Stochastic Analysis

Sheet 1

Problem 1. *Let $(X_1, X_2, \dots)$ be i.i.d. random variables with the uniform distribution on the interval $(0, 1)$. Let $(Y_1, Y_2, \dots)$ be i.i.d. random variables such that each Y_i has the standard Gaussian distribution $\mathcal{N}(0, 1)$. Construct explicitly a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and such sequences of random variables on this space. You may also present a construction in the canonical setup.*

Problem 2. *Let $(X_n)_{n\geq 1}$ be Gaussian random variables that converge in probability to a random variable X. Show that X is Gaussian, that the mean of X is the limit of the means of X_n, and that the variance of X is the limit of the variances of X_n. Does this imply that $X_n \to X$ also in L^2?*

Problem 3. *The lost problem*

Sheet 2

Problem 4. *Let $(X_t)_{t\in T}$ be a Gaussian process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Denote by*

$$H = \overline{\operatorname{span}}\{\, X_t : t \in T \,\}$$

the closure in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ of the linear span of the random variables X_t.

Show that H is a Gaussian space, that is, every finite collection of elements of H is jointly Gaussian.

Problem 5. *Let $(B_t)_{t\geq 0}$ be a* pre-Brownian motion*, that is, a centered Gaussian process such that*

$$\mathbb{E}[B_s B_t] = \min\{s, t\}, \qquad s, t \geq 0.$$

Show that B is a pre-Brownian motion if and only if one of the following equivalent conditions holds:

(a) $B_0 = 0$, and for all $0 \leq s < t$, the increment $B_t - B_s$ is independent of $\sigma(B_r : r \leq s)$ and is distributed as $\mathcal{N}(0, t-s)$.

(b) $B_0 = 0$, the increments $(B_t - B_s)_{0\leq s<t}$ are independent, and each increment $B_t - B_s$ has distribution $\mathcal{N}(0, t-s)$.

(It is fine if you just show a) or b).)

Problem 6. *Let $(B_t)_{t\geq 0}$ be a pre-Brownian motion. Show that each of the following processes is also a pre-Brownian motion:*

(i) $(-B_t)_{t\geq 0}$.

(ii) For $\lambda > 0$, the scaled process $B^{(\lambda)}_t := \lambda^{-1} B_{\lambda^2 t}$.

(iii) For $s \geq 0$, the shifted process $B^{(s)}_t := B_{t+s} - B_s$.

Problem 7. *Let $(B_t)_{t\geq 0}$ be a pre-Brownian motion with continuous sample paths, hence a Brownian motion. Construct another stochastic process $(\widetilde{B}_t)_{t\geq 0}$ which is a pre-Brownian motion but whose paths are not continuous.*

Sheet 3

Problem 8. *Use a version of Kolmogorov's continuity theorem to show that the sample paths of Brownian motion are almost surely α-Hölder continuous for every $\alpha \in (0, \frac{1}{2})$.*

Problem 9. *Let $(B_t)_{t\in[0,1]}$ be a Brownian motion, and define the process $(R_t)_{t\in[0,1]}$ by*

$$R_t := B_t - tB_1.$$

(i) Compute the finite-dimensional distributions of the process $(R_t)_{t\in[0,1]}$. (If you observe that this is a Gaussian process, it is sufficient that you determine the covariance structure.)

(ii) Show that the process $(R_t)_{t\in[0,1]}$ has the same finite-dimensional distributions as the time-reversed process $(R_{1-t})_{t\in[0,1]}$.

Problem 10. *There are two further standard ways to define the Brownian bridge on $[0, 1]$.*

(i) As a centered Gaussian process $(\beta_t)_{t\in[0,1]}$ with covariance

$$\mathbb{E}[\beta_s\beta_t] = \min\{s, t\} - st, \qquad s, t \in [0, 1].$$

(ii) As the regular conditional law of Brownian motion $(B_t)_{t\in[0,1]}$ given $B_1 = 0$ (with $B_0 = 0$ understood), that is, a process $(\widetilde{\beta}_t)_{t\in[0,1]}$ whose finite-dimensional distributions satisfy

$$\mathcal{L}\big((\widetilde{\beta}_{t_1}, \dots, \widetilde{\beta}_{t_k})\big) = \mathcal{L}\big((B_{t_1}, \dots, B_{t_k}) \,\big|\, B_1 = 0\big), \qquad 0 < t_1 < \cdots < t_k < 1.$$

Conclude that the three objects above all have the same finite-dimensional distributions, in particular

$$\mathcal{L}\big((R_{t_1}, \dots, R_{t_k})\big) = \mathcal{L}\big((B_{t_1}, \dots, B_{t_k}) \,\big|\, B_1 = 0\big)$$

for all $0 < t_1 < \cdots < t_k < 1$. You may justify the conditioning either by an explicit Gaussian computation or via the limit

$$\mathbb{E}\big[f(B_{t_1}, \dots, B_{t_k}) \,\big|\, B_1 = 0\big] = \lim_{\varepsilon\downarrow 0} \mathbb{E}\big[f(B_{t_1}, \dots, B_{t_k}) \,\big|\, |B_1| \le \varepsilon\big]$$

for bounded continuous f.

Problem 11. *Let $\Omega := \mathbb{R}^{[0,1]}$ be the set of all functions $\omega : [0, 1] \to \mathbb{R}$. Equip Ω with the product topology (equivalently, the topology of pointwise convergence), and let $\mathcal{B}$ be its Borel σ-algebra. Denote by*

$$\mathcal{C} := \{\, \omega \in \Omega : \omega \text{ is continuous on } [0, 1] \,\}$$

the subset of continuous functions. Show that $\mathcal{C} \notin \mathcal{B}$. In other words, the set of continuous functions is not measurable with respect to the Borel σ-algebra generated by the product topology on $\mathbb{R}^{[0,1]}$.

Sheet 4

Problem 12. *Let $(B_t)_{t>0}$ be a standard Brownian motion, and define the process*

$$\widehat{B}_t := t\, B_{1/t}, \qquad t > 0,$$

and $\widehat{B}_0 := 0$.

Verify that $(\widehat{B}_t)_{t\geq 0}$ is again a Brownian motion. In particular, show that it is a centered Gaussian process with covariance

$$\mathbb{E}[\widehat{B}_s \widehat{B}_t] = \min\{s, t\}, \qquad s, t \geq 0.$$

Pay attention to the behaviour at $t = 0$ and justify the continuity of the sample paths at 0.

Problem 13. *Let $(B_t)_{t\in[0,1]}$ be a Brownian motion. For each $n \in \mathbb{N}$, consider the equidistant partition of $[0, 1]$*

$$\pi_n := \left\{0, \tfrac{1}{n}, \tfrac{2}{n}, \ldots, 1\right\},$$

and define the quadratic variation along this partition by

$$Q_n := \sum_{k=1}^{n} (B_{k/n} - B_{(k-1)/n})^2.$$

Show that $\mathbb{E}[Q_n] = 1$ and compute $\mathrm{Var}(Q_n)$. Conclude that $Q_n \to 1$ in L^2, and hence also in probability.

Problem 14. *Let $(B_t)_{t\in[0,1]}$ be a Brownian motion, and let Q_n be as in the previous problem. Show that $Q_{2^m} \to 1$ almost surely as $n \to \infty$.*

(Hint: For $m \in \mathbb{N}$, consider the filtration $\mathcal{F}_m := \sigma(B_{k/2^m} : 0 \leq k \leq 2^m)$, and define

$$M_m := Q_{2^m} - 1.$$

Show that $(M_m, \mathcal{F}_m)_{m\geq 0}$ is a martingale. Compute its second moments and apply the martingale convergence theorem to conclude almost sure convergence along the dyadic partitions.)

Problem 15. *Let $(B_t)_{t\geq 0}$ be a Brownian motion. Define*

$$X_t := \frac{1}{t} \int_0^t B_s \, ds, \qquad t > 0.$$

Use Blumenthal's zero–one law to determine

$$\mathbb{P}\left(\lim_{t\downarrow 0} X_t = 0\right).$$

Sheet 5

Problem 16. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space, and let τ be a stopping time taking values in a countable set $S \subset [0, \infty]$. Recall that*

$$\mathcal{F}_\tau := \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}.$$

Show that for every $A \in \mathcal{F}_\tau$ there exists a family $(A_t)_{t\in S}$ with $A_t \in \mathcal{F}_t$ for all $t \in S$ such that

$$A = \bigcup_{t\in S} (A_t \cap \{\tau = t\}).$$

Problem 17. *Let $(\mathcal{F}_t)_{t\geq 0}$ be a right-continuous filtration. For a random time $\tau : \Omega \to [0, \infty]$, show that the following statements are equivalent.*

(a) τ is a stopping time.

(b) For every $t \geq 0$, the event $\{\tau < t\}$ belongs to $\mathcal{F}_t$.

Problem 18. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space with right-continuous filtration, and let $(\tau_n)_{n\in\mathbb{N}}$ be a sequence of stopping times such that $\tau_n \downarrow \tau$ almost surely for some stopping time τ. Show that*

$$\mathcal{F}_\tau = \bigcap_{n=1}^{\infty} \mathcal{F}_{\tau_n}.$$

Problem 19. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space with right-continuous filtration. Show that a random time $\tau : \Omega \to [0, \infty]$ is a stopping time if and only if there exists a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$, each taking values in a countable subset of $[0, \infty]$, such that $\tau_n \downarrow \tau$ almost surely. A convenient choice is*

$$\tau_n(\omega) := \inf\{k2^{-n} : k \in \mathbb{N},\ k2^{-n} \geq \tau(\omega)\}.$$

Sheet 6

Remark 1. *Let $D \subset [0,1]$ and $f : D \to \mathbb{R}$ be a function. For real numbers $c < d$, the number of upcrossings of the interval (c,d) by f is defined as*

$$U_{c,d}(f) := \sup\Big\{ k \in \mathbb{N} : \exists\, t_1 < s_1 < \cdots < t_k < s_k \text{ in } D \text{ with } f(t_i) \le c,\ f(s_i) \ge d \Big\}.$$

Problem 20. *Let D be a countable dense subset of the interval $[0,1]$, and let $f : D \to \mathbb{R}$ be a bounded function. Show the following implication. Assume there exists a function $F : [0,1] \to \mathbb{R}$ which extends f on D, has left limits everywhere, and is right-continuous. Prove that for every interval (c,d) with rational endpoints $c < d$, the function f has only finitely many upcrossings of the interval (c,d).*

Problem 21. *With the same assumptions on D and f as above, suppose that for every interval (c,d) with rational endpoints $c < d$, the function f has only finitely many upcrossings of the interval (c,d). Show that there then exists a function $F : [0,1] \to \mathbb{R}$ extending f on D, which has left limits at every point $t \in [0,1]$ and right limits at every point $t \in [0,1]$.*

Problem 22. *Given a finite sequence $x_0, \dots, x_N \in \mathbb{R}$ show that the number of upcrossings of the interval $[0,1]$ can be bounded in the form $\sum_{k<N} h_k(x_0,\dots,x_k)(x_{k+1} - x_k) + |x_N|$.*

(Hint: you may want to think of a buy low, sell high strategy.)

Problem 23. *Let $(X_k)_{k=0}^N$ be a submartingale in discrete time with respect to some filtration. Denote by $\overline{X}_k := \max\{X_0, \dots, X_k\}$ the running supremum. Prove the following form of Doob's maximal inequality: for every $\lambda > 0$,*

$$\lambda\, \mathbb{P}\Big(\overline{X}_N > \lambda\Big) \le \mathbb{E}[(X_N)^+].$$

Sheet 7

Problem 24. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $(X_n)_{n\in\mathbb{N}}$ be a sequence of integrable random variables.*

(a) Recall (and write down) the definition of uniform integrability for the family $\{X_n : n \in \mathbb{N}\}$.

(b) Assume that there exists an integrable random variable $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ such that $|X_n| \leq X$ almost surely for all $n \in \mathbb{N}$. Show that the family $\{X_n : n \in \mathbb{N}\}$ is uniformly integrable.

Problem 25. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $(X_n)_{n\in\mathbb{N}}$ be a sequence of integrable random variables.*

(a) Suppose there exists a convex function $\varphi : [0, \infty) \to [0, \infty)$ such that

- *φ is nondecreasing and $\varphi(0) = 0$,*
- *$\lim_{x\to\infty} \frac{\varphi(x)}{x} = \infty$ (that is, φ is superlinear),*
- *$\mathbb{E}[\varphi(|X_n|)] < \infty$ for all $n \in \mathbb{N}$, and*

$$\sup_{n\in\mathbb{N}} \mathbb{E}[\varphi(|X_n|)] < \infty.$$

Show that the family $\{X_n : n \in \mathbb{N}\}$ is uniformly integrable.

(b) Prove the converse: assume that the family $\{X_n : n \in \mathbb{N}\}$ is uniformly integrable. Show that there exists a function φ with the properties listed in part (a) such that

$$\sup_{n\in\mathbb{N}} \mathbb{E}[\varphi(|X_n|)] < \infty.$$

(In other words, show that the criterion in part (a) is equivalent to uniform integrability; this is the de la Vallée Poussin theorem.)

Problem 26. *Let $\mathcal{B}(\mathbb{R}^2)$ denote the Borel σ-algebra on $\mathbb{R}^2$, and let $A \in \mathcal{B}(\mathbb{R}^2)$ be a Borel set. Consider the projection of A onto the first coordinate,*

$$\pi_1(A) := \{\, x \in \mathbb{R} : \exists\, y \in \mathbb{R} \text{ with } (x, y) \in A \,\} \subset \mathbb{R}.$$

Use the Debut theorem to show that $\pi_1(A)$ is a Lebesgue-measurable subset of $\mathbb{R}$. (Hint: Work on a suitable filtered probability space carrying a process taking values in $\mathbb{R}$, and represent $\pi_1(A)$ as the set of times at which the corresponding space-time process enters an appropriate measurable set.)

Problem 27. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space, and let $M = (M_t)_{t\geq 0}$ be a martingale with continuous sample paths such that*

$$M_0 = x \in \mathbb{R}_+, \qquad M_t \geq 0 \text{ for all } t \geq 0, \qquad M_t \to 0 \text{ as } t \to \infty \text{ almost surely.}$$

For $y > x$, set

$$\tau_y := \inf\{t \geq 0 : M_t \geq y\}$$

(with the usual convention $\inf \emptyset = \infty$).

Show that

$$\mathbb{P}\left(\sup_{t\geq 0} M_t \geq y\right) = \frac{x}{y}.$$

Sheet 8

Problem 28. *Let $(X_n)_{n=0,\dots,N}$ be a submartingale in discrete time with respect to a filtration $(\mathcal{F}_n)_{n=0,\dots,N}$, and assume $X_n \in L^1$ for all n. Complete either (a) or (b):*

(a) State and prove the Doob decomposition: show that there exist a predictable process $(A_n)_{n=0,\dots,N}$ with $A_0 = 0$ and A_n increasing, and a martingale $(M_n)_{n=0,\dots,N}$ such that

$$X_n = M_n + A_n, \qquad n = 0, \dots, N.$$

Give an explicit formula for A and M in terms of X.

(b) Let $(M_n)_{n=0,\dots,N}$ be a martingale in discrete time with $M_n \in L^2$ for all n. Apply the Doob decomposition to the submartingale $(M_n^2)_{n=0,\dots,N}$ and determine its decomposition explicitly. That is, show that

$$M_n^2 = \big(M_0^2 + \sum_{k=1}^{n} \big(M_k^2 - \mathbb{E}[M_k^2 \mid \mathcal{F}_{k-1}]\big)\big) \; + \; \sum_{k=1}^{n} \mathbb{E}\Big[(M_k - M_{k-1})^2 \,\Big|\, \mathcal{F}_{k-1}\Big], \qquad n = 0, \dots, N,$$

and interpret the increasing predictable term as the discrete quadratic variation of M.

Problem 29. *Let $(M_t)_{t\geq 0}$ be a martingale with respect to its natural filtration (or any filtration with respect to which it is a martingale). Assume additionally that $(M_t)_{t\geq 0}$ is a Gaussian process and that $M_0 = 0$.*

Show that M has independent increments, that is, for all $s, t \geq 0$ the increment $M_{t+s} - M_t$ is independent of $\sigma(M_r : 0 \leq r \leq t)$. Conclude in particular that for every $0 \leq t_0 < t_1 < \cdots < t_k$ the increments $M_{t_1} - M_{t_0}, \dots, M_{t_k} - M_{t_{k-1}}$ are independent.

Problem 30. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. Let $(M_t^n)_{0\leq t\leq T}$, $n \geq 1$, be a sequence of continuous martingales such that the terminal values $(M_T^n)_{n\geq 1}$ form a Cauchy sequence in $L^2(\Omega)$.*

Show that there exists a continuous martingale $(M_t)_{0\leq t\leq T}$ such that $M_T^n \to M_T$ in L^2. (Hint: Use completeness of L^2 to obtain $M_T \in L^2$ as the limit, define $M_t := \mathbb{E}[M_T \mid \mathcal{F}_t]$, and then prove that M admits a continuous modification, for instance by Doob's L_2-inequality.)

Sheet 9

Problem 31. *Let $(M_t)_{t\geq 0}$ and $(N_t)_{t\geq 0}$ be continuous square-integrable martingales and consider the quadratic covariation*

$$\langle M, N\rangle = \tfrac{1}{4}\big(\langle M+N\rangle - \langle M-N\rangle\big).$$

Show that for every $t \geq 0$,

$$|\langle M, N\rangle_t| \leq \sqrt{\langle M\rangle_t}\,\sqrt{\langle N\rangle_t}; \tag{1}$$

this is the Kunita–Watanabe inequality in its basic form.

(Hint: For fixed t, consider the quadratic variation of the martingale $M + \lambda N$ at time t as a function of $\lambda \in \mathbb{R}$, and use that quadratic variations are nonnegative.)

Problem 32. *Let $(M_t)_{t\geq 0}$ and $(N_t)_{t\geq 0}$ be continuous square integrable martingales, and let H and K be predictable processes such that the stochastic integrals $H \cdot M$ and $K \cdot N$ are well defined.*

Starting from the inequality in the previous problem, derive the general Kunita–Watanabe inequality

$$\left|\int_0^t H_s K_s \, d\langle M, N\rangle_s\right| \leq \left(\int_0^t H_s^2 \, d\langle M\rangle_s\right)^{1/2} \left(\int_0^t K_s^2 \, d\langle N\rangle_s\right)^{1/2}, \qquad t \geq 0.$$

You may freely use the basic properties of the Itô integral and quadratic (co)variation established in the lecture, in particular the identities

$$\langle H\cdot M,\, K\cdot N\rangle_t = \int_0^t H_s K_s \, d\langle M, N\rangle_s, \qquad \langle H\cdot M\rangle_t = \int_0^t H_s^2 \, d\langle M\rangle_s,$$

and the corresponding formula for $\langle K\cdot N\rangle_t$.

Problem 33. *Let $(M_k)_{k=0,\dots,n}$ and $(N_k)_{k=0,\dots,n}$ be square-integrable martingales with respect to a filtration $(\mathcal{F}_k)_{k=0,\dots,n}$. Define the discrete predictable brackets by*

$$\langle M\rangle_k := \sum_{i=1}^{k} \mathbb{E}\Big[(\Delta M_i)^2 \mid \mathcal{F}_{i-1}\Big], \qquad \langle M, N\rangle_k := \sum_{i=1}^{k} \mathbb{E}[\Delta M_i \, \Delta N_i \mid \mathcal{F}_{i-1}].$$

Show that for every $k \leq n$,

$$|\langle M, N\rangle_k| \leq \sqrt{\langle M\rangle_k}\,\sqrt{\langle N\rangle_k}.$$

(Hint: First apply the conditional Cauchy–Schwarz inequality to each term $\mathbb{E}[\Delta M_i \, \Delta N_i \mid \mathcal{F}_{i-1}]$, then use the usual Cauchy–Schwarz inequality for finite sums.)

Sheet 10

Problem 34. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. Let $M = (M_t)_{t\geq 0}$ be a square-integrable martingale and let τ be a stopping time. Let H be a progressive process such that $H \in L^2(M)$ (i.e. the stochastic integral $H \cdot M$ is well defined in L^2).*

Show that

$$H \cdot M^\tau \;=\; \mathbf{1}_{[0,\tau]} H \cdot M,$$

where $M^\tau = (M_{t\wedge\tau})_{t\geq 0}$ and $\mathbf{1}_{[0,\tau]}H$ denotes the process $(\mathbf{1}_{\{t\leq\tau\}}H_t)_{t\geq 0}$.

Problem 35. *Let $M = (M_t)_{0\leq t\leq T}$ be a bounded martingale. Let $\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ be a partition of $[0,T]$ and denote its mesh by $|\pi| := \max_i(t_{i+1} - t_i)$.*

Show that, as $|\pi| \to 0$,

$$\sum_{i=0}^{n-1} M_{t_i}\,(M_{t_{i+1}} - M_{t_i}) \;\longrightarrow\; \int_0^T M_t\, dM_t$$

in L^2 (and hence also in probability).

Problem 36. *Let $M = (M_t)_{0\leq t\leq T}$ be as in the previous problem.*

Show that, as $|\pi| \to 0$,

$$\sum_{i=0}^{n-1} M_{t_{i+1}}\,(M_{t_{i+1}} - M_{t_i}) \;\longrightarrow\; \int_0^T M_t\, dM_t \;+\; \langle M\rangle_T$$

in L^2 (and hence also in probability).

(Hint: Relate the right-endpoint sum to the left-endpoint sum from the previous problem and identify the difference.)

Problem 37. *A continuous semimartingale is a process $X = (X_t)_{t\geq 0}$ which can be written as*

$$X_t = X_0 + M_t + A_t, \qquad t \geq 0,$$

where M is a continuous local martingale with $M_0 = 0$ and A is a continuous finite variation process with $A_0 = 0$.

For the purpose of this exercise, call a continuous adapted process X locally a semimartingale *if there exists a sequence of stopping times $(\tau_n)_{n\geq 1}$ with $\tau_n \uparrow \infty$ almost surely such that for each n there exist a continuous local martingale $M^{(n)}$ and a continuous finite variation process $A^{(n)}$ with*

$$X_{t\wedge\tau_n} = X_0 + M_t^{(n)} + A_t^{(n)} \qquad \text{for all } t \geq 0.$$

Show that X is a (global) continuous semimartingale if and only if it is locally a semimartingale in the above sense.

Sheet 10

Problem 38. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. Let $M = (M_t)_{t\geq 0}$ be a square-integrable martingale and let τ be a stopping time. Let H be a progressive process such that $H \in L^2(M)$ (i.e. the stochastic integral $H \cdot M$ is well defined in L^2).*

Show that

$$H \cdot M^\tau \;=\; \mathbf{1}_{[0,\tau]} H \cdot M,$$

where $M^\tau = (M_{t\wedge\tau})_{t\geq 0}$ and $\mathbf{1}_{[0,\tau]}H$ denotes the process $(\mathbf{1}_{\{t\leq\tau\}}H_t)_{t\geq 0}$.

Problem 39. *Let $M = (M_t)_{0\leq t\leq T}$ be a martingale which is bounded in L^2, i.e. $\sup_{t\in[0,T]} \mathbb{E}[M_t^2] < \infty$. Let $\pi = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ be a partition of $[0,T]$ and denote its mesh by $|\pi| := \max_i(t_{i+1} - t_i)$.*

Show that, as $|\pi| \to 0$,

$$\sum_{i=0}^{n-1} M_{t_i}\,(M_{t_{i+1}} - M_{t_i}) \;\longrightarrow\; \int_0^T M_t\, dM_t$$

in L^2 (and hence also in probability).

(Hint: First prove the statement for simple predictable integrands and use the density of such integrands in $L^2(M)$. You may use the L^2-isometry and Doob's inequality as needed.)

Problem 40. *Let $M = (M_t)_{0\leq t\leq T}$ be as in the previous problem, and let $\pi = \{0 = t_0 < \cdots < t_n = T\}$ be a partition of $[0,T]$.*

Show that, as $|\pi| \to 0$,

$$\sum_{i=0}^{n-1} M_{t_{i+1}}\,(M_{t_{i+1}} - M_{t_i}) \;\longrightarrow\; \int_0^T M_t\, dM_t \;+\; \langle M\rangle_T$$

in L^2 (and hence also in probability).

(Hint: Relate the right-endpoint sum to the left-endpoint sum from the previous problem and identify the difference.)

Problem 41. *A continuous semimartingale is a process $X = (X_t)_{t\geq 0}$ which can be written as*

$$X_t = X_0 + M_t + A_t, \qquad t \geq 0,$$

where M is a continuous local martingale with $M_0 = 0$ and A is a continuous finite variation process with $A_0 = 0$.

For the purpose of this exercise, call a continuous adapted process X locally a semimartingale *if there exists a sequence of stopping times $(\tau_n)_{n\geq 1}$ with $\tau_n \uparrow \infty$ almost surely such that for each n there exist a continuous local martingale $M^{(n)}$ and a continuous finite variation process $A^{(n)}$ with*

$$X_{t\wedge\tau_n} = X_0 + M_t^{(n)} + A_t^{(n)} \qquad \text{for all } t \geq 0.$$

Show that X is a (global) continuous semimartingale if and only if it is locally a semimartingale in the above sense.

Sheet 11

Problem 42. *Let $(M_t)_{t\geq 0}$ be a continuous local martingale with $M_0 = 0$. Define the* stochastic exponential *of M by*

$$\mathcal{E}(M)_t := \exp\Big(M_t - \tfrac{1}{2}\langle M\rangle_t\Big), \qquad t \geq 0.$$

(a) Use Itô's formula to show that $(\mathcal{E}(M)_t)_{t\geq 0}$ is a continuous local martingale.

(b) Observe that $\mathcal{E}(M)$ satisfies the 'stochastic differential equation'

$$\mathcal{E}(M)_t = 1 + \int_0^t \mathcal{E}(M)_s \, dM_s, \qquad t \geq 0.$$

Problem 43. *Let $(B_t)_{t\geq 0}$ be a Brownian motion, fix $T > 0$, and let $f \in C_b^2(\mathbb{R})$. Define, for $0 \leq t \leq T$,*

$$M_t := \mathbb{E}[f(B_T) \mid \mathcal{F}_t] \, .$$

(a) Use the Markov property of Brownian motion to show that there exists a function

$$u : [0,T] \times \mathbb{R} \to \mathbb{R}$$

such that

$$M_t = u(t, B_t), \qquad 0 \leq t \leq T,$$

and identify u explicitly as

$$u(t,x) := \mathbb{E}[f(x + B_{T-t})] \, .$$

(b) Show that u is of class $C^{1,2}$ and compute its derivatives with respect to t and x.

(c) Apply Itô's formula to $u(t, B_t)$ and show that there exists a predictable process $(H_t)_{0\leq t\leq T}$ such that

$$M_t = M_0 + \int_0^t H_s \, dB_s, \qquad 0 \leq t \leq T.$$

Identify H_t explicitly in terms of u.

(d) Deduce that

$$f(B_T) = \mathbb{E}[f(B_T)] + \int_0^T H_s \, dB_s.$$

In the following exercises, let $(B_t)_{t\geq 0}$ be a Brownian motion. Let $b, \sigma : \mathbb{R} \to \mathbb{R}$ be functions of class C_b^2, and assume that the 'stochastic differential equation'

$$X_t = x + \int_0^t b(X_s) \, ds + \int_0^t \sigma(X_s) \, dB_s$$

admits a unique solution $(X_t)_{t\geq 0}$ with continuous sample paths. You may further assume that $(X_t)_{t\geq 0}$ is a time-homogeneous Markov process.

Problem 44. *Fix $T > 0$ and let $f \in C_b^2(\mathbb{R})$. Define*

$$u(t,x) := \mathbb{E}_x[f(X_t)], \qquad t \geq 0, \ x \in \mathbb{R},$$

where $\mathbb{E}_x$ denotes expectation for the process started at $X_0 = x$.

(a) Show that for every $x \in \mathbb{R}$, the process

$$M_t := u(T - t, X_t), \qquad 0 \leq t \leq T,$$

is a martingale.

(b) Apply Itô's formula to $u(T - t, X_t)$ and show that u satisfies

$$\partial_t u(t,x) = b(x)\partial_x u(t,x) + \tfrac{1}{2}\sigma^2(x)\partial_{xx} u(t,x), \qquad t > 0,$$

with initial condition $u(0,x) = f(x)$.

This equation is the Kolmogorov backward equation.

Problem 45. *Assume in addition that for every $t > 0$ the law of X_t admits a smooth density $p_t(x, y)$ with respect to Lebesgue measure.*

(a) Show that for every $\varphi \in C_c^\infty(\mathbb{R})$,

$$\frac{d}{dt}\mathbb{E}_x[\varphi(X_t)] = \mathbb{E}_x\Big[b(X_t)\varphi'(X_t) + \tfrac{1}{2}\sigma^2(X_t)\varphi''(X_t)\Big].$$

(b) Rewrite the identity in part (a) in terms of the density $p_t(x, y)$ and deduce that

$$\partial_t p_t(x, y) = -\partial_y\big(b(y)p_t(x, y)\big) + \tfrac{1}{2}\partial_{yy}\big(\sigma^2(y)p_t(x, y)\big).$$

This equation is the Kolmogorov forward equation.