PS Combinatorics

(Modul: “Kombinatorik” (MALK))

Markus Fulmek

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Exercise 1: *Show that the number of all unlabeled ordered rooted trees with n vertices, where every* inner *vertex has* 2 *or* 3 *branches, equals*

$$\frac{1}{n}\sum_{j}\binom{n}{j}\binom{j}{3j-n+1}.$$

<u>Hint:</u> *Find an equation for the generating function and use Langrange's inversion formula.*

Exercise 2: *How many ways are there to (properly) parenthesize n pairwise non–commuting elements of a monoid? And how does this number change if the n elements are pairwise commuting?*
For example, consider 6 non–commuting *elements $x_1, x_2, \ldots, x_6$.* Two different *ways to parenthesize them properly would be*

$$((x_2x_5)((x_1(x_4x_6))x_3)) \text{ and } ((x_3(x_1(x_4x_6)))(x_5x_2)).$$

However, these would be equivalent *for* commuting *elements.*

<u>Hint:</u> *Translate parentheses to* labeled binary trees*: The outermost pair of parentheses corresponds to the root, and the elements of the monoid correspond to the leaves.*

Exercise 3: *Develop a theory for* weighted *generating functions (for labeled and unlabeled species). I.e., let $\mathcal{A}$ be some species with weight function ω which assigns to every object $A \in \mathcal{A}$ some element in a ring R (for instance, $R = \mathbb{Z}[y]$, the ring of polynomials in y with coefficients in $\mathbb{Z}$). So the generating function to be considered is*

$$\sum_{A \in \mathcal{A}} z^{\|A\|} \cdot \omega(A).$$

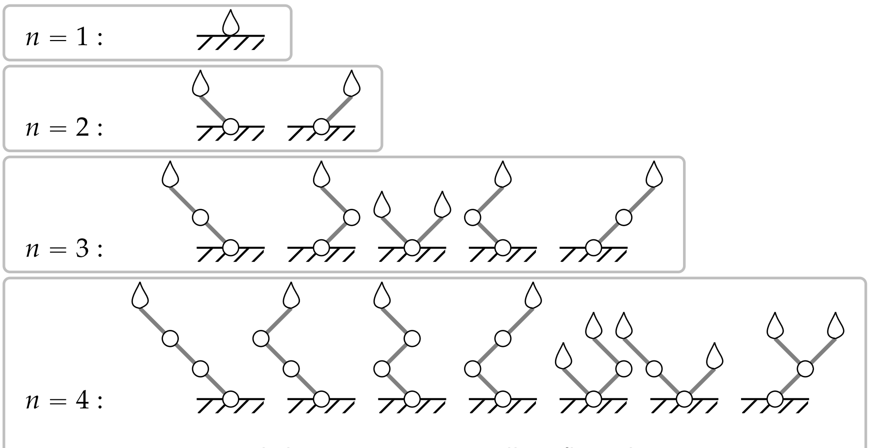
How should we define the weight function for sums, products and composition of species, so that the corresponding assertions for generating functions remain valid?

Exercise 4: *Let $f(m,n)$ be the number of all paths from $(0,0)$ to (m,n) in $\mathbb{N} \times \mathbb{N}$, where each single steps is either $(1,0)$ (step to the right) or $(0,1)$ (step to the left) or $(1,1)$ (diagonal step upwards). Use the language of species to show that*

$$\sum_{m,n \geqslant 0} f(m,n)\, x^m y^n = \frac{1}{1-x-y-x\cdot y}.$$

Exercise 5: *Determine the number of all* unlabeled ordered binary *rooted trees with n vertices and k leaves.*

<u>Hint:</u> *Consider the generating function in* 2 *variables z and y, where every rooted tree W with n vertices and k leaves is assigned $\omega(W) := z^n y^k$. The following picture shows these trees for $n = 1, 2, 3, 4$:*



. . . and these 7 trees, vertically reflected.

I.e., the first terms of the generating function are:

$$T(z, y) := \sum_W \omega(W) = z \cdot y + z^2 \cdot 2y + z^3 \left(y^2 + 4y\right) + z^4 \left(6y^2 + 8y\right) + \cdots$$

Find an equation for this generating function T, from which the series expansion can be derived.

Exercise 6: *Determine the number of all* labeled unordered *rooted trees with n vertices and k leaves.*

<u>Hint:</u> Consider the exponential generating function in 2 variables z and y (as in Exercise 5) and use Lagrange's inversion formula.

Exercise 7: *Prove Cayley's formula (the number of labeled trees on n vertices equals n^{n-2}) as follows: Take a labeled tree on n vertices and tag two vertices S and E. View S and E as the starting point and ending point of the unique path p connecting S and E in the tree. Now orient all edges belonging to p "from S to E", and all edges not belonging to p "towards p". Now travel along p from S to E and write down the labels of the vertices: Whenever a new maximal label is encountered, close a circle (by inserting an oriented edge from the vertex before this new maximum to the start of the "current circle") and start a new circle. Interpret the resulting directed graph as a function $[n] \to [n]$ (i.e., a directed edge from a to b indicates that the function maps a to b).*

Exercise 8: *Show that the number of all graphs on n vertices, m edges and k components equals the coefficient of $u^n \alpha^m \beta^k / n!$ in*

$$\left(\sum_{n \geqslant 0} (1 + \alpha)^{\binom{n}{2}} \frac{u^n}{n!} \right)^{\beta}.$$

<u>Hint:</u> Find a connection between the generating function of all labeled graphs (weight $\omega(G) := u^{|V(G)|} \alpha^{|E(G)|}$) and the generating function of all connected *labeled graphs.*