **4** (1964), 129–142.

University of California, Los Angeles, California, USA
E-mail address: `matthias@math.ucla.edu`