1. Let $B$ be a Brownian motion wrt. a filtration $(\mathcal{F}_t)_{t \geq 0}$. Show that the process $(B^2_t - t)_{t \geq 0}$ is a martingale.

2. Let $X = M + A$, where $M$ is a continuous martingale and $A$ is a continuous process which has finite variation, say $TV(A)_t$ is bounded for every $t$. Show that $\langle X \rangle = \langle M \rangle$.

3. Assume that $X, X_1, X_2$ are semi-martingales and that $H, H_1, H_2$ continuous adapted processes, $\alpha_1, \alpha_2 \in \mathbb{R}$. Show that the stochastic integral is linear, i.e.

$$\int_0^T \alpha_1 H_1 + \alpha_2 H_2 dX = \alpha_1 \int_0^T H_1 dX + \alpha_2 \int_0^T H_2 dX \quad (1)$$

$$\int_0^T H d(\alpha_1 X_1 + \alpha_2 X_2) = \alpha_1 \int_0^T H dX_1 + \alpha_2 \int_0^T H dX_2 \quad (2)$$

4. A process $H = (H_t)_{t \in [0,T]}$ is called simple if

$$H_t(\omega) = \sum_{i=1}^n H_i I_{(s_i, s_{i+1}]}(\omega),$$

where $0 \leq s_1 \leq \ldots \leq s_n$ and each $H_i$ is $\mathcal{F}_{s_i}$-measurable and bounded. Let $X$ be a continuous adapted process.

For simple integrands we set

$$I_t := \int_0^u H_t dX_t := \sum_{i=1}^n H_i (X_{s_{i+1}} \wedge u - X_{s_i} \wedge u).$$

Show that that for all $t \geq 0$ and all sequences $(\mathcal{D}_n)_{n \in \mathbb{N}}$ of partitions of $[0, t]$, i.e.

$$\mathcal{D}_n := \{0 = t_{0,n}^0 < t_{1,n}^0 < \cdots < t_{n,n}^0 = t\}$$

with $|\mathcal{D}_n| := \sup_{i \in \{1, \ldots, k_n\}} |t_{i,n}^n - t_{i-1,n}^n| \to 0$ as $n \to \infty$ it holds

$$I_t = \lim_{n \to \infty} \sum_{i=1}^{k_n} H_{t_i}^n (X_{t_i}^n - X_{t_{i-1,n}}^n)$$

in probability.