VO Combinatorics

(Module: “Combinatorics” (MALK))

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The present (draft of) lecture notes is based on lectures "Combinatorics" held since summer term 2012. It contains selected material from the following text books:

- Species: *Combinatorial Species and tree–like Structures* by F. Bergeron, G. Labelle and P. Leroux [**1**],
- Partially ordered sets: *Enumerative Combinatorics I* by R. Stanley [**7**],
- Asymptotics: *Analytic Combinatorics* by P. Flajolet and R. Sedgewick [**3**].

The reader is assumed to be familiar with basic concepts from Discrete Mathematics (see, for instance, lecture notes [**4**]). Moreover, some knowledge from Analysis (power series), Linear Algebra, Algebra (basic group theory) and Complex Analysis (contour integrals) is a necessary precondition for the material presented here.

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CHAPTER 1

Generating functions and Lagrange's Inversion Formula

1.1. A short recap: formal power series

Recall that the set $\mathbb{C}[[z]]$ of formal power series (in some variable z) with complex coefficients

$$\mathbb{C}[[z]] = \left\{ \sum_{n \geqslant 0} c_n \cdot z^n : \; c_n \in \mathbb{C} \right\}$$

is a commutative algebra with unity over $\mathbb{C}$.

The notion "formal" refers to the fact that we do not actually "compute the infinite sum $c(\alpha)$" for some concrete number α, but view $c(z)$ as a convenient notation for the sequence of coefficients $(c_n)_{n \geqslant 0}$. For some formal power series $c(z) = c_0 + c_1 \cdot z + c_2 \cdot z^2 + \cdots$, however, we adopt the intuitive notation $c(0)$ for the constant term c_0, i.e.: $c(0) := c_0$.

THEOREM 1.1.1. *The formal power series $a(z) = a_0 + a_1 \cdot z + a_2 \cdot z^2 + \cdots$ possesses a multiplicative inverse if and only if $a(0) = a_0 \neq 0$.*
This inverse (if existing) is unique. □

DEFINITION 1.1.2. *Let $a(z)$ and $b(z)$ be two formal power series, let $b(z) = 0$ (i.e., $b(z) = b_1 \cdot z + b_2 \cdot z^2 + \ldots$, $b_0 = 0$). Then the composition $(a \circ b)(z)$ of a and b is defined by*

$$(a \circ b)(z) := \sum_{i \geqslant 0} a_i \left(b(z)\right)^i .$$

Note that the condition $b(0) = 0$ is *necessary*, since otherwise we could encounter *infinite* sums for the coefficients of the composition: we do not deal with infinite sums in the calculus of formal power series.

THEOREM 1.1.3. *Let $a(z) = a_1 \cdot z + a_2 \cdot z^2 + \cdots$ be a formal power series with $a(0) = 0$. Then there is a unique* compositional inverse *$b(z) = b_1 \cdot z + b_2 \cdot z^2 + \cdots$, i.e.*

$$(a \circ b)(z) = (b \circ a)(z) = z,$$

if and only if $a_1 \neq 0$. □

1.2. Lagrange's Inversion formula

Let $f(z)$ be a formal power series with vanishing constant term (i.e., $f(0) = 0$): By Theorem 1.1.3 we know that there *is* a (unique) compositional inverse $F(z)$, i.e.,

$$F(f(z)) = f(F(z)) = z.$$

But how can we *find* this inverse? In fact, there is a quite general and very useful formula which we shall derive in the following: For that, we consider a slightly more general problem, which involves an extension of the calculus of formal power series.

DEFINITION 1.2.1. *A formal* Laurent series *(in some variable z) is a (formal) sum*

$$\sum_{n\geqslant N} a_n \cdot z^n$$

with coefficients $a_n \in \mathbb{C}$, *for some* $N \in \mathbb{Z}$*: N may be smaller than zero, i.e., there may be* negative *powers of z.*

We denote the set of all formal Laurent series by $\mathbb{C}((z))$

Addition, multiplication and composition for formal Laurent series are defined exactly as for formal power series.

THEOREM 1.2.2. *A formal Laurent series* $a(z)$ *possesses a (unique) multiplicative inverse if and only if* $a(z) \neq 0$. □

This implies that the algebra of formal Laurent series is, in fact, a *field*.

In the following we shall always assume that the formal power series $f(z)$ (with $f(z) = 0$) under consideration "starts with z^1, i.e., is of the form

$$f(z) = f_1 \cdot z^1 + f_2 \cdot z^2 + \cdots$$

where $f_1 \neq 0$ (we might express this as $\frac{f(z)}{z} \neq 0$): The general case

$$g(z) = g_m \cdot z^m + g_{m+1} \cdot z^{m+1} + \cdots \text{ for } m > 1$$

can be easily reduced to this case.

LEMMA 1.2.3. *Consider some formal power series* $f(z) = f_1 \cdot z + f_2 \cdot z^2 + \cdots$ *where* $f_1 = f(0) \neq 0$. *Then we have for all* $n \in \mathbb{Z}$

$$\left[\!\left[z^{-1}\right]\!\right] \left(f^{n-1}(z) \cdot f'(z)\right) = [n = 0].$$

(Here, we made use of Iverson's notation*:* ["*some assertion*"] *equals* 1 *if "some assertion" is true, otherwise it equals* 0[1]*.)*

PROOF. Observe that for $n \neq 0$ we have $n \cdot \left(f^{n-1}(z) \cdot f'(z)\right) = \left(f^n(z)\right)'$: For $n > 0$, this is a formal power series; and since the coefficient of z^{-1} equals 0 for *every* formal power series, the assertion is true for all $n > 0$.

If $n = -m < 0$, then $f^n(z) = c_{-m} \cdot z^{-m} + \cdots + c_{-1} \cdot z^{-1} + c_0 + c_1 \cdot z^1 + \ldots$ and

$$\left(f^n(z)\right)' = (-m) \cdot c_{-m} \cdot z^{-m-1} + \cdots + (-1) \cdot c_{-1} \cdot z^{-2} + c_1 + \ldots,$$

so the assertion is also true for all $n < 0$.

[1]Iverson's notation is a generalization of *Kronecker's delta*: $\delta_{i,j} = [i = j]$.

For $n = 0$, we simply compute:

$$\begin{aligned}
\left[\!\left[z^{-1}\right]\!\right] f^{-1}(z) \cdot f'(z) &= \left[\!\left[z^{-1}\right]\!\right] \frac{f_1 + 2 \cdot f_2 \cdot z + \cdots}{f_1 \cdot z + f_2 \cdot z^2 + \cdots} \\
&= \left[\!\left[z^{-1}\right]\!\right] \frac{1}{z} \frac{f_1 + 2 \cdot f_2 \cdot z + \cdots}{f_1 + f_2 \cdot z + \cdots} \\
&= \left[\!\left[z^{-1}\right]\!\right] \frac{1}{z} \frac{1 + 2 \cdot \frac{f_2}{f_1} \cdot z + \cdots}{1 - \underbrace{z\left(-\frac{f_2}{f_1} - \cdots\right)}_{=:h(z)}} \\
&= \left[\!\left[z^{-1}\right]\!\right] \frac{1}{z} \left(1 + 2\frac{f_2}{f_1} z + \cdots\right)\left(1 + h(z) + h(z)^2 + \cdots\right) \\
&= 1.
\end{aligned}$$

□

THEOREM 1.2.4. *Let $g(z)$ be a formal Laurent series and $f(z) = f_1 \cdot z + f_2 \cdot z^2 + \cdots$ be a formal power series with $f(0) = 0$, $f_1 \neq 0$. Suppose there is an expansion of $g(z)$ in powers $f(z)$, i.e.,*

$$g(z) = \sum_{k \geqslant N} c_k \cdot f^k(z). \tag{1.1}$$

Then the coefficients c_n are given by

$$c_n = \frac{1}{n} \left[\!\left[z^{-1}\right]\!\right] g'(z) \cdot f^{-n}(z) \quad \text{for } n \neq 0. \tag{1.2}$$

An alternative expression for these coefficients is

$$c_n = \left[\!\left[z^{-1}\right]\!\right] g(z) \cdot f'(z) \cdot f^{-n-1}(z) \quad \text{for } n \in \mathbb{Z}. \tag{1.3}$$

PROOF. For the first claim (1.2) we take the derivative of (1.1),which gives

$$g'(z) = \sum_k k \cdot c_k \cdot f'(z) \cdot f^{k-1}(z).$$

Now we multiply both sides by $f^{-n}(z)$:

$$g'(z) \cdot f^{-n}(z) = \sum_k k \cdot c_k \cdot f'(z) \cdot f^{k-n-1}(z).$$

Taking the coefficients of z^{-1} on both sides, by Lemma 1.2.3 we obtain $n \cdot c_n$ on the right–hand side and the claim follows immediately.

For the second claim we multiply (1.1) by $f'(z) \cdot f^{-n-1}(z)$, which gives

$$g(z) \cdot f^{-n-1}(z) \cdot f'(z) = \sum_{k \geqslant N} c_k \cdot f^{k-n-1}(z) \cdot f'(z),$$

and the claim follows, again, by taking the coefficients of z^{-1} on both sides. □

Now *Lagrange's inversion formula* is the following special case of Theorem 1.2.4:

COROLLARY 1.2.5 (Lagrange's inversion formula). *Let $f(z) = f_1 \cdot z + f_2 \cdot z^2 + \cdots$ be a formal power series with $f_1 \neq 0$, let $F(z)$ be the compositional inverse of f. Then we have for every $k \in \mathbb{N}$:*

$$[\![z^n]\!]\, F^k(z) = \frac{k}{n} \left[\!\!\left[z^{-k} \right]\!\!\right] f^{-n}(z) \quad \textit{for } n \neq 0. \tag{1.4}$$

An alternative expression for these coefficients is

$$[\![z^n]\!]\, F^k(z) = \left[\!\!\left[z^{-k-1} \right]\!\!\right] f'(z)\, f^{-n-1}(z). \tag{1.5}$$

PROOF. Since $F(z)$ is the compositional inverse of $f(z)$, we have $F^k(f(z)) = z^k$, i.e. (writing $F_n^{(k)} := [\![z^n]\!]\, F^k(z)$):

$$\sum_n F_n^{(k)} \cdot f(z)^n = z^k$$

By setting $g(z) = z^k$ in Theorem 1.2.4 and applying (1.2), we obtain

$$F_n^{(k)} = \frac{1}{n} \left[\!\!\left[z^{-1} \right]\!\!\right] k \cdot z^{k-1} \cdot f^{-n}(z) = \frac{k}{n} \left[\!\!\left[z^{-k} \right]\!\!\right] f^{-n}(z),$$

which is equivalent to (1.4). In the same way, applying (1.3) gives (1.5). □

EXAMPLE 1.2.6 (Catalan numbers). *Let F_n be the number of triangulations of an $(n+1)$–gon (with vertices labelled $1, 2, \ldots, n$ successively): Setting $F_1 = 1$ (i.e., assuming that there is a single triangulation of a 2–gon), the sequence $(F_n)_{n \geqslant 1}$ starts like this:*

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, \ldots$$

For $n > 1$ and any such triangulation, consider the triangle containing the edge which connects the vertices 1 and 2: This triangle "cuts" the $(n+1)$–gon in two parts, namely a $(k+1)$–gon and an $(n-k+1)$–gon, where $1 \leqslant k \leqslant n-1$. This leads to the recursion

$$F_n = \sum_{k=1}^{n-1} F_k \cdot F_{n-k} \, \textit{for } n > 1,$$

which together with the initial condition $F_1 = 1$ implies the following functional equation for the generating function $F(z) = F_1 \cdot z + F_2 \cdot z^2 + \cdots = \sum_{n \geqslant 1} F_n \cdot z^n$:

$$(F(z))^2 = F(z) - z.$$

This may be rewritten as

$$z = F(z) - F^2(z).$$

Stated otherwise: $F(z)$ is the compositional inverse of the (quite short) formal power series $z - z^2$. Applying Lagrange's Inversion formula (1.4) *(for $k = 1$) to $f(z) = z - z^2$*

we obtain:

$$\begin{aligned}
[\![z^n]\!]\; F(z) &= \frac{1}{n}\left[\!\!\left[z^{-1}\right]\!\!\right]\; (z - z^2)^{-n} \\
&= \frac{1}{n}\left[\!\!\left[z^{-1}\right]\!\!\right]\; z^{-n}(1-z)^{-n} \\
&= \frac{1}{n}\left[\!\!\left[z^{n-1}\right]\!\!\right]\; (1-z)^{-n} \\
&= \frac{1}{n}\binom{-n}{n-1}(-1)^{n-1} \\
&= \frac{1}{n}\binom{2n-2}{n-1}.
\end{aligned}$$

Clearly, $F_n = C_{n-1}$, i.e., we get (once again) the well–known Catalan–numbers*.*

CHAPTER 2

Species: Enumeration of combinatorial objects

The theory of *species* provides a general framework for labelled and unlabelled combinatorial objects and their generating functions. One can tackle this theory very abstractly; however here we would like to approach the theme by means of a couple of illustrative examples. (Much more material than treated here can be found in the textbook [**1**].)

2.1. Motivating Examples

EXAMPLE 2.1.1. *The numbers*

$$B_n := \#\left(\textit{partitions of } [n]\right)$$

are called Bell numbers*. By considering the block which contains the* largest *element* n*, one immediately obtains the recursion*

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{n-k} \cdot B_k \quad \textit{for } n \geqslant 0, \tag{2.1}$$

with initial condition $B_0 = 1$*. The sequence* $(B_n)_{n\geqslant 0}$ *starts like this:*

$$1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, \ldots$$

We consider the exponential generating function *(see e.g.* [**4**, Section 2.2]*) of these numbers*

$$B(z) := \sum_{n\geqslant 0} B_n \cdot \frac{z^n}{n!}.$$

If we multiply (2.1) *with* $\frac{z^n}{n!}$ *and (formally*[1]*) sum over all* $n \geqslant 0$ *summieren, then we obtain*

$$\underbrace{\sum_{n\geqslant 0} B_{n+1} \cdot \frac{z^n}{n!}}_{=\frac{\mathbf{d}B(z)}{\mathbf{d}z}} = \sum_{n\geqslant 0} \left(\sum_{k=0}^{n} \binom{n}{n-k} B_k \right) \frac{z^n}{n!} = \underbrace{\sum_{n\geqslant 0} \sum_{k=0}^{n} \frac{B_k}{k!} \frac{1}{(n-k)!} z^n}_{=B(z)\cdot e^z}.$$

We can solve the differential equation

$$\frac{\mathbf{d}B}{B\,\mathbf{d}z} = \frac{\mathbf{d}\log B}{\mathbf{d}z} = e^z$$

very easily:

$$\log(B) = e^z + C \implies B(z) = e^{e^z + C},$$

[1] D.h.: The infinite sum is for the time being only a *notation* for the equations (2.1)! The equation for the series simply says that the coefficients of z^n on the left- and right-hand sides are equal for all $n \geqslant 0$.

and from $B(0) = B_0 = 1$ we immediately get $C = -1$, thus altogether

$$B(z) = \mathrm{e}^{\mathrm{e}^{z}-1}. \tag{2.2}$$

At this point one can make the following observation: The exponential function e^z emerges as the exponential generating function $\mathbf{egf}_{\mathtt{sets}}$ of the number of all subsets of $[n]$ with n elements. Admittedly, this number is combinatorially not very interesting (it is namly constantly equal to 1 for all $n \geqslant 0$), but of course we can then also interpret $\mathrm{e}^z - 1$ as the exponential generating function $\mathbf{egf}_{\mathtt{sets}_1}$ of the number of all nonempty *subsets of $[n]$ with n elements (this number is merely negligibly more complicated: 1 for $n > 0$ und 0 für $n = 0$, oder shorter mit* Iverson's notation*: $[n > 0]$). The exponential generating function $\mathbf{egf}_{\mathtt{partitions}}$ thus satisfies*

$$\mathbf{egf}_{\mathtt{partitions}} = \mathbf{egf}_{\mathtt{sets}}\left(\mathbf{egf}_{\mathtt{sets}_1}\right).$$

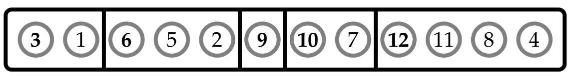
And partitions are indeed "sets of nonempty sets"! This certainly might only be an accidental analogy — but we will see that there is a "system" behind this.

In a partition of $[n]$ the ordering of the blocks does not matter — we can thus assume that the blocks

- have their largest elements "marked",
- and that they are ordered corresponding to their "markings".

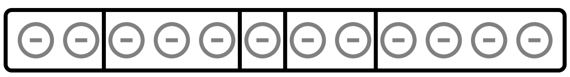
For instance, the blocks of the following partition appear in their "canonical" order like this:

$\{1,3\} \cup \{2,5,6\} \cup \{9\} \cup \{7,10\} \cup \{4,8,11,12\}$



Here one could have the idea to "forget the numbering". One could imagine this graphically for the above example like this:

$2 + 3 + 1 + 2 + 4$



Without the numbering these "blocks of indistinguishable elements" only depend on the following:

- the *cardinality* of the single blocks
- and the *ordering* of the blocks *of different size*

Very obviously we can interpret this as *compositions* of the natural number n.

EXAMPLE 2.1.2. *The same idea as in Example 2.1.1 readily leads to the recursion for the number K_n of compositions of n:*

$$K_{n+1} = \sum_{k=0}^{n} 1 \cdot K_k \quad \text{für } n \geqslant 0, \tag{2.3}$$

again with the initial condition $K_0 = 1$. We consider now the (ordinary) generating function *of these numbers*

$$K(z) := \sum_{n \geqslant 0} K_n \cdot z^n.$$

If we multiply (2.3) *by* z^n *and (formally) sum over all* $n \geqslant 0$, *then we obtain*

$$\underbrace{\sum_{n\geqslant 0} K_{n+1}\cdot z^n}_{=\frac{K(z)-K_0}{z}} = \sum_{n\geqslant 0}\left(\sum_{k=0}^{n} 1\cdot K_k\right) z^n = K(z)\cdot\frac{1}{1-z},$$

which apparently leads to the equation

$$K-1 = K\frac{z}{1-z}$$

for $K(z)$*; with the solution*

$$K(z) = \frac{1-z}{1-2z} = 1+\frac{z}{1-2z}.$$

The last equation leads immediately to the explicit (and well-known) representation

$$K_n = 2^{n-1} \quad \textit{for } n > 0.$$

Also here one can make the analogue observation to Example 2.1.1: The geometric series $\frac{1}{1-z}$ *emerges as the (ordinary) generating function* $\mathbf{gf}_{\texttt{sets}}$ *of the number of all multisets with n elements, which are all equal to* $\circ$ *(this number is again constantly equal to* 1 *for all* $n \geqslant 0$*), and* $\frac{1}{1-z}-1$ *emerges as the (ordinary) generating function* $\mathbf{gf}_{\texttt{sets}_1}$ *of the number of all* nonempty *sets with n elements, which are all equal to* $\circ$ *(this number is again* $[n > 0]$*).*

Since the elements are not labelled here (and are therefore "identical"), we are actually not dealing with sets *but with (very simple — after all there is only* one *element which can occur arbitrarily often)* multisets*: To make the notation not too complicated, we keep the term* sets*. Since the elements are all identical, their order is irrelevant:* (Multi-)sets *of indistinguishable elements are thus the same "combinatorial objects" as* sequences*! And for the (ordinary) generating function* $\mathbf{gf}_{\texttt{compositions}}$ *indeed again*

$$\mathbf{gf}_{\texttt{compositions}} = \mathbf{gf}_{\texttt{sequences}}\left(\mathbf{gf}_{\texttt{sets}_1}\right)$$

holds. And indeed, also compositions are "sequences of nonempty sets" (in the sense that there is only one *(multi-)set mit k indistinguishable elements, which we can confidently identify with the number k). We thus see again the same "analogy" as in Example 2.1.1.*

2.2. Species, labelled and unlabelled

It is commonly known that generating functions can be thought of in the following way: Given a family $\mathcal{A}$ of combinatorial objects, specify a weight function $\omega : \mathcal{A} \to R$ (R is a ring, $\omega(O)$ typically assumes the values z^n, where n is a "characteristic value" of O, such as size, length, number of parts, etc.) and form the formal sum

$$\sum_{O\in\mathcal{A}} \omega(O),$$

which we (also formally) can write out as follows:

$$\sum_{x\in R} c_x \cdot x,$$

where c_x is the number of objects in $\mathcal{A}$ with assigned weight x: Of course, the number c_x should be defined for any particular (finite) x, in order that the sum makes sense. This observation immediately leads to the following definition:

DEFINITION 2.2.1. *We denote a family $\mathcal{A}$ of combinatorial objects, of which every single object consists of a finite number n of "*atoms*" (and which moreover may possess a more or less complicated "structure"), where the atoms*

- *either may be* labelled *by the first n natural numbers [see Example 2.1.1],*
- *or may be completely indistinguishable ("unlabelled") [see Example 2.1.2],*

as labelled species *or as* unlabelled species *(generally:* species*). The number n of atoms, an object A of a species consists of, is called the* size *of A: $\|A\| = n$.*
This can be expressed a bit more abstractly as follows: To a species $\mathcal{A}$ belongs a size function $\#_{\mathcal{A}} : \mathcal{A} \to \mathbb{N}_0$, *which is given by $\#_{\mathcal{A}}(A) = \|A\|$.*
For a species we always *assume that the subfamilies*

$$\mathcal{A}_n := \{A \in \mathcal{A} : \|A\| = n\}$$

for all $n \in \mathbb{N}$ are finite*, i.e.,*

$$a_n := |\mathcal{A}_n| < \infty \; \forall n \in \mathbb{N}.$$

In particular is a species $\mathcal{A}$ always countable*.*
Of course, the size function delivers a weight function

$$\omega_{\mathcal{A}}(A) := z^{\|A\|} \quad \text{for unlabelled species,}$$

$$\omega_{\mathcal{A}}(A) := \frac{z^{\|A\|}}{\|A\|!} \quad \text{for labelled species.}$$

The generating function *of a (unlabelled or labelled) species $\mathcal{A}$ is then the formal power series*

$$\mathbf{gf}_{\mathcal{A}}(z) = \sum_{A \in \mathcal{A}} \omega_{\mathcal{A}}(A) = \sum_{A \in \mathcal{A}} z^{\|A\|} = \sum_{n \geqslant 0} a_n z^n,$$

$$\mathbf{egf}_{\mathcal{A}}(z) = \sum_{A \in \mathcal{A}} \omega_{\mathcal{A}}(A) = \sum_{A \in \mathcal{A}} \frac{z^{\|A\|}}{\|A\|!} = \sum_{n \geqslant 0} a_n \frac{z^n}{n!}$$

2.3. Unlabelled species and the enumeration of trees

EXAMPLE 2.3.1. *A totally simple species consists only of a single object, which itself consists of a single* atom*:*

$$\texttt{Atom} = \{\circ\}$$

the generating function is

$$\mathbf{gf}_{\texttt{atom}}(z) = z.$$

EXAMPLE 2.3.2. *Another simple species is (multi-)*sets *(of atoms, which are indistinguishable as unlabelled objects):*

$$\texttt{sets} = \{\emptyset, \{\circ\}, \{\circ, \circ\}, \dots\}$$

the generating function is

$$\mathbf{gf}_{\texttt{sets}}(z) = 1 + z + z^2 + \cdots = \frac{1}{1-z}.$$

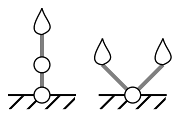
A simple "modification" would be the species $\texttt{sets}_1$ *of* nonempty *sets (of atoms); the generating function is then*

$$\mathbf{gf}_{\texttt{sets}_1}(z) = z + z^2 + z^3 + \cdots = \frac{z}{1-z} = \frac{1}{1-z} - 1.$$

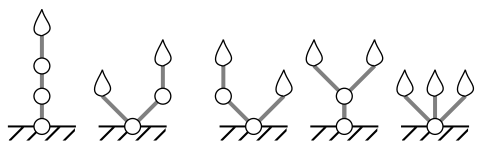
REMARK 2.3.3. *Since the atoms here are* unlabelled *(i.e., not distinguishable), their ordering does not matter: Therefore the species* `sets` *(of atoms) is equal to the species* `sequences` *(of atoms). This however depends on the* indistinguishability *of the elements (atoms): sequences of* distinguishable *elements are of course* not *the same combinatorial objects as* (multi-)sets *of these elements!*

DEFINITION 2.3.4. *A* tree *is a simply connected graph without circles.*
A rooted tree *is a tree with a special vertex, which is referred to as* root*. A rooted tree always has a "natural orientation" ("from the root away"), it thus emerges as a* directed graph*: Vertices with outgoing degree 0 are called* leaves*, all other vertices are called* inner vertices*.*
Every tree is of course planar *(is embeddable in the plane such that two arbitrary embedded edges intersect in at most one vertex): A rooted tree* in a fixed embedding in the plane*, where the order of the subtrees is relevant, is called* planar *oder* ordered rooted tree*.*
The size *of a tree (in the sense of Definition 2.2.1) can be given by the number of its vertices: Trees are thus a further example of species.*

For instance, there are the following 2 planar rooted trees with 3 vertices:



With 4 vertices there are the following 5 planar rooted trees, although the second and the third of them would be viewed as the same "normal" (i.e., "not-planar") rooted tree:



REMARK 2.3.5 (Here always: rooted trees!). *Especially for computer sciences algorithms (e.g., search algorithms) and data structures which can be "composed" of trees are interesting objects. Since these trees often have a "distinguished vertex" (a* root*), we will in the following always consider* rooted trees *(if not explicitly speficied otherwise) but for simplicity talk about* trees*.*

As we have seen in the introductory Example 2.1.2, the combinatorial construction "form finite sequences of objects" transfers to the algebraic construction "insert the generating function of the objects into the geometric series". We shall examine this "transfer"

combinatorial construction $\rightarrow$ algebraic construction in further examples:

DEFINITION 2.3.6 (Disjoint union). *Given two species $\mathcal{A}$ and $\mathcal{B}$, we can consider their* disjoint *union $\mathcal{C} = \mathcal{A} \dot\cup \mathcal{B}$; the corresponding size function is*

$$\#_{\mathcal{C}}(x) := \begin{cases} \#_{\mathcal{A}}(x) & x \in \mathcal{A}, \\ \#_{\mathcal{B}}(x) & x \in \mathcal{B}. \end{cases}$$

Apparently this construction transfers to generating functions as follows:

$$\mathbf{gf}_{\mathcal{C}} = \mathbf{gf}_{\mathcal{A}} + \mathbf{gf}_{\mathcal{B}}.$$

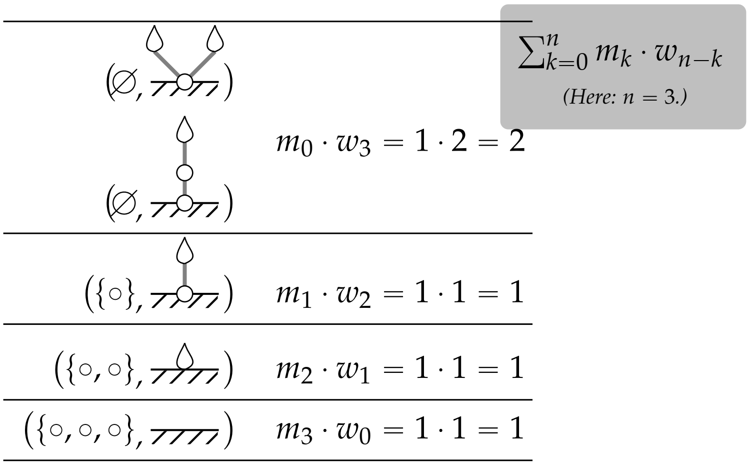
DEFINITION 2.3.7 (Cartesian Product). *Given two species $\mathcal{A}$ and $\mathcal{B}$, we can consider their cartesisian product $\mathcal{C} = \mathcal{A} \times \mathcal{B}$; the corresponding size function is*

$$\#_{\mathcal{C}}((x, y)) := \#_{\mathcal{A}}(x) + \#_{\mathcal{B}}(y).$$

Apparently this construction transfers to generating functions as follows:

$$\mathbf{gf}_{\mathcal{C}} = \mathbf{gf}_{\mathcal{A}} \cdot \mathbf{gf}_{\mathcal{B}}.$$

We visualize this graphically by means of the objects of size 3 *in the product* `sets` $\times$ `planarrootedtrees`*:*



A special case is the k–fold cartesian product of a species $\mathcal{A}$ whose objects are sequences *of length k of objects from $\mathcal{A}$:*

$$\mathcal{A}^k = \underbrace{\mathcal{A} \times \mathcal{A} \times \cdot \times \mathcal{A}}_{k \text{ times}} : \ \mathbf{gf}_{\mathcal{A}^k} = (\mathbf{gf}_{\mathcal{A}})^k.$$

If $k = 0$, we obtain the single-element species $\mathcal{A}^0 = \{\epsilon\}$ which only contains the empty sequence $\epsilon = ()$ with $\|\epsilon\| = 0$.

For instance, we can view the combinatorial objects "compositions of a number n with exactly k parts" as species $\mathcal{K}$ (with size function n): This apparently corresponds to

$$\mathcal{K} = \texttt{sets}_1{}^k, \text{ thus } \mathbf{gf}_{\mathcal{K}} = \left(\mathbf{gf}_{\texttt{sets}_1}\right)^k = \left(\frac{z}{1-z}\right)^k.$$

DEFINITION 2.3.8 (Sequences). *Given a species $\mathcal{A}$ which does not contain any object of size* 0*, we can consider* arbitrarily long *finite sequences of objects from $\mathcal{A}$, i.e., the species $\mathcal{A}^\star$*

$$\mathcal{A}^\star := \bigcup_{k\geqslant 0} \mathcal{A}^k$$

According to the previous examples this transfers to generating functions as follows:

$$\mathbf{gf}_{\mathcal{A}^\star} = \sum_{k\geqslant 0} \left(\mathbf{gf}_{\mathcal{A}}\right)^k = \frac{1}{1-\mathbf{gf}_{\mathcal{A}}}.$$

We have already seen a concrete examples for this in Example 2.1.2:

$$\texttt{compositions} = \texttt{sets}_1^\star.$$

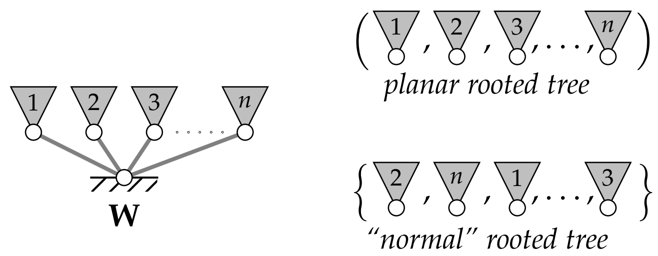
REMARK 2.3.9. *Why did we have to assume here that $\mathcal{A}$ contains no objects of size* 0*? Because otherwise $\mathcal{A}^\star$ would contain* infinitely *many objects of size* 0*, in contradiction to our assumption for species:*

$$\|\,()\,\| = \|\,(\epsilon)\,\| = \|\,(\epsilon,\epsilon)\,\| = \|\,(\epsilon,\epsilon,\epsilon)\,\| = \cdots = 0.$$

APPLICATION 2.3.10. *If one removes the root from a* nonempty *rooted tree* **W***, a (possibly empty) "list" of connected components arises which themselves consist of* nonempty *rooten trees: If the rooted tree* **W** *is*

- ordered*, then one should also view the "list" as ordered (i.e., as* sequence*) — then from the (ordered) sequence one can of course again uniquely reconstruct the original (ordered) rooted tree* **W***,*
- unordered*, then one should also view the "list" as unordered (i.e., as* multiset*) — then from the (unordered) multiset one can of course again uniquely reconstruct the original (unordered) rooted tree.*

The following graphic illustrates this simple thought:



If we denote the number of planar rooted trees with n vertices with f_n, then we obtain by definition the generating function of the species of planar rooted trees with at least one vertex:

$$f(z) = \sum_{n\geqslant 1} f_n \cdot z^n = z + z^2 + 2z^3 + 5z^4 + \cdots$$

From the Application 2.3.10 we first get

$$\texttt{planarrootedtrees}_1 = \texttt{atom} \times \texttt{planarrootedtrees}_1^\star$$

and hence then the following functional equation:

$$f = z \cdot \left(f^0 + f^1 + f^2 + \cdots\right) = \frac{z}{1-f} \iff f^2 - f + z = 0.$$

This quadratic equation for f has two solutions of which the following has the "right" (positive) coefficients:

$$f(z) = \frac{1-\sqrt{1-4z}}{2} = \sum_{n\geqslant 0} \frac{1}{n+1}\binom{2n}{n} z^{n+1}. \tag{2.4}$$

Before we start the next venturesome calculation, we remind ourselves of the ...

Transfer principle for formal power series

If an identity for analytic functions also makes sense for the corresponding formal power series, then it is automatically also an identity for formal power seriesn.
Conversely: If an identity for formal power series also makes sense for the corresponding analytic functions (this means that there exists a nontrivial common radius of convergence for all involved series), then it is automatically also an identity for analytic functions.

EXAMPLE 2.3.11 (Sets). *Let $\mathcal{A}$ be a species which does not contain any object of size* 0. *If we in sequences of $\mathcal{A}$ "forget about the order", we obtain the species $\mathcal{C} = \texttt{sets}(\mathcal{A})$ (as said: actually multisets — elements can occur repeatedly): the size function for an object*

$$\{A_1, A_2, \dots, A_k\} \in \mathcal{C}$$

is given (just as in sequences) by

$$\#_C(\{A_1, A_2, \dots, A_k\}) = \sum_{i=1}^{k} \#_A(A_i).$$

Thus we obtain for the generating function

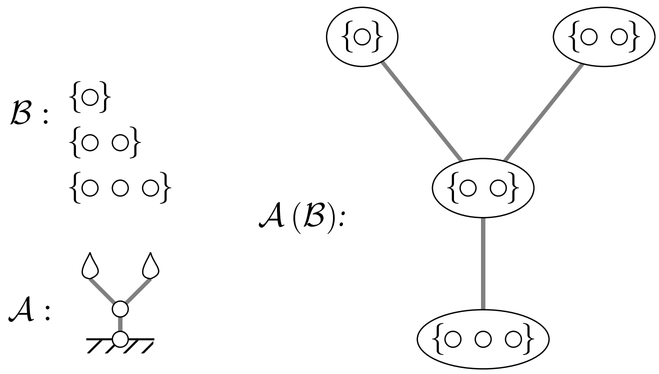
$$\begin{aligned}
\mathbf{gf}_{\mathcal{C}}(z) &= \sum_{C\in\mathcal{C}} z^{\#_C(C)} \quad \leftarrow C = \Big\{\underbrace{A_1,\dots,A_1}_{k_1},\underbrace{A_2,\dots,A_2}_{k_2},\dots\Big\} \\
&= \left(\sum_{k_1\geqslant 0}\left(z^{\|A_1\|}\right)^{k_1}\right)\cdot\left(\sum_{k_2\geqslant 0}\left(z^{\|A_2\|}\right)^{k_2}\right)\cdots \\
&= \prod_{A\in\mathcal{A}}\sum_{k\geqslant 0}\left(z^{\|A\|}\right)^{k} \\
&= \prod_{i\geqslant 1}\left(\frac{1}{1-z^i}\right)^{a_i} \quad \leftarrow a_i := |\{A\in\mathcal{A} : \|A\| = i\}| < \infty \\
&= e^{a_1\left(z+\frac{z^2}{2}+\frac{z^3}{3}+\cdots\right)+a_2\left(z^2+\frac{z^4}{2}+\frac{z^6}{3}+\cdots\right)+\cdots} \quad \leftarrow \log\frac{1}{1-z} = \sum\frac{z^n}{n} \\
&= \exp\left(\mathbf{gf}_{\mathcal{A}}(z) + \frac{\mathbf{gf}_{\mathcal{A}}(z^2)}{2} + \frac{\mathbf{gf}_{\mathcal{A}}(z^3)}{3} + \cdots\right).
\end{aligned}$$

For instance we obtain the species `numberpartitions` *(i.e., "unordered" compositions) as* `sets` $(\texttt{sets}_1)$.

EXAMPLE 2.3.12 (Composition). *Given two species $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{B}$ contains no empty object (of size zero; compare with Remark 2.3.9):*

$$\nexists B \in \mathcal{B} : \#_{\mathcal{B}}(B) = 0.$$

Then we construct the composition of species $\mathcal{A}(\mathcal{B})$ *as follows: We imagine all the atoms of an object from $\mathcal{A}$ "blown up" and insert in each such atom an object from $\mathcal{B}$. The following graphic visualizes the construction for $\mathcal{A} =$* `rootedtrees` *und $\mathcal{B} =$* $\texttt{sets}_1$*:*



However, unfortunately this construction does not *translate to the composition of the respective generating functions! Foir instance, we have:*

$$\texttt{sets}(\texttt{sets}_1) = \texttt{numberpartitions},$$

but

$$\frac{1}{1-\frac{z}{1-z}} \neq \prod_{i=1}^{\infty} \frac{1}{1-z^i}.$$

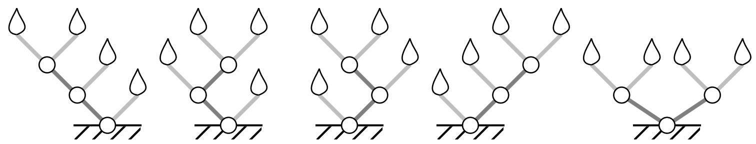
(The problem is apparently: the construction as such is not unique — in the above visualization we could also have distributed the objects from $\mathcal{B}$ differently among the atoms of the objects from $\mathcal{A}$. We would need to, say, first construct all possible *"insertions of objects in atoms", and then figure out which of those are actually different!)*

2.3.1. Enumeration of binary trees. We now want to show the usefulness of the viewpoint of species on typical enumeration problems.

DEFINITION 2.3.13. *A rooted tree with the property that every vertex has outdegree $\leqslant 2$ is called* binary tree. *With other words: On each vertex of a binary tree two subtrees are hanging which may also be empty. (More generally, a rooted tree with the property that evey vertex has outdegree $\leqslant m$ is called an m–*ary tree.*) An* ordered binary tree *is a binary tree where at every inner vertex the* order *of its* two *subtrees matters (i.e., there is* always *a left and a right subtree;* even *if one of the subtrees is* empty*! Note that a* binary ordered tree *is* not the same concept *as a* planar binary tree*!)*

A binary tree with the property that each inner vertex has outdegree $=$ 2 is called complete binary tree. *(More generally, a rooted tree with the property that each inner vertex has outdegree m is called a* complete m–ary tree.*)*

We would like to determine the number of complete binary *ordered* trees with n inner vertices: This is the same as the number of *ordered* binary trees with n vertices. The following graphic shows the objects of this species $\mathcal{B}$ with size (number of vertices) $n = 3$:



The "combinatorial decomposition" of this species is simple: An ordered binary tree is

- either *empty*
- or consists of a *root* (an atom) and a *pair of subtrees*.

Formally thus:

$$\mathcal{B} = \{\epsilon\} \,\dot\cup\, \left(\texttt{atom} \times \mathcal{B}^2\right).$$

This translates to the following equation for the generating function:

$$\mathbf{gf}_{\mathcal{B}} = 1 + z \cdot (\mathbf{gf}_{\mathcal{B}})^2 .$$

This quadratic equation for $\mathbf{gf}_{\mathcal{B}}$ has again two solutions of which

$$\mathbf{gf}_{\mathcal{B}} = \frac{1-\sqrt{1-4z}}{2z} = \sum_{n\geqslant 0} \frac{1}{n+1}\binom{2n}{n} z^n \tag{2.5}$$

is the "right" one.

This simple idea of course also works generally for the species of unordered m–ary trees $\mathcal{T}_m$:

$$\mathbf{gf}_{\mathcal{T}_m} = 1 + z \cdot \left(\mathbf{gf}_{\mathcal{T}_m}\right)^m . \tag{2.6}$$

For $m > 2$ it is however simpler to determine the coefficients of the generating function with the Lagrange inversion formula; see Corollary 1.2.5, which we shall repeat here:

COROLLARY 2.3.14 (Lagrange–inversion formula). *Let $f(z)$ be a formal power series with vanishing constant term and $F(z)$ the corresponding compositional inverse series (i.e., $F(f(z)) = f(F(z)) = z$). Then one has*

$$[\![z^n]\!]\; F^k(z) = \frac{k}{n}\left[\!\!\left[z^{-k}\right]\!\!\right] f^{-n}(z) \quad \textit{for } n \neq 0, \tag{2.7}$$

or

$$[\![z^n]\!]\; F^k(z) = \left[\!\!\left[z^{-k-1}\right]\!\!\right] f'(z) f^{-n-1}(z). \tag{2.8}$$

After all, (2.6) just means that $\left(\mathbf{gf}_{\mathcal{T}_m} - 1\right)$ is the compositional inverse of

$$\frac{z}{(1+z)^m}.$$

According to (2.7) one thus has for $n > 0$

$$\begin{aligned}
[\![z^n]\!]\, \mathbf{gf}_{\mathcal{T}_m} &= \frac{1}{n} [\![z^{-1}]\!] \left(\frac{(1+z)^{m\cdot n}}{z^n}\right) \quad \leftarrow\text{binomial theorem!} \\
&= \frac{1}{n}\binom{m\cdot n}{n-1} \quad \leftarrow = \frac{(m\cdot n)(m\cdot n-1)\cdots(m\cdot n-n+2)}{n\cdot(n-1)!} \\
&= \frac{1}{n\cdot(m-1)+1}\binom{m\cdot n}{n}.
\end{aligned}$$

Exercise 1: *Show that the number of all unlabelled ordered nonempty rooted trees with n vertices, where every* inner *vertex has* 2 *or* 3 *branches, equals*

$$\frac{1}{n}\sum_j \binom{n}{j}\binom{j}{3j-n+1}.$$

Hint: *Find an equation for the generating function and use Langrange's inversion formula.*

Exercise 2: *How many ways are there to (properly) parenthesize n pairwise non–commuting elements of a monoid? And how does this number change if the n elements are pairwise commuting?*

For example, consider 6 non–commuting *elements $x_1, x_2, \ldots, x_6$.* Two different *ways to parenthesize them properly would be*

$$((x_2x_5)((x_1(x_4x_6))x_3)) \text{ and } ((x_3(x_1(x_4x_6)))(x_5x_2)).$$

However, these would be equivalent *for* commuting *elements.*

Hint: *Translate parentheses to* labelled binary trees*: The outermost pair of parentheses corresponds to the root, and the elements of the monoid correspond to the leaves.*

2.4. Bijective combinatorics on rooted trees

It is noticeable that the generating functions for nonempty planar rooted trees (2.4) and ordered binary trees (2.5) are equal up to factor z, and the numbers of these objects with fixed size (number of vertices) is essentially a *Catalan number* $C_k = \frac{1}{k+1}\binom{2k}{k}$: This can also be shown using *bijections*.

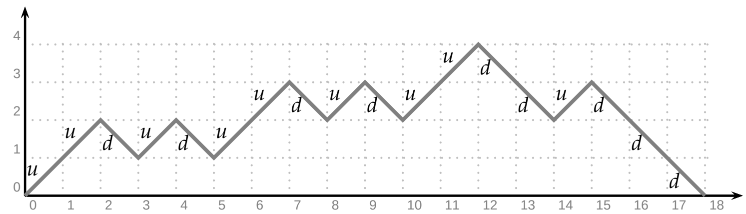
2.4.1. Dyck paths of length $2n$.

DEFINITION 2.4.1. *The* integer lattice *is the infinite directed graph with vertex set $\mathbb{Z}\times\mathbb{Z}$ and edge set (step set)*

$$\{((x,y),(x+1,y+1)) : (x,y)\in\mathbb{Z}\times\mathbb{Z}\} \cup \{((x,y),(x+1,y-1)) : (x,y)\in\mathbb{Z}\times\mathbb{Z}\}.$$

(The directed edges hence always go either "one step right–up": u or "one step right–down": d.)

A path in the integer lattice which starts in $(0,0)$ and ends in $(2n,0)$ and which never *comes below the x–axis, is called* Dyck path *of length $2n$.*



It is a well known fact that the number of Dyck paths of length $2n$ *is equal to the Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$ *(see e.g.* [**4**, Anhang A.2.2]*).*

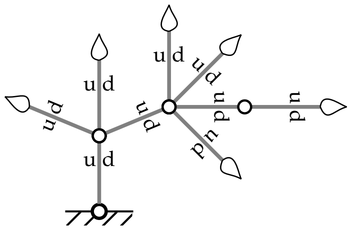
2.4.1.1. *Planar trees ↔ Dyck paths.* From the graphic it becomes apparent that a Dyck path can be coded as as word of length $2n$ over the alphabet $\{u, d\}$, where

- the number of u's is equal to the number of d's, i.e. n,
- and in each "initial part" of the word at least as many u's occur as d's.

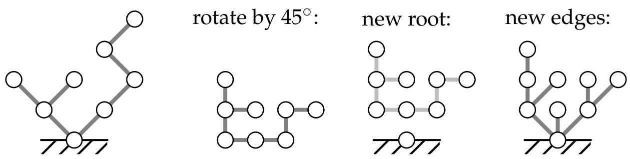
For the above graphic this "code word" is:

$$uududuududuuddudd d.$$

This immediately translates to a coding for ordered trees (*plant louse coding*). Each tree on $n + 1$ vertices has exactly n edges: We imagine a "plant louse" which sequentially walks along the edges of the ordered rooted tree (always starting with the left subtree).

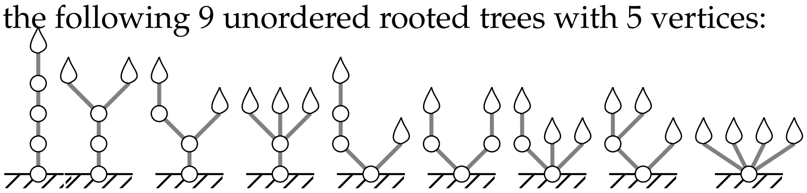


2.4.1.2. *Ordered binary trees ↔ Planar trees.* The corresponding *rotation correspondence* is best made clear in a graphic:



2.5. Unordered rooted trees

There are the following 9 unordered rooted trees with 5 vertices:



According to Application 2.3.10, by "tearing out the root" from an unordered nonempty rooted tree, a (multi–)set of rooted trees arises, i.e., for the species of unordered rooted trees $\mathcal{T}$ one has:

$$\mathcal{T} = \{\circ\} \times \texttt{sets}(\mathcal{T}). \tag{2.9}$$

This transfers as follows to the generating function (see Example 2.3.11):

$$\begin{aligned}\mathbf{gf}_{\mathcal{T}}(z) &= z\left(\frac{1}{1-z}\right)^{t_1}\cdot\left(\frac{1}{1-z^2}\right)^{t_2}\cdot\left(\frac{1}{1-z^3}\right)^{t_3}\cdots \text{ (Cayley)}\\ &= z\cdot\exp\left(\mathbf{gf}_{\mathcal{T}}(z)+\frac{1}{2}\mathbf{gf}_{\mathcal{T}}\left(z^2\right)+\frac{1}{3}\mathbf{gf}_{\mathcal{T}}\left(z^3\right)+\cdots\right) \text{ (Polya).}\end{aligned}$$

From the formula of Cayley the coefficients t_i can be recursively computed:

$$\begin{aligned}\mathbf{gf}_{\mathcal{T}}(z) &= t_1 z + t_2 z^2 + t_3 z^3 + \cdots \quad \leftarrow \tbinom{-t}{n}=(-1)^n\tbinom{t+n-1}{n}\\ &= z\left(1+t_1 z+\binom{t_1+1}{2}z^2+\cdots\right)\cdot\left(1+t_2 z^2+\binom{t_2+1}{2}z^4+\cdots\right)\cdots\\ &= z+t_1 z^2+\left(\binom{t_1+1}{2}+t_2\right)z^3+\left(\binom{t_1+2}{3}+t_1 t_2+t_3\right)z^4+\cdots\end{aligned}$$

Concretely one obtains:

$$\begin{aligned}\mathbf{gf}_{\mathcal{T}}(z) = z + z^2 + 2z^3 + 4z^4 + 9z^5 + 20z^6 + 48z^7 + 115z^8\\ + 286z^9 + 719z^{10} + 1842z^{11} + \cdots\end{aligned}$$

2.6. labelled species

EXAMPLE 2.6.1. *The labelled species* `atom` *again only consists of a single element, for which there is clearly just one labeling: "○ is labelled by 1". The generating function is hence again*

$$\mathbf{egf}_{\mathtt{atom}}(z) = \frac{z}{1!} = z.$$

EXAMPLE 2.6.2. *A further species is* `sets` *(of atoms): In the* labelled *objects the atoms are distinguishable but, since in a set the order does not matter, there is exactly one labelled set with n elements:*

$$\mathtt{sets} = \{\emptyset, \{1\}, \{1,2\}, \ldots\},$$

the (exponential) generating function thus is

$$\mathbf{egf}_{\mathtt{sets}}(z) = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots = \exp(z).$$

The (exponential) generating function of $\mathtt{sets}_1$ *then is*

$$\frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \exp(z) - 1.$$

EXAMPLE 2.6.3. *Since the atoms here are* labelled, *the (labelled) species* `sets` *(of atoms) is not anymore equal to the (labelled) species* `sequences` *(of atoms): The latter is simply einfach equal to the (labelled) species* `permutations` *with the generating function*

$$\mathbf{egf}_{\mathtt{permutations}}(z) = 1 + \frac{1!}{1!}z + \frac{2!}{2!}z^2 + \cdots = \frac{1}{1-z}.$$

The species `cycles` *of the* cyclic *permutations (i.e. of the permutations whose unique cycle decomposition consists of a single cycle) apparently has the generating function*

$$\mathbf{egf}_{\texttt{cycles}}(z) = \frac{0!}{1!}z + \frac{1!}{2!}z^2 + \frac{2!}{3!}z^3 + \cdots = \log\frac{1}{1-z}.$$

Just as with the unlabelled species there is also here a "transfer"

combinatorial construction → algebraic construction.

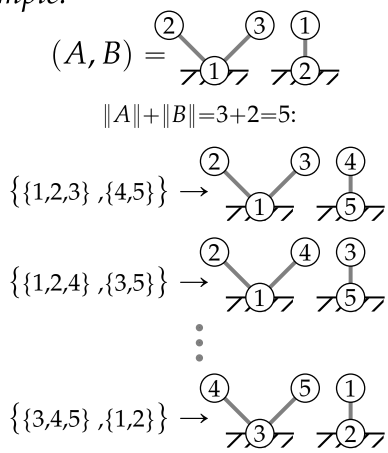
EXAMPLE 2.6.4 (Disjoint union). *Given two labelled species $\mathcal{A}$ and $\mathcal{B}$, we can (just as with the unlabelled species) consider their* disjoint *union $\mathcal{C} = \mathcal{A} \dot\cup \mathcal{B}$, and also here one has for the generating functions:*

$$\mathbf{egf}_{\mathcal{C}} = \mathbf{egf}_{\mathcal{A}} + \mathbf{egf}_{\mathcal{B}}.$$

EXAMPLE 2.6.5 (Product). *Given two labelled species $\mathcal{A}$ and $\mathcal{B}$, we define their product $\mathcal{C} = \mathcal{A} \star \mathcal{B}$ as follows: We first consider the cartesian product $\mathcal{A} \times \mathcal{B}$. From a pair of objects $(A, B) \in \mathcal{A} \times \mathcal{B}$ (where $\|A\| = m$ and $\|B\| = n$) we construct $\binom{m+n}{n}$ ddifferent labelled objects according to the following rule:*

- *Consider all decompositions of the set $[m+n]$ in two disjoint parts*
$$[m+n] = X \dot\cup Y \text{ with } |X| = m, |Y| = n,$$
- *For* every *such decomposition replace the numbers from A resp. B according to the* order-preserving bijections[2]
$$[m] \to X$$
$$[n] \to Y$$

We illustrate this by an example:



The size function is of course again given by

$$\#_{\mathcal{C}}((A, B)) := \#_{\mathcal{A}}(A) + \#_{\mathcal{B}}(B).$$

This construction indeed transfers again to generating functions:

$$\mathbf{egf}_{\mathcal{A}} \cdot \mathbf{egf}_{\mathcal{B}} = \sum_{n\geqslant 0}\left(\sum_{k=0}^{n}\frac{n!}{n!}\frac{a_k}{k!}\frac{b_{n-k}}{(n-k)!}\right)z^n = \sum_{n\geqslant 0}\frac{c_n}{n!}z^n = \mathbf{egf}_{\mathcal{C}},$$

[2]Between any two finite ordered sets X and Y with $|X| = |Y|$ there of course exists exactly one *order-preserving* bijection.

since $c_n = \sum_{k=0}^{n} \binom{n}{k} \cdot a_k \cdot b_{n-k}$ *by construction.*
For instance:

- $\texttt{sets} \star \texttt{sets} = \texttt{subsets\_of\_a\_given\_set}$, *which one could also interpret as "functions to* $\{0,1\}$*". That the cardinality of the power set of an n–element set equals* 2^n *is now also readily seen from the generating function:*

$$\left(\mathbf{egf}_{\texttt{sets}}(z)\right)^2 = \left(\mathrm{e}^z\right)^2 = \mathrm{e}^{2z} = \sum_{n\geqslant 0} \frac{2^n}{n!} z^n.$$

- $\texttt{sets}_1 \star \texttt{sets}_1$ *are thus to be viewed as "*surjective *functions to* $\{0,1\}$*".*

A special case is again the k–fold product of a (labelled) species $\mathcal{A}$*:*

$$\mathcal{A}^k := \underbrace{\mathcal{A} \star \mathcal{A} \star \cdots \star \mathcal{A}}_{k \text{ times}} : \ \mathbf{egf}_{\mathcal{A}^k} = \left(\mathbf{egf}_{\mathcal{A}}\right)^k.$$

For instance:

- $\texttt{sets}^k = \texttt{ordered\_decomposition\_in\_k\_blocks}$, *which one could also interpret as "functions to* $[k]$*".*
- ${\texttt{sets}_1}^k$ *are thus to be viewed as "*surjective *functions to* $[k]$*".*

EXAMPLE 2.6.6 (Sequences). *Also for a* labelled *species* $\mathcal{A}$ *which contains no object of size* 0 *we can consider arbitrary finite sequences of objects from* $\mathcal{A}$*, i.e. the species* $\mathcal{A}^\star$

$$\mathcal{A}^\star := \bigcup_{k\geqslant 0} \mathcal{A}^k.$$

Again this transfers to generating functions as follows:

$$\mathbf{egf}_{\mathcal{A}^\star} = \sum_{k\geqslant 0} \left(\mathbf{egf}_{\mathcal{A}}\right)^k = \frac{1}{1 - \mathbf{egf}_{\mathcal{A}}}.$$

For instance, we obtain for the species of surjections $\tau : [n] \to [k]$ *(with* $\|\tau\| = n$*)*

$$\texttt{surjections} = {\texttt{sets}_1}^\star.$$

It immediately follows: The exponential generating function of the numbers[3]

$$\sum_{k=0}^{n} S_{n,k} \cdot k!$$

(where $S_{n,k}$ *denotes the* Stirling numbers *of the second kind) thus is*

$$\frac{1}{1 - (\mathrm{e}^z - 1)} = \frac{1}{2 - \mathrm{e}^z}.$$

EXAMPLE 2.6.7 (Sets). *If in an object from* $\mathcal{A}^\star$ *we "forget about the order", we obtain the species*

$$\mathcal{C} = \texttt{sets}\left(\mathcal{A}\right).$$

[3]These are the numbers of all surjections from $[n] \to [k]$, where $k = 0, 1, \ldots, n$ is arbitrary.

By the labeling the k $\mathcal{A}$–objects of an object from $\mathcal{A}^k$ is anyway *distinguishable (i.e.: a* multiset *of labelled objects is always a* set*); for each set with k $\mathcal{A}$–objects there are $k!$ ordered k–tuples of these objects in $\mathcal{A}^k$. Thus we obtain for the generating function*

$$\mathbf{egf}_{\mathcal{C}}(z) = \sum_{k\geqslant 0} \frac{(\mathbf{egf}_{\mathcal{A}}(z))^k}{k!} = \mathbb{e}^{\mathbf{egf}_{\mathcal{A}}(z)}.$$

For instance we obtain the species

$$\texttt{setpartitions} = \texttt{sets}(\texttt{sets}_1);$$

the exponential generating function of the Bell numbers *hence is*

$$\mathbb{e}^{\mathbb{e}^z-1}$$

(see also the introductory Example 2.1.1).

REMARK 2.6.8. *As we have seen, in many cases there is a clear difference between labelled and unlabelled species. But for* ordered *trees nothing new arises by the labeling, because the vertices of ordered trees* anyway *have a "natural ordering" (root, successor of the root from left to right, successor of the successor of the root from left to right, ...), such that therefore holds:*

$$\#(\textit{unlabelled ordered trees}) \cdot n! = \#(\textit{labelled ordered trees})$$

(where n denotes the number of the vertices), i.e., the generating functions for unlabelled and labelled ordered trees are equal.

EXAMPLE 2.6.9 (Composition). *Given two species $\mathcal{A}$ and $\mathcal{B}$, such that $\mathcal{B}$ contains no "empty object" (with size zero):*

$$\nexists B \in \mathcal{B} : \#_{\mathcal{B}}(B) = 0.$$

Then we can construct the composition of species $\mathcal{A}(\mathcal{B})$ *as follows:*

- *Take a k–tuple $(B_1, B_2, \ldots, B_k)$ of objects from $\mathcal{B}^k$.*
- *Order the k–tuple by the smallest number in each component B_i: De facto we thus consider k–element* sets *(compare with Example 2.6.7)*

  $$\left\{B_{j_1} < B_{j_2} < \cdots < B_{j_k}\right\},$$

  i.e. a species with the generating function

  $$\frac{(\mathbf{egf}_{\mathcal{B}}(z))^k}{k!}.$$

- *Take an $\mathcal{A}$–object A of size $\#_{\mathcal{A}}(A) = k$ and imagine the atoms of A "blown up": In the labelled atoms now insert in the "canonical order" the objects $B_{j_1}, \cdots, B_{j_k}$ (i.e., in the A–atom with number 1 one inserts B_{j_1}, in the A–atom with number 2 one inserts B_{j_2}, etc.).*
- *The size of the thus produced object from $C \in \mathcal{C} := \mathcal{A}(\mathcal{B})$ is then of course given by*

  $$\#_{\mathcal{C}}(C) = \#_{\mathcal{B}}(B_1) + \cdots + \#_{\mathcal{B}}(B_k).$$

Here the construction indeed translates to the composition of the generating functions:

$$\begin{aligned}\mathbf{egf}_{\mathcal{C}}(z) &= \sum_{k\geqslant 0}\sum_{\substack{A\in\mathcal{A}\\ \|A\|=k}}\frac{1}{k!}\sum_{\left\{B_{j_1},\dots,B_{j_k}\right\}\in\mathcal{B}^k}\frac{z^{\|B_{j_1}\|+\dots+\|B_{j_k}\|}}{\left(\|B_{j_1}\|+\dots+\|B_{j_k}\|\right)!}\\ &= \sum_{k\geqslant 0} a_k\frac{\left(\mathbf{egf}_{\mathcal{B}}(z)\right)^k}{k!} = \mathbf{egf}_{\mathcal{A}}\left(\mathbf{egf}_{\mathcal{B}}(z)\right).\end{aligned}$$

Exercise 3: *Develop a theory for* weighted *generating functions (for labelled and unlabelled species). I.e., let $\mathcal{A}$ be some species with weight function ω which assigns to every object $A\in\mathcal{A}$ some element in a ring R (for instance, $R=\mathbb{Z}[y]$, the ring of polynomials in y with coefficients in $\mathbb{Z}$). So the generating function to be considered is*

$$\sum_{A\in\mathcal{A}} z^{\|A\|}\cdot\omega(A).$$

How should we define the weight function for sums, products and composition of species, so that the corresponding assertions for generating functions remain valid?

Exercise 4: *Let $f(m,n)$ be the number of all paths from $(0,0)$ to (m,n) in $\mathbb{N}\times\mathbb{N}$, where each single steps is either $(1,0)$ (step to the right) or $(0,1)$ (step upwards) or $(1,1)$ (diagonal step upwards). Use the language of species to show that*

$$\sum_{m,n\geqslant 0} f(m,n)\,x^m y^n = \frac{1}{1-x-y-x\cdot y}.$$

EXAMPLE 2.6.10 (Cayley's theorem). *Let $\mathcal{T}$ denote the species of nonempty unorderd* labelled *rooted trees. With the same consideration as in Section 2.5 we obtain the same "species equation"* (2.9) *also for the* labelled *rooted trees, thus*

$$\mathcal{T} = \{\circ\}\times\texttt{sets}(\mathcal{T}).$$

This equation here translates according to Example 2.6.7 resp. 2.6.9 to

$$\mathbf{egf}_{\mathcal{T}}(z) = z\cdot \mathrm{e}^{\mathbf{egf}_{\mathcal{T}}(z)},$$

hence $\mathbf{egf}_{\mathcal{T}}(z)$ is the compositional inverse of $z\cdot\mathrm{e}^{-z}$. We can very easily determine their coefficients using the Lagrange inversion formula (Corollary 1.2.5: (2.7) *for $k=1$):*

$$\begin{aligned}[\![z^n]\!]\,\mathbf{gf}_{\mathcal{T}}(z) &= \frac{1}{n}\left[\!\!\left[z^{-1}\right]\!\!\right]\,\left(z\cdot\mathrm{e}^{-z}\right)^{-n}\\ &= \frac{1}{n}\left[\!\!\left[z^{-1}\right]\!\!\right]\,z^{-n}\mathrm{e}^{nz}\\ &= \frac{1}{n}\left[\!\!\left[z^{n-1}\right]\!\!\right]\,\mathrm{e}^{nz}\\ &= \frac{1}{n}\frac{n^{n-1}}{(n-1)!}\end{aligned}$$

for $n>0$. This means, the number of labelled unordered rooted trees on $n>0$ vertices is

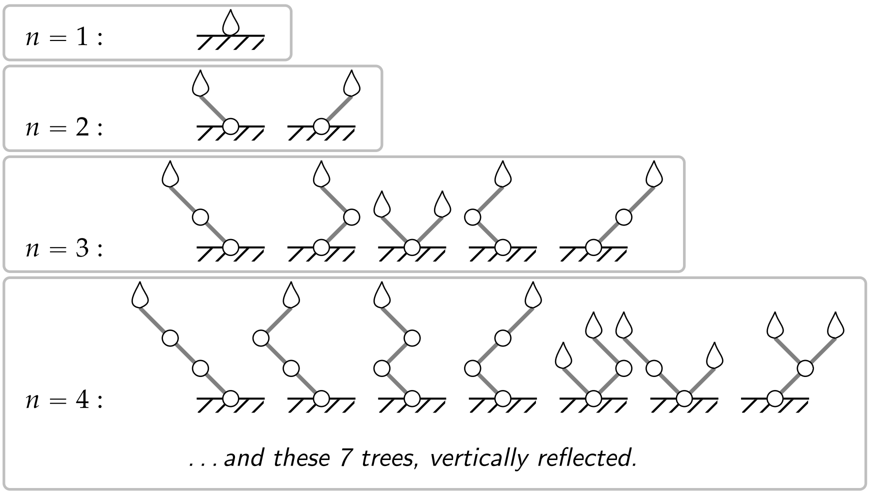
$$t_n = n^{n-1}.$$

If we "forget" about the root, we then obtain Cayley's formula for the number of all labelled trees *(not rooted trees!):*

$$\#\left(\textit{labelled treeson } n \textit{ vertices}\right) = n^{n-2}. \tag{2.10}$$

Exercise 5: *Determine the number of all* unlabelled ordered binary *rooted trees with n vertices and k leaves.*

Hint: Consider the generating function in 2 variables z and y, where every rooted tree W with n vertices and k leaves is assigned $\omega(W) := z^n y^k$. The following picture shows these trees for $n = 1, 2, 3, 4$:



I.e., the first terms of the generating function are:

$$T(z, y) := \sum_W \omega(W) = z \cdot y + z^2 \cdot 2y + z^3 \left(y^2 + 4y\right) + z^4 \left(6y^2 + 8y\right) + \cdots$$

Find an equation for this generating function T, from which the series expansion can be derived.

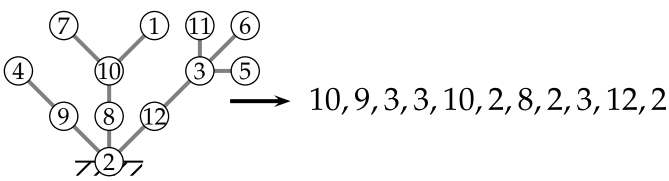
Exercise 6: *Determine the number of all* labelled unordered *rooted trees with n vertices and k leaves.*

Hint: Consider the exponential generating function in 2 variables z and y (as in Exercise 5) and use Lagrange's inversion formula.

2.6.1. Combinatorial proof of Cayley's theorem. The *Prüfer correspondence* delivers a bijection between

- unordered, labelled rooted trees with n vertices,
- and sequences of numbers of length $n-1$ from $[n]$.

For a given rooted tree this works as follows: Look for the leaf with the smallest number (labeling), note down the number of the inner vertex, on which this leaf is hanging, and remove the leaf: Repeat this until only the root is left. In a small example:



Conversely one recovers from the *Prüfer code*

$$(p(v_1), p(v_2), \dots, p(v_{n-1}))$$

the corresponding rooted tree again by proceeding as follows: Let v_1 be the *smallest* number from $[n]$, which does *not* appear in the Prüfer code. The numbers which do not appear in the Prüfer code correspond to the leaves in the original rooted tree; v_1 is thus the number of the first removed leaf. By construction this leaf was hanging on the vertex with the number $p(v_1)$, we thus note down

$$(v_1, p(v_1)) \text{ } (v_1 \text{ is hanging on } p(v_1))$$

and continue with the remaining Prüfer code

$$(p(v_2), p(v_3), \dots, p(v_{n-1}))$$

recursively: Let v_2 be the smallest number in $[n] \setminus \{v_1\}$ which does not appear in the remaining Prüfer code; we thus note down

$$(v_2, p(v_2)) \text{ } (v_2 \text{ is hanging on } p(v_2)),$$

etc. For the above example the following sequence of notations arises:

$$((1,10),(4,9),(5,3),(6,3),(7,10),(9,2),(10,8),(8,2),(11,3),(3,12),(12,2)),$$

and it is clear that the root has number 2: With this the rooted tree can be uniquely reconstructed.

Exercise 7: *Prove Cayley's formula (the number of labelled trees on n vertices equals n^{n-2}) as follows: Take a labelled tree on n vertices and tag two vertices S and E. View S and E as the starting point and ending point of the unique path p connecting S and E in the tree. Now orient all edges belonging to p "from S to E", and all edges not belonging to p "towards p". Now travel along p from S to E and write down the labels of the vertices: Whenever a new maximal label is encountered, close a cycle (by inserting an oriented edge from the vertex before this new maximum to the start of the "current cycle") and start a new cycle. Interpret the resulting directed graph as a function $[n] \to [n]$ (i.e., a directed edge from a to b indicates that the function maps a to b).*

Exercise 8: *Show that the number of all graphs on n vertices, m edges and k components equals the coefficient of $u^n \alpha^m \beta^k / n!$ in*

$$\left(\sum_{n \geqslant 0} (1+\alpha)^{\binom{n}{2}} \frac{u^n}{n!} \right)^{\beta}.$$

<u>Hint:</u> *Find a connection between the generating function of all labelled graphs (weight $\omega(G) := u^{|V(G)|} \alpha^{|E(G)|}$) and the generating function of all* connected *labelled graphs.*

Exercise 9: *Show that the number of labelled* unicyclic *graphs (i.e., connected graphs with exactly* one *cycle) on n vertices equals*

$$\frac{1}{2} \sum_{j=3}^{n} \binom{n}{j} j! \, n^{n-1-j}.$$

<u>Hint:</u> *Find a representation as a composition of species.*

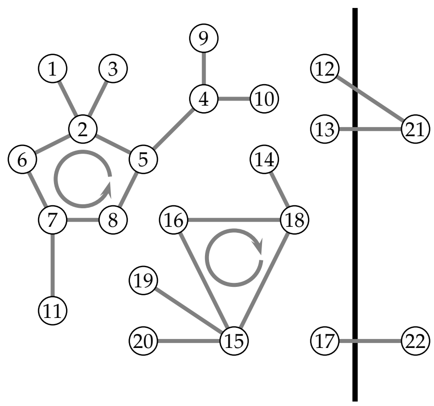
Exercise 10: *Let* $T(u)=\sum_{n\geqslant 0}(n+1)^{n-1}u^n/n!$. *Prove the identity*

$$\frac{T^j(u)}{1-uT(u)}=\sum_{l\geqslant 0}(l+j)^l\,\frac{u^l}{l!}.$$

Hint: For a bijective *proof consider the species* $\mathcal{W}$ *of labelled trees, where the vertex with the largest label (i.e.:* n*, if the tree has* n *vertices) is tagged as the* root*, but this label is* erased*, and the root does not contribute to the* size *of the tree (i.e: if* t *has* n *vertices (including the root), then we have* $\|t\|_{\mathcal{W}}=n-1$*): Obviously,* $T=\mathbf{gf}_W$.

Now consider functions $f:[l]\to[l+j]$*. Visualize such function* f *as a directed graph with vertex set* $[l+j]$ *and directed edges* $(x,f(x))$*. The following graphic illlustrates this for the case* $l=20$*,* $j=2$ *and*

$$(f(n))_{n=1}^{l}=(2,6,2,5,2,7,8,5,4,4,7,21,21,18,16,18,22,15,15,15):$$



The components of this graph are rooted trees or unicyclic graphs; all *vertices in* $[l+j]\setminus[l]$ *appear as roots of corresponding trees.*

Exercise 11: *The* derivative $\mathcal{A}'$ *of a labelled species* $\mathcal{A}$ *is defined as follows: Objects of species* $\mathcal{A}'$ *with size* $n-1$ *are objects of* $\mathcal{A}$ *with size* n*, whose atoms are numbered from* 1 *to* $n-1$ *(*not *from* 1 *to* n*), such that there is one atom without a label. A typical element of* `Sequences`$'$ *is*

$$(3,1,2,5,\circ,4),$$

where $\circ$ *indicates the unlabelled atom.*

Show: The generating function of $\mathcal{A}'$ *is precisely the derivative of the generating function of* $\mathcal{A}$*. Moreover, show:*

(1) $(\mathcal{A}+\mathcal{B})'=\mathcal{A}'+\mathcal{B}'$.
(2) $(\mathcal{A}\star\mathcal{B})'=\mathcal{A}'\star\mathcal{B}+\mathcal{B}'\star\mathcal{A}$.
(3) $(\mathcal{A}\circ\mathcal{B})'=\left(\mathcal{A}'\circ\mathcal{B}\right)\star\mathcal{B}'$.

These equations are to be understood as size–preserving bijections.

Exercise 12: *Show the following identities for labelled species (i.e., give a size–preserving bijection between the families of combinatorial objects — this is* more than *an identity for the corresponding generating functions):*

(1) $\mathtt{oPar}'=\mathtt{oPar}^2\star\mathtt{Sets}$*, where* `oPar` *denotes the species of* ordered *set partitions (i.e., the order of the blocks of the partitions matters).*

(2) $\texttt{Polyp}' = \texttt{Sequences}(\texttt{Atom}) \star \texttt{Sequences}(2\texttt{Atom})$, *where* Polyp *denotes the species*

$$\texttt{Cycles}\left(\texttt{Sequences}_{\geqslant 1}\right);$$

i.e., an object of Polyp *is a cycle, where there is a nonempty sequence attached to each atom of the cycle.*

Exercise 13: *Let $\mathcal{A}$ be the (labelled) species of (unordered) rooted trees, $\mathcal{U}$ the (labelled) species of trees (without root) and $\mathcal{F}$ the (labelled) species of rooted forests. Show the following equations (i.e., give a size–preserving bijection between the families of combinatorial objects — this is* more than *an identity for the corresponding generating functions):*

(1) $\mathcal{A}' = \mathcal{F} \star \texttt{Sequences}(\mathcal{A})$,
(2) $\mathcal{U}'' = \mathcal{F} \star \mathcal{A}'$,
(3) $\mathcal{A}'' = (\mathcal{A}')^2 + (\mathcal{A}')^2 \star \texttt{Sequences}(\mathcal{A})$.

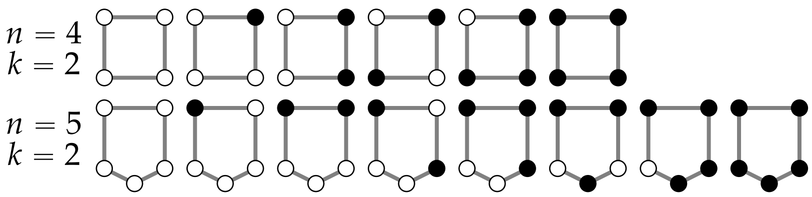
Exercise 14: *Compute all derivatives of* $\texttt{Sets}^2$ *and of* Sequences.

2.7. Pólya's theorem

We start with two motivating examples.

EXAMPLE 2.7.1. *Given a cube, how many* essentially different *possibilities exist to colour its sides with the colours red and blue? ("Essentially different" shall mean here: We view two colourings, which turn into each other by a rotation, as equal.)*
There are in total 10 possibilities: all sides red, all sides blue, one side red, one side blue, 2 opposite sides red, 2 opposite sides blue, 2 adjacent sides blue, 2 adjacent sides red, 3 in a corner colliding sides blue, 3 in a row connected sides blue.

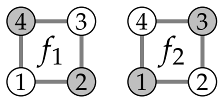
EXAMPLE 2.7.2. *Given pearles in k colours. How many possibilities exist to form a necklace of length n, where 2 necklaces are viewed as* equal *ansehen, if they can be turned into each other by a rotation (but* not *by a reflection!)?*



If we want to describe Example 2.7.2 "abstractly", then we can do this as follows: Suppose an object on n (labelled) atoms is given. These atoms are mapped into a set of values $R = [k]$ of k colours (in the example: $k = 2$). We ask about the number of such mappings, where we consider 2 mappings f_1, f_2 as equivalent if there exists a *cyclic* permutation $\sigma \in \mathfrak{S}_n$ of labelled atoms, such that

$$f_1 \circ \sigma = f_2.$$

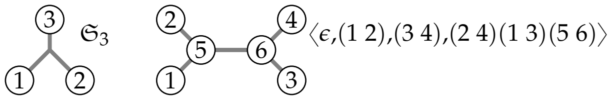
For instance:



$f_1 \circ (1\ 2\ 3\ 4) = f_2.$

This problem can be generalised as follows:

DEFINITION 2.7.3. *Let a finite set* $[n] = \{1, 2, \ldots, n\}$ *and a group* $G \subseteq \mathfrak{S}_n$ *of permutations be given. The corresponding idea is: We have a (labelled) object* O *on* n *atoms which we identify with the elements from* $[n]$*, and* G *is the group of symmetry mappings, i.e., the group of permutations of atomes of* O*, which leave* O *invariant; for instance:*



Further, let $R = [k]$ *be a given set of k colours. We are interested in the number of all* nonequivalent *mappings* $f : [n] \to R$*, where* $f_1 \sim f_2$ *if* $f_1 \circ g = f_2$ *for any* $g \in G$*. With other words: We search for the number of all* equivalence classes *of mappings under the equivalence relation*

$$f_1 \sim f_2 :\Longleftrightarrow \exists g \in G : f_1 \circ g = f_2.$$

We call these equivalence classes of "colourings" patterns*.*

Let each element $r \in R$ *be assigned a weight* $\omega(r)$ *(usually a monomial, e.g. simply* z^r*), the weight of a function be* $\omega(f) := \prod_{i=1}^{n} \omega(f(i))$*. It is clear:* $f_1 \sim f_2 \implies \omega(f_1) = \omega(f_2)$*; the weight of a pattern shall be* $\omega(f)$*, where* f *is any representative of the pattern (i.e. of the equivalence class).*

As always we are interested in the generating function of all patterns, i.e. in the sum

$$\sum_{\text{pattern}} \omega(f).$$

EXAMPLE 2.7.4. *We consider the example of the cube with* $\omega(\text{read}) = x$*,* $\omega(\text{blue}) = 1$*: The weight of a pattern is hence* $x^{\#(\text{red faces})}$*. The generating function is then:*

$$x^6 + x^5 + 2x^4 + 2x^3 + 2x^2 + x + 1.$$

LEMMA 2.7.5 (Burnside's Lemma, Theorem of Cauchy–Frobenius). *Consider the problem from Definition 2.7.3. Let* $\psi_\alpha(g)$ *be the number of all functions* f *with weight* $\omega(f) = \alpha$ *and* $f \circ g = f$*. Then the number of* all *patterns with weight* α *is equal to*

$$\frac{1}{|G|} \sum_{g \in G} \psi_\alpha(g).$$

PROOF. Consider all pairs (g, f) with $f \circ g = f$, $\omega(f) = \alpha$: By *double counting* one immediately has

$$\sum_{g \in G} \psi_\alpha(g) = \sum_{f : \omega(f) = \alpha} \eta(f),$$

where $\eta(f)$ denotes the number of all g with $f \circ g = f$.

For fixed f, $G_f := \{g \in G : f \circ g = f\}$ is a subgroup of G.

Now consider all $f \circ h$ with $h \in G$: Considering this as *set*, this is simply the equivalence class $\overline{f}$ of f: How often does each element occur?

$$f \circ h_1 = f \circ h_2 \iff f \circ \left(h_1 \circ h_2^{-1}\right) = f$$
$$\iff h_1 \circ h_2^{-1} \in G_f \iff h_1 \in G_f \circ h_2.$$

I.e., for the elements of a fixed right coset of G_f one always gets the same functions. Hence we have:

$$|G| = \left|\overline{f}\right| \cdot \left|G_f\right| \implies \eta\,(f) = \left|G_f\right| = |G| / \left|\overline{f}\right|.$$

Wir thus obtain:

$$\sum_{f:\omega(f)=\alpha} \eta\,(f) = \sum_{f:\omega(f)=\alpha} \frac{|G|}{\left|\overline{f}\right|} = |G| \cdot \sum_{f:\omega(f)=\alpha} \frac{1}{\left|\overline{f}\right|}$$
$$= |G| \cdot \sum_{\overline{f}:\omega\left(\overline{f}\right)=\alpha} \sum_{f \in \overline{f}} \frac{1}{\left|\overline{f}\right|} = |G| \cdot \sum_{\overline{f}:\omega(f)=\alpha} 1$$
$$= \#\,(\text{pattern with weight } \alpha) \cdot |G|\,.$$

□

DEFINITION 2.7.6. *The* cycle index *of a group $G \subseteq \mathfrak{S}_n$ is*

$$\mathrm{P}_G\,(x_1, x_2, \ldots, x_n) := \frac{1}{|G|} \sum_{g \in G} \left(x_1^{c_1} \cdot x_2^{c_2} \cdots x_n^{c_n}\right) = \frac{1}{|G|} \sum_{g \in G} \left(x_{|z_1|} \cdot x_{|z_2|} \cdots\right),$$

where $z_1 \cdot z_2 \cdots = g$ denotes the disjoint cylce decomposition of the permutation g and $|z_i|$ denotes the length of the cycle z_i, and wehre $(c_1, c_2, \ldots)$ denotes the cycle type *of g (i.e., c_i is equal to the number of cycles of length i in g).*

EXAMPLE 2.7.7. *The rotation group of the cube can be described by permutations of its* faces*:*

- *the identity delivers x_1^6,*
- *rotation around the axis, which is determined by the midpoints of two opposite faces, delivers $3 \cdot x_2^2 \cdot x_1^2$ (for $180°$) and $6 \cdot x_4 \cdot x_1^2$ (for $\pm 90°$),*
- *rotation around the axis, which is determined by the midpoints of two diagonally apposite (parallel) edges, delivers (for $180°$) $6 \cdot x_2^3$,*
- *rotation around the spatial diagonals (for $\pm 120°$) delivers $8 \cdot x_3^2$.*

In total thus:

$$\mathrm{P}_G = \frac{1}{24}\left(x_1^6 + 3x_1^2 x_2^2 + 6x_1^2 x_4 + 6x_2^3 + 8x_3^2\right).$$

EXAMPLE 2.7.8. *A rotation of a chain of length n corresponds to a permutation π with*

$$\pi\,(i) = i + m \pmod{n}.$$

A cycle of length d of π is a (minimal) sequence

$$l = (i, i + m, \ldots, i + (d-1)\,m)\,,$$

such that $i + d \cdot m \equiv i \pmod{m}$, i.e., $d \cdot m \equiv 0 \pmod{n}$ and because of the minimality *of l, we have $d = \frac{n}{\gcd m,n}$.*

In π there are therefore only *cycles of length d, and in fact exactly $\frac{n}{d}$ many: The weight is $\omega(\pi) = x_d^{n/d}$. The only question left is: How many m's are there with* $\gcd m, n = n/d$, $0 \leqslant m < n$? *From the numbers $i \cdot \frac{n}{d}$*

$$\left(0, 1 \cdot \frac{n}{d}, 2 \cdot \frac{n}{d}, \ldots, (d-1) \cdot \frac{n}{d}\right)$$

these are those where i and d are relative prime:

$$\gcd n, i \cdot \frac{n}{d} = \frac{n}{d} \iff \gcd d, i = 1.$$

Their number is $\varphi(d)$, the Eulerian φ–function:

$$d = p_1^{k_1} \cdots p_l^{k_l} : \ \varphi(d) = d\left(1 - \frac{1}{p_1}\right) \cdots \left(1 - \frac{1}{p_l}\right).$$

Thus:

$$\mathrm{P}_G = \frac{1}{n} \sum_{d|n} \varphi(d)\, x_d^{\frac{n}{d}}. \tag{2.11}$$

THEOREM 2.7.9 (Pólya's theorem). *Let $G \subseteq \mathfrak{S}_n$. The generating function for all patterns $\overline{f}$, $f : [n] \to R$, is*

$$\sum_{m \text{ pattern}} \omega(m) = \mathrm{P}_G\left(\sum_{r \in R} \omega(r), \sum_{r \in R} \omega(r)^2, \sum_{r \in R} \omega(r)^3, \ldots\right)$$

PROOF. Consider $\alpha = \omega(f)$ for a function $f : [n] \to R$. By Lemma 2.7.5 the number of all patterns with weight α is equal to

$$\frac{1}{|G|} \sum_{g \in G} \psi_\alpha(g),$$

where $\psi_\alpha(g) = \#(\{h : \omega(h) = \alpha \text{ and } h \circ g = h\})$.

We multiply this with α and sum over all α:

$$\begin{aligned}
\sum_{m \text{ pattern}} \omega(m) &= \sum_{\alpha} \frac{1}{|G|} \sum_{g \in G} \psi_\alpha(g)\, \alpha \\
&= \frac{1}{|G|} \sum_{g \in G} \sum_{\alpha} \psi_\alpha(g)\, \alpha \\
&= \frac{1}{|G|} \sum_{g \in G} \sum_{f :\ f \circ g = f} \omega(f) \\
&= \ldots
\end{aligned}$$

Think: If $g = z_1 \circ z_2 \circ \cdots \circ z_l$ denotes the disjoint cycle decomposition of g, then $f \circ g = f$ if and only if the function f is *constant* on all cycles $z_1, z_2, \ldots z_l$. Thus:

$$\begin{aligned}
\ldots &= \frac{1}{|G|} \sum_{g \in G} \sum_{f:[l] \to R} \omega\left(f(1)\right)^{|z_1|} \cdot \omega\left(f(2)\right)^{|z_2|} \cdots \omega\left(f(l)\right)^{|z_l|} \\
&= \frac{1}{|G|} \sum_{g \in G} \left(\sum_{r \in R} \omega(r)^{|z_1|} \right) \cdot \left(\sum_{r \in R} \omega(r)^{|z_2|} \right) \cdots \left(\sum_{r \in R} \omega(r)^{|z_l|} \right) \\
&= \mathrm{P}_G \left(\sum_{r \in R} \omega(r)^1, \sum_{r \in R} \omega(r)^2, \ldots \right). \quad \leftarrow x_{|z_i|} \mapsto \left(\sum_{r \in R} \omega(r)^{|z_i|} \right)
\end{aligned}$$

□

REMARK 2.7.10. *If the set R of colours contains only* one *element which is simply assigned the weight z, then there is of courses only* one *pattern, and the generating function of this single pattern is simply z^n, where n is the size of the object — and indeed also*

$$\mathrm{P}_G\left(z, z^2, \ldots\right) = z^n$$

holds.

EXAMPLE 2.7.11. *For the cube we indeed obtain:*

$$\frac{1}{24}\Big(6\left(x^4+1\right)(x+1)^2 + 8\left(x^3+1\right)^2 + 3\left(x^2+1\right)^2(x+1)^2 + 6\left(x^2+1\right)^3 + (x+1)^6\Big) = x^6 + x^5 + 2x^4 + 2x^3 + 2x^2 + x + 1.$$

EXAMPLE 2.7.12. *For a chain of length n which is coloured with k colours, we assign the weight x_i to the colour i and obtain:*

$$\frac{1}{n} \sum_{d|n} \varphi(d) \left(x_1^d + x_2^d + \cdots + x_k^d \right)^{n/d}.$$

For the special case $k = 2$ this yields:

$$\begin{aligned}
\frac{1}{n} \sum_{d|n} \varphi(d) \left(x_1^d + x_2^d \right)^{n/d} &= \frac{1}{n} \sum_{d|n} \varphi(d) \sum_{l} \binom{n/d}{l} x_1^{d \cdot l} x_2^{(n/d - l) \cdot d} \quad \leftarrow m := d \cdot l \\
&= \frac{1}{n} \sum_{m} x_1^m x_2^{n-m} \sum_{d | \gcd n, m} \varphi(d) \binom{n/d}{m/d}.
\end{aligned}$$

For k colours one more generally has:

$$\begin{aligned}
&\frac{1}{n} \sum_{d|n} \varphi(d) \sum_{l_1 + \ldots l_k = n/d} \binom{n/d}{l_1, \ldots, l_k} x_1^{d \cdot l_1} \cdots x_k^{d \cdot l_k} = \quad \leftarrow m_i := d \cdot l_i \\
&\qquad \frac{1}{n} \sum_{m_1 + \cdots m_k = n} x_1^{m_1} \cdots x_k^{m_k} \sum_{d | \gcd n, m_1, \ldots, m_k} \varphi(d) \binom{n/d}{m_1/d, \ldots, m_k/d}.
\end{aligned}$$

Exercise 15: *How many* different *necklaces of n pearls in k colours are there? (This should be understood "as in real life", where rotations* and *reflections of necklaces are considered equal; in contrast to the presentation in the lecture course.)*

2.8. A generalisation

We can apparently generalise Definition 2.7.3, by considering *also on the set of colours R* a permutation group H which acts on R: How many patterns are there which *under the action* of H are yet different? I.e, we now consider two patterns f_1, f_2 as equivalent if there is a $g \in G$ (G is the perrmutation group on the atoms) and a $h \in H$ (H is permutation group on the colours) such that

$$f_1 \sim f_2 :\iff f_1 \circ g = h \circ f_2$$

EXAMPLE 2.8.1. *In Example 2.7.1 (colourings of the faces of the cube with colours $R = \{blue, red\}$), let the group be $H = \mathfrak{S}_R$: I.e., there are now only 6 possibilities for "essentially different" patterns:*

- *6 faces have the same colour,*
- *exactly 5 faces have the same colour,*
- *exactly 2 opposite faces have the same colour,*
- *exactly 2 adjacent faces have the same colour,*
- *exacly 3 in a corner colliding faces have the same colour,*
- *eactly 3 faces in a row have the same colour.*

If we define the weight of a pattern by $x^{\max(\#(red),\#(blue))}$, then the generating function is:

$$x^6 + x^5 + 2x^4 + 2x^3.$$

THEOREM 2.8.2.

$$\sum_{m \text{ pattern}} \omega(m) = \frac{1}{|G| \cdot |H|} \sum_{\substack{g \in G \\ h \in H}} \sum_{f : f \circ g = h \circ f} \omega(f). \tag{2.12}$$

PROOF. Obviously there holds:

$$\begin{aligned} f \circ g_1 = f \circ g_2 &\implies g_1 \in G_f \cdot g_2, \\ h_1 \circ f = h_2 \circ f &\implies h_1 \in h_2 \cdot H_f. \end{aligned}$$

$$\begin{aligned} \sum_{\substack{f,g,h \\ f \circ g = h \circ f}} \omega(f) &= \sum_f \omega(f) \frac{|G| \cdot |H|}{|G_f| \cdot |H_f|} \\ &= |G| \cdot |H| \cdot \sum_{w \text{ pattern}} \omega(m). \end{aligned}$$

□

2.9. Cycle index series

We now want to apply these considerations to labelled species: For that we (essentially) “sum” the cycle indices for all objects.

DEFINITION 2.9.1. *Let $\mathcal{F}$ be a labelled species: Then the symmetric group $\mathfrak{S}_n$ acts on the subfamilies*

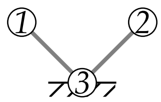
$$\mathcal{F}_n := \{F \in \mathcal{F} : \|F\| = n\}$$

“by permutation of the labeling of the atoms”. We denote the action *of such a permutation σ on F with $\sigma(F)$; if such a permutation σ fixes (leaves invariant) an object F, then we write for it: $\sigma(F) = F$. It is clear: For each object $F \in \mathcal{F}_n$*

$$\mathrm{sym}_{\mathcal{F}}(F) := \{\sigma \in \mathfrak{S}_n : \sigma(F) = F\}$$

is a subgroup[4] *of $\mathfrak{S}_n$: We refer to it as* symmetry group *of the object F.*

EXAMPLE 2.9.2. *The permutation $\sigma = (12)(3)$ (in cycle notation) obviously fixes the following* unordered *rooted tree:*



The same *object, viewed as* ordered *rooted tree, would however aber* only *be fixed by the identity (compare also with Remark 2.6.8).*

DEFINITION 2.9.3 (Cycle index series). *Let $\mathcal{F}$ be a labelled species. For a permutation $\sigma \in \mathfrak{S}_n$ we define:*

$$\mathrm{fix}_{\mathcal{F}}(\sigma) := \{F \in \mathcal{F}_n : \sigma(F) = F\}.$$

$|\mathrm{fix}_{\mathcal{F}}(\sigma)|$ is hence the number of objects F in $\mathcal{F}$ for which σ belongs to the symmetry group of F.

The cycle type of a permutation σ we denote by

$$c(\sigma) := (c_1, c_2, \ldots, c_n).$$

(I.e., c_i is the number of cycles of length i in σ.)

Further, for a sequence $\mathbf{x} := (x_1, x_2, \ldots)$ of infinitely many variables we define:

$$\mathbf{x}^{c(\sigma)} := x_1^{c_1} \cdot x_2^{c_2} \cdots x_n^{c_n}.$$

[4]algebraically: the stabilizer–subgroup of F.

The cycle index series *of a species $\mathcal{F}$ is then the formal power series (in infinitely many variables)*

$$
\begin{aligned}
Z_{\mathcal{F}}(\mathbf{x}) &= \sum_{\substack{\sigma,F\\ \sigma(F)=F}} \frac{\mathbf{x}^{c(\sigma)}}{\|F\|!} \\
&= \sum_{n\geqslant 0}\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\left|\mathrm{fix}_{\mathcal{F}}(\sigma)\right|\cdot\mathbf{x}^{c(\sigma)} \qquad (2.13)\\
&= \sum_{n\geqslant 0}\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\sum_{\substack{F\in\mathcal{F}_n:\\ \sigma(F)=F}}\mathbf{x}^{c(\sigma)} \\
&= \sum_{n\geqslant 0}\frac{1}{n!}\sum_{F\in\mathcal{F}_n}\sum_{\substack{\sigma\in\mathfrak{S}_n:\\ \sigma(F)=F}}\mathbf{x}^{c(\sigma)}. \qquad (2.14)
\end{aligned}
$$

REMARK 2.9.4. *Unordered labelled rooted trees differ from ordered labelled rooted trees "only" by the property that the first have a nontrivial symmetry group (i.e..: not only consisting of the identity), compare with Example 2.9.2. With other words: For the species of* ordered *labelled rooted trees*

$$W_1 \sim W_2 :\Longleftrightarrow \text{“}W_1 \text{ and } W_2 \text{ are identical as unordered objects”}$$

is clearly an equivalence relation *whose equivalence classes can be naturally identified with the* unordered *labelled rooted trees. "From the view of the* unordered *trees" this equivalence relation is given by its symmetry group G (which consists of "rearrangements" of subtrees):*

$$W_1 \sim W_2 :\Longleftrightarrow \exists\, \sigma \in G : \sigma(W_1) = W_2.$$

This can be generalized as follows: Let $\widehat{\mathcal{F}}$ be a labelled specied which contains only the trivial *symmetry group (i.e., all $n!$ relabelings of an object $\widehat{F} \in \widehat{\mathcal{F}}$ are viewed as different):*

$$\mathrm{sym}_{\widehat{\mathcal{F}}}\left(\widehat{F}\right) = \{\mathrm{id}\} \text{ for all } \widehat{F} \in \widehat{\mathcal{F}}.$$

Let G be a "larger" (casually spoken) symmetry group for the objects from $\widehat{\mathcal{F}}$; i.e.,

$$\{\mathrm{id}\} \sqsubset G = G_{\widehat{F}} \sqsubset \mathfrak{S}_n \text{ for all } \widehat{F} \in \widehat{\mathcal{F}}_n \subseteq \widehat{\mathcal{F}}.$$

Then one can consider the species $\mathcal{F}$ of the equivalence classes *of $\widehat{\mathcal{F}}$ with respect to this symmetry group, whose objects $F \in \mathcal{F}$ are determined by the prpoerty that for any two representatives $F_1, F_2 \in F \subseteq \widehat{\mathcal{F}}$*

$$F_1 \sim F_2 :\Longleftrightarrow \exists\, \sigma \in G : \sigma(F_1) = F_2 \qquad (2.15)$$

holds. For a $\widehat{F} \in \widehat{\mathcal{F}}_n$ let

$$
\begin{aligned}
\mathfrak{S}_n\left(\widehat{F}\right) &:= \left\{\sigma\left(\widehat{F}\right) : \sigma \in \mathfrak{S}_n\right\} \text{ (all “relabelings” of } \widehat{F}\text{)},\\
\mathfrak{S}_n(F) &:= \{\sigma(F) : \sigma \in \mathfrak{S}_n\} \text{ (all equivalence classes w.r.t. (2.15)}.
\end{aligned}
$$

Then "permutation of the labelings from $[n]$" delivers a group action *from $\mathfrak{S}_n$ to $\mathfrak{S}_n\left(\widehat{F}\right)$ which induces an action on the* equivalence classes *$\mathfrak{S}_n(F)$. Clearly, G_F is the* stabilizer[5] *of F:*

$$G_F = \mathrm{sym}_{\mathcal{F}}(F),$$

those there holds (see also the proof of Lemma 2.7.5):

$$|\mathfrak{S}_n(F)| = \frac{n!}{\mathrm{sym}_{[\mathcal{F}]}(F)} = \frac{n!}{|G_F|}.$$

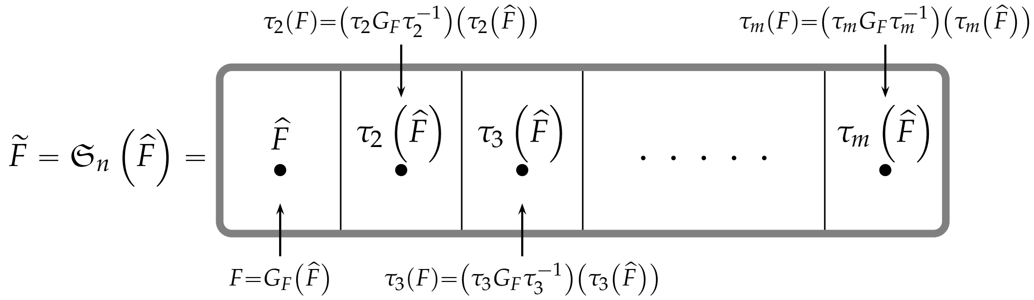
For $F_1, F_2 \in \mathfrak{S}_n\left(\widehat{F}\right)$ there holds (by definition) $F_2 = \tau(F_1)$ for a $\tau \in \mathfrak{S}_n$. Then there holds[6] for any such $\tau \in \mathfrak{S}_n$:

$$\tau\left(\mathrm{sym}_{\mathcal{F}}(F_1)\right)\tau^{-1} = \mathrm{sym}_{\mathcal{F}}(F_2).$$

Algebraically this means: For two elements in the same orbit the corresponding stabilizers are conjugated *subgroups; in particular they are* isomorphic. *Combinatorially this means that the symmetries do not depend on the concrete labeling of the atoms (for our rooted trees this is obviously clear).*

Then we can also consider the species $\widetilde{\mathcal{F}}$ of the corresponding unlabelled *structures whose objects are the* equivalence classes *of $\widehat{\mathcal{F}}$ under the equivalence relation "is obtained by an EMarbitrary relabeling of", i.e. with the symmetry group equal to the* full *symmetric group.*

Graphical illustration of the general situation ($G_F := \mathrm{sym}_{\mathcal{F}}(F)$):



Casually spoken: Generally labelled species $\mathcal{F}$ appear "in the middle between the corresponding unlabelled *species $\widetilde{\mathcal{F}}$ and the corresponding* totally unsymmetric *species $\widehat{\mathcal{F}}$"; expressed by the corresponding symmetry groups:*

$$\mathrm{sym}_{\widehat{\mathcal{F}}}\left(\widehat{F}\right) = \{\mathrm{id}\} \sqsubset G_F = \mathrm{sym}_{\mathcal{F}}(F) \sqsubset \mathfrak{S}_n = \mathrm{sym}_{\widetilde{\mathcal{F}}}\left(\widetilde{F}\right).$$

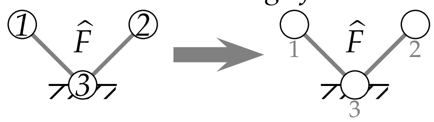
EXAMPLE 2.9.5. *To make this completely concrete, we consider again the unordered labelled rooted tree from Example 2.9.2:*

---

[5]Because G_F *fixes* by *definition* the equivalence class F.

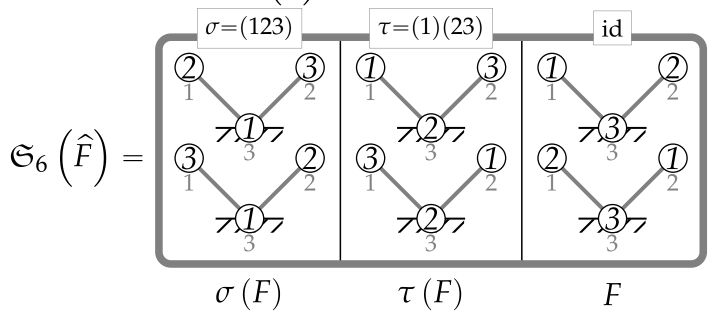
[6]By definition of a group action.

"Fix numbering of $\widehat{F}$":



$$G_F = \{\mathrm{id}, (3)\,(12)\} \simeq \mathbb{Z}_2.$$

Consider $\mathfrak{S}_6\left(\widehat{F}\right)$, *i.e.*, all *reorderings of* $\widehat{F}$*:*



With this considerations in the background we can now formulate:

LEMMA 2.9.6. *With the notation introduced in Remark 2.9.4 there holds: The cycle indikator series of a* labelled *species F is the sum over the cycle indices of all objects of the corresponding* unlabelled *species* $\widetilde{\mathcal{F}}$*. This means:*

$$\sum_{\widetilde{F}\in\widetilde{\mathcal{F}}} \mathrm{P}_{G_F}(\mathbf{x}) = \mathrm{Z}_{\mathcal{F}}(\mathbf{x})\,. \tag{2.16}$$

PROOF. The proof consists only of a longer transformation:

$$\begin{aligned}
\sum_{\widetilde{F}\in\widetilde{\mathcal{F}}} \mathrm{P}_{G_F}(\mathbf{x}) &= \sum_{\widetilde{F}\in\widetilde{\mathcal{F}}} \frac{1}{|G_F|} \sum_{\gamma\in G_F} \mathbf{x}^{c(\gamma)} \quad \leftarrow\text{Definition 2.7.6}\\
&= \sum_{n\geqslant 0} \sum_{\gamma\in\mathfrak{S}_n} \sum_{\substack{\widetilde{F}\in\widetilde{\mathcal{F}}_n\\ \gamma\in G_F}} \frac{1}{|G_F|}\cdot \mathbf{x}^{c(\gamma)} \quad \leftarrow\text{"regrouping of terms"}\\
&= \sum_{n\geqslant 0} \frac{1}{n!} \sum_{\gamma\in\mathfrak{S}_n} \sum_{\substack{\widetilde{F}\in\widetilde{\mathcal{F}}_n\\ \gamma\in G_F}} \frac{n!}{|G_F|}\cdot \mathbf{x}^{c(\gamma)} \quad \leftarrow G_F \text{ is stabilizer of } F\\
&= \sum_{n\geqslant 0} \frac{1}{n!} \sum_{\gamma\in\mathfrak{S}_n} \sum_{\substack{F\in\mathcal{F}_n:\\ \gamma(F)=F}} \mathbf{x}^{c(\gamma)} \quad \leftarrow \tfrac{n!}{|G_F|} = |\mathfrak{S}_n(F)|\\
&= \sum_{n\geqslant 0} \frac{1}{n!} \sum_{\gamma\in\mathfrak{S}_n} |\mathrm{fix}_{\mathcal{F}}(\gamma)| \cdot \mathbf{x}^{c(\gamma)}\\
&= \mathrm{Z}_{\mathcal{F}}(\mathbf{x})\,.
\end{aligned}$$

□

THEOREM 2.9.7. *There holds:*

$$\mathrm{Z}_{\mathcal{F}}(z,0,0,\dots) = \mathbf{egf}_{\mathcal{F}}(z)\,, \tag{2.17}$$

$$\mathrm{Z}_{\mathcal{F}}\left(z,z^2,z^3,\dots\right) = \mathbf{gf}_{\widetilde{\mathcal{F}}}(z)\,, \tag{2.18}$$

where $\widetilde{\mathcal{F}}$ *is the* unlabelled *species associated to* $\mathcal{F}$ *(with the same symmetries).*

PROOF. Equation (2.17) is obtained by a simple calculation from Definition (2.13):

$$\begin{aligned}
Z_{\mathcal{F}}(z,0,0,\dots) &= \sum_{n\geqslant 0}\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\left|\mathrm{fix}_{\mathcal{F}}(\sigma)\right|\cdot z^{c_1}\cdot 0^{c_2}\cdot 0^{c_3}\cdots\\
&= \sum_{n\geqslant 0}\frac{1}{n!}\left|\mathrm{fix}_{\mathcal{F}}(\mathrm{id}_n)\right|\cdot z^n\\
&= \sum_{n\geqslant 0}\frac{1}{n!}\cdot f_n\cdot z^n = \mathbf{egf}_{\mathcal{F}}(z).
\end{aligned}$$

Equation (2.18) is a direct corollary from Lemma 2.9.6, since $\mathrm{P}_{G_F}\left(z,z^2,\dots\right)$ is simply equal to z^n fory any $F\in\mathcal{F}_n$.

□

EXAMPLE 2.9.8. *For the species* `atom` *we obtain* $Z_{\texttt{atom}} = x_1$.
For the species `sets` *we obtain according to* (2.13)

$$Z_{\texttt{sets}} = \sum_{n\geqslant 0}\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\mathbf{x}^{c(\sigma)},$$

because there is exactly one set on n elements which (of course) is fixed by any permutation σ *of its atoms:*

$$\mathrm{fix}_{\texttt{sets}}(\sigma) = 1.$$

It is well-known that the number of all permutations of $\mathfrak{S}_n$ *of cycle type* $(c_1,c_2,\dots,c_n)$ *is exactly*[7]

$$\frac{n!}{1^{c_1}\cdot c_1!\cdot 2^{c_2}\cdot c_2!\cdots n^{c_n}\cdot c_n!}. \tag{2.19}$$

Therefore we further obtain:

$$\begin{aligned}
Z_{\texttt{sets}} &= \sum_{n\geqslant 0}\sum_{1\cdot c_1+2\cdot c_2+\cdots+n\cdot c_n=n}\frac{x_1^{c_1}\cdot x_2^{c_2}\cdots x_n^{c_n}}{1^{c_1}\cdot c_1!\cdot 2^{c_2}\cdot c_2!\cdots n^{c_n}\cdot c_n!}\\
&= \sum_{1\cdot c_1+2\cdot c_2+\cdots<\infty}\frac{\left(\frac{x_1}{1}\right)^{c_1}}{c_1!}\cdot\frac{\left(\frac{x_2}{2}\right)^{c_2}}{c_2!}\cdots\\
&= \mathbb{e}^{x_1+\frac{x_2}{2}+\frac{x_3}{3}+\cdots}.
\end{aligned}$$

Combined with (2.18) *thsi yields again the (ordinary) generating function of the (unlabelled) species* `sets`*:*

$$Z_{\texttt{sets}}\left(z,z^2,\dots\right) = \frac{1}{1-x} = \mathbf{gf}_{\texttt{sets}}(z).$$

[7] Simple counting argument: Consider an arbitrary permutation τ as a cycle decomposition (the first c_1 elemente are the fixpoints, the following $2\cdot c_2$ elements are the 2-cycles, etc.): In how many ways is one and the same permutation σ of cycle type $(c_1,\dots,c_k)$ obtained in this manner?

For the species `cycles` *we obtain (compare with Observation 2.9.4):*

$$\begin{aligned}
Z_{\texttt{cycles}} &= \sum_{n\geqslant 1}\frac{1}{n!}\sum_{A\in\texttt{cycles}_n}\sum_{\substack{\sigma\in\mathfrak{S}_n:\\ \sigma(A)=A}}\mathbf{x}^{c(\sigma)} \quad\leftarrow(2.14)\\
&= \sum_{n\geqslant 1}\frac{1}{n!}(n-1)!\sum_{\substack{\sigma\in\mathfrak{S}_n:\\ \sigma(A)=A}}\mathbf{x}^{c(\sigma)} \quad\leftarrow\text{Def. cycle index}\\
&= \sum_{n\geqslant 1}\frac{1}{n}\sum_{d|n}\varphi(d)\,x_d^{n/d} \quad\leftarrow(2.11)\\
&= \sum_{d\geqslant 1}\sum_{k\geqslant 1}\frac{1}{k\cdot d}\varphi(d)\,x_d^k \quad\leftarrow n=k\cdot d\\
&= \sum_{d\geqslant 1}\frac{\varphi(d)}{d}\log\frac{1}{1-x_d}.
\end{aligned}$$

For the species `permutations` *we think: Let $\sigma\in\mathfrak{S}_n$ be of cycle type $(c_1,c_2,\dots,c_n)$. How does a permutation π look like which is fixed by σ?*

$$\begin{array}{ccc}
i & \xrightarrow{\sigma} & \sigma(i)\\
\pi\big\downarrow & & \pi\big\downarrow\\
\pi(i) & \xrightarrow{\sigma} & \sigma(\pi(i))=\pi(\sigma(i))
\end{array}$$

This means: π is fixed by σ, if and only if

$$\pi=\sigma^{-1}\circ\pi\circ\sigma \iff \pi^{-1}=\sigma^{-1}\circ\pi^{-1}\circ\sigma \iff \pi\circ\sigma=\sigma\circ\pi.$$

Suppose i and $\pi(i)=j$ belong to cycles of different length in σ: W.l.o.g. let the σ–cycle of i be the shorter *one (otherwise consider π^{-1} instaed of π); let k be its length:*

$$\begin{array}{ccccccc}
i & \xrightarrow{\sigma} & \sigma(i) & \xrightarrow{\sigma} & \cdots & \xrightarrow{\sigma} & \sigma^k(i)=i\\
\pi\big\downarrow & & \pi\big\downarrow & & \pi\big\downarrow & & \pi\big\downarrow\\
j & \xrightarrow{\sigma} & \sigma(j) & \xrightarrow{\sigma} & \cdots & \xrightarrow{\sigma} & \sigma^k(j)\neq j
\end{array}$$

Hence in π always only those *numbers are permuted which in σ belong to cycles of* equal *length! Hoe many possibilities are there? We consider the c_k cycles of length k of σ:*

$$\underbrace{(i_1\dots)}_{k}\underbrace{(i_2\dots)}_{k}\dots\underbrace{(i_{c_k}\dots)}_{k}.$$

Let i be the smallest number in these c_k k–cycles: The permutation π can map i to $c_k\cdot k$ numbers: Thereby, at the same time, also the images

$$\pi(i),\pi(\sigma(i)),\dots,\pi\left(\sigma^{k-1}(i)\right)$$

are already determined! For the next smaller number j there are thus only $(c_k-1)\cdot k$ possibilities leftn, etc.: It is clear that in this manner all permutations π, which are *fixed by σ, are correctly counted; the number of these permutations thus is in total*

$$c_1! \cdot 1^{c_1} \cdot c_2! \cdot 2^{c_2} \cdots$$

On the other side there are exactly

$$\frac{n!}{1^{c_1} \cdot c_1! \cdot 2^{c_2} \cdot c_2! \cdots}$$

permutations $\sigma \in \mathfrak{S}_n$ of cycle type $(c_1, c_2, \dots)$ (see (2.19)). Thus we obtain:

$$\begin{aligned} Z_{\texttt{permutations}} &= \sum_{n \geqslant 0} \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \left|\text{fix}_{\texttt{permutations}}(\sigma)\right| \cdot \mathbf{x}^{c(\sigma)} \\ &= \sum_{n \geqslant 0} \frac{1}{n!} \sum_{(c_1, \dots, c_n)} \frac{n! \cdot (c_1! \cdot 1^{c_1} \cdots c_n! \cdot n^{c_n})}{1^{c_1} \cdot c_1! \cdots c_n! \cdot n^{c_n}} \cdot x_1^{c_1} \cdots x_n^{c_n} \\ &= \sum_{(c_1, c_2, \dots)} x_1^{c_1} \cdot x_2^{c_2} \cdots = \prod_{i \geqslant 1} \frac{1}{1 - x_i}. \end{aligned}$$

The following theorems are concerned with product and composition of species with symmetries: We actually would first need to *define* the symmetries for these constructions — but how these symmetries are to be *meant*[8], will become clear in the proofs.

THEOREM 2.9.9. *Let $\mathcal{A}$ and $\mathcal{B}$ be two species. Then there hold:*

$$Z_{\mathcal{A} \dot\cup \mathcal{B}} = Z_{\mathcal{A}} + Z_{\mathcal{B}}, \tag{2.20}$$

$$Z_{\mathcal{A} \star \mathcal{B}} = Z_{\mathcal{A}} \cdot Z_{\mathcal{B}}. \tag{2.21}$$

If $\mathcal{A}$ contains no object of size zero, then there also holds:

$$Z_{\mathcal{A}^\star} = \frac{1}{1 - Z_{\mathcal{A}}}. \tag{2.22}$$

PROOF. Equation (2.20) immediately follows from Definition (2.13).
For (2.21) we consider an element $C = (A, B) \in \binom{n}{k} \cdot (\mathcal{A} \times \mathcal{B})$, where $n = \|A\| + \|B\|$ and $k = \|A\|$. A permutation σ which leaves such a pair invariant, decomposes[9] in $\sigma = (\sigma_1, \sigma_2)$ with

$$\sigma((A, B)) = (\sigma_1(A), \sigma_2(B)).$$

Therefore

$$|\text{fix}_C(\sigma)| = \binom{n}{k} \cdot |\text{fix}_{\mathcal{A}}(\sigma_1)| \cdot |\text{fix}_{\mathcal{B}}(\sigma_2)|,$$

and the cycle type of σ of course satisfies

$$c(\sigma) = c(\sigma_1) + c(\sigma_2).$$

For (2.22) we only need to remember

$$\mathcal{A}^\star = \dot\bigcup_{k \geqslant 0} \mathcal{A}^k.$$

The claim then follows from (2.20) and (2.21). □

[8]The interpretations being dealt with are anyway the most obvious ones.

[9]I.e.: This is how the action on the product is *meant*!

THEOREM 2.9.10. *Let $\mathcal{A}, \mathcal{B}$ be two species, let $\mathcal{B}$ contain no object of size* 0. *With the short notation*

$$\mathbf{x}_m := (x_m, x_{2m}, x_{3m}, \dots)$$

we have:

$$\mathrm{Z}_{\mathcal{A}(\mathcal{B})}(\mathbf{x}) = \mathrm{Z}_{\mathcal{A}}(\mathrm{Z}_{\mathcal{B}}(\mathbf{x}_1), \mathrm{Z}_{\mathcal{B}}(\mathbf{x}_2), \mathrm{Z}_{\mathcal{B}}(\mathbf{x}_3), \dots).$$

PROOF. By Definition (2.14) there holds:

$$\mathrm{Z}_{\mathcal{A}(\mathcal{B})}(x_1, x_2, x_3, \dots) = \sum_{n \geqslant 0} \frac{1}{n!} \sum_{\substack{(C,\sigma): \\ C \in \mathcal{A}(\mathcal{B}):\, \|C\|=n \\ \sigma \in \mathfrak{S}_n:\, \sigma(C)=C}} \mathbf{x}^{c(\sigma)}.$$

Indeed, each object $C \in \mathcal{A}(\mathcal{B})$ by construction "consists"

- of an $\mathcal{A}$–object A of size k
- and of k $\mathcal{B}$–objects $B_1, \dots, B_k$;

is thus, so to say, "of the form"

$$C = (A; B_1, B_2, \dots, B_k).$$

A permutation σ, which fixes this object, can be described as follows:

- σ corresponds "macroscopically" to a permutation $\rho \in \mathfrak{S}_k$, which fixes A fixiert (notation: $[\rho \mid \sigma]$),
- Each cycle $(i_1, \dots, i_l)$ of ρ induces a cycle of σ which maps the atoms of B_{i_m} to the ataoms of $B_{i_{m+1}}$ (the subindices are of course considered modulo the cykle length lt): In particular the objects $\left(B_{i_1}, \dots, B_{i_l}\right)$, which correspond to this cycle, are identical as *unlabelled* objects ("structures", compare with the considerations in Observation 2.9.4).

Each pair $(C, \sigma) \in \mathcal{A}(\mathcal{B}) \times \mathfrak{S}_n$ from the region of smmation defines an *ordered* set partition of $[n] = B_1 \dot\cup B_2 \cdots \dot\cup B_k$: We can achieve by a relabeling τ of the atoms of $C \to C'$ another arbitrary ordered set partition (with the same *sequence* of block sizes, of course) and we obtaine another pair

$$\left(C', \sigma' = \tau \circ \sigma \circ \tau^{-1}\right)$$

from the region of summation, which delivers the same cycle type:

$$c(\sigma) = c\left(\sigma'\right).$$

There are (independent from the order of the blocks B_i)

$$\frac{n!}{\|B_1\|! \cdots \|B_k\|!}$$

such ordered set partitions. Of these — up to relabeling — identical objects we of course always only need to consider one; e.g. the one with the "canonical" set partition

$$\{\{1, 2, \dots, \|B_1\|\}, \{\|B_1\| + 1, \dots, \|B_1\| + \|B_2\|\}, \dots\}.$$

On the other hand, each relabeling ν of the Atome from $A \to A'$ first induces a permutation $\rho' = \nu \circ \rho \circ \nu^{-1}$ for which also $\rho'(A') = A'$ holds, and if we *reorder* the blocks B_i accordingly

$$B_i \mapsto B_{\nu(i)},$$

we obtain in this way again simply *the same* object C (with *the same* permutation σ which fixes it): Clearly there are $k!$ such relabelings.
Therefor we first obtain:

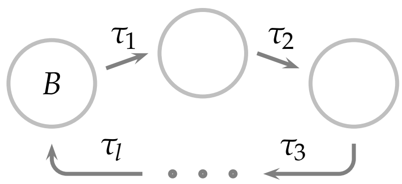
$$Z_{\mathcal{A}(\mathcal{B})}(\mathbf{x}) = \sum_{n\geqslant 0} \frac{1}{n!} \sum_{\substack{(C,\sigma):\\ C=(A;B_1,B_2,\dots,B_k)\\ (B_1,\dots,B_k)\text{ “canonically”}\\ \sigma\in\mathfrak{S}_n:\ \sigma(C)=C}} \sum_{\rho:[\rho|\sigma]} \frac{n!}{\|B_1\|!\cdots\|B_k\|!} \cdot \frac{1}{k!} \cdot \mathbf{x}^{c(\sigma)}$$

$$= \sum_{k\geqslant 0} \frac{1}{k!} \sum_{\substack{A\in\mathcal{A}_k\\ \rho(A)=A\\ \rho=\zeta_1\circ\zeta_2\dots\zeta_m}} \sum_{\sigma:[\rho|\sigma]} \sum_{B_1:\zeta_1} \frac{1}{\|B_1\|!^{\ell(\zeta_1)}} \cdots \sum_{B_m:\zeta_m} \frac{1}{\|B_m\|!^{\ell(\zeta_m)}} \cdot \mathbf{x}^{c(\sigma)}.$$

Two things are to be made clear:

- How many “completed” permutations σ belong to a fixed ρ?
- How does the cycle type $c(\sigma)$ of the “completed” permutation σ arise from the permutation ρ or its cycles?

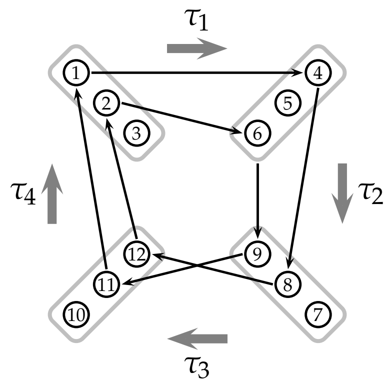
For this we consider a fixed cycle ζ of ρ with $\ell(\zeta) = l$ which “sends around in a circle” the $\mathcal{B}$–object B:



This cycle is “completed” by the maps (“relabeling of the original object B”) $\tau_1, \tau_2, \dots, \tau_l$ to cycles of the “full” permutation σ. Here it should be observed that

$$\tau := \tau_1 \circ \tau_2 \cdots \tau_l \tag{2.23}$$

gives a permutation which *fixes* B. For instance, in the following schematic graphic the cycle length is $l = 4$ and the permutation (in cycle notation) is $\tau = (12)(3)$:



From the graphic it is immediately clear: If the cycle decomposition of τ is given by

$$\tau = \zeta_1 \circ \zeta_2 \cdots \zeta_s,$$

then the corresponding cycles of the "composed permutation" σ have the lengths $(\ell(\zeta_1) \cdot l, \ell(\zeta_2) \cdot l, \dots)$.
Further is follows: For a fixed permutation τ which fixes B, there are

$$(\|B\|!)^{l-1}$$

"completions" $(\tau_1, \tau_2, \dots, \tau_l)$. For the first $(l-1)$ τ_i's can be chosen arbitrarily; τ_l is then uniquely determined by (2.23).
In total we can thus continue the calculation in the following way:

$$\begin{aligned} Z_{\mathcal{A}(\mathcal{B})} &= \sum_{k\geqslant 0} \frac{1}{k!} \sum_{\substack{A\in\mathcal{A}_k\\ \rho(A)=A\\ \rho=\zeta_1\circ\zeta_2\dots\zeta_m}} \sum_{\substack{B_1\in\zeta_1\\ \ell(\zeta_1)=l_1\\ \tau_1(B_1)=B_1}} \frac{\left(x_{l_1}, x_{2l_1}, \dots\right)^{c(\tau_1)}}{\|B_1\|!} \cdots \sum_{\substack{B_m\in\zeta_m\\ \ell(\zeta_m)=l_m\\ \tau_m(B_m)=B_m}} \frac{\left(x_{l_m}, x_{2l_m}, \dots\right)^{c(\tau_m)}}{\|B_m\|!} \\ &= Z_{\mathcal{A}}\left(Z_{\mathcal{B}}(x_1, x_2, \dots), Z_{\mathcal{B}}(x_2, x_4, \dots), \dots\right). \end{aligned}$$

□

EXAMPLE 2.9.11. *We have*

$$\begin{aligned} Z_{\texttt{permutations}} &= Z_{\texttt{sets}}\left(Z_{\texttt{cycles}}(x_1, x_2, \dots), Z_{\texttt{cycles}}(x_2, x_4, \dots),\right) \\ &= e^{\sum_{n\geqslant 1} \frac{\varphi(n)}{n} \log\frac{1}{1-x_n} + \frac{1}{2}\sum_{n\geqslant 1}\frac{\varphi(n)}{n}\log\frac{1}{1-x_{2n}} + \cdots} \\ &= e^{\sum_{n\geqslant 1} \log\frac{1}{1-x_n} \sum_{d|n} \frac{\varphi(d)}{n}} \quad \leftarrow \sum_{d|n} \frac{\varphi(d)}{n} = 1 \\ &= \prod_{n\geqslant 1} \frac{1}{1-x_n}. \end{aligned}$$

Exercise 16: *Determine the cycle index series of the species* `Fixfree` *of permutations* without *fixed points.*

Hint: Show the relation `Sets` $\cdot$ `Fixfree` = `Permutations`.

Exercise 17: *Determine the cycle index series for the species "set partitions".*

Exercise 18: *Given some arbitrary species $\mathcal{A}$, show the formula*

$$Z_{\mathcal{A}'}(x_1, x_2, \dots) = \left(\frac{\partial}{\partial x_1} Z_{\mathcal{A}}\right)(x_1, x_2, \dots).$$

Exercise 19: *Show:*

$$\prod_{k=1}^{\infty} \frac{1}{1-x^k} = \exp\left(\sum_{n\geqslant 1} \frac{1}{n} \frac{x^n}{1-x^n}\right).$$

Hint: `Permutations` = `Sets` (`Cycles`).

CHAPTER 3

Partially ordered sets

3.1. Definition and Examples

DEFINITION 3.1.1. *A* partially ordered set *(or short* poset*) is a set P together with a relation $\leqslant$ ("less than or equal to") that satisfies the following conditions:*

(1) $\forall x \in P : x \leqslant x$ *(reflexivity)*
(2) $\forall x, y \in P : x \leqslant y$ *and* $y \leqslant x \implies x = y$ *(antisymmetry)*
(3) $\forall x, y, z \in P : x \leqslant y$ *and* $y \leqslant z \implies x \leqslant z$ *(transitivity)*

The following notation is commonly used ("strictly less than"):

$$x < y :\Longleftrightarrow x \leqslant y \text{ and } x \neq y.$$

In addition, we use the following notation:

$$x \gtrdot y :\Longleftrightarrow x > y and \nexists z : x > z > y$$

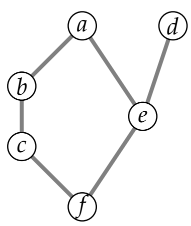
We say: "x covers *y".*
Two elements x and y are said to be comparable *if $x \leqslant y$ or $y \leqslant x$, otherwise they are said to be* incomparable.
In case two posets P, Q are considered, we denote the

- *order relation on P by $\leqslant_P$,*
- *while the order relation on Q is denoted by $\leqslant_Q$,*

in order to avoid confusions. (However, we simply use $\leqslant$ for the usual order on $\mathbb{N}$.)

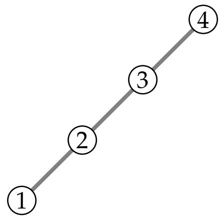
For *finite* posets P (to be more precise: if P does not have too many elements ;-), *Hasse–diagrams* are a useful description of the poset, where the order relations are represented as the edges of a graph with vertex set P. The following figure illustrates this for $P = \{a, b, c, d, e, f\}$ with the following covering relations (which of course uniquely determine a finite poset!)

$$b \lessdot a, c \lessdot b, e \lessdot a, e \lessdot d, f \lessdot c, f \lessdot e.$$

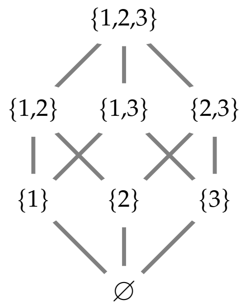


EXAMPLE 3.1.2. *Examples of posets:*

(1) $P = [n]$ linearly ordered*: An order relation is said to be* linear*, if* any two *elements x, y are comparable (the Hasse–diagram looks like a line then):*

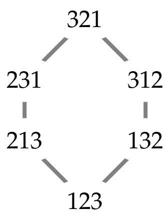


(2) $P = 2^{[n]}$ *(Power set of* $[n]$*) ordered by* set inclusion*: This poset is said to be the* Boolean algebra B_n*. E.g., for* $n = 3$*:*

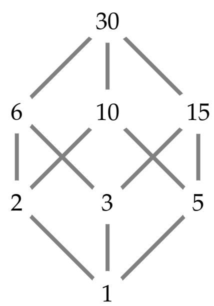


(3) $P = \mathfrak{S}_n$ *with* weak order*:* $\sigma < \pi$ *if and only if the permutation* π *is obtained from the permutation* σ *by interchanging two* adjacent *elements* and *the number of* inversions *is increased by this transposition.*

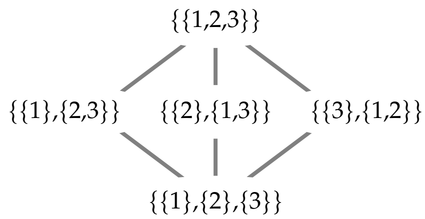
$$\sigma = \cdots \pi_i < \pi_{i+1} \cdots$$
$$\pi = \cdots \pi_{i+1} > \pi_i \cdots$$



(4) $P = T_n := \{d \in \mathbb{N} : d \mid n\}$*, ordered by* $a \leqslant b :\Longleftrightarrow a \mid b$*: This poset is said to be the* divisor lattice*. E.g.,* $n = 30$*:*



(5) $P = \mathbb{P}_n$*, the family of all (set–)partitions of* $[n]$*, ordered by "refinement": For two partitions* π, τ *we have* $\pi \leqslant \tau$*, if every* block *of* π *is fully contained in some block of* τ *(i.e., "*π *is constructed from* τ *by subdividing blocks"). E.g., for* $n = 3$*:*



(6) $P = V_{n,q}$, *the family of subspaces of a finite vectorspace $(GF(q))^n$ over a finite field $GF(q)$, ordered by "subspace–relation" (that is set inclusion).*

DEFINITION 3.1.3. *Given two posets P, Q with order relation $\leqslant_P$ (for P) and $\leqslant_Q$ (for Q). A map $\phi : P \to Q$ is said to be* order-preserving, *if, for all $x, y \in P$, we have:*

$$x \leqslant_P y \implies \phi(x) \leqslant_Q \phi(y).$$

The posets P and Q are said to be isomorphic *if there exists a* bijective *map $\phi : P \to Q$ such that ϕ and ϕ^{-1}* both *are* order-preserving *(such a map ϕ is said to be an* order isomorphism*), that is*

$$x \leqslant_P y \iff \phi(x) \leqslant_Q \phi(y).$$

EXAMPLE 3.1.4. *Let $P = \{0,1\}^n$ be the set of all $\{0,1\}$–vectors of length n, ordered by*

$$(v_1, \ldots, v_n) \leqslant (w_1, \ldots, w_n) :\iff v_i \leqslant w_i \text{ for } i = 1, 2, \ldots, n.$$

The interpretation of these vectors as characteristic functions of subsets induces an order-preserving bijection $P \to B_n$ on the Boolean algebra B_n (see example 3.1.2).

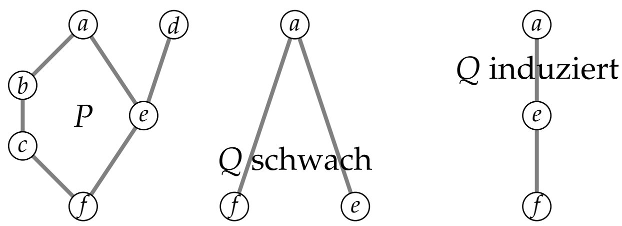
Exercise 20: *Let P be some finite poset and $f : P \to P$ an order–preserving bijection. Show that f^{-1} is* also *order–preserving.*
Show that this is not true in general *for infinite posets.*

DEFINITION 3.1.5. *Let P be a poset: A partially ordered subset $Q \subseteq P$ is said to be*

- *a* weak subposet *of P if $x \leqslant_Q y \implies x \leqslant_P y$ for all $x, y \in Q$,*
- *an* induced subposet *of P if $x \leqslant_Q y \iff x \leqslant_P y$ for all $x, y \in Q$.*

There is an analogy to the difference between *subgraphs* and *induced* subgraphs; the following figure illustrates the difference:

$$Q = \{a, e, f\} \subset P = \{a, b, c, d, e, f\}:$$



DEFINITION 3.1.6. *Let P be a poset and $x, y \in P$ with $x \leqslant y$: An induced subset*

$$[x, y] := \{z : x \leqslant z \leqslant y\}$$

is said to be an interval *(in particular: $[x, x] = \{x\}$).*
A poset that has only intervals of finite cardinality is said to be locally finite.

DEFINITION 3.1.7 (Chains and Antichains). *Let P be a poset: An element $x \in P$ with*

$$\nexists\, y \in P : y < x$$

is said to be a minimal element *of P, while an element $x \in P$ with*

$$\nexists\, y \in P : y > x$$

is said to be a maximal element *of P.*

An element $x \in P$ with $x \leqslant y$ (resp. $x \geqslant y$) for all $y \in P$ is said to be the minimum *(resp.* maximum*). If P has a minimum (maximum), then it is of course* uniquely *determined: We denote it by $\hat{0}$ (resp. $\hat{1}$).*

If P contains a minimum $\hat{0}$, then we refer to an element that covers $\hat{0}$ as atom *of P.*

If P contains a maximum $\hat{1}$, then we refer to an element that is covered by $\hat{1}$ as coatom *of P.*

A subset C of P that is linearly ordered *as induced subposet is said to be a* chain *in P. A subset A of P where any two elements are incomparable is said to be an* antichain *(or* Sperner family*) of P.*

If a chain $C = \{x_1 < x_2 < \cdots < x_m\}$ is finite*, then $\ell(C) := |C| - 1$ is the* length *of C.*

A (finite) chain is said to be saturated*, if "its elements cover each other (in P)", that is if*

$$C = \{x_1 \lessdot x_2 \lessdot \cdots \lessdot x_m\}.$$

A saturated chain C is said to be maximal*, if there exists no saturated chain D with $C \subseteq D$ and $\ell(D) > \ell(C)$.*

Exercise 21: *(a) Find a (finite) poset P where*

- *the length of a* longest *chain is l,*
- every *element of P belongs to a chain of length l,*

which nevertheless has a maximal *chain of length $< l$.*

(b) Let P be a (finite) poset with connected Hasse–diagram, where the longest *chain has length l (there might be several longest chains). Moreover, assume that for all $x, y \in P$ such that $y \gtrdot x$ (y covers x), x and y belong to a chain of length l: Show that under this assumption* all *maximal chains have length l.*

DEFINITION 3.1.8 (Order ideal). *A subset $I \subseteq P$ with the property*

$$x \in I \text{ and } y \in P \text{ with } y \leqslant x \implies y \in I$$

is said to be an order ideal *of P. For an arbitrary subset $S \subseteq P$, the set*

$$\langle S \rangle := \{y \in P : \exists x \in S : y \leqslant x\}$$

is always an order ideal; it is the order ideal generated *by S. If S consists of a single element x, then $\langle S \rangle = \langle \{x\} \rangle$ is the principal order ideal generated by x.*

A subset $I \subseteq P$ with the property

$$x \in I \text{ and } y \in P \text{ with } y \geqslant x \implies y \in I$$

is said to be a dual order ideal *of P (*dual principal order ideal *is defined in a similar manner).*

The set of all order ideals of P, ordered by inclusion, is itself a poset, which is denoted by $\mathcal{J}(P)$ in this text: Such a poset always has a minimum $\hat{0}$ (the empty set) and always has a maximum $\hat{1}$ (P itself).

PROPOSITION 3.1.9. *Let P be a poset, $I \in \mathcal{J}(P)$ be a order ideal of P, and x a maximal element in I.*

Then also $I \setminus \{x\}$ is an order ideal of P.

In case that P is finite, there is a bijection between order ideals I and antichains A:

$$I \mapsto A := \{x : x \text{ maximal in } I\}$$
$$A \mapsto I = \langle A \rangle := \{y : \exists x \in A : y \leqslant x\}$$

DEFINITION 3.1.10 (Linear extension). *An order-preserving bijection from a finite poset P with $|P| = n$ to $[n]$ is said to be a* linear extension *of P.*

PROPOSITION 3.1.11. *The number of linear extensions of a finite poset P equals the number of maximal chains in $\mathcal{J}(P)$.*

PROOF. Let $\sigma : P \to [n]$ be a linear extension. Set $I_0 := \emptyset$ and $I_k := \sigma^{-1}([k])$ for $k = 1, 2, \dots, n$. Then the subset $I_k \subseteq P$ is always an *order ideal* of P (since σ is order-preserving) with $|I_k| = k$ (as σ is bijective); this implies in $\mathcal{J}(P)$:

$$\hat{0} = I_0 \lessdot I_1 \lessdot \cdots \lessdot I_n = P = \hat{1}.$$

This means the following: Every linear extension σ of P is associated with a maximal chain in $\mathcal{J}(P)$.

Conversely, let C be a maximal chain in $\mathcal{J}(P)$: The length of such a chain is obviously n; indeed, one can obtain each such maximal chain algorithmically as follows: Start with $I_0 = \emptyset$; if $I_{k-1} \neq P$ has already been constructed, choose a *minimal* element x_k from $P \setminus I_{k-1}$ and set $I_k := I_{k-1} \cup \{x_k\}$. For this chain C, we define $\sigma : P \to [n]$:

$$\sigma(x_k) = k;$$

The map σ is of course bijective; in addition, the map is order-preserving, since

$$x_i \leqslant_P x_j \implies i \leqslant j.$$

(As $i > j \implies x_i \nleqslant_P x_j$.)

In summary, we observe the following: There exists a bijection between the linear extensions of P and the maximal chains in $\mathcal{J}(P)$. □

DEFINITION 3.1.12. *A (finite) poset is said to be* graded *of rank n, if each maximal chain has the same length n. For such a poset P, there exists a uniquely determined rank function $\rho : P \to \{0, 1, \dots, n\}$ such that*

- *$\rho(x) = 0$, if x is a minimal element of P,*
- *$\rho(y) = \rho(x) + 1$, if $y \gtrdot x$.*

In case $\rho(x) = i$, we say: x has rank i. For such a poset P, let p_i be the number of elements of rank i: Then the polynomial

$$F(P,q) := \sum_{i=0}^{n} p_i \, q^i$$

is said to be the rank generating function *of P.*

Exercise 22: *Consider the "zigzag–poset" Z_n with elements $x_1, x_2, \ldots, x_n$ and cover relations*

$$x_{2i-1} \lessdot x_{2i} \text{ for } i \geqslant 1, 2i \leqslant n \text{ and } x_{2i} \gtrdot x_{2i+1} \text{ for } i \geqslant 1, 2i+1 \leqslant n$$

a) How many order ideals are there in Z_n?
b) Let $W_n(q)$ be the rank generating function of the lattice of order ideals $J(Z_n)$ of Z_n. For instance, $W_0(q) = 1$, $W_1(q) = 1 + q$, $W_2(q) = 1 + q + q^2$, $W_3(q) = 1 + 2q + q^2 + q^3$. Show:

$$W(q,z) := \sum_{n=0}^{\infty} W_n(q)\, z^n = \frac{1 + (1+q)z - q^2 z^3}{1 - (1+q+q^2)z^2 + q^2 z^4}.$$

c) Let e_n be the number of all linear extensions of Z_n. Show:

$$\sum_{n=0}^{\infty} e_n \frac{z^n}{n!} = \tan z + \frac{1}{\cos z}.$$

3.2. Construction of posets

Given two posets, there are several possibilities to construct new posets from them. The simplest poset (disregarding the empty poset), which consists of a single element, is denoted by $\circ$ in the following.

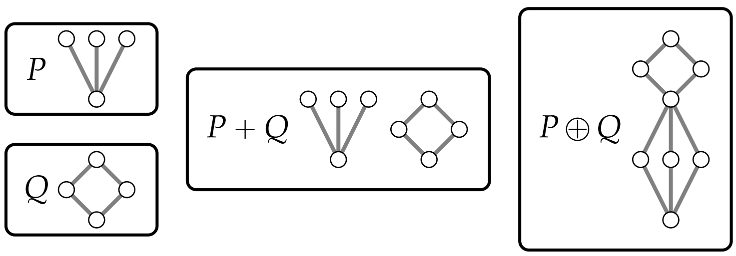
DEFINITION 3.2.1. *Let P and Q be posets, and suppose that P and Q are* disjoint *as sets. The direct sum* direct sum *of P and Q is denoted by $P + Q$: It is the poset on the union $P \cup Q$ such that $x \leqslant y$ if (and only if)*

- *$x, y \in P$ and $x \leqslant_P y$*
- *or $x, y \in Q$ and $x \leqslant_Q y$.*

Moreover, we denote the ordinal sum *of P and Q by $P \oplus Q$: It is the poset on the union $P \cup Q$ such that $x \leqslant y$ if (and only if)*

- *$x, y \in P$ and $x \leqslant_P y$*
- *or $x, y \in Q$ and $x \leqslant_Q y$*
- *r $x \in P$ and $y \in Q$.*

Using Hasse diagrams, the direct sum and the ordinal sum can be illustrated as follows:



We obviously have

$$P + Q \simeq Q + P,$$

however, in general,

$$P \oplus Q \not\simeq Q \oplus P.$$

Both, the direct sum and the ordinal sum, are associative.
An antichain with n elements is obviously isomorphic to the n-fold direct sum

$$n \cdot \circ := \underbrace{\circ + \circ + \cdots \circ}_{n \text{ times}},$$

and a chain with n elements is isomorphic to the n-fold ordinal sum

$$\underbrace{\circ \oplus \circ \oplus \cdots \circ}_{n \text{ times}}.$$

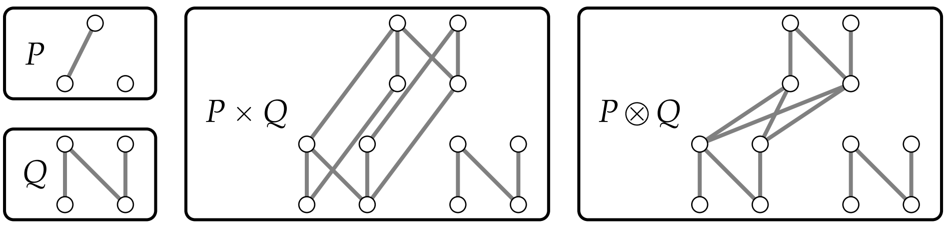
DEFINITION 3.2.2. *Let P and Q be posets.*
We denote the direct product *of P and Q by $P \times Q$, which is defined as the poset on the cartesian product $P \times Q$ such that $(x, y) \leqslant (x', y')$ if (and only if)*

$$x \leqslant_P x' \text{ and } y \leqslant_Q y'.$$

Moreover, we denote the ordinal product *of P and Q by $P \otimes Q$, which is defined as the poset on the cartesian product $P \times Q$ such that $(x, y) \leqslant (x', y')$ if (and only if)*

$$x = x' \text{ and } y \leqslant_Q y', \text{ oder } x <_P x'.$$

Using Hasse diagrams, the direct sum and the ordinal sum can be illustrated as follows:



Again, we have

$$P \times Q \simeq Q \times P,$$

but, in general,

$$P \otimes Q \not\simeq Q \otimes P.$$

The direct product is associative.

Exercise 23: *Let P, Q be graded posets, let r and s be the maximal ranks of P and Q, respectively, and let $F(P, q)$ and $F(Q, q)$ be the corresponding rank generating functions. Show:*

a) If $r = s$ (otherwise maximal chains would be of different lengths), then $F(P + Q, q) = F(P, q) + F(Q, q)$.

b) $F(P \oplus Q, q) = F(P, q) + q^{r+1} F(Q, q)$.

c) $F(P \times Q, q) = F(P, q) \cdot F(Q, q)$.

d) $F(P \otimes Q, q) = F\left(P, q^{s+1}\right) \cdot F(Q, q)$.

DEFINITION 3.2.3. *Let P and Q be posets.*
Then Q^P denotes the set of all order-preserving maps $f : P \to Q$ with order relation

$$f \leqslant g, \text{ if } f(x) \leqslant_Q g(x) \text{ for all } x \in P.$$

EXAMPLE 3.2.4. *The Boolean lattice can also be obtained as follows, see Example 3.1.4:*

$$B_n \simeq \{0,1\}^{n\cdot\circ}.$$

Exercise 24: *Let P, Q, R be posets. Find order isomorphisms for the following relations:*
a) $P \times (Q + R) \simeq (P \times Q) + (P \times R)$.
b) $R^{P+Q} \simeq R^P \times R^Q$.
c) $\left(R^Q\right)^P \simeq R^{Q\times P}$.

Exercise 25: *Let P be a finite poset and define $G_P(q,t) := \sum_I q^{|I|} t^{m(I)}$, where the summation range is the set of all order ideals I of P, and where $m(I)$ denotes the number of maximal elements of I. (For instance: $G_P(q,1)$ is the rank generating function of $\mathcal{J}(P)$.)*
a) Let Q be a poset with n elements. Show:

$$G_{P\otimes Q}(q,t) = G_P\left(q^n, q^{-n} \cdot \left(G_Q(q,t) - 1\right)\right),$$

where $P \otimes Q$ denotes the ordinal *product.*
b) Let P be a poset with p elements. Show:

$$G_P\left(q, \frac{q-1}{q}\right) = q^p.$$

3.3. Lattices

DEFINITION 3.3.1 (Lattice). *Let P be a poset, and let $x, y \in P$: An element z with*

$$z \geqslant x \text{ and } z \geqslant y$$

is said to be an upper bound *of x and y; z is said to be the* least upper bound *or* supremum *of x and y, if, for every upper bound w of x and y, we have*

$$w \geqslant z.$$

If there exists a least upper bound z of x and y, then it is of course unique; we write $x \vee y$ and say "x sup y".
Analogously, we define the greatest lower bound *or* infimum *(if it exists): We write $x \wedge y$ and say "x inf y".*
A poset in which each pair (x,y) *of elements has a least upper bound as well as a greatest lower bound is said to be a* lattice.
A finite *lattice has a (unique) minimum $\hat{0}$ and a (unique) maximum $\hat{1}$.*
An element x in a lattice that covers $\hat{0}$ is said to be an atom.
An element x in a lattice that is covered by $\hat{1}$ is said to be a coatom.

EXAMPLE 3.3.2. *Examples of finite lattices are as follows:*

1. $[n]$ *with the usual linear order.*
2. *the power set $2^{[n]}$, ordered by set inclusion.*
3. $\mathfrak{S}_n$ *with the weak order.*

(4) *The set* $T_n = \{d \in \mathbb{N} : d \mid n\}$ *of (positive) divisors of a number* $n \in \mathbb{N}$*, ordered by divisibility* ($d_1 \leqslant d_2 :\Longleftrightarrow d_1 \mid d_2$)*.*
(5) *The set* Π_n *of (set–)partitions of* $[n]$*, ordered by "refinement of partitions".*
(6) *The set* $V_{n,q}$ *of subspaces of an n–dimensional vector space over a finite field, ordered by the subset relation (which is again set inclusion).*

COROLLARY 3.3.3. *In each lattice, we have the following basic rules:*

(1) $\wedge$ *and* $\vee$ *are commutative and associative operations in* L*.*
(2) $\forall x \in L : x \vee x = x \wedge x = x$ *(*$\wedge$ *and* $\vee$ *are idempotent operations in* L*).*
(3) $x \wedge y = x \iff x \vee y = y \iff x \leqslant y$*.*
(4) $x \wedge (x \vee y) = x = x \vee (x \wedge y)$ (rule of absorption)*.*

PROOF. (1): Commutativity is of course obvious.
Associativity for $\vee$ can be shown as follows: $x \vee (y \vee z)$ is greater than or equal

- to x and $y \vee z$,
- and therefore also to x, y and z,
- and therefore also to $x \vee y$ and z,
- and therefore also to $(x \vee y) \vee z$;

this implies $x \vee (y \vee z) \geqslant (x \vee y) \vee z$, and the same is true also for the other direction of the inequality sign.
The argument is the same for $\wedge$.
(2) This is trivial.
(3) Using $x \leqslant y$, we conclude $x \vee y = y$, and $x \vee y = y$ implies $y \geqslant x$. The argument is the same for $\wedge$.
(4) This follows immediately from (3). □

COROLLARY 3.3.4. *Let* L *be a lattice, and let* $a' \leqslant a$ *in* L*. Then we have following for all* $b \in L$

$$a' \wedge b \leqslant a \wedge b \tag{3.1}$$
$$a' \vee b \leqslant a \vee b. \tag{3.2}$$

PROOF. The statement is an immediate consequence of

$$a' \wedge b \leqslant \begin{cases} b \\ a' \leqslant a \end{cases} \qquad \text{resp. } a \vee b \geqslant \begin{cases} b \\ a \geqslant a' \end{cases} \quad .$$

□

COROLLARY 3.3.5 (Modular inequality). *Let* L *be a lattice, and let* $x, y, z \in L$*. Then we have*

$$x \leqslant z \implies x \vee (y \wedge z) \leqslant (x \vee y) \wedge z. \tag{3.3}$$

PROOF. Using the definition, we can immediately conclude:

$$\begin{aligned} x \leqslant x \vee y \text{ and } x \leqslant z &\implies x \leqslant (x \vee y) \wedge z, \\ y \wedge z \leqslant y \leqslant x \vee y \text{ and } y \wedge z \leqslant z &\implies y \wedge z \leqslant (x \vee y) \wedge z. \end{aligned}$$

The two inequalities yield (3.3). □

It is possible to give an axiomatic characterization of lattices that is based on the operations $\wedge$ and $\vee$.

THEOREM 3.3.6. *Let P be a set with binary operations $\vee$ and $\wedge$ that satisfy items 1, 2 and 4 from Corollary 3.3.3. Then P is a lattice, where $\leqslant$ is defined by*

$$x \leqslant y :\Longleftrightarrow \; x \wedge y = x.$$

PROOF. First we show the following: The relation defined above is indeed a partial order.

- Reflexivity follows from item 2 in Corollary 3.3.3:
$$x \wedge x = x \implies x \leqslant x.$$
- Antisymmetry follows directly from the definition (and from item 1 in Corollary 3.3.3: commutativity):
$$x \leqslant y \text{ and } y \leqslant x \iff x \wedge y = x \text{ and } y \wedge x = y \implies x = y.$$
- Transitivity: Using
$$x \leqslant y \text{ and } y \leqslant z \iff x \wedge y = x \text{ and } y \wedge z = y,$$
we can conclude (employ item 1 in Corollary 3.3.3: associativity)
$$x \wedge z = (x \wedge y) \wedge z = x \wedge (y \wedge z) = x \wedge y = x,$$
that is $x \leqslant z$.

Next we show: For each $x, y \in P$, there exists an infimum. The notation already suggests the right claim: $x \wedge y$ *is* the infimum, since

- $x \wedge y \leqslant x$: This follows directly from Corollary 3.3.3, item 1+2:
$$x \wedge (x \wedge y) = (x \wedge x) \wedge y = x \wedge y.$$
- For z with $z \leqslant x$ and $z \leqslant y$, we have (Corollary 3.3.3: associativity):
$$z \wedge (x \wedge y) = (z \wedge x) \wedge y = z \wedge y = z \implies z \leqslant x \wedge y.$$

Finally, we show the symmetry relation

$$x \wedge y = x \iff x \vee y = y.$$

Both directions follow from item 4 in Corollary 3.3.3:

- $x \vee y = (x \wedge y) \vee y = y,$
- $x \wedge y = x \wedge (x \vee y) = x.$

Using this symmetry relation it becomes evident that (as expected) also $x \vee y$ is the supremum. This concludes the proof. □

PROPOSITION 3.3.7. *Let P be a* finite *poset with $\hat{1}$ such that there exists the infimum $x \wedge y$ for all $x, y \in P$. Then P is already a lattice.*

PROOF. It suffices to show that supremum $x \vee y$ exists for all $x, y \in P$. For this purpose, it makes sense to define

$$x \vee y := \bigwedge_{z \geqslant x,y} z.$$

The set over which the infimum is taken is not empty (as $\hat{1} \geqslant x, y$) and also finite, which guarantees that $x \vee y$ is well-defined. It also has the property of a supremum: As $a, b \geqslant x, y \implies x, y \leqslant a \wedge b$, it can be proven (induction) that $x, y \leqslant \bigwedge_{z \geqslant x,y} z$, and, for *every* upper bound $a \geqslant x, y$, we obviously have $a \geqslant \bigwedge_{z \geqslant x,y} z$. □

REMARK 3.3.8. *Let P and Q be lattices. Then the direct product $P \times Q$ is also a lattice, and we have*

$$(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \vee y_2),$$
$$(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \wedge y_2).$$

Similarly, Q^P is a lattice with

$$(f \vee g)(x) = f(x) \vee g(x),$$
$$(f \wedge g)(x) = f(x) \wedge g(x).$$

Exercise 26: *Let L be a finite lattice. Show that the following three conditions are equivalent for all $x, y \in L$:*
(a) L is graded (i.e.: all maximal chains have in L the same length), and for the rank function $\mathbf{rk}$ of L there holds

$$\mathbf{rk}(x) + \mathbf{rk}(y) \geqslant \mathbf{rk}(x \wedge y) + \mathbf{rk}(x \vee y).$$

(b) If y covers the element $x \wedge y$, then $x \vee y$ covers the element x.
(c) If x and y both cover element $x \wedge y$, then $x \vee y$ covers both elements x and y.
(A lattice L obeying one of these conditions is called semimodular.*)*

Hint: Employ an indirect proof for (c) $\implies$ (a): For the first assertion in (a), if there are intervals which are not graded, *then we may choose an interval $[u, v]$ among them which is* minimal *with respect to set–inclusion (i.e., every sub–interval* is *graded). Then there are two elements $x_1, x_2 \in [u, v]$, which both cover u, and the length of all maximal chains in $[x_i, v]$ is ℓ_i, such that $\ell_1 \neq \ell_2$. Now apply (b) or (c) to x_1, x_2.*
For the second assertion in (a), take a pair $x, y \in L$ with

$$\mathbf{rk}(x) + \mathbf{rk}(y) < \mathbf{rk}(x \wedge y) + \mathbf{rk}(x \vee y), \tag{3.4}$$

such that the length of the interval $[x \wedge y, x \vee y]$ is minimal, and under all such pairs, also $\mathbf{rk}(x) + \mathbf{rk}(y)$ is minimal. Since it is impossible that both *x and y cover $x \wedge y$ (why?), w.l.o.g. there is an element x' with $x \wedge y < x' < x$. Show that $X = x, Y = x' \vee y$ is a pair such that $\mathbf{rk}(X) + \mathbf{rk}(Y) < \mathbf{rk}(X \wedge Y) + \mathbf{rk}(X \vee Y)$, but where the length of the interval $[X \wedge Y, X \vee Y]$ is less than the length of $[x \wedge y, x \vee y]$.*

Exercise 27: *Let L be a finite semimodular lattice. Show that the following two conditions are equivalent:*
a) For all elements $x, y, z \in L$ with $z \in [x, y]$ (i.e., $x \leqslant y$) there is an element $u \in [x, y]$, such that $z \wedge u = x$ and $z \vee u = y$ (u is a "complement" of z in the interval $[x, y]$).
b) L is atomic, *i.e.:* Every *element can be represented as the supremum of atoms.*
(A finite semimodular lattice obeying one of these conditions is called geometric.*)*

Exercise 28: *Let G be a (labelled) graph on n vertices. A partition of the vertex–set $V(G)$ is called connected if every block of the partition corresponds to a* connected *induced subgraph of G. The set of all connected partitions is a subposet of the poset of partitions of $V(G)$, and thus a poset itself. (If G is the complete graph, the poset of connected partitions of $V(G)$ is the same as the poset of* all *partitions of $V(G)$.)*

Show that the poset of connected partitions of G is a geometric lattice.

Exercise 29: *A lattice L is called* modular *if it is graded and for all $x, y \in L$ there holds:*

$$\mathbf{rk}(x) + \mathbf{rk}(y) = \mathbf{rk}(x \wedge y) + \mathbf{rk}(x \vee y). \tag{3.5}$$

(In particular, the lattice $L(V)$ of subspaces of a finite vector space is modular.)

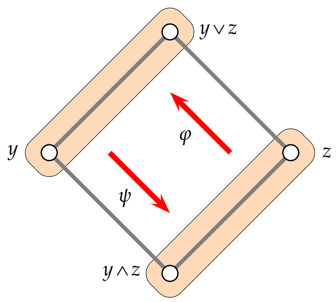
Show: A finite lattice L is modular if and only if for all $x, y, z \in L$ with $x \leqslant z$ there holds:

$$x \vee (y \wedge z) = (x \vee y) \wedge z. \tag{3.6}$$

Hint: *Show that* (3.6) *implies the* Diamond Property*: The mappings*

$$\psi : [y, y \vee z] \to [y \wedge z, z], \psi(x) = x \wedge z$$
$$\varphi : [y \wedge z, z] \to [y, y \vee z], \varphi(x) = x \vee y$$

are order preserving bijections *with $\phi \circ \psi = \mathrm{id}$, see the following picture:*



Exercise 30: *Show: The lattice Π_n of all partitions of an n–element set is* not *modular.*

3.3.1. Distributive lattices.

A particularly important class of lattices are the distributive lattices.

DEFINITION 3.3.9. *A lattice L is said to be* distributive*, if, for all $a, b, c \in L$, we have the following laws of* distributivity*:*

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c),$$
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

REMARK 3.3.10. *It suffices to require either of these laws as the other then follows. We show how to deduce the second law from the first law:*

$$\begin{aligned}(a \vee b) \wedge (a \vee c) &= ((a \vee b) \wedge a) \vee ((a \vee b) \wedge c) &&\leftarrow \text{1. dist.l.}\\ &= a \vee ((a \vee b) \wedge c) &&\leftarrow \text{absorption}\\ &= a \vee ((a \wedge c) \vee (b \wedge c)) &&\leftarrow \text{1. dist.l.}\\ &= a \vee (b \wedge c) &&\leftarrow \text{ass.l. and absorption}\end{aligned}$$

REMARK 3.3.11. *Let L_1 and L_2 be two distributive lattices. Then also the direct product $L_1 \times L_2$ is a distributive lattice.*

EXAMPLE 3.3.12. *Examples for distributive lattices are:*

(1) *Linear order*
(2) *Boolean lattice B_n: Here we have $A \vee B = A \cup B$ and $A \wedge B = A \cap B$; the laws of distributivity are the set theoretic identities*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

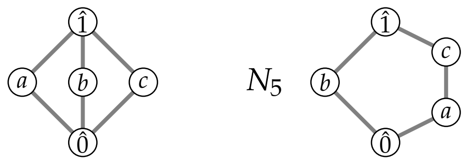
(3) *Divisor lattice $T_n = \{d \in \mathbb{N} : d \mid n\}$: Indeed, let $n = p_1^{k_1} \cdots p_r^{k_r}$ be the unique* prime factor decomposition *of n, then*

$$T_n \simeq [0, k_1] \times \cdots [0, k_r] .$$

(4) *For an arbitrary poset P, the lattice of order ideals $\mathcal{J}(P)$ is always distributive.*

On the other hand, simple examples of lattices that are not *distributive are the following (given by their Hasse diagram):*

M_5 N_5



$a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$ $a \vee (b \wedge c) \neq (a \vee b) \wedge (a \vee c)$

THEOREM 3.3.13 (Structure theorem for finite distributive lattices, Birkhoff's Theorem). *Let L be a finite distributive lattice. Then there exists a poset P such that*

$$L \simeq \mathcal{J}(P) .$$

DEFINITION 3.3.14. *Let L be lattice. An element $x \neq \hat{0} \in L$ is said to be* supremum-irreducible, *if x* cannot *be represented as follows*

$$x = y \vee z,$$

where $y < x$ and $z < x$.

LEMMA 3.3.15. *Let L be distributive lattice, and let $a \in L$ be supremum–irreducible. Then we have*

$$a \leqslant x \vee y \implies (a \leqslant x \text{ oder } a \leqslant y) .$$

PROOF. As $a = a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y)$, it follows from the assumption that

$$a \wedge x = a \text{ oder } a \wedge y = a,$$

which implies $a \leqslant x$ or $a \leqslant y$. □

PROOF OF 3.3.13. Let P be the set of supremum–irreducible elements of L, interpreted as a poset with the order relation "inherited" from L.
We define a map $\varphi : L \to \mathcal{J}(P)$ by

$$\varphi\left(\hat{0}\right) = \emptyset;\ \varphi(x) := I(x) := \left\{x' \in P : x' \leqslant x\right\} \text{ for } x \neq \hat{0}. \tag{3.7}$$

Claim: φ is *bijective* with inverse map

$$\psi : I \mapsto \bigvee_{y \in I} y$$

for $I \neq \emptyset \in \mathcal{J}(P)$ (Special case: $\psi(\emptyset) = \hat{0}$).
For the special case $x = \hat{0}$ resp. $y = \emptyset$ we have $\varphi(x) = y$ and $\psi(y) = x$. It remains to show:

$$\forall x \in L, x \neq \hat{0} : \ (\psi \circ \varphi)(x) = x, \tag{3.8}$$

$$\forall y \in \mathcal{J}(P), y \neq \emptyset : \ (\varphi \circ \psi)(y) = y. \tag{3.9}$$

We first show (3.8). For this purpose, we observe that $x \neq \hat{0}$ is *always* representable as supremum of supremum–irreducible elements: In case x is itself supremum–irreducible, this is of course trivial:

$$x = \bigvee_{y = x} y.$$

Therefore suppose that x is *not* supremum–irreducible, then, by definition, there exist elements $x' < x$ and $x'' < x$ with $x = x' \vee x''$. The same considerations can also be applied to x' and x'', etc.: Since L is finite, the algorithm must stop at some point, and we obtain the required decomposition into supremum–irreducible elements

$$x = x_1 \vee x_2 \vee \cdots \vee x_k \text{ with } x_i < x \text{ for } i = 1, \ldots, k.$$

This implies

$$\{x_1, \ldots, x_k\} \subseteq \varphi(x) = I(x),$$

that is

$$x = x_1 \vee x_2 \vee \cdots \vee x_k \leqslant \bigvee_{y \in I(x)} y \leqslant x,$$

and we can conclude $\bigvee_{y \in I(x)} y = x$, and so (3.8) is proven.
Now we prove (3.9): Suppose $y = \{x_1, x_2 \ldots, x_k\} \in \mathcal{J}(P)$. Let $x = \psi(y) = x_1 \vee x_2 \vee \cdots \vee x_k$. It is obvious that $y \subseteq \varphi(x)$; it remains to show $\varphi(x) \subseteq y$, that is that *each* supremum–irreducible element x' with $x' \leqslant x$ is included in y:

$$x' = x' \wedge x = \left(x' \wedge x_1\right) \vee \cdots \vee \left(x' \wedge x_k\right),$$

and, since x' is supremum–irreducible, there is an i with $x' = x' \wedge x_i$: However, this means that $x' \leqslant x_i$, and thus (3.9) is proven.
Claim: φ is order preserving, i.e.:

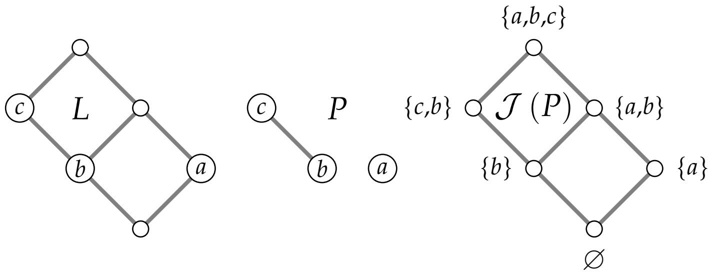
$$x_1 \leqslant x_2 \iff I(x_1) \subseteq I(x_2).$$

Consider for $x_1 \leqslant x_2$ an arbitrary element $x' \in I(x_1)$: $x' \leqslant x_1 \implies x' \leqslant x_2 \implies x' \in I(x_2)$, therefore $I(x_1) \subseteq I(x_2)$. Conversely, $I(x_1) \subseteq I(x_2)$ implies

$$\bigvee_{y \in I(x_1)} y \leqslant \left(\bigvee_{y \in I(x_1)} y\right) \vee \left(\bigvee_{y \in I(x_2) \setminus I(x_1)} y\right) = \bigvee_{y \in I(x_2)} y.$$

The two claims imply Theorem 3.3.13. □

EXAMPLE 3.3.16. *The following example illustrates the situation:*



THEOREM 3.3.17. *Let L be a finite distributive lattice. Then L has a rank function* **rk***; it is given by*

$$\mathbf{rk}\,(x) = |I\,(x)|\,,$$

where $I\,(x)$ is the ideal defined in (3.7) *(of the poset of supremum-irreducible elements of L). In particular, it follows that the rank* $\mathbf{rk}\,(L)$ *of L is equal to the number of supremum-irreducible elements of L.*

PROOF. By Birkhoff's Theorem (Theorem 3.3.13), we have

$$L \simeq \mathcal{J}\,(P)\,,$$

where P is the poset of supremum-irreducible elements in L. Consequently, it suffices to show that all maximal chains in $\mathcal{J}\,(P)$ have the same length.

Let

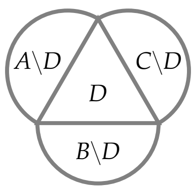
$$\hat{0} = \emptyset = I_0 \lessdot I_1 \lessdot \cdots \lessdot I_r = \hat{1} = P$$

be a maximal chain in P. We claim: $|I_k| = |I_{k-1}| + 1$. Indeed, if $S := I_k \backslash I_{k-1} = \{x_1, \ldots, x_m\}$, then, using $S \neq \emptyset$, there exists a maximal element x_i (with respect to the order of P) in S. But then $I_k \backslash \{x_i\}$ is also an order ideal of P (see Corollary 3.1.9), and, since $I_k \gtrdot I_{k-1}$, we can conclude $I_{k-1} = I_k \backslash \{x_i\}$. Hence all maximal chains in $\mathcal{J}\,(P)$ have the same length $|P|$. □

3.4. Incidence algebra and Möbius inversion

Let A, B and C be three sets with

$$A \cap B = A \cap C = B \cap C = A \cap B \cap C =: D.$$



Obviously (following the *inclusion-exclusion principle*) we have:

$$|A \cup B \cup C| = 1 \cdot |A| + 1 \cdot |B| + 1 \cdot |C| + (-2) \cdot |D|\,.$$

Evidently, the coefficients that can occur (in our case $1, 1, 1, -2$) only depend on the order relation (set inclusion) of the involved sets $A \cup B \cup C$, A, B, C and D: Indeed, it is possible to compute these coefficients quite elegantly from the order relation as we will see in the following.

3.4.1. The incidence algebra of locally finite posets.

DEFINITION 3.4.1. *Let P be a* locally finite *poset (recall that this means that for all $x, y \in P$ with $x \leqslant y$ the interval $[x, y]$ is* finite*). We consider the set $\mathcal{I}(P)$ of all functions*

$$f : P \times P \to \mathbb{C}$$

with the property

$$f(x,y) \equiv 0 \text{ if } x \nleqslant y \tag{3.10}$$

(that is to say that the function f is only "defined on intervals") with the ordinary addition and scalar multiplication of functions

$$f, g \in \mathcal{I}(P) : (f+g)(x,y) := f(x,y) + g(x,y),$$
$$f \in \mathcal{I}(P), \lambda \in \mathbb{C} : (\lambda \cdot f)(x,y) := \lambda \cdot (f(x,y))$$

*and the following (non-commutative) multiplication (*convolution*):*

$$(f \star g)(x,y) := \sum_{z \in P} f(x,z)\, g(z,y).$$

The range of summation is finite (!), as P is assumed to be locally finite, and, by (3.10)*, the sum can also be written as:*

$$\sum_{z \in P} f(x,z)\, g(z,y) = \sum_{x \leqslant z \leqslant y} f(x,z)\, g(z,y).$$

Evidently, $\mathcal{I}(P)$ is a vector space *over $\mathbb{C}$ under the addition and scalar multiplication. As for the multiplication (convolution), it satisfies the law of associativity:*

$$\begin{aligned}
((f \star g) \star h)(x,y) &= \sum_{z \in P} (f \star g)(x,z)\, h(z,y) \\
&= \sum_{z \in P} \left(\sum_{u \in P} f(x,u)\, g(u,z) \right) h(z,y) \\
&= \sum_{u \in P} \sum_{z \in P} f(x,u)\, g(u,z)\, h(z,y) \\
&= \sum_{u \in P} f(x,u)\, (g \star h)(u,y) \\
&= (f \star (g \star h))(x,y).
\end{aligned}$$

Furthermore, we have the laws of distributivity

$$f \star (g+h) = f \star g + f \star h,$$
$$(f+g) \star h = f \star h + g \star h$$

(as can be shown easily). Finally, with

$$\delta \in \mathcal{I}(P) : \delta(x,y) = \begin{cases} 1 & : x = y, \\ 0 & \text{otherwise} \end{cases},$$

we obviously have

$$\delta \star f = f \star \delta = f,$$

that is, $\mathcal{I}(P)$ is a (non-commutative) algebra *with unit δ: We refer to it as the* incidence algebra *of P.*

EXAMPLE 3.4.2. *For $P = [n]$, $\mathcal{I}(P)$ is equal to the algebra of $n \times n$ upper triangular matrices over $\mathbb{C}$: Indeed, each value $f(x,y)$ of an element $f \in \mathcal{I}(P)$ can be interpreted as the entry in row x and columns y of a corresponding triangular matrix M, and, conversely, each such matrix defines uniquely a function $f \in \mathcal{I}(P)$. It can also be seen easily that the multiplication in $\mathcal{I}(P)$ corresponds to matrix multiplication.*

THEOREM 3.4.3. *Let P be a locally finite poset. An element $f \in \mathcal{I}(P)$ has an inverse element f^{-1} (i.e.: $\exists\, f^{-1} \in \mathcal{I}(P)$ with $f \star f^{-1} = f^{-1} \star f = \delta$) if and only if $f(x,x) \neq 0\ \forall x \in P$. If an inverse exists, then it is uniquely determined.*

PROOF. One implication ($\implies$) is obvious:

$$1 = \left(f^{-1} \star f\right)(x,x) = \sum_{x \leqslant z \leqslant x} f(x,z)\, f^{-1}(z,x) \implies f(x,x) \neq 0.$$

As for the other implication ($\impliedby$), we consider an f with $f(x,x) \neq 0\ \forall x \in P$. We define $g(x,x) := \frac{1}{f(x,x)}$, and, for $x < y$, recursively ("inductively with respect to the length of the intervals $[x,y]$"):

$$\sum_{x \leqslant z \leqslant y} f(x,z)\, g(z,y) = 0,$$

i.e., $g(x,y)\, f(x,x) = -\sum_{x<z\leqslant y} f(x,z)\, g(z,y)$ and, therefore,

$$g(x,y) = -\frac{1}{f(x,x)} \sum_{x<z\leqslant y} f(x,z)\, g(z,y).$$

The so constructed g satisfies $f \star g = \delta$; analogously, we can also find an h with $g \star h = \delta$: Then we have

$$f = f \star \delta = f \star (g \star h) = (f \star g) \star h = \delta \star h = h,$$

and so we also have $g \star f = \delta$, and $f^{-1} := g$ is the inverse. □

3.4.2. The zeta function of locally finite posets.

DEFINITION 3.4.4. *Let P be a locally finite poset. The function $\zeta \in \mathcal{I}(P)$:*

$$\zeta(x,y) = \begin{cases} 1 & \text{for } x \leqslant y \\ 0 & \text{sonst} \end{cases}$$

is said to be the zeta function *of P.*

REMARK 3.4.5. *From an abstract point of view, the zeta function of a poset "is" essentially the order relation of the poset: Indeed,* each *relation $\sim$ on P "is" a subset of the cartesian product $P \times P$ (namely, the set of all pairs (x,y) with $x \sim y$), and the zeta function is from this point of view simply the* characteristic function *of the order relation.*

For $P = [n]$ (see Example 3.4.2), the zeta function corresponds to the upper triangular matrix that has only 1's above the main diagonal.

The following functions are useful: For instance,

$$\zeta^2(x,y) = \sum_{x \leqslant z \leqslant y} \underbrace{\zeta(x,z)\,\zeta(z,y)}_{1} = \#(\text{elements in } [x,y]),$$

and, more general,

$$\zeta^k(x,y) = \sum_{x_0 = x \leqslant x_1 \leqslant \cdots \leqslant x_{k-1} \leqslant x_k = y} \underbrace{\zeta(x,x_1)\,\zeta(x_1,x_2)\cdots\zeta(x_{k-1},y)}_{1},$$

this is the number of "multichains" of length k from x to y. In order to count "ordinary" chains of length k from x to y, it is useful to consider $(\zeta - \delta)$:

$$(\zeta - \delta)(x,y) = \begin{cases} 1 & \text{für } x < y \\ 0 & \text{otherwise} \end{cases}$$

The number of interest in then $(\zeta - \delta)^k(x,y)$.

REMARK 3.4.6. *Although the algebra is not commutative, each element of course commutes with the unit δ. This implies the following special case of the* Binomial Theorem*:*

$$(\zeta - \delta)^k(x,y) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j \zeta^{k-j}(x,y).$$

This identity is also a special case of the inclusion-exclusion principle*: Indeed, if we delete precisely j of those elements in a "multichain"*

$$(x = x_0, x_1, \ldots, x_k = y)$$

of length k from x to y that are repeated *after their* first *appearance, then we are left with a "multichain" of length $(k-j)$, and, conversely, there are $\binom{k}{j}$ possibilities to construct a "multichain" of length k from a "multichain" of length $(k-j)$: Each "multichain" of length $(k-j)$ consists of $k+1-j$ elements, and* each *of these elements can be repeated (also multiple times!): Since we need to add a total of j "repetitions", this corresponds to a multiset with j elements selected from a set of with $(k+1-j)$ elements, and therefore we have*

$$\binom{(k+1-j)+j-1}{j} = \binom{k}{j}.$$

An additional example is $(2\delta - \zeta)$,

$$(2\delta - \zeta) = \begin{cases} 1 & \text{if } x = y, \\ -1 & \text{if } x < y, \\ 0 & \text{otherwise.} \end{cases}$$

According to Theorem 3.4.3, this function is invertible.

Now we aim at counting *all* chains (of any length) from x to y. Let l be the length of the interval $[x,y]$, then, according to considerations above, the number is

$$\sum_{k=0}^{l} (\zeta - \delta)^k(x,y),$$

and, using a well-known trick (*telescoping series*) for the geometric series, we have

$$(\delta - (\zeta - \delta)) \star \sum_{k=0}^{l} (\zeta - \delta)^k = \delta - \underbrace{(\zeta - \delta)^{l+1}}_{=0} = \delta,$$

and so the number is given by $(2\delta - \zeta)^{-1}(x, y)$.

3.4.3. Möbius inversion.

DEFINITION 3.4.7. *The zeta function ζ of a locally finite poset P is according to Theorem 3.4.3 invertible: $\mu := \zeta^{-1}$ is said to be the* Möbiusfunction *of P. By definition, we have:*

$$\sum_{x \leqslant z \leqslant y} \underbrace{\zeta(x, z)}_{=1} \cdot \mu(z, y) = \sum_{x \leqslant z \leqslant y} \mu(z, y) = 0 \text{ for all } x < y, \tag{3.11}$$

$$\sum_{x \leqslant z \leqslant y} \mu(x, z) \cdot \underbrace{\zeta(z, y)}_{=1} = \sum_{x \leqslant z \leqslant y} \mu(x, z) = 0 \text{ for all } x < y \tag{3.12}$$

and $\mu(x, x) \equiv 1$.

THEOREM 3.4.8 (Möbius inversion). *Let P be a locally finite poset such that every principal order ideal is finite. Let $f, g : P \to \mathbb{C}$ be functions on P with values in $\mathbb{C}$. Then the following assertions are equivalent:*

$$g(x) = \sum_{y \leqslant x} f(y) \;\; \forall x \in P \tag{3.13}$$

$$f(x) = \sum_{y \leqslant x} g(y) \mu(y, x) \;\; \forall x \in P. \tag{3.14}$$

REMARK 3.4.9. *This is a generalisation of Möbius inversion in number theory: The natural numbers, ordered by the divisor relation, is a locally finite poset such that every principal ideal (the principal ideal of n is the set of divisors of n) is finite:*

$$g(n) = \sum_{d \mid n} f(d) \iff f(n) = \sum_{d \mid n} \mu\left(\frac{n}{d}\right) g(d)$$

(the interval $[d, n]$ in this poset is isomorphic to $[1, n/d]$, that is $\mu(d, n) = \mu(1, n/d) =: \mu(n/d)$).

PROOF OF THE MÖBIUS INVERSION. For the proof, we only need to perform two simple calculations:

$$\begin{aligned}
\sum_{y\leqslant x} g(y)\cdot\mu(y,x) &= \sum_{y\leqslant x}\sum_{z\leqslant y} f(z)\cdot\mu(y,x) \quad \leftarrow\text{by (3.13)}\\
&= \sum_{y\leqslant x}\mu(y,x)\sum_{z\leqslant y} f(z)\cdot\zeta(z,y) \quad \leftarrow\text{definition } \zeta\\
&= \sum_{z\in P} f(z)\sum_{z\leqslant y\leqslant x}\zeta(z,y)\cdot\mu(y,x)\\
&= \sum_{z\in P} f(z)\,\delta(z,x)\\
&= f(x),
\end{aligned}$$

and, conversely:

$$\begin{aligned}
\sum_{x\leqslant s} f(x) &= \sum_{x\leqslant s}\sum_{y\leqslant x} g(y)\cdot\mu(y,x) \quad \leftarrow\text{by (3.14)}\\
&= \sum_{x\leqslant s}\zeta(x,s)\sum_{y\leqslant x} g(y)\cdot\mu(y,x) \quad \leftarrow\text{definition } \zeta\\
&= \sum_{y\in P} g(y)\sum_{y\leqslant x\leqslant s}\mu(y,x)\cdot\zeta(x,s)\\
&= \sum_{y\in P} g(y)\,\delta(y,s)\\
&= g(s).
\end{aligned}$$

□

REMARK 3.4.10. *From an abstract point of view, the key observation is that the equations* (3.13) *and* (3.14) *are equivalent to the "symmetric" equations*

$$\begin{aligned}
g(x) &= \sum_{y\in P} f(y)\cdot\zeta(y,x) \;\; \forall x\in P,\\
f(x) &= \sum_{y\in P} g(y)\cdot\mu(y,x) \;\; \forall x\in P,
\end{aligned}$$

and that $\mathcal{I}(P)$ acts on the vector space $\mathbb{C}^P$ of all functions $P\to\mathbb{C}$ as algebra of linear mappings *(from the right) with*

$$(f\xi)(x) := \sum_{y\in P} f(y)\,\xi(y,x) \text{ für } f\in\mathbb{C}^P, \xi\in\mathcal{I}(P).$$

Möbius inversion is then nothing else but

$$f\zeta = g \iff f = g\mu.$$

We have the following dual version of Theorem 3.4.8:

THEOREM 3.4.11. *Let P be a locally finite poset such that every dual principal order ideal is finite. Let $f,g: P\to\mathbb{C}$ be functions on P with values in $\mathbb{C}$. Then the following* *assertions are equivalent:*

$$g(x) = \sum_{y \geqslant x} f(y) \;\; \forall x \in P \tag{3.15}$$

$$f(x) = \sum_{y \geqslant x} \mu(x,y)\, g(y) \;\; \forall x \in P. \tag{3.16}$$

EXAMPLE 3.4.12. *Let $S_1, S_2, \ldots, S_n$ be sets. Consider the poset that consists of all intersections of selections of these sets, ordered by inclusion. Here the* empty *intersection is defined as*

$$\bigcap_{i \in \varnothing} S_i := S_1 \cup S_2 \cup \cdots \cup S_n = \hat{1}.$$

We aim at deriving a formula for $|S_1 \cup S_2 \cup \cdots \cup S_n|$ and define for this purpose functions $g, f : P \to \mathbb{C}$ as $g(T) := |T|$ and $f(T) := |T \setminus (\bigcup_{T' < T} T')|$ (in words: $f(T)$ is the number of elements in T that are not contained in any $T' \in P$ with $T' \subsetneq T$).
First we observe that in general we have the following (think for a moment!):

$$g(T) = \sum_{T' \subseteq T} f(T').$$

Using Möbius inversion, we obtain

$$f(T) = \sum_{T' \subseteq T} g(T') \cdot \mu(T', T).$$

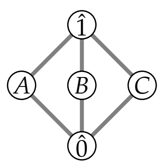
In particular, we have

$$0 = f(\hat{1}) = \sum_{T' \subseteq \hat{1}} g(T') \cdot \mu(T', \hat{1}) = \sum_{T' \subseteq \hat{1}} \mu(T', \hat{1})\, |T'|,$$

we leads us to the following formulation of the inclusion-exclusion principle:

$$|S_1 \cup S_2 \cup \cdots \cup S_n| = |\hat{1}| = -\sum_{T' \subsetneq \hat{1}} \mu(T', \hat{1})\, |T'|.$$

In the introductory example (see the beginning of Abschnitt 3.4) we then have the following:



$$\mu(\hat{1}, \hat{1}) = 1$$
$$\mu(., \hat{1}) = -1$$
$$\mu(\hat{0}, \hat{1}) = 2$$

3.4.4. Calculation of Möbius functions of several concrete posets. Möbius inversion is of course only useful if we can compute the Möbius function.

EXAMPLE 3.4.13. *Let $P = \mathbb{N}$ (with the natural order). It follows immediately from the defining equation* (3.11) *that*

$$\mu(x,y) = \begin{cases} 1 & x = y, \\ -1 & y = x + 1, \\ 0 & \text{otherwise.} \end{cases}$$

The Möbius inversion implies the "discrete analogue of the fundamental theorem of calculus"

$$g(n) = \sum_{i=1}^{n} f(i)\ \forall n \geqslant 1 \iff f(n) = \begin{cases} g(1) & \text{for } n = 1, \\ g(n) - g(n-1) & \text{for } n > 1. \end{cases}$$

From an abstract point of view, this means that the summation operator

$$(\Sigma\, c)(n) := \sum_{i=1}^{n} c(i)$$

and the difference operator

$$(\Delta\, c)(n) := \begin{cases} c(1) & n = 1 \\ c(n) - c(n-1) & n > 1 \end{cases}$$

are inverse operators on the vector space $\mathbb{C}^{\mathbb{N}}$ *of all functions* $c : \mathbb{N} \to \mathbb{C}$*.*

THEOREM 3.4.14. *Let* P *and* Q *be locally finite posets. For the direct product* $P \times Q$*, we have*

$$\mu_{P\times Q}((x_1, y_1), (x_2, y_2)) = \mu_P(x_1, y_1) \cdot \mu_Q(x_2, y_2).$$

PROOF. Recall that in the direct product $P \times Q$ we have

$$(x_1, y_1) \leqslant (x_2, y_2) :\iff x_1 \leqslant x_2 \text{ and } y_1 \leqslant y_2.$$

We check whether the given function has the defining property of the Möbius function:

$$\sum_{(x_1,y_1)\leqslant(x_3,y_3)\leqslant(x_2,y_2)} \mu_P(x_3, x_2) \cdot \mu_Q(y_3, y_2) =$$

$$\sum_{x_1\leqslant x_3\leqslant x_2}\ \sum_{y_1\leqslant y_3\leqslant y_2} \mu_P(x_3, x_2) \cdot \mu_Q(y_3, y_2) = \delta_P(x_1, x_2) \cdot \delta_Q(y_1, y_2)$$

$$= \delta_{P\times Q}((x_1, y_1), (x_2, y_2)).$$

□

EXAMPLE 3.4.15 (Classical Möbius function). *The divisor lattice* $T_n := \{d : d \mid n\}$ *(order relation = divisor relation) for* n *with prime factor decomposition* $n = p_1^{k_1} \cdot p_2^{k_2} \cdot p_m^{k_m}$ *is isomorphic to the product of linear posets*

$$T_n \simeq [0, k_1] \times \cdots \times [0, k_m].$$

By Theorem 3.4.14, we obtain:

$$\begin{aligned} \mu_{T_n}(x, y) &= \mu_{T_n}\left(p_1^{\alpha_1} \cdots p_m^{\alpha_m}, p_1^{\beta_1} \cdots p_m^{\beta_m}\right) \\ &= \mu_{[0,k_1]}(\alpha_1, \beta_1) \cdots \mu_{[0,k_m]}(\alpha_m, \beta_m) \\ &= \begin{cases} (-1)^{\sum_{i=1}^{m}(\beta_i - \alpha_i)} & \text{if } (\beta_i - \alpha_i) \in \{0, 1\}\ \forall i, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Since we have the following isomorphy

$$[x, y] = \left[1, \frac{y}{x}\right]$$

in the divisor lattice, the Möbius function satisfies

$$\mu(d, d \cdot k) = \mu(1, k) =: \mu(k),$$

and we obtain the classical Möbius function *appearing in number theory*

$$\mu(k) = \begin{cases} (-1)^m & k = p_1 \cdots p_m \ (i \neq j \implies p_i \neq p_j), \\ 0 & \textit{sonst.} \end{cases}$$

EXAMPLE 3.4.16 (Euler's φ–function). Euler's φ–function*, which also appears in number theory, is defined as*

$$\phi(n) := |\{x \in [n] : \mathrm{ggT}(x, n) = 1\}|.$$

(In words: $\phi(n)$ is the number of positive x no greater than n that are coprime to n. For instance, if $n = 12$ these numbers are $1, 5, 7, 11$, and so we have $\varphi(12) = 4$.)
Now the following holds

$$n = \sum_{d \mid n} \phi(d).$$

This is because $[n]$ is the disjoint union

$$[n] = \dot{\bigcup}_{d \mid n} \underbrace{\left\{ y \cdot \frac{n}{d} : y \leqslant d \text{ and } \mathrm{ggT}(y, d) = 1 \right\}}_{\text{Number: } \varphi(d)}.$$

Using Möbius inversion (set $g(n) = n$ and $f(n) = \phi(n)$), we obtain

$$\begin{aligned} \phi(n) &= \sum_{d \mid n} \mu(d, n)\, d \\ &= \sum_{d \mid n} \mu\left(\frac{n}{d}, n\right) \frac{n}{d} \\ &= \sum_{d \mid n} \mu(1, d) \frac{n}{d} \\ &= n \sum_{d \mid n} \mu(d) \frac{1}{d} \\ &= n \prod_{p \mid n} \left(1 - \frac{1}{p}\right). \end{aligned}$$

The last step can be justified as follows: Let $n = p_1^{k_1} \cdots p_m^{k_m}$, then the penultimate sum ranges over all d that are products of distinct *primes, that is $d = p_{i_1} \cdots p_{i_j}$ (the Möbius function is 0 for all other divisors) with $\mu(d) = (-1)^j$.*
For example, we have $\varphi(12) = 12 \cdot \left(1 - \frac{1}{2}\right) \cdot \left(1 - \frac{1}{3}\right) = 4$.

EXAMPLE 3.4.17 (Boolean lattice). *The power set $2^{[n]}$ of $[n]$ can be interpreted as family of characteristic functions (see Example 3.1.4):*

$$A \subseteq [n] \leftrightarrow f_A : [n] \to \{0, 1\}, f_A(x) = [x \in A]^1.$$

[1]Iversons notation.

This implies that the following posets are isomorphic:

$$2^{[n]} \simeq \{0,1\}^n .$$

The Möbius function of the (linearly order) poset $\{0,1\}$ *is obviously*

$$\mu(0,0) = \mu(1,1) = 1;\ \mu(0,1) = -1.$$

This implies for $T \subseteq S \subseteq [n]$ *immediately*

$$\mu(T,S) = \prod_{i=1}^{n} \mu([i \in T], [i \in S]) = (-1)^{|S \setminus T|} .$$

The *inclusion-exclusion principle* can also be formulated as follows: Let E be a set of properties that might or might not be fulfilled for individual elements of a ground set A (e.g., A could be the set of permutations of $[n]$, and $E = \{E_1, \dots, E_n\}$ could consist of the following n properties

$$E_i := \text{"has } i \text{ as fixpoint"}, i = 1, 2, \dots, n.)$$

For an arbitrary subset $S \subseteq E$, let

$$f_{=}(S) := \#(x \in A, \text{ that have } \textit{precisely} \text{ the properties in } S)$$

(that is these elements have no properties "outside" of S), and, for an arbitrary subset $T \subseteq E$, let

$$f_{\geqslant}(T) := \#(x \in A, \text{ that have } \textit{at least} \text{ the properties in } T)$$

(that is these elements can also have properties "outside" of T; e.g., the number of permutations in $\mathfrak{S}_n$, that have *at least* the k different fixpoints $i_1, i_2, \dots, i_k$, is equal to $(n-k)!$). Evidently, we have

$$f_{\geqslant}(T) = \sum_{S \supseteq T} f_{=}(S) .$$

Using Möbius inversion, we have:

$$f_{=}(T) = \sum_{S \supseteq T} (-1)^{|S \setminus T|} f_{\geqslant}(S) .$$

For the special case $T = \varnothing$, we have:

$$f_{=}(\varnothing) = \sum_{S} (-1)^{|S|} f_{\geqslant}(S) . \tag{3.17}$$

E.g., the number of permutations in $\mathfrak{S}_n$ without fixpoint (i.e., the well-known number of *derangements*) is:

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k (n-k)! = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

From this point of view, (3.17) is indeed the inclusion-exclusion principle:

COROLLARY 3.4.18 (Inklusion–exclusion). *Let $A_1, A_2, \ldots, A_n$ be n (not necessarily disjoint!) sets. Then we have:*

$$\begin{aligned} |A_1 \cup \cdots \cup A_n| = |A_1| + \cdots + |A_n| - |A_1 \cap A_2| - \cdots \\ - |A_{n-1} \cap A_n| + |A_1 \cap A_2 \cap A_3| + \cdots - \cdots \\ + (-1)^{n-1} |A_1 \cap A_2 \cap \cdots \cap A_n| \end{aligned}$$

PROOF. In the context of Example 3.4.17: Let $A = A_1 \cup \cdots \cup A_n$ be sets, and let $E = \{E_1, \ldots, E_n\}$ be the following n properties

$$E_i := \text{“is contained in } A_i\text{”}, i = 1, 2, \ldots, n.$$

Then we obviously have $f_{=}(\varnothing) = 0$, and the assertion follows immediately from (3.17). □

COROLLARY 3.4.19. *Let L be a finite distributive lattice. The rank function on L satisfies*

$$\begin{aligned} \mathbf{rk}(x_1 \vee \cdots \vee x_n) = \mathbf{rk}(x_1) + \cdots + \mathbf{rk}(x_n) - \mathbf{rk}(x_1 \wedge x_2) - \cdots \\ - \mathbf{rk}(x_{n-1} \wedge x_n) + \mathbf{rk}(x_1 \wedge x_2 \wedge x_3) + \cdots - \cdots \\ + (-1)^{n-1} \mathbf{rk}(x_1 \wedge \cdots \wedge x_n). \end{aligned}$$

PROOF. According to Birkhoff's Theorem (Theorem 3.3.13), there exists a poset P such that

$$L \simeq \mathcal{J}(P).$$

In the isomorphy, each $x \in L$ corresponds to an order ideal I_x of P, and $\mathbf{rk}(x) = |I_x|$. In addition $I_{x \wedge y} = I_x \cap I_y$: This follows immediately from Corollary 3.4.18. □

3.4.5. Möbius algebra of a locally finite lattice.

DEFINITION 3.4.20. *Let L be a locally finite poset. We consider the complex vector space of the* formal *linear combinations $\sum_{x \in L} c_x x$ with coefficients $c_x \in \mathbb{C}$. With the bilinear (that is: “extended distributively”) multiplication*

$$x \cdot y := x \wedge y \; \forall x, y \in L,$$

this vector space becomes a commutative algebra $\mathcal{A}(L)$. The algebra is frequently called Möbiusalgebra. *If L is finite, then L is obviously a basis of the vector space of idempotent elements ($x \cdot x = x \wedge x = x$).*
For each $x \in L$, we define the vector $\delta_x \in \mathcal{A}(L)$ as linear combination:

$$\delta_x := \sum_{\hat{0} \leqslant y \leqslant x} \mu(y, x)\, y.$$

For the special vectors $\delta_x \in \mathcal{A}(L)$, Möbius inversion gives

$$\sum_{y \leqslant x} \delta_y = \sum_{y \leqslant x} \sum_{s \leqslant y} \mu(s, y)\, s = \sum_{s \leqslant x} s \sum_{s \leqslant y \leqslant x} \mu(s, y) = x. \tag{3.18}$$

In other words: The elements $\{\delta_y : y \in L\}$ span the vector space $\mathcal{A}(L)$; under the assumption that L is finite, they are also a basis.

LEMMA 3.4.21. *Let L be a locally finite lattice. In $\mathcal{A}(L)$ we have:*

$$\delta_x \cdot \delta_y = [x = y]\, \delta_x. \tag{3.19}$$

PROOF. First we note that

$$\begin{aligned}
\delta_x \cdot \delta_y &= \left(\sum_{z_1 \leqslant x} \mu(z_1, x)\, z_1 \right) \cdot \left(\sum_{z_2 \leqslant y} \mu(z_2, y)\, z_1 \right) \\
&= \sum_{z_1 \leqslant x} \sum_{z_2 \leqslant y} \mu(z_1, x)\, \mu(z_2, y)\, z_1 \wedge z_2.
\end{aligned}$$

By (3.18), $z_1 \wedge z_2 = \sum_{s \leqslant z_1 \wedge z_2} \delta_s$, this is further equal to

$$\sum_{s \leqslant x \wedge y} \delta_s \left(\underbrace{\sum_{s \leqslant z_1 \leqslant x} \mu(z_1, x)}_{=[s=x]} \right) \left(\underbrace{\sum_{s \leqslant z_2 \leqslant y} \mu(z_2, y)}_{=[s=y]} \right) = [x = y]\, \delta_x.$$

□

COROLLARY 3.4.22. *Let L be a finite lattice with at least 2 elements, and let $\hat{1} \neq a \in L$. Then we have*

$$\sum_{x : x \wedge a = \hat{0}} \mu\left(x, \hat{1}\right) = 0. \tag{3.20}$$

PROOF. According to (3.18) and (3.19), we have on the one hand:

$$a \cdot \delta_{\hat{1}} = \left(\sum_{b \leqslant a} \delta_b \right) \cdot \delta_{\hat{1}} = 0 \in \mathcal{A}(L) \text{ if } a \neq \hat{1}.$$

On the other hand, by definition of δ_x, we have

$$a \cdot \delta_{\hat{1}} = a \cdot \sum_{x \in L} \mu\left(x, \hat{1}\right) x = \sum_{x \in L} \mu\left(x, \hat{1}\right) x \wedge a.$$

Now $a \cdot \delta_{\hat{1}}$ can be expanded (as any other element) in the basis L:

$$a \cdot \delta_{\hat{1}} = \sum_{x \in L} c_x x,$$

and, as $a \cdot \delta_{\hat{1}} = 0$ (zero vector in $\mathcal{A}(L)$), we have in particular $c_{\hat{0}} = 0$:

$$0 = c_{\hat{0}} = \sum_{x : x \wedge a = \hat{0}} \mu\left(x, \hat{1}\right).$$

(This sum is similar to the defining relation $\sum_x \mu\left(x, \hat{1}\right) = 0$, however, it contains less terms in general.) □

EXAMPLE 3.4.23 (Möbius function of the partition lattice). *For two partitions σ, π of $[n]$, we have $\sigma \leqslant \pi :\iff$ "all blocks of σ are contained in blocks of π". For instance:*

$$\pi_0 = 1\ 4\ 5\ 6\ 7 | 2\ 8\ 9\ 10 | 3$$
$$\sigma_0 = 1 | 4\ 5 | 6\ 7 | 2 | 8\ 9\ 10 | 3$$

In this example, it can be seen immediately that: $[\sigma_0, \pi_0] \simeq \Pi_3 \times \Pi_2 \times \Pi_1$.
More generally: Let $\pi = \{B_1, B_2, \dots, B_k\}$*, and let precisely* λ_i *blocks of* σ *be contained in* B_i*, then we have* $[\sigma, \pi] \simeq \Pi_{\lambda_1} \times \Pi_{\lambda_2} \times \cdots \times \Pi_{\lambda_k}$*; here we have* $\sigma \sim \left(\hat{0}_{\Pi_{\lambda_1}}, \dots, \hat{0}_{\Pi_{\lambda_k}}\right)$ *and* $\pi \sim \left(\hat{1}_{\Pi_{\lambda_1}}, \dots, \hat{1}_{\Pi_{\lambda_k}}\right)$*. We conclude that*

$$\mu(\sigma, \pi) = \mu\left(\hat{0}_{\Pi_{\lambda_1}}, \hat{1}_{\Pi_{\lambda_1}}\right) \cdot \mu\left(\hat{0}_{\Pi_{\lambda_2}}, \hat{1}_{\Pi_{\lambda_2}}\right) \cdots \mu\left(\hat{0}_{\Pi_{\lambda_k}}, \hat{1}_{\Pi_{\lambda_k}}\right).$$

It follows that we can compute the Möbius function of Π_n *in general, if we can determine* $\mu\left(\hat{0}_{\Pi_m}, \hat{1}_{\Pi_m}\right)$ *for all* $m \in \mathbb{N}$*. We use Corollary 3.4.22. Let* $a = 1\,2\,\dots(m-1)\,|m$*. If* $x \wedge a = \hat{0}_{\Pi_m}$*, then we either have* $x = \hat{0}_{\Pi_m}$ *or* $x = m\ i|$ *(remainder in blocks with* 1 *element) for an* $i = 1, 2, \dots, m-1$*. According to* (3.20)*, we have*

$$\mu_m := \mu\left(\hat{0}_{\Pi_m}, \hat{1}_{\Pi_m}\right) = -\sum_{x = m\ i|(remainder)} \mu\left(x, \hat{1}_{\Pi_m}\right) = -(m-1)\,\mu_{m-1}.$$

With the initial condition $\mu_1 = 1$*, we have*

$$\mu_m = \mu\left(\hat{0}_{\Pi_m}, \hat{1}_{\Pi_m}\right) = (-1)^{m-1}(m-1)! \tag{3.21}$$

For our concrete example, we obtain

$$\mu(\sigma_0, \pi_0) = (-1)^2\, 2!\, (-1)\, 1!1 = -2.$$

COROLLARY 3.4.24. *Let L be finite lattice, and let* $X \subseteq L$ *be such that*

- $\hat{1}_L \notin X$,
- $y \in L$ *and* $y \neq \hat{1}_L \implies \exists x \in X$ *with* $y \leqslant x$.

(That is, X contains all coatoms of L.) Then we have

$$\mu\left(\hat{0}_L, \hat{1}_L\right) = \sum_k (-1)^k N_k,$$

where N_k *the number of all k–subsets of X with infimum* $\hat{0}_L$*.*

PROOF. For $x \in L$, we have in $\mathcal{A}(L)$

$$\hat{1}_L - x = \sum_{y \leqslant \hat{1}_L} \delta_y - \sum_{y \leqslant x} \delta_y = \sum_{y \nleqslant x} \delta_y.$$

On the one hand, this implies

$$\prod_{x \in X}\left(\hat{1}_L - x\right) = \sum_{y: y \nleqslant x\ \forall x \in X} \delta_y \quad \leftarrow \text{by (3.19)}$$
$$= \delta_{\hat{1}_L}. \quad \leftarrow \text{by assumption}$$

On the other hand, we can also simply expand the product in order to obtain an expansion in terms of the basis L:

$$\hat{1}_L - \sum_{x \in X} x + \sum_{x, y \in X} x \wedge y - + \cdots = \underbrace{\sum_{x \in L} \mu\left(x, \hat{1}_L\right) x}_{\delta_{\hat{1}_L}}.$$

As comparison of the coefficients of $x = \hat{0}_L$ yields the assertion. ☐

REMARK 3.4.25. *In Corollary 3.4.24, it is of course natural to let X be the set of coatoms of L. In particular, we have the following: If $\hat{0}_L$ is not representable as the infimum of coatoms, then we have $\mu\left(\hat{0}_L, \hat{1}_l\right) = 0$.*

COROLLARY 3.4.26. *Let L be a finite lattice. Suppose $\hat{0}_L$ is not representable as the infimum of coatoms or $\hat{1}_L$ is not representable as the supremum of atoms, then we have*

$$\mu\left(\hat{0}_L, \hat{1}_L\right) = 0.$$

EXAMPLE 3.4.27 (Möbiusfunction of a finite distributive lattice). *For a finite distributive lattice L, Birkhoff's Theorem (Theorem 3.3.13) implies:*

$$L \simeq \mathcal{J}(P).$$

Let $I \subseteq I' \in \mathcal{J}(P)$, then we have

$$\mu\left(I, I'\right) = \begin{cases} (-1)^{\left|I' \setminus I\right|} & \text{if } [I, I'] \simeq \text{ Boolean lattice,} \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, let $I' = I \cup \{x_1, \ldots, x_n, \ldots, x_m\}$, where $I \cup \{x_1\}, \ldots, I \cup \{x_n\}$ are the atoms in $[I, I']$. Then there are only two possibilities:

Case 1: $I \cup \{x_1, \ldots, x_n\} = I'$, then we have $[I, I'] \simeq 2^{[n]}$.

Case 2: $I \cup \{x_1, \ldots, x_n\} \neq I'$, then we have $[I, I'] \not\simeq 2^{[n]}$, because in the Boolean lattice $\hat{1}$ is always *representable as the supremum of atoms.*

(See also the paper of Rota and Stanley **[6]** regarding Möbiusfunctions.)

EXAMPLE 3.4.28 (Möbiusfunction of $V_n(q) = \mathbf{GF}(q)^n$). *Let $V, W \leqslant V_n(q)$ be two subspaces with $V \leqslant W$. Then the interval $[V, W]$ is isomorphic to $\left[\left\{\vec{0}\right\}, W/V\right]$, where the latter interval only depends on the dimension* $\dim(V/W) = \dim(V) - \dim(W)$. *Therefore, it suffices to compute $\mu_n := \mu\left(\left\{\vec{0}\right\}, V_n(q)\right)$. For this purpose, we use the dualised version of Corollary 3.4.22:*

$$\sum_{x : x \vee a = \hat{1}} \mu\left(\hat{0}, x\right) = 0. \tag{3.22}$$

Let a be an atom: From $x \vee a = \hat{1}$, we conclude that

- *either $a \leqslant x \implies x = \hat{1}$,*
- *or $a \nleqslant x \implies a \wedge x = \hat{0}$.*

In $V_n(q)$ we have in general

$$\mathbf{rk}(x) + \mathbf{rk}(c) = \mathbf{rk}(x \wedge c) + \mathbf{rk}(x \vee c)$$

Therefore, $a \wedge x = \hat{0}$ implies immediately $\mathbf{rk}(x) + 1 = 0 + n$, and so: If x is a coatom, then (3.22) *implies*

$$\mu\left(\hat{0}, \hat{1}\right) = -\sum_{a \nleqslant x, x \text{ coatom}} \mu\left(\hat{0}, x\right).$$

Let a be a fixed atom in $V_n(q)$, i.e., a subspace of dimension 1. $V_n(q)$ has $\left[{n \atop n-1}\right]_q = [n]_q = q^{n-1} + q^{n-2} + \cdots + q + 1$ coatoms, that is subspaces [2] *of dimension $n-1$, among these $\left[{n-1 \atop n-2}\right]_q = [n-1]_q = q^{n-2} + \cdots + q + 1$ include the subspace a. Moreover, we have*

$$\mu_n := \mu_{V_n(q)}\left(\hat{0}, \hat{1}\right) = -q^{n-1}\mu_{n-1}.$$

This implies, using the initial condition $\mu_1 = -1$, that

$$\mu_n = (-1)^n q^{\binom{n}{2}}.$$

As a small application, we compute the number of spanning sets in $V_n(q)$. Let $f(W)$ be the number of subsets that span W, and let $g(W)$ be the number of non-empty subspaces of W. Evidently, we have:

$$g(W) = \sum_{T \leqslant W} f(T).$$

Using Möbius inversion, we obtain:

$$f(W) = \sum_{T \leqslant W} \mu(T, W)\, g(T),$$

in particular

$$f(V_n(q)) = \sum_T \mu(T, V_n(q))\, g(T) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{n-k} q^{\binom{n-k}{2}} \left(2^{q^k} - 1\right).$$

Exercise 31: *Prove the "NBC–Theorem" ("Non–broken circuit theorem") of G.–C. Rota: Let L be geometric lattice. We assume that the atoms of L are* labelled *(with natural numbers $1, 2, \ldots$). A set B of atoms is called* independent, *if $\mathbf{rk}\left(\bigvee B\right) = |B|$, otherwise it is called* dependent. *A set C of atoms is called a* circuit, *if C is a* minimal *dependent set. A* broken circuit *is a set corresponding to a circuit from which its largest atom (with respect to the labeling of atoms) was removed. A* non–broken circuit *is a set B of atoms which does not contain a broken circuit. Then Rota's Theorem states:*

$$\mu\left(\hat{0}, x\right) = (-1)^{\mathbf{rk}(x)} \cdot \#\left(\text{non–broken circuits } B \text{ with } \bigvee B = x\right).$$

Exercise 32: *Show: The Möbius function $\mu(x, y)$ of a semimodular lattice is alternating, i.e.*

$$(-1)^{\text{length of } [x,y]} \mu(x, y) \geqslant 0.$$

Moreover, show that the Möbius function of a geometric lattice is strictly *alternating, i.e.*

$$(-1)^{\text{length of } [x,y]} \mu(x, y) > 0.$$

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[2]Direct counting argument: There are $[n]_q = \frac{q^n - 1}{q - 1}$ subspaces of dimension 1 in $V_n(q)$. A basis of a k-dimensional subspace W can be extended to a basis of $V_n(q)$ by adding $n-k$ vectors; the span of these $n-k$ vectors is isomorphic to $V_{n-k}(q)$. An arbitrary k-dimensional subspace of $V_n(q)$ can be written in $[n]_q \cdot [n-1]_q \cdots [n-k+1]_q$ ways as an (ordered) direct sum of 1-dimensional subspaces; this observation leads directly to the number of k–dimensional subspaces: $\left[{n \atop k}\right]_q = \frac{[n]_q \cdot [n-1]_q \cdots [n-k+1]_q}{[k]_q \cdot [k-1]_q \cdots [1]_q}$.

<u>Hint:</u> Use the following formula for the Möbius function of a lattice:

$$\sum_{x : x \vee a = \hat{1}} \mu\left(\hat{0}, x\right) = 0 \text{ for all } a \in L. \tag{3.23}$$

Show that if a is an atom, then there follows:

$$\mu\left(\hat{0}, \hat{1}\right) = -\sum_{\substack{x \text{ coatom} \\ x \not\geqslant a}} \mu\left(\hat{0}, x\right). \tag{3.24}$$

CHAPTER 4

Asymptotic enumeration

So far, we used formal power series in an entirely algebraic way, retrieving informations about their coefficients — either explicit formulae or recursions — by elementary computations.
In this chapter we will see, that we can obtain even more information by analytical methods (a lot more about these methods can be found in [**3**].)

EXAMPLE 4.0.1. *We already computed the exponential generating function of the* Bell–numbers B_n*, which enumerate all (set–)partitions of an* n*–element set* (2.2)*:*

$$B(z) = \sum \frac{B_n}{n!} z^n = \mathrm{e}^{\mathrm{e}^z - 1} = \sum \frac{1}{m!} (\mathrm{e}^z - 1)^m = \sum_{m,j} \frac{1}{m!} \binom{m}{j} (-1)^{m-j} \mathrm{e}^{jz},$$

which immediately yields the formula:

$$B_n = \sum_{m,j} \frac{1}{m!} \binom{m}{j} (-1)^{m-j} j^n.$$

Now we may ask: What is the "order of magnitude" of the number B_n, for n large? The analogous question "what is the order of magnitude of $n!$?" is answered by *Stirling's Formula*

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{\mathrm{e}}\right)^n \text{ for } n \to \infty. \tag{4.1}$$

We want to find similar "asymptotic" results also in other cases.

4.1. Landau's notation

DEFINITION 4.1.1. *Let* S *be an arbitrary set and let* f, g *be real– or complex–valued functions* $S \to \mathbb{R}$ *(or* $S \to \mathbb{C}$*). Then the meaning of the* O–notation *"*f *is a Big–O of* g *on* S*" is:*

$$f(z) = O(g(z)) :\Longleftrightarrow \exists\, C > 0 : \forall z \in S : |f(z)| \leqslant C \cdot |g(z)|. \tag{4.2}$$

In many cases, the set of interest S *is an "appropriately chosen" (but not necessarily fixed) neighbourhood of some point* ζ *(*$\zeta = \infty$ *is also possible): If, for instance,* f, g *are actually* sequences *(i.e., functions* $\mathbb{N} \to \mathbb{C}$*), then*

$$f(n) = O(g(n)) \text{ for } n \to \infty$$

means that there is *a neighbourhood* S *of* ∞ *(i.e.,* $S = \{n \in \mathbb{N} : n \geqslant a\}$ *for some* a*), for which* (4.2) *holds.*
Similarly, the meaning of the o–notation *"*f *is a Small–O of* g *for* $z \to \zeta$*" is:*

$$f(z) = o(g(z)) \text{ for } z \to \zeta :\Longleftrightarrow \lim_{z \to \zeta} \frac{f(z)}{g(z)} = 0. \tag{4.3}$$

Moreover, we shall use the following notation for asymptotic equivalence*:*

$$f(z) \sim g(z) \text{ for } z \to \zeta :\Longleftrightarrow \lim_{z \to \zeta} \frac{f(z)}{g(z)} = 1. \tag{4.4}$$

REMARK 4.1.2. *The O–notation and o–notation must not be misunderstood as "equation": For instance, the meaning of*

$$n^3 + 2n^2 - 1 = O\left(n^3\right)$$

is "$n^3 + 2n^2 - 1$ belongs to the class *of all functions $f : \mathbb{N} \to \mathbb{C}$, for which there is some $C > 0$ such that $|f(n)| \leqslant C\left|n^3\right|$. In spoken language this misunderstanding is avoided by saying "…is* a *Big–O of …" and not "…is equal to Big–O of …". But since O–notation and o–notation are commonly used, we stick to it.*

REMARK 4.1.3. *The "$\sim$"–notation for asymptotic equivalence basically is superfluous, since it can be replaced by the o–notation:*

$$f(x) \sim g(x) \iff f(x) = g(x)(1 + o(1)) \iff f(x) = \mathrm{e}^{o(1)} \cdot g(x).$$

But, again, this notation is commonly used, so we stick to it.

EXAMPLE 4.1.4. *O–Notation:*

$$\begin{aligned} n^3 + 2n^2 - 1 &= O\left(n^3\right) \ (n \to \infty) \\ z^2 &= O(z) \ (z \to 0) \\ z &= O\left(z^2\right) \ (z \to \infty) \\ (\log z)^5 &= O\left(\sqrt{z}\right) \ (z \to \infty) \end{aligned}$$

EXAMPLE 4.1.5. *o–Notation:*

$$\begin{aligned} n^3 + 2n^2 - 1 &= o\left(n^4\right) \ (n \to \infty) \\ z^2 &= o(z) \ (z \to 0) \\ \frac{1}{n} &= o(1) \ (n \to \infty) \\ n! &= \sqrt{2\pi n}\left(\frac{n}{\mathrm{e}}\right)^n (1 + o(1)) \ (n \to \infty) \\ o(f(z) \cdot g(z)) &= f(z) \cdot o(g(z)) \ (z \to \zeta) \end{aligned}$$

The meaning of the last "equation" is (cf. Remark 4.1.2): The class of functions ϕ, for which

$$\lim_{z \to \zeta} \frac{\phi(z)}{f(z) \cdot g(z)} = 0$$

holds, is contained *in the class of functions $f \cdot \psi$, for which*

$$\lim_{z \to \zeta} \frac{\psi(z)}{g(z)} = 0$$

holds (simply set $\psi(z) := \frac{\phi(z)}{f(z)}$).

EXAMPLE 4.1.6. *Asymptotic equivalence:*

$$z+1 \sim z \ (z \to \infty)$$
$$\sinh z \sim \frac{1}{2}e^{z} \ (z \to \infty)$$
$$n! \sim e^{-n} n^{n} \sqrt{2\pi n} \ (n \to \infty)$$
$$\pi(z) \sim \frac{z}{\log(z)} \ (z \to \infty)$$

The idea behind these notations is, of course, to replace *a* complicated *function by a* simpler *one, which is asymptotically equivalent.*

Exercise 33: *Let $f, g : \mathbb{N} \to \mathbb{R}^{+}$ be two functions (from the natural numbers to the nonnegative reals). Which of the following rules is valid for $n \to \infty$ (under which preconditions)?*

$$O(f(n)) + O(g(n)) = O(f(n) + g(n)) \qquad O(f(n)) - O(g(n)) = O(f(n) - g(n))$$
$$O(f(n)) \cdot O(g(n)) = O(f(n) \cdot g(n)) \qquad \frac{O(f(n))}{O(g(n))} = O(f(n)/g(n))$$
$$O(f(n))^{O(g(n))} = O\left(f(n)^{g(n)}\right) \qquad \exp(O(f(n))) = O(\exp(f(n)))$$
$$\sqrt{O(f(n))} = O\left(\sqrt{f(n)}\right) \qquad g(O(f(n))) = O(gf(n))$$
$$e^{f(n)+O(g(n))} = e^{f(n)}(1 + O(g(n))) \qquad \log(f(n) + g(n)) = \log(f(n)) + O(g(n)/f(n)).$$

(The "equations" should be interpreted as follows: $O(f(n)) + O(g(n))$ is the class of all functions of the form $f^{\star}(n) + g^{\star}(n)$, where $f^{\star}(n) = O(f(n))$ and $g^{\star}(n) = O(g(n))$; the first "equation" means, that this class is contained *in the class $O(f(n) + g(n))$.)*

Exercise 34: *Same question as in the preceding exercise, where $O(.)$ is replaced by $o(.)$.*

Exercise 35: *Let f_1, f_2, g_1, g_2 be functions $\mathbb{N} \to \mathbb{C}$, such that $f_1(n) \sim f_2(n)$ and $g_1(n) \sim g_2(n)$ for $n \to \infty$. Which of the following rules are valid for $n \to \infty$ (under which preconditions)?*

$$f_1(n) + g_1(n) \sim f_2(n) + g_2(n) \qquad f_1(n) - g_1(n) \sim f_2(n) - g_2(n)$$
$$f_1(n) \cdot g_1(n) \sim f_2(n) \cdot g_2(n) \qquad \frac{f_1(n)}{g_1(n)} \sim \frac{f_2(n)}{g_2(n)}$$
$$f_1(n)^{g}{}_1(n) \sim f_2(n)^{g_2(n)} \qquad \exp(f_1(n)) \sim \exp(f_2(n)))$$
$$\sqrt{f_1(n)} \sim \sqrt{f_2(n)} \qquad g_1(f_1(n)) \sim g_2(f_2(n))$$
$$\log(f_1(n)) \sim \log(f_2(n))).$$

EXAMPLE 4.1.7. *Here are some typical generating functions that we want to examine in the following:*

- *Derangements:* $\sum D_n \frac{z^n}{n!} = e^{-z} \cdot \frac{1}{1-z}$,
- *Catalan numbers:* $\sum C_n z^n = \frac{1-\sqrt{1-4z}}{2z}$,
- *Bell–numbers:* $\sum B_n \frac{z^n}{n!} = e^{e^{z}-1}$.

For the Derangements (i.e., the number of fixed–point–free permutations) we obtained an exact formula by the principle of inclusion–exclusion (cf. the presentation preceding Corollary 3.4.18), from which we may directly derive an asmyptotic formula:

$$D_n = n!\left(\frac{1}{0!}-\frac{1}{1!}+\frac{1}{2!}-\cdots+(-1)^n\frac{1}{n!}\right)\sim\frac{n!}{e}.$$

For the Catalan numbers, we may easily derive an asmyptotic formula by Stirling's formula (4.1)*:*

$$C_n = \frac{(2n)!}{(n+1)\,n!^2}\sim\left(\frac{2n}{e}\right)^{2n}\left(\frac{e}{n}\right)^{2n}\frac{1}{n\sqrt{\pi n}} = 4^n\cdot\frac{1}{\sqrt{\pi}n^{3/2}}.$$

The asymptotics for the Catalan numbers shows the typical feature: If a generating function f has singularities, let ρ be a singularity of minimal absolute value. Then the coefficients of f show the following asymptotic behaviour:

$$[\![z^n]\!]\,f(z)\sim\left(\frac{1}{|\rho|}\right)^n\cdot\Theta(n)$$

with $\lim_{n\to\infty}\sqrt[n]{\Theta(n)} = 1$; the type of the singularity determines $\Theta(n)$. This kind of asymptotic analysis is called *singularity analysis* (cf. Abschnitt 4.3). If there are no (or no "*simple*") singularities, then the *saddle point method* proves to be useful in many cases (cf. Abschnitt 4.4).

4.2. Recapitulation: Elements of complex analysis

We will quickly recapitulate some basic concepts from complex analysis, which we shall need in the following:

DEFINITION 4.2.1. *A* domain G *is an open connected subset of the complex numbers* $\mathbb{C}$*:* $G \subseteq \mathbb{C}$*. A function* $f : G \to \mathbb{C}$ *is called* analytic *at* $z_0 \in G$ *if there is a neighbourhood* $U \subseteq G$ *of* z_0 *in which the series expansion*

$$f(z) = \sum_{n\geqslant 0} c_n (z-z_0)^n$$

holds (for certain coefficients $c_n \in \mathbb{C}$*). A function* $f : G \to \mathbb{C}$ *is called analytic (on G) if* f *is analytic for* all $z \in G$*.*

For a function f, which is analytic at z_0, we have:

- There is a positive radius of convergence R, such that the series converges for every point in the *interior* of the disk

  $$\mathbf{D}(z_0,R) := \{z\in\mathbb{C} : |z-z_0|\leqslant R\}$$

  with center z_0 and radius R.
- $f(z)$ is analytic in the *interior* $\{z\in\mathbb{C} : |z-z_0| < R\}$ of this *disk of convergence*.
- The series *diverges* for all z with $|z-z_0| > R$.

DEFINITION 4.2.2. *A function $f : G \to \mathbb{C}$ is called* (complex*!) differentiable at z_0, if the limit*

$$\lim_{\delta \to 0} \frac{f(z_0 + \delta) - f(z_0)}{\delta} \; (\delta \text{ is a complex number!})$$

exists; f is called differentiable on G if f is differentiable for all *$z \in G$.*

THEOREM 4.2.3. *A function $f : G \to \mathbb{C}$ is analytic on G if and only if f is differentiable on G.*

Let f and g be two analytic functions, let $\lambda \in \mathbb{C}$. Then $f + g, \lambda \cdot f$ and $f(g)$ are also analytic; and $\frac{f}{g}$ is analytic at all points ζ with $g(\zeta) \neq 0$. □

DEFINITION 4.2.4. *A function $h : G \to \mathbb{C}$ is called* meromorphic *at z_0 if $h(z)$ can be written as a quotient of two analytic functions f, g in a neighbourhood U of z_0:*

$$\forall z \in U \setminus \{z_0\} : \; g(z) \neq 0 \text{ and } h(z) = \frac{f(z)}{g(z)}.$$

In this case we also have

$$h(z) = \sum_{n \geqslant -m} c_n (z - z_0)^n$$

for all $z \neq z_0$ in the disk centered at z_0. If m is the largest natural number, for which $c_{-m} \neq 0$ in this series expansion, then z_0 is called a pole *of order m. The coefficient c_{-1} of $(z - z_0)^{-1}$ in this series expansion is called the* residue *of h at the point z_0, we also denote it as* $\operatorname{Res}(h, z_0)$*:*

$$\operatorname{Res}(h, z_0) = \left[\!\left[(z - z_0)^{-1} \right]\!\right] h(z).$$

DEFINITION 4.2.5. *Let $\phi : [0, 1] \to G$ be a differentiable function which determines a* contour Γ *in the domain G (with starting point $\phi(0)$ and end point $\phi(1)$); let $f : G \to \mathbb{C}$ be a function. Then the* contour integral $\int_\Gamma f(z)\, d(z)$ *is defined as*

$$\int_0^1 f(\phi(t))\, \phi'(t)\, \mathbf{d}t.$$

The contour Γ is called closed *if $\phi(0) = \phi(1)$.*

REMARK 4.2.6. *The contour integral only depends on the* contour Γ*, not on the concrete* parametrization $\phi(t)$*.*

THEOREM 4.2.7 (Cauchy's Theorem). *Let $f : G \to \mathbb{C}$ be analytic on G, and let Γ be an* arbitray *closed contour in G. Then we have*

$$\int_\Gamma f(z)\, \mathbf{d}z = 0.$$

THEOREM 4.2.8 (Residue Theorem). *Let $h(z)$ be meromorphic on G, let Γ be a closed contour in G. Then we have*

$$\frac{1}{2\pi \mathbb{i}} \int_\Gamma h(z)\, \mathbf{d}z = \sum_s n_s \cdot \operatorname{Res}(h, s),$$

where the sum runs over all poles *s of h, and n_s denotes the* winding number[1] *of Γ with respect to s.*

COROLLARY 4.2.9 (Cauchy's integral formula). *Let $f(z) = \sum_{n\geqslant 0} f_n z^n$ be analytic in a disk centered at 0, and let Γ be a contour in the interior of this disk, which winds around 0* exactly once *(in positive orientation) (i.e., the winding number with respect to 0 is 1), then we have*

$$[\![z^n]\!]\, f(z) = f_n = \frac{1}{2\pi\mathrm{i}} \int_\Gamma \frac{f(z)}{z^{n+1}} \mathbf{d}z. \tag{4.5}$$

DEFINITION 4.2.10. *Let $f(z) = \sum_{n\geqslant 0} c_n (z - z_0)^n$ be an analytic function with radius of convergence R; let ζ be a point in the boundary*

$$\partial\mathbf{D}(z_0, R) = \{z \in \mathbb{C} : |z - z_0| = R\}$$

of the disk of convergence. If there is a neighbourhood U of ζ and an analytic function $\hat{f}$ on U, such that

$$\hat{f}\Big|_{U\cap\mathbf{D}(z_0,R)} = f|_{U\cap\mathbf{D}(z_0,R)},$$

then this $\hat{f}$ is called an analytic continuation *of f in ζ. If there is such analytic continuation, then it is* unique.

ζ is called a singularity *of f, if there is* no *analytic continuation of f in ζ f.*

We denote the set of singularities of f by $\mathrm{Sing}(f)$. $\mathrm{Sing}(f) \cap \partial\mathbf{D}(z_0, R)$ *is always closed; the case*

$$\mathrm{Sing}(f) = \partial\mathbf{D}(z_0, R)$$

is possible: Then $\partial\mathbf{D}(z_0, R)$ is called the natural boundary *of f.*

EXAMPLE 4.2.11. *The function*

$$\sqrt{1+z} = (1+z)^{\frac{1}{2}} = \sum_{k\geqslant 0} \binom{\frac{1}{2}}{k} z^k$$

has no analytic continuation at $\zeta = -1$, so -1 is a singularity of this function.

The functions $\frac{1}{1-z}$ and $\frac{\mathrm{e}^{-z}}{1-z}$ both have a singularity (more precisely: a pole) at $\zeta = 1$.

Exercise 36: *Let f, g be complex functions which are analytic on some given domain. Which of the following rules are valid (under which preconditions)?*

$$\mathrm{Sing}(f \pm g) \subseteq \mathrm{Sing}(f) \cup \mathrm{Sing}(g) \qquad \mathrm{Sing}(f \cdot g) \subseteq \mathrm{Sing}(f) \cup \mathrm{Sing}(g)$$

$$\mathrm{Sing}(f/g) \subseteq \mathrm{Sing}(f) \cup \mathrm{Sing}(g) \cup \mathrm{Null}(g) \qquad \mathrm{Sing}(f \circ g) \subseteq \mathrm{Sing}(g) \cup g^{(-1)}(\mathrm{Sing}(f))$$

$$\mathrm{Sing}\left(\sqrt{f}\right) \subseteq \mathrm{Sing}(f) \cup \mathrm{Null}(f) \qquad \mathrm{Sing}(\log f) \subseteq \mathrm{Sing}(f) \cup \mathrm{Null}(f)$$

$$\mathrm{Sing}\left(f^{(-1)}\right) \subseteq f(\mathrm{Sing}(f)) \cup f(\mathrm{Null}(f'))$$

Here, $\mathrm{Sing}(f)$ *denotes the set of singular points of f, and* $\mathrm{Null}(f)$ *denotes the set of zeroes of f.*

[1]Loosely speaking, the winding number enumerates how often the contour Γ "winds around" s: In the examples we consider here, the winding number will always be equal to 1.

THEOREM 4.2.12. *Let $f(z) = \sum f_n \cdot z^n$ be analytic at 0 with radius of convergence $R < \infty$. Then there exists a singularity ζ in the boundary of the disk of convergence (i.e., $|\zeta| = R$).*

PROOF. Indirect: Suppose not; then there exists an analytic continuation of f at all points with $|z| = R$. Therefore, for all z with $|z| = R$ there is a neighbourhood U_z, such that f has an analytic continuation on U. Now $\bigcup_{z:|z|=R} U_z$ is an open cover of $\partial\mathbf{D}(0,R)$: since the boundary of the disk is *compact*, there exists a finite subcover. But this implies that f is analytic on $\mathbf{D}(0,R_1)$ with $R_1 > R$, and for some R_0 with $R < R_0 < R_1$ we have, according to (4.5):

$$f_n = [\![z^n]\!]\, f(z) = \frac{1}{2\pi \mathbb{i}} \int_{|z|=R_0} \frac{f(z)}{z^{n+1}} \mathbf{d}z.$$

We may parametrize the circle with center 0 and radius R_0 as follows: $z(\theta) = R_0 \cdot \mathbb{e}^{\mathbb{i}\theta}$ and $\frac{\mathbf{d}}{\mathbf{d}\theta} z(\theta) = \mathbb{i} z(\theta)$. Then we obtain for this contour:

$$f_n = \frac{1}{2\pi} \int_0^{2\pi} \frac{f\left(R_0 \cdot \mathbb{e}^{\mathbb{i}\theta}\right)}{R_0^n \cdot \mathbb{e}^{n \cdot \mathbb{i}\theta}} \mathbf{d}\theta.$$

From this we get immediately the upper bound

$$|f_n| \leqslant \frac{1}{R_0^n} \cdot \max_{|z|=R_0} |f(z)|, \tag{4.6}$$

i.e., $f_n = O\left(R_0^{-n}\right)$ $(n \to \infty)$. Let q with $R < q < R_0$, then $f(z) = \sum f_n z^n$ converges for $|z| < q$ according to the well–known *root test*[2], a contradiction (since $q > R$). ☐

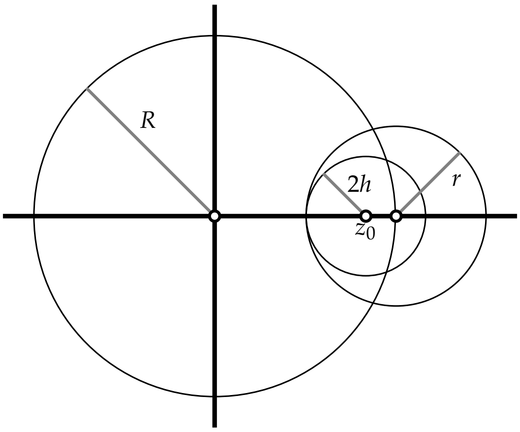
THEOREM 4.2.13 (Pringsheim). *Let $f(z) = \sum f_n z^n$ be analytic at 0 with radius of convergence R. Assume that $f_n \geqslant 0$ for all $n \geqslant 0$. Then the (real) number R is a singularity of f.*

PROOF. Indirect: Suppose, $f(z)$ is analytic at $\zeta = R$. Then there is a radius of convergence r such that $f(z)$ is analytic for all z with $|z - R| < r$. Set $h = \frac{r}{3}$ and $z_0 = R - h$. Then we have

$$f(z) = \sum_m g_m (z - z_0)^m$$

for certain coefficients g_m and all z with $|z - z_0| \leqslant 2h$.

[2]From $|f_n| \leqslant \frac{C}{R_0^n}$ we get for $|z| < q < R_0$: $\sqrt[n]{|f_n z^n|} < \frac{q}{R_0}\sqrt[n]{C}$, in particular $\limsup_{n\to\infty} \sqrt[n]{|f_n z^n|} < 1$ (since $\frac{q}{R_0} < 1$).



Equivalently, we have:

$$f(z_0+z)=\sum_m g_m\cdot z^m \text{ for } |z|\leqslant 2h=\frac{2r}{3}$$

and therefore

$$\begin{aligned} g_m=[\![z^m]\!]\, f(z+z_0) &= \frac{1}{m!}\left(\frac{\mathbf{d}}{\mathbf{d}z}\right)^m f(z+z_0)\,|_{z=0} \\ &= \frac{1}{m!} f^{(m)}(z_0) \\ &= \frac{1}{m!}\sum_n f_n\cdot n\cdot(n-1)\cdots(n-m+1)\, z_0^{n-m} \\ &= \sum_{n\geqslant m}\binom{n}{m} f_n\cdot z_0^{n-m}. \end{aligned}$$

So we have

$$f\underbrace{(z_0+2h)}_{=R+h}=\sum_m\left(\sum_{n\geqslant m}\binom{n}{m}f_n\cdot z_0^{n-m}\right)(2h)^m.$$

The right–hand side is a converging double sum (since $R+h$ lies in the interior of $\mathbf{D}(R,r)$) and has only *non–negative* summands: Therefore it is *absolutely convergent*, and we may reorder the summands arbitrarily.

$$\begin{aligned} f(R+h) &= \sum_n f_n\sum_{m=0}^{n}\binom{n}{m} z_0^{n-m}(2h)^m \\ &= \sum_n f_n(z_0+2h)^n=\sum_n f_n(R+h)^n, \end{aligned}$$

a contradiction, since $R+h>R$. □

4.3. Singularity analysis

EXAMPLE 4.3.1. *Most of the power series we are interested in are generating functions stemming from some enumeration problem and thus cleary having non–negative coefficients: So Pringsheim's Theorem is applicable in these cases.*

(1) *For the* Derangement–numbers D_n *we know*

$$\sum \frac{D_n}{n!} z^n = \frac{e^{-z}}{1-z}$$

and $\frac{D_n}{n!} \sim \frac{1}{e}$*; therefore the radius of convergence is* 1 *— and* $\zeta = 1$ *is, indeed, a singularity.*

(2) *Let* S_n *be the number of* surjections $[n] \to [k]$ *(*$k \leqslant n$ *arbitrary): From the combinatoriel point of view it is clear that we encounter the species* `sequences` $(\texttt{setn}_1)$ *(cf. Example 2.6.6), therefore we have the generating function*

$$\sum \frac{S_n}{n!} z^n = \frac{1}{1 - (e^z - 1)} = \frac{1}{2 - e^z}.$$

The singularities of this function obviously are $\log 2 + 2\pi i k, k \in \mathbb{Z}$*; the only real singularity is* $\log 2$*, which is, indeed, the radius of convergence.*

(3) *The generating function for the Catalan numbers is*

$$\frac{1 - \sqrt{1-4z}}{2z}$$

with singularity $\zeta = \frac{1}{4}$*; hence the radius of convergence is* $\frac{1}{4}$*, too.*

(4) *The generating function for the species* `cycles` *ist*

$$\log \frac{1}{1-z} = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots$$

with singularity $\zeta = 1$*; hence the radius of convergence is* 1*, too.*

4.3.1. The Exponential Growth Formula.

DEFINITION 4.3.2. *A singularity* ζ *of* f *of* minimal absolute value *is called* dominant singularity.

DEFINITION 4.3.3. *We say that a sequence of numbers* $(a_n)_{n=0}^{\infty}$ *is of* exponential order K^n *and introduce the short notation* $a_n \bowtie K^n$*:*

$$a_n \bowtie K^n :\Longleftrightarrow \limsup_{n\to\infty} \sqrt[n]{|a_n|} = K, \tag{4.7}$$

i.e., $\forall \epsilon > 0$ *we have*

$$|a_n| > (K - \epsilon)^n \text{ for infinitely many } n, \tag{4.8}$$

$$|a_n| < (K + \epsilon)^n \text{ for fast all } n. \tag{4.9}$$

Stated otherwise:

$$a_n = K^n \cdot \Theta(n),$$

where Θ is *subexponential*, i.e.

$$\limsup_{n\to\infty} \sqrt[n]{|\Theta(n)|} = 1.$$

Typical examples for subexponential sequences are

$$1, n^3, n^{-3/2}, \log n, (-1)^n, e^{\sqrt{n}}, e^{\log^2 n}, n^2 \sin n\frac{\pi}{2}, \ldots$$

THEOREM 4.3.4. *Let $f(z) = \sum_n f_n \cdot z^n$ be analytic at 0, let R be the absolute value of a dominant singularity. Then we have:*

$$f_n \bowtie \left(\frac{1}{R}\right)^n.$$

PROOF. This follows immediately from *Cauchy–Hadamard's formula* for the radius of convergence R of a power series $\sum_n f_n z^n$:

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|f_n|}}. \tag{4.10}$$

(For the absolute value of a dominant singularity clearly equals the radius of convergence.) ☐

Exercise 37: *Let $p(n)$ be the number of (integer) partitions of n. We know that*

$$\sum_{n=0}^{\infty} p(n) z^n = \prod_{i=1}^{\infty} \frac{1}{1-z^i}.$$

What are the (dominant) singular points of this generating function? What does this imply for the asymptotic behaviour of $p(n)$ for $n \to \infty$?

We sum up these results:

PRINCIPLE 4.3.5 (First principle of singularity analysis). *The (absolute values of) singularities of $f(z) = \sum f_n \cdot z^n$ determine the exponential growth of the coefficients f_n.*

As examples illustrating this principle, we already considered the Derangement–numbers D_n, the number of surjections S_n and the Catalan numbers C_n.

DEFINITION 4.3.6. *Let G be a domain, let $z_0 \in G$. Let $f : G \backslash z_0 \to \mathbb{C}$ be an analytic function. Then the point z_0 is called an* isolated singularity *of f. There are precisely three possibilities: The singularity z_0 may be:*

- *a* removable singularity, *if there is an analytic continuation of f on the whole of G*[3].
- *a* pole, *if z_0 is not a removable singularity is and there is some natural number m, such that $(z - z_0)^m \cdot f(z)$ has a removable singularity at z_0. If m is the* minimal *such number, then z_0 is called a* pole *of m–th order.*
- *an* essential singularity, *if z_0 is neither removable nor a pole.*

EXAMPLE 4.3.7. *The following functions $\mathbb{C} \backslash \{0\} \to \mathbb{C}$ have an isolated singularity at 0:*

- *For $\frac{\sin z}{z}$, 0 is a* removable *singularity,*
- *for $\frac{1}{z^m}$, 0 is a* pole *of m–th order,*
- *for $e^{\frac{1}{z}}$, 0 is an* essential *singularity.*

[3] Accoriding to *Riemann's Theorem* this is possible, if f is bounded in a neighbourhood of z_0. Analytic continuations are always unique.

REMARK 4.3.8. *The logarithm is a* covering map *which is not analytic on* any *domain containing* 0*, so* 0 *is not an isolated singularity of the logarithm (and thus of the root function* $\sqrt[n]{z} = z^{1/n} = \exp \frac{1}{n} \log z$*).*

PROPOSITION 4.3.9. *Let* $f(z)$ *be analytic on* $|z| < R$ *with* $0 \leqslant R < \infty$*. Then we have for arbitrary* $r \in (0, R)$*:*

$$|f_n| \leqslant \frac{\max_{|z|=r} |f(z)|}{r^n},$$

i.e.

$$|f_n| \leqslant \inf_{0<r<R} \frac{\max_{|z|=r} |f(z)|}{r^n}.$$

If $f(z)$ only has *non–negative* coefficients, then we have

$$f_n \leqslant \frac{f(r)}{r^n} \implies f_n \leqslant \inf_{0<r<R} \frac{f(r)}{r^n}.$$

PROOF. The fundamental bound is derived from Cauchy's integral formula (4.5) (see the proof of Theorem 4.2.12): Setting $z(\theta) = r \cdot \mathbb{e}^{\mathbb{i}\theta}$ and $\frac{\mathbf{d}}{\mathbf{d}\theta} z(\theta) = \mathbb{i} z(\theta)$, we obtain

$$f_n = \frac{1}{2\pi\mathbb{i}} \int_{|z|=r} \frac{f(z)}{z^{n+1}} \mathbf{d}z = \frac{1}{2\pi} \int_0^{2\pi} \frac{f\left(r \cdot \mathbb{e}^{\mathbb{i}\theta}\right)}{r^n \mathbb{e}^{n \cdot \mathbb{i}\theta}} \mathbf{d}\theta$$

□

REMARK 4.3.10. *If* $f(z)$ *has only* non–negative *coefficients, then the* inf *is assumed at*

$$\left(\frac{f(r)}{r^n}\right)' = 0 \iff \frac{f'(r)}{r^n} - n\frac{f(r)}{r^{n+1}} = 0 \iff r\frac{f'(r)}{f(r)} = n.$$

We shall make use of this simple observation when we will deal with the saddle point method.

Now we want to explain the following principle:

PRINCIPLE 4.3.11 (Second principle of singularity analysis). *The nature of the dominant singularities determines the subexponential part* $\Theta(n)$ *in*

$$f_n \sim K^n \cdot \Theta(n).$$

4.3.2. Rational functions. From the well–known *partial fraction decomposition* of a rational function f (expressed as the quotient of two polynomials p and q, which do not have a common root)

$$f(z) = \frac{p(z)}{q(z)} = \frac{p(z)}{C(z-z_1)^{e_1} \cdots (z-z_k)^{e_k}} = \underbrace{R(z)}_{\text{polynomial}} + \sum_{i=1}^{k} \sum_{l=1}^{e_i} \frac{\gamma_{i,l}}{(z-z_i)^l}$$

together with the *binomial Theorem*

$$(z-z_i)^{-l} = (-z_i)^{-l} \sum_{n \geqslant 0} \binom{-l}{n} \left(-\frac{z}{z_i}\right)^n = (-1)^l \sum_{n \geqslant 0} z^n \binom{n+l-1}{l-1} z_i^{-n-l}$$

we immediately obtain the following exact formula for the coefficients $f_n = [\![z^n]\!]\, f(z)$, $n > \deg(R)$:

$$f_n = \sum_{i=1}^{k} \sum_{l}^{e_i} (-1)^l \gamma_{i,l} \binom{n+l-1}{l-1} z_i^{-n-l}$$
$$= \sum_{i=1}^{k} z_i^{-n} \underbrace{\sum_{l}^{e_i} (-1)^l \gamma_{i,l} \binom{n+l-1}{l-1} z_i^{-l}}_{\text{polynomial in } n}$$

From this, we derive immediately the asymptotics:

THEOREM 4.3.12. *Let $1, \ldots, a_m$ be the dominant singularities (poles) of the rational function $f(z)$; let $e_1, \ldots, e_m$ be the corresponding orders of these poles. Then we have:*

$$f_n = \sum_{i=1}^{m} a_i^{-n} \underbrace{\text{polynomial}_i(n)}_{\text{of degree } e_i - 1} + O\left(b^{-n}\right)$$

where b is a non–dominant *singularity of minimal absolute value.*

EXAMPLE 4.3.13. *The generating function of the* Fibonacci–numbers F_n *is*

$$\sum_{n \geqslant 0} F_n \cdot z^n = \frac{1}{1 - z - z^2}.$$

The singularities of this rational function are $\frac{-1 \pm \sqrt{5}}{2}$. The partial fraction decomposition is

$$\frac{1}{1 - z - z^2} = \frac{1 + \sqrt{5}}{2\sqrt{5}} \frac{1}{1 - \frac{1+\sqrt{5}}{2} z} - \frac{1 - \sqrt{5}}{2\sqrt{5}} \frac{1}{1 - \frac{1-\sqrt{5}}{2} z},$$

from which we immediately obtain the asymptotics:

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^{n+1} + O\left(\left(\underbrace{\frac{1 - \sqrt{5}}{2}}_{\simeq 0.618034} \right)^{n+1} \right).$$

Further simple examples:

$$\frac{1}{(1 - 2z)^2} = \sum_{n \geqslant 0} (n+1)\, 2^n z^n \text{ (one pole of second order)},$$
$$\frac{1}{1 - z^2} = \sum_{n \geqslant 0} \frac{1}{2} \left((1)^n + (-1)^n \right) z^n \text{ (two dominant poles)}.$$

4.3.3. Meromorphic functions. Meromorphic functions can be treated in much the same way as rational functions:

THEOREM 4.3.14. *Let $f(z)$ be meromorphic for $|z| \leqslant R$ with poles at $a_1, \dots, a_m$; let $e_1, \dots, e_m$ be the orders of these poles. Let f be analytic at 0 and for all z with $|z| = R$. Then we have:*

$$f_n = \sum_{i=1}^{m} a_i^{-n} \underbrace{\text{polynomial}_i(n)}_{\text{of degree } e_i - 1} + O\left(R^{-n}\right)$$

PROOF. The proof proceeds by "subtraction of singularities".
Consider the pole a_1. In a neighbourhood of a_1 we may expand $f(z)$ int a *Laurent series*:

$$f(z) = \underbrace{c_{-e_1}(z-a_1)^{-e_1} + \cdots c_{-1}(z-a_1)^{-1}}_{=:S_{a_1}(z)} + \underbrace{\sum_{n \geqslant 0} c_n (z-a_1)^n}_{=:H_{a_1}(z)}.$$

Here $S_{a_1}(z)$ is the "singular part" at a_1, while $H_{a_1}(z)$ is analytic at a_1. This "subtraction of singular parts" can, of course, be repeated for all the remaining poles, whence we obtain

$$f(z) = \underbrace{\sum_{i=1}^{m} S_{a_i}(z)}_{=:S(z)} + \underbrace{\left(f(z) - \sum_{i=1}^{m} S_{a_i}(z)\right)}_{=:H(z)},$$

where $S(z)$ is a rational function with poles $a_1, \dots, a_m$ and $H(z)$ is an analytic function in the disk $|z| \leqslant R + \epsilon$ for some $\epsilon > 0$ (since f is analytic for all z with $|z| = R$, the "usual" compactness argument yields the claim). Since we have for the coefficients h_n of $H(z)$

$$h_n = O\left(R^{-n}\right)$$

(cf. (4.6)), the assertion follows. □

EXAMPLE 4.3.15. *We already computed the exponential generating function of the numbers S_n of all surjections $[n] \to [k]$ (arbitrary k):*

$$f(z) = \frac{1}{2 - e^z}.$$

This function has poles at $\log 2 + k \cdot 2\pi i$, $k \in \mathbb{Z}$; by "subtracting" the singular part at $\log 2$ we obtain

$$f(z) = -\frac{1}{2}\left(\frac{1}{z - \log 2}\right) + H(z) = \frac{1}{2\log 2}\left(\frac{1}{1 - \frac{1}{\log 2} z}\right) + H(z)$$

(since $\lim_{z \to \log 2} \frac{z - \log 2}{2 - e^z} = -\frac{1}{2}$, by de L'Hospital's rule), where $H(z)$ is analytic for all z with $|z| \leqslant 2\pi$. Hence we get from Theorem 4.3.14

$$f_n = \frac{1}{2}\left(\frac{1}{\log 2}\right)^{n+1} + O\left((2\pi)^{-n}\right),$$

which implies for the numbers S_n:

$$S_n = \frac{n!}{2}\left(\frac{1}{\log 2}\right)^{n+1} + O\left(n!\,(2\pi)^{-n}\right) \tag{4.11}$$

EXAMPLE 4.3.16. *The (exponential) generating function of* `cycles`* *(i.e.,* `sequences (cycles)`*) is*

$$f(z) = \frac{1}{1 - \log \frac{1}{1-z}}.$$

Obviously there is one pole at $z = 1$ and another at

$$\log \frac{1}{1-z} = 1 \iff \frac{1}{1-z} = \mathbb{e} \iff z = \frac{\mathbb{e} - 1}{\mathbb{e}} = 1 - \frac{1}{\mathbb{e}} \simeq 0.632121,$$

hence the dominant *singularity is $\zeta = 1 - \mathbb{e}^{-1}$. Again, by de L'Hospital's rule, we get*

$$\lim_{z\to\zeta} \frac{z - \zeta}{1 - \log \frac{1}{1-z}} = \lim_{z\to\zeta} \frac{1}{\frac{-1}{1-z}} = \zeta - 1 = -\mathbb{e}^{-1},$$

i.e.,

$$f(z) = -\frac{\mathbb{e}^{-1}}{z - \left(1 - \mathbb{e}^{-1}\right)} + H(z).$$

$H(z)$ is analytic in the (closed!) disk with radius $R = 1 - \epsilon$ ($\epsilon > 0$), therefore we obtain the asympotics

$$f_n = \frac{1}{\mathbb{e}} \left(\frac{1}{1 - \mathbb{e}^{-1}} \right)^{n+1} + O\left((1-\epsilon)^{-n}\right) \simeq \frac{1}{\mathbb{e}} 1.58198^{n+1} + O\left(1.0001^n\right).$$

EXAMPLE 4.3.17. *The number $C_{n,k}$ of surjections $[n] \to [k]$ can be expressed in terms of the* Stirling–numbers of the second kind*:*

$$C_{n,k} = k! \cdot S_{n,k}.$$

We want to investigate the following question: If we assume uniform probability distribution *on the set of all surjections $[n] \to [k]$ (n fixed, k arbitray), what is the* expected value *of the cardinality of the surjection's image? Our approach is to consider the generating function in two variables*

$$f(t, z) := \sum_{k\geqslant 0} \sum_{n\geqslant 0} C_{n,k} \cdot \frac{z^n}{n!} \cdot t^k = \sum_{k\geqslant 0} t^k \left(\mathbb{e}^z - 1\right)^k = \frac{1}{1 - t\left(\mathbb{e}^z - 1\right)}.$$

Then we have

$$\left.\frac{\mathbf{d}f}{\mathbf{d}t}\right|_{t=1} = \sum_{n\geqslant 0} \left(\sum_{k=0}^{n} k \cdot C_{n,k} \right) \cdot \frac{z^n}{n!} = \frac{\mathbb{e}^z - 1}{\left(2 - \mathbb{e}^z\right)^2}.$$

The poles (order: 2) of this function are $\log 2 + k \cdot 2\pi\mathbb{i}$, $k \in \mathbb{Z}$, *and the* Laurent–series expansion *is*

$$\frac{\mathbb{e}^z - 1}{\left(2 - \mathbb{e}^z\right)^2} = \frac{1}{4\left(z - \log 2\right)^2} + \frac{1}{4\left(z - \log 2\right)} + H(z),$$

where $H(z) = -\frac{7}{48} + \frac{1}{48}(z - \log 2) + O\left((z - \log 2)^2\right)$ is analytic in the (closed!) disk of radius $R = \sqrt{4\pi^2 + (\log 2)^2} - \epsilon$ ($\epsilon > 0$; we may choose $R \simeq 6.3213$). From *this we obtain immediately*

$$[\![z^n]\!]\ \frac{e^z-1}{(2-e^z)^2} = \frac{n+1}{4(\log 2)^2}\left(\frac{1}{\log 2}\right)^n - \frac{1}{4\log 2}\left(\frac{1}{\log 2}\right)^n + O\left(R^{-n}\right)$$
$$= \frac{1}{4}(n+1-\log 2)\left(\frac{1}{\log 2}\right)^{n+2} + O\left(R^{-n}\right).$$

We must combine this result with (4.11)*...*

$$[\![z^n]\!]\ \frac{1}{2-e^z} = \frac{1}{2}\left(\frac{1}{\log 2}\right)^{n+1} + O\left(R^{-n}\right),$$

since the expected value we are interested is the quotient of coefficients:

$$\frac{\frac{1}{4}(n+1-\log 2)\left(\frac{1}{\log 2}\right)^{n+2} + O\left(R^{-n}\right)}{\frac{1}{2}\left(\frac{1}{\log 2}\right)^{n+1} + O\left(R^{-n}\right)} = \frac{n+1}{2\log 2} - \frac{1}{2} + O\left(R^{-n}\right).$$

(Since $\log 2 \sim 0.693147 < 1$*; cf* (4.16) *in the following Lemma.)*

The following Lemma states some "typical manipulations" concerning the O–notation; in particular the transformation from the preceding example:

LEMMA 4.3.18. *Let $f(n)$ and $h(n)$ be arbitratry sequences (i.e., functions $\mathbb{N} \to \mathbb{C}$). Then we have*

$$O(f(n)) \cdot h(n) = O(f(n) \cdot h(n)). \tag{4.12}$$

If in particular $h(n) = O(1)$, then we simply have

$$O(f(n)) \cdot h(n) = O(f(n)). \tag{4.13}$$

Let $r : \mathbb{N} \to \mathbb{C}$ with $\lim_{n\to\infty} r(n) = 0$ (i.e., $r = o(1)$). Then we have:

$$\frac{1}{1+O(r(n))} = 1 + O(r(n)) \text{ for } n \to \infty. \tag{4.14}$$

Moreover, let $g : \mathbb{N} \to \mathbb{C}$, such that $\frac{1}{g(n)} = O(1)$. Then we have:

$$\frac{1}{1+\frac{O(r(n))}{g(n)}} = 1 + \frac{O(r(n))}{g(n)} \text{ for } n \to \infty. \tag{4.15}$$

Finally, let $f : \mathbb{N} \to \mathbb{C}$ with $\frac{f(n)}{(g(n))^2} = O(1)$. Then we have

$$\frac{f(n)+O(r(n))}{g(n)+O(r(n))} = \frac{f(n)}{g(n)} + O(r(n)) \tag{4.16}$$

PROOF. (4.12): Let $\zeta(n) \in O(f(n))$ be some fixed (but arbitrarily chosen) element from $O(f(n))$. Then we have by definition

$$|\zeta(n) \cdot h(n)| = |\zeta(n)| \cdot |h(n)|$$
$$\leqslant K \cdot |f(n)| \cdot |h(n)|$$
$$= K \cdot |f(n) \cdot h(n)|.$$

(4.13): Simply continue the above chain of inequalities with "$\leqslant K \cdot K' \cdot |f(n)|$", which holds by definition.

(4.14): By definition of the limit and of O, respectively, there is some $C > 0$ and some $n_0 \in \mathbb{N}$, such that for a fixed (but arbitrarily chosen) element $\zeta(n) \in O(r(n))$ in the denominator of (4.14) there holds:

$$|\zeta(n)| \leqslant C \cdot |r(n)| \leqslant \frac{1}{2} \text{ for all } n \geqslant n_0.$$

But this implies that for all $n \geqslant n_0$

$$\begin{aligned}\left|\frac{1}{1+\zeta(n)}\right| = \frac{1}{|1+\zeta(n)|} &\leqslant \frac{1}{1-|\zeta(n)|}\\ &\leqslant \frac{1}{1-C\cdot|r(n)|} \quad \leftarrow 1-C|r(n)|\leqslant 1-\zeta(n)\\ &= 1+\frac{C\cdot|r(n)|}{1-C\cdot|r(n)|} \quad \leftarrow 1-C\cdot|r(n)|\geqslant \tfrac{1}{2}\\ &\leqslant 1+(2\cdot C)\cdot|r(n)|,\end{aligned}$$

which proves (4.14).

(4.15): By assumption there is some $\epsilon > 0$, such that $|g(n)| > \epsilon$ for $n > N$. Let $\zeta(n)$ be as above, but now choose n_0 such that

$$|\zeta(n)| \leqslant C \cdot |r(n)| \leqslant \frac{\epsilon}{2} \text{ for all } n \geqslant n_0.$$

Then we have $|C\cdot|r(n)|/g(n)| \leqslant 1/2$ for all $n \geqslant \max(n_0, N)$, and

$$\begin{aligned}\left|\frac{1}{1+\frac{\zeta(n)}{g(n)}}\right| \leqslant \frac{1}{1-\frac{|\zeta(n)|}{|g(n)|}} &\leqslant \frac{1}{1-\frac{C\cdot|r(n)|}{|g(n)|}}\\ &= 1+\frac{C\cdot|r(n)|/|g(n)|}{1-C\cdot|r(n)|/|g(n)|} \quad \leftarrow C\cdot|r(n)|/|g(n)|\leqslant \tfrac{1}{2}\\ &\leqslant 1+(2\cdot C)\cdot\frac{|r(n)|}{|g(n)|},\end{aligned}$$

which proves (4.15).

(4.16): Finally, we have

$$\frac{f(n)+O(r(n))}{g(n)+O(r(n))} = \frac{f(n)}{g(n)+O(r(n))} + \frac{O(r(n))}{g(n)+O(r(n))}.$$

For the first term on the right–hand side we have

$$\begin{aligned}\frac{f(n)}{g(n)+O(r(n))} &= \frac{f(n)}{g(n)}\left(\frac{1}{1+\frac{O(r(n))}{g(n)}}\right)\\ &= \frac{f(n)}{g(n)}\left(1+\frac{O(r(n))}{g(n)}\right) \quad \leftarrow \text{by (4.15)}\\ &= \frac{f(n)}{g(n)} + \frac{f(n)\,O(r(n))}{(g(n))^2}\\ &= \frac{f(n)}{g(n)} + O(r(n)) \quad \leftarrow \text{by assumption and (4.13)}.\end{aligned}$$

For the second term on the right–hand side we have according to (4.13)

$$\frac{O(r(n))}{g(n)+O(r(n))}=O(r(n)),$$

since also $|g(n)+O(r(n))|>\epsilon/2$ for n "large enough", so $\frac{1}{g(n)+O(r(n))}=O(1)$. Of course we have $O(r(n))+O(r(n))=O(r(n))$, so also (4.16) is proved. □

4.3.4. Several dominant singularities.

EXAMPLE 4.3.19 (Cancellation). *Consider the sum of rational functions* $\frac{1}{1+z^2}$ *and* $\frac{1}{1-z^3}$*:*

$$\frac{1}{1+z^2}+\frac{1}{1-z^3}=\frac{2+z^2-z^3}{(1+z^2)(1-z^3)}=\frac{2-z^2+z^3+z^4+z^8+z^9-z^{10}}{1-z^{12}}.$$

The coefficients of z^n *vanish for* $n\equiv 1,5,6,7,11 \pmod{12}$.

EXAMPLE 4.3.20 (Nonperiodic fluctuations). *Consider the rational function*

$$\frac{1}{1-\frac{6}{5}z+z^2}=1+\frac{6}{5}z+\frac{11}{25}z^2-\frac{84}{125}z^3-\frac{779}{625}z^4-\frac{2574}{3125}z^5+\cdots$$

with partial fraction decomposition

$$\frac{1}{1-\frac{6}{5}z+z^2}=\frac{\frac{1}{2}+\frac{3\mathbb{i}}{8}}{1-\left(\frac{3}{5}-\frac{4\mathbb{i}}{5}\right)z}+\frac{\frac{1}{2}-\frac{3\mathbb{i}}{8}}{1-\left(\frac{3}{5}+\frac{4\mathbb{i}}{5}\right)z}.$$

The roots of the denominators have absolute value 1 *and are inverses of one another, i.e.*

$$\frac{3}{5}+\frac{4\mathbb{i}}{5}=\mathbb{e}^{\mathbb{i}\theta} \text{ and } \frac{3}{5}-\frac{4\mathbb{i}}{5}=\mathbb{e}^{-\mathbb{i}\theta}.$$

So the coefficient of z^n *may be written as*

$$\begin{aligned}\frac{\mathbb{e}^{\mathbb{i}n\theta}+\mathbb{e}^{-\mathbb{i}n\theta}}{2}+\frac{3}{4}\cdot\frac{\mathbb{e}^{\mathbb{i}n\theta}-\mathbb{e}^{-\mathbb{i}n\theta}}{2\mathbb{i}}&=\cos(n\cdot\theta)+\frac{3}{4}\sin(n\cdot\theta)\\&=\cos(n\cdot\theta)+\frac{\cos\theta}{\sin\theta}\sin(n\cdot\theta)\\&=\frac{\sin((n+1)\cdot\theta)}{\sin\theta}.\end{aligned}$$

4.3.5. "Standardized" singularity analysis. We shall now present the "(quasi–) mechanical algorithm" of (standard) singularity analysis for determining the asymptotics of the coefficients of some generating function $f(z)$, as developped by Philippe Flajolet and Andrew Odlyzko. It consists of the following steps:

Normalization of the singularity: If ρ is the "relevant singularity" of $f(z)$, consider $\hat{f}(z):=f(\rho\cdot z)$: then 1 is the "relevant singularity" of $\hat{f}(z)$.

Asymptotics for "standard functions": For "standard functions" of the form

$$(1-z)^{-\alpha}\left(\frac{1}{z}\log\frac{1}{1-z}\right)^{\beta}$$

there is a general result describing the asymptotics of the coefficients.

Transfer–Theorems: If we know some asymptotic bound for a function, we may immediately derive an asymptotic bound for its coefficients:

$$f(z) = O\left((1-z)^{-\alpha}\left(\log\frac{1}{1-z}\right)^{\beta}\right) \implies [\![z^n]\!]\, f(z) = O\left(n^{\alpha-1}(\log n)^{\beta}\right)$$

4.3.5.1. *The Gamma–function.*

DEFINITION 4.3.21 (Gamma–function). *The* Gamma–function $\Gamma(s)$ *is defined by* Euler's Integral Representation

$$\Gamma(s) = \int_0^\infty e^{-t}\cdot t^{s-1}\mathbf{d}t. \tag{4.17}$$

for $\Re(s) > 0$. *The Gamma–function fulfils the functional equation*

$$\Gamma(s+1) = s\cdot\Gamma(s). \tag{4.18}$$

It follows that $\Gamma(s)$ *has an analytic continuation to* $\mathbb{C}\setminus\{0,-1,-2,\dots\}$*: We have for all* $m \in \mathbb{N}_0$

$$\Gamma(s) \sim \frac{(-1)^m}{m!}\frac{1}{s+m} \text{ for } s \to -m. \tag{4.19}$$

For the Gamma–function we also have Euler's Reflection Formula*:*

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}. \tag{4.20}$$

From the functional equation (4.18) we immediately obtain

$$\Gamma(s+n) = s^{\overline{n}}\cdot\Gamma(s) = s\cdot(s+1)\cdots(s+n-1)\cdot\Gamma(s). \tag{4.21}$$

In particular, from (4.17) we see immediately $\Gamma(1) = 1$, so we have for $n \geqslant 0 \in \mathbb{Z}$

$$\Gamma(n+1) = n!$$

and likewise with $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$\Gamma\left(n+\frac{1}{2}\right) = \left(\frac{1}{2}\right)^{\overline{n}}\cdot\sqrt{\pi} = \frac{(2n)!\cdot\sqrt{\pi}}{n!\cdot 4^n}. \tag{4.22}$$

For the Gamma–function, *Stirling's formula* gives an asymptotic approximation:

$$\Gamma(s+1) \sim s^s e^{-s}\sqrt{2\pi s}\left(1+\frac{1}{12s}+\frac{1}{288s^2}-\frac{139}{51840s^3}+\cdots\right). \tag{4.23}$$

REMARK 4.3.22 (Ψ–function). *The* Ψ*–function is defined as the logarithmic derivative of the Gamma–function:*

$$\Psi(z) = \frac{\mathbf{d}}{\mathbf{d}z}\log\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}.$$

The following formula gives evaluations of the Ψ–function at rational numbers $z = \frac{p}{q}$*; cf.* [**2**, section 1.7.3, equation (29)]*:*

$$\Psi\left(\frac{p}{q}\right) = -\gamma - \log q - \frac{\pi}{2}\cot\left(\frac{\pi p}{q}\right) + \sum_{n=1}^{\frac{q}{2}\prime} \cos\left(2\pi\frac{p\,n}{q}\right)\log\left(2 - 2\cos\left(2\pi\frac{n}{q}\right)\right), \quad (4.24)$$

where the prime ($\frac{q}{2}\prime$*) in the upper bound of the summation means that for even q the last summand (with index* $n = \frac{q}{2}$*) is multiplied by* $\frac{1}{2}$*. (Here,* γ *denotes the* Euler–Mascheroni constant*.) For example:*

$$\Psi\left(\frac{1}{2}\right) = -\gamma - \log 2 - \frac{\pi}{2}\cot\left(\frac{\pi}{2}\right) + \frac{1}{2}\cos\pi\log\left(2 - 2\cos\left(\pi\right)\right) = -\gamma - 2\log 2, \quad (4.25)$$

since $\cot\left(\frac{\pi}{2}\right) = 0$*,* $\cos\pi = -1$ *and* $\log 4 = 2\log 2$*.*

REMARK 4.3.23. *For the following Theorem it is useful is to recall the* polar coordinates *for complex numbers: Let* $z \in \mathbb{C}$ *with* $\Re(z) = x$ *and* $\Im(z) = y$*, i.e.,* $z = x + \mathrm{i}y$*. Then we may express z in polar coordinates* (r, ϕ)*, i.e.* $z = r \cdot (\cos\phi + \mathrm{i}\sin\phi)$ *with* $r = |z|$ *and* $\phi = \phi_0 + 2\pi k$ *for* $k \in \mathbb{Z}$*, where* ϕ_0 *denotes the main value (in the interval* $-\pi < \arg(z) \leqslant \pi$*)of the argument* $\arg(z)$*. (for* $r = 0$ *we may choose the angle* ϕ *arbitrarily.) We have:*

$$\mathrm{e}^{x+\mathrm{i}y} = \mathrm{e}^{x} \cdot \left(\cos y + \mathrm{i}\sin y\right), \quad (4.26)$$

$$\log\left(x + \mathrm{i}y\right) = \log\left|x + \mathrm{i}y\right| + \mathrm{i}\arg\left(x + \mathrm{i}y\right). \quad (4.27)$$

In particular, this means that the (complex) logarithm (the inverse function of the exponential function) is a multi–valued *function, which might be "forced" to be single–valued by constraining its range (for instance, by considering only the main values of the argument).*

Moreover, we have the following connection of the trigonometricn functions with the exponential function:

$$\cos\theta = \frac{\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}}{2},$$

$$\sin\theta = \frac{\mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta}}{2\mathrm{i}}.$$

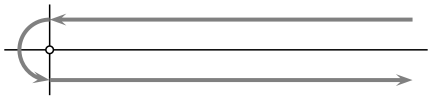
THEOREM 4.3.24 (Presentation with Hankel–contour integral). *For the Gamma–function we have*

$$\frac{1}{\Gamma(s)} = \frac{1}{2\pi\mathrm{i}}\int_{\mathcal{H}} (-t)^{-s} \cdot \mathrm{e}^{-t}\mathbf{d}t, \quad (4.28)$$

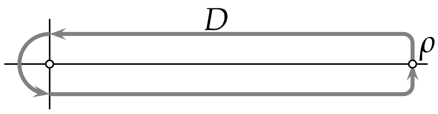
where $\mathcal{H} = \mathcal{H}^{-} + \mathcal{H}^{\circ} + \mathcal{H}^{+}$ *is a contour which cosists of the following parts*

- $\mathcal{H}^{-} = \{w - \mathrm{i} : w \geqslant 0\}$,
- $\mathcal{H}^{\circ} = \left\{-\mathrm{e}^{\mathrm{i}\phi} : -\frac{\pi}{2} \leqslant \phi \leqslant \frac{\pi}{2}\right\}$,
- $\mathcal{H}^{+} = \{w + \mathrm{i} : w \geqslant 0\}$

See the following graphical illustration:



PROOF. Let D be a contour, starting at a point $\rho > 0$ on the real axis, winding once around zero in the anti–clockwise orientation and returning zu ρ.



For $z \in \mathbb{C} \setminus \mathbb{N}$ with $\Re(z) > 0$, consider the integral

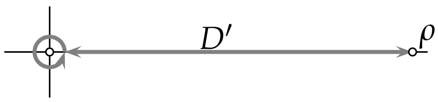
$$\int_D (-t)^{z-1} \mathrm{e}^{-t} \mathbf{d}t. \tag{4.29}$$

The factor

$$(-t)^{z-1} = \mathrm{e}^{(z-1)\cdot\log(-t)} = \mathrm{e}^{(z-1)\cdot\log(|-t|)+\arg(-t)}$$

in the integrand is unique if we require $\log(-t) \in \mathbb{R}$ for $t < 0$: I.e., for all points $t \in D$ we have $-\pi \leqslant \arg(z) \leqslant \pi$.

The integrand is not analytic in the interior of D (since the logarithm is analytic only in the *slitted complex plane*, i.e., in *no* neighbourhood of 0), but according to Cauchy's Theorem 4.2.7 we may "deform" the contour D to another contour D' without changing the value of the integral:



In the first (straight) part of this new contour D' we have $\arg(-t) = -\pi$, i.e.,

$$(-t)^{z-1} = \mathrm{e}^{(z-1)(\log t - \mathrm{i}\pi)},$$

and in the last (straight) part we have $\arg(-t) = \pi$, i.e.,

$$(-t)^{z-1} = \mathrm{e}^{(z-1)(\log t + \mathrm{i}\pi)}.$$

(For both parts we have $\log t \in \mathbb{R}$.) Parametrizing the small circle as $-t(\theta) = \delta \mathrm{e}^{\mathrm{i}\theta}$ we obtain:

$$\int_D (-t)^{z-1} \mathrm{e}^{-t} \mathbf{d}t = \int_\rho^\delta \mathrm{e}^{-\mathrm{i}\pi(z-1)} t^{z-1} \mathrm{e}^{-t} + \int_\delta^\rho \mathrm{e}^{\mathrm{i}\pi(z-1)} t^{z-1} \mathrm{e}^{-t}$$
$$+ \mathrm{i} \int_{-\pi}^{\pi} \left(\delta \mathrm{e}^{\mathrm{i}\theta}\right)^{z-1} \mathrm{e}^{\delta \mathrm{e}^{\mathrm{i}\theta}} \delta \mathrm{e}^{\mathrm{i}\theta} \mathbf{d}\theta =$$
$$-2\mathrm{i}\sin(\pi z) \int_\delta^\rho t^{z-1} \mathrm{e}^{-t} \mathbf{d}t + \mathrm{i}\delta^z \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}z\theta + \delta \mathrm{e}^{\mathrm{i}\theta}} \mathbf{d}\theta.$$

(Since $\sin(\pi(z-1)) = \sin(\pi z - \pi) = -\sin(\pi z)$.)

This equality holds for all positive real numbers $\delta \leqslant \rho$: for $\delta \to 0$ we have $\delta^z \to 0$ (since $\Re(z) > 0$) and $\int_{-\pi}^{\pi} e^{iz\theta+\delta e^{i\theta}} \mathbf{d}\theta \to \int_{-\pi}^{\pi} e^{iz\theta} \mathbf{d}\theta$ (since the integrand converges uniformly); so we obtain:

$$\int_D (-t)^{z-1} e^{-t} \mathbf{d}t = -2i \sin(\pi z) \int_0^\rho t^{z-1} e^{-t} \mathbf{d}t.$$

This holds for all positive real numbers ρ: $\mathcal{H}$ is obviously the "limiting case of the contour" D for $\rho \to \infty$, so there holds

$$\int_{\mathcal{H}} (-t)^{z-1} e^{-t} \mathbf{d}t = -2i \sin(\pi z) \int_0^\infty t^{z-1} e^{-t} \mathbf{d}t,$$

i.e.,

$$\Gamma(z) = -\frac{1}{2i \sin(\pi z)} \int_{\mathcal{H}} (-t)^{z-1} e^{-t} \mathbf{d}t. \tag{4.30}$$

The contour $\mathcal{H}$ is not close to 0, therefore we do not need the assumption $\Re(z) > 0$ any more[4]: The integral in (4.30) is a (single–valued) analytic function, which equals $\Gamma(z)$ for $\Re(z) > 0$; so *Hankel's formula* (4.30) for the Gamma–function holds for *all* $z \in \mathbb{C} \backslash \mathbb{Z}$. Replacing z by $1-z$ and applying Euler's Reflection Formula (4.20), we obtain the assertion. □

4.3.5.2. *Asymptotics for "standard functions".* According to the binomial theorem, for the "*standard function*" $(1-z)^{-\alpha}$ (where $\alpha \in \mathbb{C}$ is arbitrary) we obtain

$$[\![z^n]\!]\ (1-z)^{-\alpha} = (-1)^n \binom{-\alpha}{n} = \binom{n+\alpha-1}{n}.$$

If $\alpha \in \{0, -1, -2, \dots\}$, then the asymptotics of these coefficients is trivial (since in that case $(1-z)^{-\alpha}$ is a polynomial, and the coefficients are all 0 for $n > -\alpha$). Otherwise, we can write

$$\binom{n+\alpha-1}{n} = \frac{\alpha \cdot (\alpha+1) \cdots (\alpha+n-1)}{n!} = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}.$$

Hence, we could determine the asymptotics by means of *Stirling's formula for the gamma function* (4.23)

$$\Gamma(s+1) = s^s e^{-s} \sqrt{2\pi s} \left(1 + O\left(\frac{1}{s}\right)\right).$$

However, it is more convenient to apply Cauchy's integral formula (4.5):

THEOREM 4.3.25 (Asymptotics for coefficients of standard functions, 1). *Let $\alpha \in \mathbb{C} \backslash \{0, -1, -2, \dots\}$. We consider the "standard function"*

$$f(z) = (1-z)^{-\alpha}.$$

Then, asymptotically as $n \to \infty$, we have

$$[\![z^n]\!]\ f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)} \left(1 + \sum_{k=1}^{\infty} \frac{e_k(\alpha)}{n^k}\right), \tag{4.31}$$

[4]An analytic continuation for the logarithm exists only in the *slitted* complex plane!

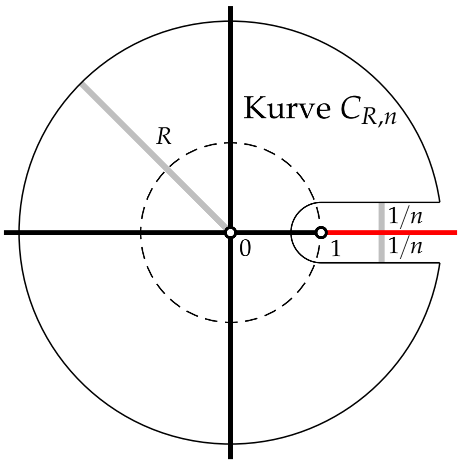
where $e_k(\alpha)$ is a polynomial of degree $2k$ in α, which is divisible by $\alpha^{\underline{k+1}} = \alpha \cdot (\alpha-1)\cdots(\alpha-k)$. The first terms read:

$$[\![z^n]\!]\ (1-z)^{-\alpha} \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1+\frac{\alpha\cdot(\alpha-1)}{2n}+\frac{\alpha\cdot(\alpha-1)\cdot(\alpha-2)\cdot(3\alpha-1)}{24n^2}\right.$$
$$\left.+\frac{\alpha^2\cdot(\alpha-1)^2\cdot(\alpha-2)\cdot(\alpha-3)}{48n^3}+O\left(\frac{1}{n^4}\right)\right). \quad (4.32)$$

PROOF. Due to Cauchy's integral formula (4.5), we have

$$[\![z^n]\!]\ (1-z)^{-\alpha} = \frac{1}{2\mathbb{i}\pi}\int_{\mathcal{C}}\frac{(1-z)^{-\alpha}}{z^{n+1}}\mathbf{d}z, \quad (4.33)$$

where $\mathcal{C}$ is an *arbitrary* closed contour (with winding number 1) around 0, which avoids the ray of real numbers $\geqslant 1$. Concretely, we choose the contour $\mathcal{C}_{R,n}$ that is described in the following figure:



For all $n > \Re(-\alpha)$, the integral becomes arbitrarily small along the "circle piece" of $\mathcal{C}_{R,n}$ if we let $R \to \infty$[5], so that "only" the "Hankel-like piece" $\mathcal{H}_n = \mathcal{H}_n^- + \mathcal{H}_n^\circ + \mathcal{H}_n^+$ of $\mathcal{C}_{R,n}$ plays a role, which consists of the subpieces

- $\mathcal{H}_n^- = \left\{w - \frac{\mathbb{i}}{n} : w \geqslant 1\right\}$,
- $\mathcal{H}_n^\circ = \left\{1 - \frac{e^{\mathbb{i}\phi}}{n} : -\frac{\pi}{2} \leqslant \phi \leqslant \frac{\pi}{2}\right\}$,
- $\mathcal{H}_n^+ = \left\{w + \frac{\mathbb{i}}{n} : w \geqslant 1\right\}$.

Performing the substitution,

$$z = 1 + \frac{t}{n},\ \mathbf{d}z = \frac{1}{n}\mathbf{d}t$$

we obtain

$$[\![z^n]\!]\ (1-z)^{-\alpha} = \frac{n^{\alpha-1}}{2\mathbb{i}\pi}\int_{\mathcal{H}}(-t)^{-\alpha}\left(1+\frac{t}{n}\right)^{-n-1}\mathbf{d}t, \quad (4.34)$$

where $\mathcal{H}$ *is* the Hankel contour from Theorem 4.3.24.

[5]The reason is that the modulus of the integrand can be estimated from above as $\left|\frac{(1-z)^{-\alpha}}{z^{n+1}}\right| \leqslant R^{\Re(-\alpha)-n-1}$, and hence the integral is less than or equal to $2\pi R^{\Re(-\alpha)-n}$.

For $|t| < n$ the power series expansion of $\log\left(1+\frac{t}{n}\right)$ converges, hence we obtain

$$\begin{aligned}\left(1+\frac{t}{n}\right)^{-n-1} &= \mathrm{e}^{-(n+1)\log(1+t/n)} \\ &= \exp\left(-(n+1)\left(\frac{t}{n}-\frac{t^2}{2n^2}+\frac{t^3}{3n^3}-+\cdots\right)\right) \\ &= \exp\left(-t+\frac{t^2-2t}{2n}+\frac{3t^2-2t^3}{6n^2}+\cdots\right) \quad \leftarrow\text{sort } n^{-k} \\ &= \mathrm{e}^{-t}\cdot\exp\left(\frac{t^2-2t}{2n}+\frac{3t^2-2t^3}{6n^2}+\cdots\right) \\ &= \mathrm{e}^{-t}\left(1+\frac{t^2-2t}{2n}+\frac{3t^4-20t^3+24t^2}{24n^2}+\cdots\right). \qquad (4.35)\end{aligned}$$

That is, for $|t| \leqslant R$ ($R < \infty$ arbitrary but fixed) the integrand of (4.34) converges for $n \to \infty$ *uniformly* to $(-t)^{-\alpha}\,\mathrm{e}^{-t}$. "Asymptotically", we therefore have

$$\left(1+\frac{t}{n}\right)^{-n-1} = \mathrm{e}^{-t}\left(1+O\left(\frac{1}{n}\right)\right) \text{ for } n \to \infty.$$

"Formally", by substitution in (4.34), we obtain

$$\begin{aligned}[\![z^n]\!]\;(1-z)^{-\alpha} &= \frac{n^{\alpha-1}}{2\mathrm{i}\pi}\int_{\mathcal{H}}(-t)^{-\alpha}\mathrm{e}^{-t}\left(1+O\left(\frac{1}{n}\right)\right)\mathbf{d}t \\ &= \frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1+O\left(\frac{1}{n}\right)\right).\end{aligned}$$

That was admittedly somewhat bold, and it needs justification. To this end, we split the Hankel contour $\mathcal{H}$

- into the part with $\Re(t) \geqslant \log^2 n$: we shall show that this part is "negligible";
- and into the part with $\Re(t) < \log^2 n$: in that case, for n large enough we have $\frac{|t|}{n} \sim \frac{\log^2 n}{n} < 1$ (because $|t| \leqslant \sqrt{\Re(t)^2+1}$ and $\log z < \sqrt{z}$ for all $z > 0$) and the expansion (4.35) "works".

To see that the first part is negligible, some finer work is required for the estimation of the integral. The subparts of the integrals over $\mathcal{H}^+$ and $\mathcal{H}^-$ can be dealt with "in once" because, to start with, we have

$$\left|\int_{\log^2 n}^{\infty}\frac{(-t\pm\mathrm{i})^{\alpha}}{\left(1+\frac{t\pm\mathrm{i}}{n}\right)^{n+1}}\mathbf{d}t\right| \leqslant \int_{\log^2 n}^{\infty}\frac{|t+\mathrm{i}|^{\alpha}}{\left|1+\frac{t+\mathrm{i}}{n}\right|^{n+1}}\mathbf{d}t.$$

From

$$\begin{aligned}z^{\alpha} &= \mathrm{e}^{\alpha\log z} = \mathrm{e}^{(\Re(\alpha)+\mathrm{i}\Im(\alpha))(\log|z|+\mathrm{i}\arg z)} \\ &= \mathrm{e}^{\Re(\alpha)\log|z|-\Im(\alpha)\arg z}\cdot\mathrm{e}^{\mathrm{i}(\Im(\alpha)\log|z|+\Re(\alpha)\arg z)}\end{aligned}$$

it follows that

$$|z^{\alpha}| = \mathrm{e}^{\Re(\alpha)\log|z|-\Im(\alpha)\arg z},$$

thus here concretely

$$\left|(t+\mathbb{i})^{\alpha}\right| = |t+\mathbb{i}|^{\Re(\alpha)} \cdot \mathbb{e}^{-\Im(\alpha)\arg(t+\mathbb{i})} \leqslant C \cdot |t|^{\Re(\alpha)}$$

since $\arg(t+\mathbb{i}) \to 0$ for $t \to \infty$ and

$$\lim_{t\to\infty} \frac{\sqrt{t^2+1}}{t} = \sqrt{\lim_{t\to\infty} \frac{t^2+1}{t^2}} = 1.$$

Furthermore, we have

$$\left|1+\frac{t+\mathbb{i}}{n}\right| \geqslant \left(1+\frac{t}{n}\right),$$

hence we may continue the estimation of the integral (we put $\alpha' = \Re(\alpha)$):

$$\left|\int_{\log^2 n}^{\infty} \frac{(-t\pm\mathbb{i})^{\alpha}}{\left(1+\frac{t\pm\mathbb{i}}{n}\right)^{n+1}}\mathbf{d}t\right| \leqslant C \cdot \int_{\log^2 n}^{\infty} \frac{t^{\alpha'}}{\left|1+\frac{t}{n}\right|^{n+1}}\mathbf{d}t.$$

Here, the integration variable $t \geqslant \log^2 n$ is real, which we assume *always* in what follows. Then we have

$$\left|1+\frac{t}{n}\right| \geqslant 1+\frac{\log^2 n}{n},$$

from which we may infer

$$\left|1+\frac{t}{n}\right|^{-n} \leqslant \left(1+\frac{\log^2 n}{n}\right)^{-n}$$

for all $n \in \mathbb{N}$.

From real analysis, we know (see for example [**5**, Satz 26.1]): for a sequence $(x_n)_n$ with $\lim_{n\to\infty} x_n = 0$, and with $x_n \neq 0$ and $x_n > -1$ for all n, we have

$$\lim_{n\to\infty} (1+x_n)^{1/x_n} = \mathbb{e}.$$

We choose the special sequence[6] $\frac{\log^2 n}{n}$ and obtain

$$\lim_{n\to\infty}\left(1+\frac{\log^2 n}{n}\right)^{n/\log^2 n} = \mathbb{e},$$

or, equivalently (the real logarithm is continuous):

$$\lim_{n\to\infty} \frac{n\log\left(1+\frac{\log^2 n}{n}\right)}{\log^2 n} = 1$$

This implies that, at least, for $\epsilon > 0$ we have

$$n\log\left(1+\frac{\log^2 n}{n}\right) \geqslant (1-\epsilon)\log^2 n, \tag{4.36}$$

if n is large enough.

[6] Apply de l'Hospital's rule twice: $\lim_{n\to\infty} \frac{\log^2 n}{n} = \lim_{n\to\infty} \frac{2\log n}{n} = \lim_{n\to\infty} \frac{2}{n} = 0$.

Now we claim: for fixed k, we may choose an$n$ such that for $t \geqslant \log^2 n$ we have

$$\frac{t^{\alpha'}}{\left(1+\frac{t}{n}\right)^{n+1}} \leqslant \frac{1}{n^k t^2}. \tag{4.37}$$

For, if we take the logarithm on both sides of this claim, then we obtain the equivalent inequality

$$\left(\alpha'+2\right)\log t - (n+1)\log\left(1+\frac{t}{n}\right) \leqslant -k\log n \tag{4.38}$$

(since the real logarithm is a monotone increasing function), and if we substitute for $t \to \log^2 n$, then it is obvious that the following inequality holds for n large:

$$\left(\alpha'+2\right)\log\log^2 n - \underbrace{(n+1)\log\left(1+\frac{\log^2 n}{n}\right)}_{\geqslant(1-\epsilon)\log^2 n \text{ by (4.36)}} \leqslant -k\log n$$

(since $\log^2 n$ grows faster than $\log\log^2 n$ and $\log n$). If we consider the derivative of (4.38), then we see that

$$\frac{2+\alpha'}{t} - \frac{n+1}{n}\frac{1}{1+t/n} = \frac{2+\alpha'}{t} - \frac{n+1}{n+t} \leqslant 0,$$

for $t \geqslant \frac{n(2+\alpha')}{n-\alpha'-1}$, and since the right-hand side of this inequality converges to $2+\alpha'$, for n large enough we have also $\log^2 n \geqslant \frac{n(2+\alpha')}{n-\alpha'-1}$. Thus, the inequality (4.37) is established, and we may now complete our estimation:

$$\left|\int_{\log^2 n}^{\infty} \frac{(-t\pm \mathrm{i})^{\alpha}}{\left(1+\frac{t\pm\mathrm{i}}{n}\right)^{n+1}}\mathbf{d}t\right| \leqslant C\cdot\int_{\log^2 n}^{\infty}\frac{1}{n^k t^2}\mathbf{d}t = \frac{C}{n^k\log^2 n}.$$

By (4.35), for large n the integral (4.34) equals a sum of terms of the form

$$c_k\int_{\mathcal{H}:\Re(t)\leqslant\log^2 n}(-t)^{-\alpha}\,\mathrm{e}^{-t}t^k\mathbf{d}t,$$

"except for the above error term". We would of course like to argue that these integrals "essentially" are equal to

$$\int_{\mathcal{H}}(-t)^{-\alpha+k}\,\mathrm{e}^{-t}\mathbf{d}t = \frac{2\pi\mathrm{i}}{\Gamma(\alpha-k)} = \frac{2\pi\mathrm{i}\cdot(\alpha-k)\cdot(\alpha-k+1)\cdots(\alpha-1)}{\Gamma(\alpha)}$$

(by Hankel's formula (4.28) and the functional equation (4.21), respectively). In order to see this, we argue that the "cut-off remaining integral" (using the same manipulations as in the proof of Theorem 4.3.24) leads to the real integral

$$2\mathrm{i}\sin(\pi\alpha)\int_{\log^2 n}^{\infty}\mathrm{e}^{-t}(-t)^{-\alpha}\,\mathbf{d}t,$$

whose integrand clearly becomes very small if t becomes large:

$$\left|\mathrm{e}^{-t}(-t)^{-\alpha}\right| \leqslant \mathrm{e}^{-t}t^{\Re(\alpha)} \leqslant \frac{1}{n^k t^2}$$

(since $\lim_{t\to\infty} e^{-t} t^{\Re(\alpha)+2} = 0$). Hence, we can bound the error as before. If we put everything together, we arrive at the claim. □

EXAMPLE 4.3.26 (Catalan numbers). *Applying* (4.32) *to the generating function of the Catalan numbers,*

$$\sum_{n\geqslant 1} C_{n-1}\cdot z^n = \frac{1-\sqrt{1-4\cdot z}}{2},$$

we have $\alpha = -\frac{1}{2}$. *Setting* $y = 4\cdot z$, *we obtain:*

$$\begin{aligned} C_{n-1} &= -\frac{4^n}{2}[\![y^n]\!]\ (1-y)^{-\alpha} \\ &\simeq 4^{n-1}\cdot\frac{n^{-3/2}}{\sqrt{\pi}}\cdot\left(1+\frac{3}{8n}+\frac{25}{128n^2}+\frac{105}{1024n^3}+O\left(n^{-4}\right)\right). \end{aligned}$$

REMARK 4.3.27. *For the coefficients* e_k *in* (4.31) *we have*

$$e_k = \sum_{l=k}^{2k} \lambda_{k,l}\,(\alpha-1)\cdot(\alpha-2)\cdots(\alpha-l)$$

with $\lambda_{k,l} = [\![v^k t^l]\!]\ e^t\cdot(1+vt)^{-1-1/v}$.

THEOREM 4.3.28 (Asymptotics for coefficients of standard functions, 2). *Let* $\alpha\in\mathbb{C}\setminus\{0,-1,-2,\dots\}$. *We consider the "standard function"*

$$f(z) = (1-z)^{-\alpha}\left(\frac{1}{z}\log\frac{1}{1-z}\right)^{\beta}. \tag{4.39}$$

Then, asymptotically as $n\to\infty$, *we have*

$$[\![z^n]\!]\ f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}(\log n)^{\beta}\left(1+\sum_{k=1}^{\infty}\frac{C_k(\alpha)}{\log^k n}\right), \tag{4.40}$$

where $C_k = \binom{\beta}{k}\Gamma(\alpha)\left.\frac{\mathbf{d}^k}{\mathbf{d}s^k}\frac{1}{\Gamma(s)}\right|_{s=\alpha}$.

REMARK 4.3.29. *The factor* $\frac{1}{z}$ *in* (4.39) *is "just" there, in order that* $f(z)$ *is a power series, also in the case where* $\beta\notin\mathbb{Z}$ *(the logarithm is analytic in a neighbourhood of* $z=1$*):*

$$\begin{aligned} \left(\frac{1}{z}\log\frac{1}{1-z}\right)^{\beta} &= \exp\left(\beta\log\left(\frac{1}{z}\log\frac{1}{1-z}\right)\right) \\ &= \exp\left(\beta\log\left(1+\frac{z}{2}+\frac{z^2}{3}+\frac{z^3}{4}+\cdots\right)\right) \\ &= \exp\left(\beta\left(\frac{z}{2}+\frac{5z^2}{24}+\frac{z^3}{8}+\cdots\right)\right) \end{aligned}$$

REMARK 4.3.30. *The formulae in Theorems 4.3.25 and 4.3.28 actually remain valid also in the case* $\alpha\in\{0,-1,-2,\dots\}$ *if we interpret them as* limit cases, *i.e., if we set*

$$\frac{1}{\Gamma(\alpha)} = \lim_{z\to\alpha}\frac{1}{\Gamma(z)} = 0.$$

In (4.40)*, the limit should be taken* after *cancelling the Gamma–factors* $\Gamma(\alpha)$*, so only the* first *term vanishes: For instance, we have*

$$[\![z^n]\!]\ \frac{z}{\log(1-z)^{-1}} = -\frac{1}{n\cdot\log^2 n} + \frac{2\cdot\gamma}{n\cdot\log^3 n} + O\left(\frac{1}{n\cdot\log^4 n}\right).$$

SKETCH OF PROOF. In analogy to the proof of Theorem 4.3.25, we choose a "Hankel-type" contour in Cauchy's integral formula. After the substitution $z \to 1+\frac{t}{n}$, we obtain an integral over the "ordinary" Hankel contour $\mathcal{H}$ with integrand

$$\begin{aligned}\frac{1}{n}f\left(1+\frac{t}{n}\right)\left(1+\frac{t}{n}\right)^{-n-1} &= \frac{1}{n}\left(-\frac{t}{n}\right)^{-\alpha}\left(\log\left(-\frac{n}{t}\right)\right)^{\beta}\left(1+\frac{t}{n}\right)^{-\beta-n-1}\\ &\sim n^{\alpha-1}(-t)^{-\alpha}(\log n)^{\beta}\left(\frac{1-\frac{\log(-t)}{\log n}}{1-\frac{-t}{n}}\right)^{\beta}\mathrm{e}^{-t}.\end{aligned}$$

For $|t| \leqslant \log n$, we have

$$\left(1+\frac{t}{n}\right)^{-\beta} = \sum_{k\geqslant 0}\binom{-\beta}{k}\frac{t^k}{n^k} = 1 + O\left(\frac{\log n}{n}\right).$$

Hence, these terms are negligible in our $\frac{1}{\log^k n}$-expansion. Consequently, we may simplify further:

$$\frac{1}{n}f\left(1+\frac{t}{n}\right)\left(1+\frac{t}{n}\right)^{-n-1} \sim n^{\alpha-1}(\log n)^{\beta}\,\mathrm{e}^{-t}(-t)^{-\alpha}\left(1-\frac{\log(-t)}{\log n}\right)^{\beta},$$

and expand the part which is independent of t by the binomial theorem:

$$\mathrm{e}^{-t}(-t)^{-\alpha}\left(1-\beta\frac{\log(-t)}{\log n}+\frac{\beta(\beta-1)}{2!}\left(\frac{\log(-t)}{\log n}\right)^2-\dots\right)..$$

Analogously to the proof of Theorem 4.3.25, we must show that this expansion is correct "up to an asymptotically negligible rest"; this allows us to integrate termwise, where we interchange differentiation

$$\frac{\mathbf{d}}{\mathbf{d}s}(-t)^{-s} = \frac{\mathbf{d}}{\mathbf{d}s}\exp(-s\log(-t)) = -\log(-t)\exp(-s\log(-t))$$

and integration:

$$\begin{aligned}\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{H}}(-t)^{-s}\,\mathrm{e}^{-t}\log^k(-t)\,\mathbf{d}t &= (-1)^k\frac{\mathbf{d}^k}{\mathbf{d}s^k}\left(\frac{1}{2\pi\mathrm{i}}\int_{\mathcal{H}}(-t)^{-s}\,\mathrm{e}^{-t}\mathbf{d}t\right)\\ &= (-1)^k\frac{\mathbf{d}^k}{\mathbf{d}s^k}\frac{1}{\Gamma(s)}. \quad \leftarrow\text{by (4.28)}\end{aligned}$$

This establishes the claim. □

EXAMPLE 4.3.31. *A typical application of Theorem 4.3.28 is* ($\alpha = \frac{1}{2}, \beta = -1$)*:*

$$[\![z^n]\!]\ \frac{1}{\sqrt{1-z}}\frac{1}{\frac{1}{z}\log\frac{1}{1-z}} = \frac{1}{\sqrt{n\pi}\log n}\left(1-\frac{\gamma+2\log 2}{\log n}+O\frac{1}{\log^2 n}\right),$$

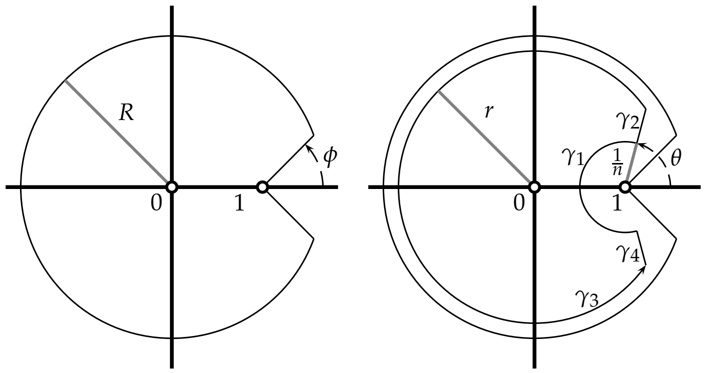
since $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, $\left(\Gamma(z)^{-1}\right)' = -\frac{\Psi(z)}{\Gamma(z)}$ *and* $\Psi\left(\frac{1}{2}\right) = -\gamma - 2\log 2$ *(see* (4.25)*).*

4.3.5.3. *Transfer theorems.*

DEFINITION 4.3.32. *For two real numbers* ϕ, R *with* $R > 1$ *and* $0 < \phi < \frac{\pi}{2}$, *we define the open domain*

$$\Delta(\phi, R) := \{z \in \mathbb{C} : |z| < R, z \neq 1, |\arg(z-1)| > \phi\}.$$

We call such a domain a Δ*-domain.*[7] *The left part in the following figure illustrates this.*



THEOREM 4.3.33. *Let* $\alpha, \beta \in \mathbb{R}$ *(not* $\mathbb{C}$*!) be arbitrary, and let* $f(z)$ *be a function that is analytic on a* Δ*-domain. If* f *in the* intersection *of a neighbourhood of* 1 *and the* Δ*-domain satisfies*

- *the property that, if*

$$f(z) = O\left((1-z)^{-\alpha}\left(\log\frac{1}{1-z}\right)^{\beta}\right),$$

  then

$$[\![z^n]\!]\, f(z) = O\left(n^{\alpha-1}(\log n)^{\beta}\right); \tag{4.41}$$

- *the property that, if*

$$f(z) = o\left((1-z)^{-\alpha}\left(\log\frac{1}{1-z}\right)^{\beta}\right),$$

  then

$$[\![z^n]\!]\, f(z) = o\left(n^{\alpha-1}(\log n)^{\beta}\right). \tag{4.42}$$

PROOF. As usual, we use Cauchy's integral formula

$$f_n = [\![z^n]\!]\, f(z) = \frac{1}{2\pi\mathbb{i}}\int_{\gamma}\frac{f(z)}{z^{n+1}}\mathbf{d}z$$

[7] Philippe Flajolet, in line with his cultural background, used to call these domains "Camembert domains".

for a contour γ in the Δ-domain, which encircles the origin once in positive direction. Concretely, we choose (see the right part of the above figure) the contour $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ which consists of the following parts:

$$\gamma_1 = \left\{ z \in \mathbb{C} : |z-1| = \frac{1}{n}, |\arg(z-1)| \geqslant \theta \right\},$$
$$\gamma_2 = \left\{ z \in \mathbb{C} : \frac{1}{n} \leqslant |z-1|, |z| \leqslant r, |\arg(z-1)| = \theta \right\},$$
$$\gamma_3 = \{ z \in \mathbb{C} : |z| = r, |\arg(z-1)| \geqslant \theta \},$$
$$\gamma_4 = \left\{ z \in \mathbb{C} : \frac{1}{n} \leqslant |z-1|, |z| \leqslant r, |\arg(z-1)| = -\theta \right\};$$

where $1 < r < R$ and $\phi < \theta < \frac{\pi}{2}$. We treat each of the individual corresponding parts of the integral separately:

$$f_n^j = \frac{1}{2\pi \mathbb{i}} \int_{\gamma_j} \frac{f(z)}{z^{n+1}} \mathbf{d}z,$$

$j = 1, 2, 3, 4$. Then we have:

$$f_n^1 = O\left(\frac{1}{n}\right) \cdot O\left(\left(\frac{1}{n}\right)^{-\alpha} (\log n)^\beta\right) = O\left(n^{\alpha-1}\left(\log n^\beta\right)\right),$$

since $f = O\left(n^\alpha (\log n)^\beta\right)$ along γ_1 by assumption, the length of the integration path is $O\left(\frac{1}{n}\right)$, and $z^{-n-1} = O(1)$ along γ_1.

Furthermore, for $r' := n \cdot r$ we have

$$\begin{aligned}
\left|f_n^2\right| &\leqslant \frac{1}{2\pi n} \int_1^{r'} \left|\frac{t}{n}\right|^{-\alpha} \left|\log \frac{n}{t}\right|^\beta \left|1 + \frac{\mathbb{e}^{\mathbb{i}\theta} t}{n}\right|^{-n-1} \mathbf{d}t \quad \leftarrow \log \tfrac{n}{t} \leqslant \log n \\
&\leqslant \frac{1}{2\pi} n^{\alpha-1} (\log n)^\beta \int_1^\infty t^{-\alpha} \left|1 + \frac{\mathbb{e}^{\mathbb{i}\theta} t}{n}\right|^{-n-1} \mathbf{d}t \quad \leftarrow \left|1+\tfrac{\mathbb{e}^{\mathbb{i}\theta} t}{n}\right| \geqslant 1 + \Re\left(\tfrac{\mathbb{e}^{\mathbb{i}\theta} t}{n}\right) \\
&\leqslant \frac{1}{2\pi} n^{\alpha-1} (\log n)^\beta \int_1^\infty t^{-\alpha} \left|1 + \frac{t\cos\theta}{n}\right|^{-n-1} \mathbf{d}t.
\end{aligned}$$

Since

$$\lim_{n\to\infty} \int_1^\infty t^{-\alpha} \left|1 + \frac{t\cos\theta}{n}\right|^{-n-1} \mathbf{d}t = \int_1^\infty t^{-\alpha} \mathbb{e}^{-t\cos\theta} \mathbf{d}t < \infty, \quad \leftarrow \text{because } 0<\theta<\tfrac{\pi}{2}$$

it follows that

$$f_n^2 = O\left(n^{\alpha-1} (\log n)^\beta\right),$$

and completely analogously this also holds for f_n^4.

Along γ_3, the function $f(z)$ is *bounded*, but $z^{-n} = O\left(r^{-n}\right)$; consequently f_n^3 is "exponentially small".

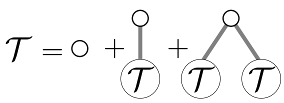
The estimation with o (instead of O) works in an analogous fashion. (It is however slightly more complicated: the straight pieces must be "decomposed" in a similar way as in the proof of Theorem 4.3.25. . . .) □

In the following examples, the well-known Gaußian integral

$$\int_{-\infty}^{\infty} e^{-\frac{s^2}{2}} \mathbf{d}s = \sqrt{2\pi}. \tag{4.43}$$

appears again and again.

EXAMPLE 4.3.34 (Unary-binary trees). *We consider the species $\mathcal{T}$ of (unlabelled non-empty) planar unary–binary trees (i.e., each internal node has either one or two "descendants"). The following figure illustrates schematically the decomposition of this species:*



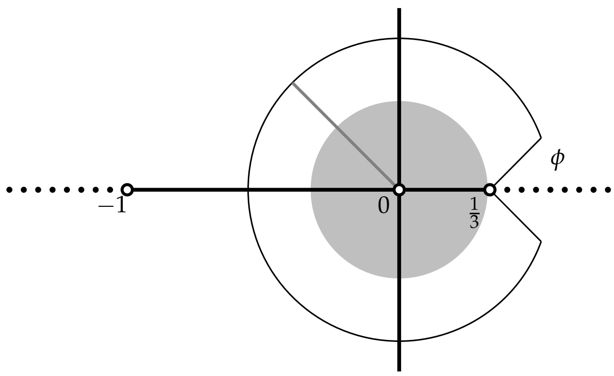
This decomposition can be directly translated into the following functional equation for the (ordinary) generating function $T(z)$:

$$T(z) = z\left(1 + T(z) + T^2(z)\right).$$

Its solution is

$$T(z) = \frac{1 - z - \sqrt{(1+z)(1-3z)}}{2z}.$$

The dominant singularity is visibly $z = \frac{1}{3}$, and the function is analytic in a Δ-domain, cf. the following figure:



With the substitution

$$1 - 3z = u \iff z = \frac{1-u}{3},$$

we obtain

$$\begin{aligned} T(u) &= \frac{1-\frac{1-u}{3}-\sqrt{\left(1+\frac{1-u}{3}\right)u}}{2\frac{1-u}{3}} \\ &= \frac{2+u-3\sqrt{\frac{1}{3}u(4-u)}}{2(1-u)} \quad \leftarrow 2+u=2(1-u)+3u \\ &= 1+\frac{3u-\sqrt{3}\sqrt{u}\,2\sqrt{\left(1-\frac{u}{4}\right)}}{2(1-u)} \quad \leftarrow 2 \text{ binom. th. "trivial"} \\ &= 1+\frac{3u-2\sqrt{3}\sqrt{u}\,(1+O(u))}{2}(1+O(u)) \\ &= 1-\sqrt{3}\sqrt{u}+\frac{3u}{2}+O\left(u^{3/2}\right). \end{aligned}$$

Altogether, this leads to the expansion (now again in z):

$$T(z) = 1-\sqrt{3}\sqrt{1-3z}+\frac{3}{2}(1-3z)+O\left((1-3z)^{3/2}\right).$$

By (4.31) *(with* $\alpha = -\frac{1}{2}$*), we have*

$$[\![s^n]\!]\,\sqrt{1-s} \sim \frac{n^{-\frac{3}{2}}}{\Gamma\left(-\frac{1}{2}\right)}\left(1+\sum_{k=1}^{\infty}\frac{e_k\left(-\frac{1}{2}\right)}{n^k}\right).$$

By (4.22) *in combination with* (4.20)*, we have*

$$\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi},$$

and by (4.41) *(with* $\alpha = -\frac{3}{2}$ *and* $\beta = 0$*), we see that the coefficient of the rest* $O\left((1-s)^{3/2}\right)$ *is of the order*

$$O\left(n^{-\frac{5}{2}}\right).$$

Altogether, we obtain

$$T_n = [\![z^n]\!]\,T(z) = 3^n\sqrt{\frac{3}{4\pi n^3}}+O\left(3^n n^{-\frac{5}{2}}\right)$$

for the coefficient T_n *of* z^n *in* $T(z)$*.*

EXAMPLE 4.3.35 (2-regular graphs). *The labelled species of* 2*-regular graphs can be interpreted as sets of cycles of length at least 3 (= cycles in the sense of graph theory), where we identify cycles if they differ only by a reflection. Hence, the corresponding* *exponential generating function equals*

$$\begin{aligned} f(z) &= \exp\left(\frac{1}{2}\left(\frac{z^3}{3}+\frac{z^4}{4}+\cdots\right)\right)\\ &= \exp\left(\frac{1}{2}\log\frac{1}{1-z}-\frac{z}{2}-\frac{z^2}{4}\right)\\ &= \frac{e^{-\frac{z}{2}-\frac{z^2}{4}}}{\sqrt{1-z}} \end{aligned}$$

We do the substitution $u = 1 - z$*:*

$$\begin{aligned} f(u) &= \frac{e^{-\frac{1-u}{2}-\frac{1-2u+u^2}{4}}}{\sqrt{u}} = \frac{e^{-\frac{3}{4}+u-\frac{u^2}{4}}}{\sqrt{u}}\\ &= \frac{e^{-\frac{3}{4}}}{\sqrt{u}}\left(1+u+O\left(u^2\right)\right)\\ &= e^{-\frac{3}{4}}(1-z)^{-\frac{1}{2}}+e^{-\frac{3}{4}}(1-z)^{\frac{1}{2}}+O\left((1-z)^{\frac{3}{2}}\right). \end{aligned}$$

We basically proceed as in the previous example and apply Theorem 4.3.25 to the standard functions

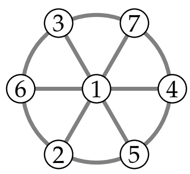
- $(1-z)^{-1/2}$*, where we use* two *terms of the asymptotic expansion in* (4.31)*,*
- and *to* $(1-z)^{1/2}$*, where we use only the dominant term in* (4.31).

This gives

$$f_n = e^{-\frac{3}{4}}\frac{1}{\sqrt{\pi n}} - e^{-\frac{3}{4}}\frac{5}{8\sqrt{\pi n^3}} + O\left(n^{-\frac{5}{2}}\right).$$

Hence, the number of 2*-regular graphs is asymptotically equal to* $n! \cdot f_n$*.*

EXAMPLE 4.3.36 (Children's rounds). *The labelled species "children's rounds" is a partition of the set of children into several "individual rounds", in each of which one child sits in the centre; see the following illustration:*



There are exactly $\frac{n!}{n-1}$ *different such "rounds" with n children (since cyclic permutations of the children that sit around in a circle are identified); the exponential generating function of such "single rounds" is therefore*

$$z^2 + \frac{z^3}{2} + \frac{z^4}{3} + \cdots,$$

and "children's rounds" are simply sets of such "single rounds". Consequently, the corresponding generating function equals

$$\begin{aligned} f(z) &= \exp\left(z^2 + \frac{z^3}{2} + \frac{z^4}{3} + \cdots\right) \\ &= \exp\left(z \log \frac{1}{1-z}\right) \\ &= (1-z)^{-z} = \frac{1}{1-z}(1-z)^{-z+1} \\ &= \frac{1}{1-z} \exp\left((1-z)\log(1-z)\right). \end{aligned}$$

This function is analytic in $\mathbb{C}\backslash\mathbb{R}_{\geqslant 1}$*, and it has a singularity at* $\zeta = 1$*. We expand the exponential function about* $z_0 = 0$*:*

$$\begin{aligned} f(z) &= \frac{1 + \log(1-z)(1-z) + \frac{(\log(1-z)(1-z))^2}{2!} + \cdots}{1-z} \\ &= \frac{1}{1-z} + \log(1-z) + \sum_{n\geqslant 2} \frac{(\log(1-z))^n (1-z)^{n-1}}{n!} \\ &= \frac{1}{1-z} - \log\frac{1}{1-z} + \frac{1}{2}\cdot(1-z)\cdot\log^2\left(\frac{1}{1-z}\right) \\ &\quad + O\left((1-z)^2 \cdot \log^3\left(\frac{1}{1-z}\right)\right) \text{ (see [3, p. 397]).} \end{aligned}$$

The part $\frac{1}{1-z} - \log\frac{1}{1-z}$ *contributes* $1 - \frac{1}{n}$ *to the coefficient. After application of Remark 4.3.30, we obtain an estimation for the rest from* (4.41) *(with* $\alpha = -1$ *and* $\beta = 2$*, respectively* $\alpha = -2$ *and* $\beta = 3$*). Thus, altogether we have*

$$\begin{aligned} f_n &= 1 - \frac{1}{n} - n^{-2}\cdot(\log n)\cdot\left(1 + O\left(\frac{1}{\log n}\right)\right) + O\left(n^{-3}\cdot\log^2 n\right) \\ &= 1 - \frac{1}{n} - n^{-2}\cdot\log n + O\left(n^{-2}\right). \end{aligned}$$

4.4. Saddle point method

Singularity analysis turned out to be extremely useful for the determination of asymptotics. However, what do we do if the generating function

- either has no singularities at all, such as e.g. e^z (sets), $e^{z+\frac{z^2}{2}}$ (involutions) or e^{e^z-1} (Bell numbers),
- or cannot be approximated by our "standard functions" about the dominant singularities, such as e.g. $e^{\frac{z}{1-z}}$ (sets of non-empty permutations)?

4.4.1. Heuristics: contour through saddle point. In such cases, the following heuristic argument may help. Starting point is again Cauchy's integral formula:

$$[\![z^n]\!]\, f(z) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z^{n+1}}\mathbf{d}z. \tag{4.44}$$

The idea is to deform the contour C around 0 with winding number 1 inside the domain G where the function f is analytic *such that*

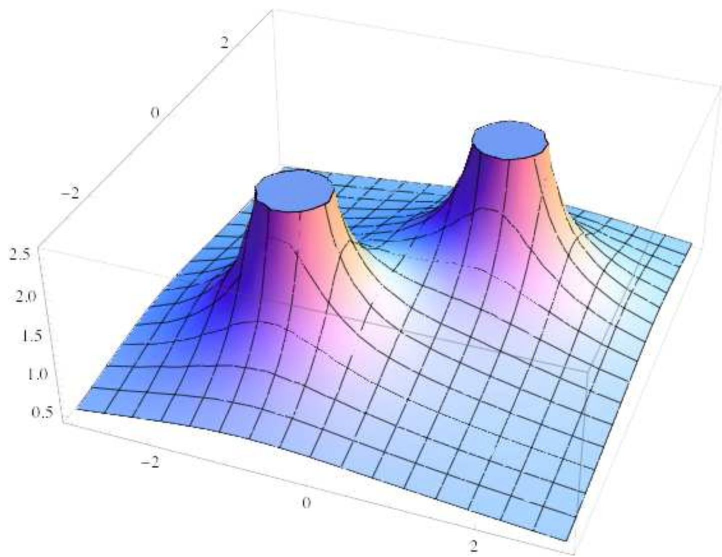
- the integrand $\frac{f(z)}{z^{n+1}}$ is so small on a part C' of the contour C such that the part $\frac{1}{2\pi \mathbb{i}} \int_{C'} \frac{f(z)}{z^{n+1}} \mathbf{d}z$ of the integral is negligible in the asymptotic considerations,
- so that the asymptotically relevant part of the integral is the one coming from the "rest" C'' of the contour C (where moreover this rest should be easily computable, at least asymptotically).

This sounds somewhat unclear and vague, so that one cannot imagine how this is going to work in practice. As a matter of fact, this vague idea can be successfully carried through in so many cases such that it acquired a name: *saddle point method*.[8]

So, to begin with, we have to estimate the modulus of the integrand in (4.44),

$$\left| \frac{f(z)}{z^{n+1}} \right|.$$

This real valued function appears graphically as a "mountain landscape" over the complex plane; according to the above vague requirement, the path described by the contour C should "almost always be close to the plane, and only once lead through a mountain pass". (See the following figure.)



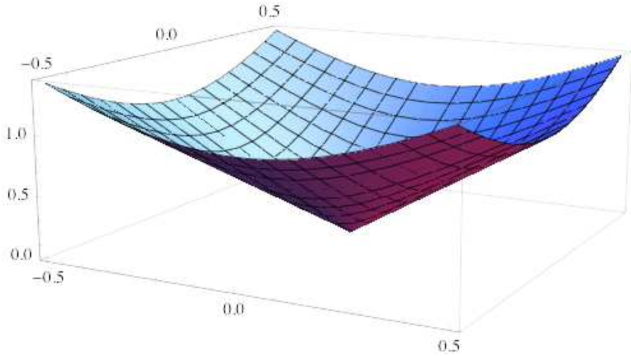
So, let us consider the function $g(z) := \frac{f(z)}{z^{n+1}}$. By Taylor's theorem, we have

$$g(z) = g(z_0) + (z - z_0)\, g'(z_o) + \frac{1}{2}(z - z_0)^2 g''(z_0) + O\left((z - z_0)^3\right) \text{ for } z \to z_0.$$

We write $|g|$ in a neighbourhood of z_0:

Case 1: $g(z_0) = 0$. In this case, the function $|g|$ has a global minimum in z_0, see the following figure:

[8]Recall that a trick that works at least three times becomes a method ...



Case 2: $g(z_0) \neq 0$ and $g'(z_0) \neq 0$. Then we write $\frac{g'(z_0)}{g(z_0)} = \lambda \mathbb{e}^{\mathbb{i}\phi}$ in polar coordinates, and we see that

$$|g(z)| = |g(z_0)| \cdot \left|1 + r \cdot \lambda \cdot \mathbb{e}^{\mathbb{i}(\phi+\theta)} + O\left(r^2\right)\right|.$$

For small r along a line segment through z_0 with angle θ, we have in first approximation:

- for $\theta = -\phi \pm \pi$, the function $|g(z)|$ grows, from a minimum $\sim |g(z_0)|(1 - r\lambda)$ to a maximum $\sim |g(z_0)|(1 + r\lambda)$;
- for $\theta = -\phi \pm \frac{\pi}{2}$, we have $|1 \pm r\lambda\mathbb{i}| = \sqrt{1 + r^2\lambda^2} = 1 + O\left(r^2\right) = 1 + o(r)$, so that the function $g(z)$ is "essentially constant".

Case 3: $g(z_0) \neq 0$ and $g'(z_0) = 0$. Let $\frac{1}{2}\frac{g''(z_0)}{g(z_0)} = \lambda \mathbb{e}^{\mathbb{i}\phi}$ in polar coordinates. Then

$$|g(z)| = |g(z_0)| \left|1 + \lambda r^2 \mathbb{e}^{\mathbb{i}(2\theta+\phi)} + O\left(r^3\right)\right|.$$

For angle θ equal to

- $\theta = -\frac{\phi}{2}$, we have $\mathbb{e}^{\mathbb{i}(2\theta+\phi)} = 1$; the function $|g(z)|$ behaves along this direction "essentially" like a $\cup$-shaped parabola;
- $\theta = -\frac{\phi}{2} + \frac{\pi}{2}$, we have $\mathbb{e}^{\mathbb{i}(2\theta+\phi)} = -1$; the function $|g(z)|$ behaves along this direction "essentially" like a $\cap$-shaped parabola.

In other words, the point z_0 is a *saddle point* of $|g(z)|$. The "strategy" of the *saddle point method* consists is to deform the the contour Γ for Cauchy's integral formula such that

- it passes through the saddle point
- in the direction of "steepest descent"[9] (i.e., in direction $-\frac{\phi}{2} + \frac{\pi}{2}$, with the above notation in polar coordinates);

hoping that the contour integral can be split into two parts:

- into one (small) part about the saddle point, which provides the "dominant" contribution to the integral;
- and in a remaining part, which is "negligible".

[9] This explains that the saddle point method is alternatively also called *"method of steepest descent"*.

REMARK 4.4.1 (Fundamental theorem of algebra). *Each polynomial $p(z)$ of degree $n > 0$ over the field $\mathbb{C}$ has a root in $\mathbb{C}$. For, suppose not, then $f(z) := \frac{1}{p(z)}$ would be bounded,*

$$|f(z)| \leqslant C \text{ for all } z \in \mathbb{C},$$

and then, by Cauchy's integral formula, for all Taylor coefficients of f we would get

$$|[\![z^n]\!]\, f(z)| \leqslant \frac{C}{2\pi}\frac{1}{R^n} \text{ for all } R > 0,$$

that is, $|[\![z^n]\!]\, f(z)| = 0$ for $n > 0$, a contradiction.

Let us see how the saddle point method works in a familiar example.

EXAMPLE 4.4.2 (Stirling's formula). *Let $f(z) = \mathrm{e}^z = \sum_{n\geqslant 0} \frac{1}{n!} z^n$. We want to find a "good" contour of integration Γ for Cauchy's integral formula*

$$\frac{1}{n!} = \frac{1}{2\pi\mathrm{i}} \int_\Gamma \frac{\mathrm{e}^z}{z^{n+1}} \mathbf{d}z.$$

So we look for a saddle point: the equation

$$\left(\frac{\mathrm{e}^z}{z^{n+1}}\right)' = 0 \iff \frac{\mathrm{e}^z}{z^{n+1}} - (n+1)\frac{\mathrm{e}^z}{z^{n+2}} = 0$$

has clearly the solution $z = n+1$ — however, we do not need to stick strictly to our "saddle point heuristic", we may equally well try $z = n$ ($n \sim n+1$ for $n \to \infty$; n is so-to-speak an "asymptotic saddle point"). For $g(z) := \left(\frac{\mathrm{e}^z}{z^{n+1}}\right)$, the values $g(n)$ and $g''(n)$ are real, *thus the angle ϕ (see the preceding considerations: "Case 3") is 0, and as contour that cuts the real axis in the point n orthogonally we simply choose the circle with radius $r = n$:*

$$z(\theta) = n \cdot \mathrm{e}^{\mathrm{i}\theta} \text{ with derivative } z'(\theta) = \mathrm{i} \cdot n \cdot \mathrm{e}^{\mathrm{i}\theta} = \mathrm{i} \cdot z. \tag{4.45}$$

We consider the contour integral

$$\begin{aligned} \frac{1}{2\pi\mathrm{i}} \int_0^{2\pi} \frac{\mathrm{e}^{z(\theta)}}{(z(\theta))^{n+1}} \cdot z'(\theta)\, \mathbf{d}\theta &= \frac{1}{2\pi\mathrm{i}} \int_0^{2\pi} \frac{\mathrm{e}^{n\cdot\mathrm{e}^{\mathrm{i}\theta}}}{n^{n+1}\mathrm{e}^{\mathrm{i}(n+1)\theta}} \cdot \mathrm{i} \cdot n \cdot \mathrm{e}^{\mathrm{i}\theta} \mathbf{d}\theta \\ &= \frac{n^{-n}}{2\pi} \int_0^{2\pi} \mathrm{e}^{n\cdot\mathrm{e}^{\mathrm{i}\theta} - \mathrm{i}\cdot n\theta} \mathbf{d}\theta \\ &= \left(\frac{n}{\mathrm{e}}\right)^{-n} \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{n\left(\mathrm{e}^{\mathrm{i}\theta} - 1 - \mathrm{i}\theta\right)} \mathbf{d}\theta. \end{aligned}$$

We split this integral now "skilfully":

$$\frac{1}{2\pi}\int_0^{2\pi} = \underbrace{\frac{1}{2\pi}\int_{-\delta}^{\delta}}_{=:I_n^0} + \underbrace{\frac{1}{2\pi}\int_{\delta}^{2\pi-\delta}}_{=:I_n^1},$$

where we choose $\delta = n^{-2/5}$.
We first consider I_n^1: $\left|\mathrm{e}^{n\cdot\mathrm{e}^{\mathrm{i}\theta}}\right| = \mathrm{e}^{\Re\left(n\cdot\mathrm{e}^{\mathrm{i}\theta}\right)} = \mathrm{e}^{n\cos\theta}$ is

- decreasing *on the interval (δ, π), the maximum is attained at δ,*
- increasing *on the interval $(\pi, 2\pi - \delta)$, the maximum is attained at $2\pi - \delta$.*

Consequently, we may bound the integrand I_n^1 from above as follows:

$$\begin{aligned}
\left|\mathrm{e}^{n\left(\mathrm{e}^{\mathrm{i}\theta}-1-\mathrm{i}\theta\right)}\right| &= \mathrm{e}^{\Re\left(n\left(\mathrm{e}^{\mathrm{i}\theta}-1-\mathrm{i}\theta\right)\right)} \\
&= \mathrm{e}^{n(\cos\theta-1)} \\
&\leqslant \mathrm{e}^{n(\cos\delta-1)} \quad \leftarrow \cos z=\sum_m z^{2m}(-1)^m\frac{1}{(2m)!} \\
&\sim \mathrm{e}^{-n\frac{\delta^2}{2}} = \mathrm{e}^{-\frac{\sqrt[5]{n}}{2}}.
\end{aligned}$$

Altogether we obtain $I_n^1 = O\left(\exp\left(-\frac{1}{2}\cdot n^{\frac{1}{5}}\right)\right)$.

Now we consider I_n^0:

$$\begin{aligned}
\int_{-\delta}^{\delta} \mathrm{e}^{n\left(\mathrm{e}^{\mathrm{i}\theta}-1-\mathrm{i}\theta\right)}\mathbf{d}\theta &= \int_{-\delta}^{\delta} \mathrm{e}^{n\left(-\frac{\theta^2}{2}+O\left(\theta^3\right)\right)}\mathbf{d}\theta \quad \leftarrow \mathrm{e}^z=\sum_n \frac{z^n}{n!} \\
&= \int_{-\delta}^{\delta} \mathrm{e}^{-n\frac{\theta^2}{2}+O\left(n^{-\frac{1}{5}}\right)}\mathbf{d}\theta \quad \leftarrow \left|\theta^3\right|\leqslant\delta^3=n^{-\frac{6}{5}} \\
&= \int_{-\delta}^{\delta} \mathrm{e}^{-n\frac{\theta^2}{2}}\cdot\left(1+O\left(n^{-\frac{1}{5}}\right)\right)\mathbf{d}\theta \\
&= \left(1+O\left(n^{-\frac{1}{5}}\right)\right)\int_{-\delta}^{\delta} \mathrm{e}^{-n\frac{\theta^2}{2}}\cdot\mathbf{d}\theta.
\end{aligned}$$

We make small a computation on the side:

$$\begin{aligned}
\int_{\delta}^{\infty} \mathrm{e}^{-n\cdot\frac{\theta^2}{2}}\mathbf{d}\theta &= \int_{0}^{\infty} \mathrm{e}^{-\frac{n}{2}\cdot\left(\delta^2+2\delta t+t^2\right)}\mathbf{d}t \quad \leftarrow \theta=t+\delta \\
&\leqslant \mathrm{e}^{-\frac{n\delta^2}{2}}\int_{0}^{\infty} \mathrm{e}^{-n\delta t}\mathbf{d}t \\
&= \mathrm{e}^{-\frac{n\delta^2}{2}}\left(-\frac{1}{n\delta}\mathrm{e}^{-n\delta t}\right)\Big|_0^{\infty} \\
&= \frac{1}{n\delta}\mathrm{e}^{-\frac{n\delta^2}{2}} = O\left(\mathrm{e}^{-n^{\frac{1}{5}}}\right). \quad \leftarrow \delta=n^{-\frac{2}{5}}
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\int_{-\delta}^{\delta} \mathrm{e}^{-n\cdot\frac{\theta^2}{2}}\mathbf{d}\theta &= \int_{-\infty}^{\infty} \mathrm{e}^{-n\cdot\frac{\theta^2}{2}}\mathbf{d}\theta + O\left(\mathrm{e}^{-n^{1/5}}\right) \quad \leftarrow s=\theta\sqrt{\tfrac{n}{2}} \\
&= \sqrt{\frac{2}{n}}\underbrace{\int_{-\infty}^{\infty} \mathrm{e}^{-s^2}\mathbf{d}s}_{\sqrt{\pi}} + O\left(\mathrm{e}^{-n^{1/5}}\right) \\
&= \sqrt{\frac{2\pi}{n}} + O\left(n^{-\frac{1}{5}}\right). \quad \leftarrow O\left(\mathrm{e}^{-x}\right)=O\left(\tfrac{1}{x}\right)
\end{aligned}$$

If we put all this together, then we obtain the following variant of Stirling's formula:

$$\frac{1}{n!} = \left(\frac{n}{\mathrm{e}}\right)^{-n}\frac{1}{\sqrt{2\pi n}}\left(1+O\left(n^{-1/5}\right)\right).$$

EXAMPLE 4.4.3. *Let u_n be the number of* ordered *set partitions; these are like ordinary (set) partitions except that the* order *of the elements in each block is relevant. In the language of species, these are*

$$\texttt{sets}\,(\texttt{permutationens}_1)\,,$$

from which we directly read off the (exponential) generating function:

$$\sum_{n\geqslant 0} u_n \frac{z^n}{n!} = \exp\left(\frac{z}{1-z}\right).$$

For Cauchy's integral formula, we have to deal with the integrand

$$f(z) = \frac{\exp\left(\frac{z}{1-z}\right)}{z^{n+1}}.$$

Its derivative is

$$f'(z) = \frac{\exp\left(\frac{z}{1-z}\right)}{z^{n+2}} \cdot \left(\frac{z}{(1-z)^2} - (n+1)\right). \leftarrow \left(\tfrac{z}{1-z}\right)' = \tfrac{1}{(1-z)} + \tfrac{z}{(1-z)^2} = \tfrac{1}{(1-z)^2}$$

If we equate $f'(z)$ to zero, we are led to the equation

$$z^2(n+1) - z(2(n+1)+1) + n + 1 = 0.$$

Its roots are

$$\begin{aligned} z_{1,2} &= \frac{2n+3 \pm \sqrt{4n^2+12n+9-4n^2-8n-4}}{2(n+1)} \leftarrow z_{1,2} = \tfrac{-b\pm\sqrt{b^2-4ac}}{2a} \\ &= \frac{2n+3\pm\sqrt{4n+5}}{2(n+1)}. \end{aligned}$$

We choose the smaller root

$$z_1 = 1 + \frac{1}{2(n+1)} - \frac{\sqrt{4(n+1)+1}}{2(n+1)},$$

and, from the series expansion

$$\sqrt{1+z} = 1 + \frac{z}{2} - \frac{z^2}{8} + \frac{z^3}{16} - \frac{5z^4}{128} + O\left(z^5\right) \quad (|z|<1)$$

of the square root about $z=0$, we obtain

$$\frac{\sqrt{4z+1}}{2z} = \frac{\sqrt{1+\frac{1}{4z}}}{\sqrt{z}} = \frac{1}{\sqrt{z}}\left(1 + O\left((4z)^{-1}\right)\right); (z\to\infty),$$

and, for $z = (n+1)$, we get

$$z_1 = 1 + \frac{1}{2(n+1)} - \frac{1}{\sqrt{n+1}}\left(1 + O\left(\frac{1}{4(n+1)}\right)\right) \quad (n\to\infty).$$

By the series expansion

$$\frac{1}{\sqrt{1+z}} = 1 - \frac{z}{2} + \frac{3z^2}{8} - \frac{5z^3}{16} + \frac{35z^4}{128} + O\left(z^5\right) \quad (|z|<1)$$

about $z = 0$, we now obtain

$$\frac{1}{\sqrt{n+1}} = \frac{1}{\sqrt{n}}\frac{1}{\sqrt{1+\frac{1}{n}}} = \frac{1}{\sqrt{n}}\left(1+O\left(\frac{1}{n}\right)\right)(n\to\infty).$$

Altogether, we have

$$z_1 = 1 - \frac{1}{\sqrt{n}} + \frac{1}{2(n+1)} + O\left(n^{-3/2}\right).$$

Again, we approach the situation "generously". We choose the "asymptotic saddle point"

$$r = 1 - \frac{1}{\sqrt{n}}$$

on the real axis, which is at the same time the radius of the circle about 0 *that we choose as integration contour (see the preceding example, in particular* (4.45)*). Then Cauchy's integral formula yields*

$$\frac{u_n}{n!} = \frac{1}{2\pi}\int_0^{2\pi}\exp\left(\frac{r\mathbb{e}^{\mathbb{i}\theta}}{1-r\mathbb{e}^{\mathbb{i}\theta}} - n\mathbb{i}\theta\right)r^{-n}\mathbf{d}\theta.$$

As in the preceding example, we split this integral "skilfully":

$$\int_0^{2\pi} = \int_{-\delta}^{2\pi-\delta} = \underbrace{\int_{-\delta}^{\delta}}_{=:I_n^0} + \underbrace{\int_{\delta}^{2\pi-\delta}}_{=:I_n^1},$$

where we choose $\delta = n^{-7/10}$.

Since $r = 1 - \frac{1}{\sqrt{n}}$, we have

$$\begin{aligned}\frac{r\mathbb{e}^{\mathbb{i}\theta}}{1-r\mathbb{e}^{\mathbb{i}\theta}} &= r\cdot\mathbb{e}^{\mathbb{i}\theta} + r^2\cdot\mathbb{e}^{2\mathbb{i}\theta} + r^3\cdot\mathbb{e}^{3\mathbb{i}\theta} + \cdots \quad \leftarrow \mathbb{e}^{\mathbb{i}\theta} = 1 + \mathbb{i}\theta - \tfrac{\theta^2}{2} + O\left(\theta^3\right)\\ &= 1\left(\underbrace{r + r^2 + r^3 + \cdots}_{\frac{r}{1-r} = \sqrt{n}-1}\right) + \mathbb{i}\theta\left(\underbrace{r + 2r^2 + 3r^3 + \cdots}_{\frac{r}{(1-r)^2} = n - \sqrt{n}}\right)\\ &\quad - \frac{\theta^2}{2}\left(\underbrace{r + 2^2r^2 + 3^2r^3 + \cdots}_{\frac{r(1+r)}{(1-r)^3} = \sqrt{n} - 3n + 2n^{3/2}}\right)\\ &\quad + O\left(\theta^3\right)\left(\underbrace{r + 2^3r^2 + 3^3r^3 + \cdots}_{\frac{r\left(1+4r+r^2\right)}{(1-r)^4} = O\left(\frac{1}{(1-r)^4}\right) = O\left(n^2\right)}\right)(n\to\infty).\end{aligned}$$

For the part I_n^0, we have $|\theta|^3 \leqslant \delta^3 = n^{-21/10}$, hence $O(\theta^3)\,O(n^2) = O\left(n^{-1/10}\right)$. Consequently, we get

$$\begin{aligned}
I_n^0 r^n &= \frac{1}{2\pi}\int_{-\delta}^{\delta} e^{\frac{r}{1-r}+i\theta\left(\frac{r}{(1-r)^2}-n\right)-\theta^2\frac{r(1+r)}{2(1-r)^3}+O\left(\frac{\theta^3}{(1-r)^4}\right)}\,\mathbf{d}\theta \\
&= \frac{1}{2\pi}\int_{-\delta}^{\delta} \exp\left(\sqrt{n}-1-i\theta\sqrt{n}-\theta^2\left(n^{3/2}+O(n)\right)+O\left(n^{-1/10}\right)\right)\mathbf{d}\theta \\
&= \frac{1}{2\pi}\int_{-\delta}^{\delta} \exp\left(\sqrt{n}-1-\theta^2 n^{3/2}+O\left(n^{-1/10}\right)\right)\mathbf{d}\theta,
\end{aligned}$$

where the last simplification follows from

$$\theta\sqrt{n} = O\left(n^{-2/10}\right),\ \theta^2 O(n) = O\left(n^{-4/10}\right).$$

As in the preceding example, we have

$$\begin{aligned}
\int_{\delta}^{\infty} e^{-\theta^2 n^{3/2}}\,\mathbf{d}\theta &= \int_0^{\infty} e^{-\left(\delta^2+2\delta t+t^2\right)n^{3/2}}\,\mathbf{d}t \quad {\scriptstyle\leftarrow \theta=\delta+t} \\
&\leqslant e^{-n^{3/2}\delta^2}\int_0^{\infty} e^{-2\delta t n^{3/2}}\,\mathbf{d}t \\
&= e^{-n^{3/2}\delta^2}\left(\frac{1}{-2\delta n^{3/2}}e^{-2\delta t n^{3/2}}\right)\Bigg|_0^{\infty} \\
&= \frac{e^{-n^{3/2}\delta^2}}{2\delta n^{3/2}} = O\left(e^{-n^{1/10}}\right), \quad {\scriptstyle\leftarrow \delta=n^{-7/10}},
\end{aligned}$$

and thus

$$I_n^0 r^n = \frac{1}{2\pi}e^{\sqrt{n}-1}\left(\underbrace{\int_{-\infty}^{\infty}\exp\left(-\theta^2 n^{3/2}\right)\mathbf{d}\theta}_{\sqrt{\pi}n^{-3/4}}\right)\left(1+O\left(n^{-1/10}\right)\right).$$

Together with the expansion

$$\begin{aligned}
r^{-n} &= \exp\left(-n\log\left(1-\frac{1}{\sqrt{n}}\right)\right) \\
&= \exp\left(-n\left(-\frac{1}{\sqrt{n}}-\frac{1}{2n}+O\left(n^{-3/2}\right)\right)\right) \\
&= \exp\left(\sqrt{n}+\frac{1}{2}+O\left(n^{-1/2}\right)\right) \\
&= e^{\sqrt{n}+\frac{1}{2}}e^{O\left(n^{-1/2}\right)} = e^{\sqrt{n}+\frac{1}{2}}\left(1+O\left(n^{-1/2}\right)\right),
\end{aligned}$$

we eventually arrive at

$$I_n^0 = \frac{e^{2\sqrt{n}-\frac{1}{2}}}{2\sqrt{\pi}n^{3/4}}\left(1+O\left(n^{-1/10}\right)\right).$$

For the remaining part of the integral, which we denoted by I_n^1, we collect the following facts: we have (of course) $e^{\frac{z}{1-z}} = e^{\frac{1}{1-z}-1}$ and (equally obviously) $\frac{1}{1-z} = \frac{1-\bar{z}}{(1-z)(1-\bar{z})}$; from this, it follows immediately that

$$\left|e^{\frac{z}{1-z}}\right| = \frac{1}{e}\left|\exp\left(\frac{1-re^{-i\theta}}{1-2r\cos\theta+r^2}\right)\right| = \frac{1}{e}\exp\left(\frac{1-r\cos\theta}{1-2r\cos\theta+r^2}\right).$$

The derivative of the function $\frac{1-rz}{1-2rz+r^2}$ ($z \in [-1,1]$, $0 < r < 1$), with respect to z is $\frac{r(1-r^2)}{\left(1-2rz+r^2\right)^2} > 0$. Consequently, it is (strictly) monotone increasing. However, this implies that, for $z = re^{i\theta}$ and $\delta \leqslant \theta \leqslant 2\pi - \delta$, we have

$$\left|e^{\frac{z}{1-z}}\right| \leqslant e^{-1}\exp\left(\frac{1-r\cos\delta}{1-2r\cos\delta+r^2}\right),$$

and

$$\exp\left(\frac{1-r\cos n^{-7/10}}{1-2r\cos n^{-7/10}+r^2}\right) = \exp\left(\sqrt{n} - n^{1/10} + O(1)\right).$$

(This is seen — after the substitutions $r = 1 - \frac{1}{\sqrt{n}}$ and $n = z^{-10}$ — from the Laurent series expansion

$$\frac{\left(z^5-1\right)\cos\left(z^7\right)+1}{z^{10}-2z^5+2\left(z^5-1\right)\cos\left(z^7\right)+2} = \frac{1}{z^5} - \frac{1}{z} + O\left(z^3\right),$$

and the "backward substitution" $z = n^{-1/10}$.) Furthermore,

$$I_n^1 \leqslant \frac{2\pi}{2\pi} r^{-n} e^{\sqrt{n}-n^{1/10}} e^{O(1)} = O\left(e^{2\sqrt{n}-n^{1/10}}\right),$$

with the expansion of r^{-n} as before. This may not look very spectacular: however, asymptotically, I_n^1 is "much smaller" than I_n^0:

$$\left|\frac{I_n^1}{I_n^0}\right| \leqslant C\left|n^{3/4}e^{-n^{1/10}}\right|,$$

hence

$$I_n^1 = o\left(I_n^0\right)\ (n \to \infty).$$

REMARK 4.4.4 (Longer asymptotic expansion). *We may take more terms in the Taylor series expansion in θ — for e^z (see Example 4.4.2) we get for instance*

$$\begin{aligned}\int_{-\infty}^{\infty} e^{n\left(e^{i\theta}-1-i\theta\right)}\mathbf{d}\theta &= \int_{-\infty}^{\infty} e^{n\left(-\frac{\theta^2}{2}-i\frac{\theta^3}{6}+\frac{\theta^4}{24}+i\frac{\theta^5}{120}+O\left(\theta^6\right)\right)}\mathbf{d}\theta \quad \leftarrow e^z=1+z+O\left(z^2\right)! \\ &= \int_{-\infty}^{\infty} e^{-n\frac{\theta^2}{2}}\left(1 - ni\frac{\theta^3}{6} + n\frac{\theta^4}{24} + ni\frac{\theta^5}{120} + nO\left(\theta^6\right)\right)\mathbf{d}\theta \\ &= \frac{\sqrt{2\pi}}{\sqrt{n}} - \underbrace{niO(\dots)}_{=0,\ \text{see below.}} + \frac{n}{24}\int_{-\infty}^{\infty}\theta^4 e^{-\theta^2\frac{n}{2}}\mathbf{d}\theta \\ &\quad + O\left(n\int_{-\infty}^{\infty}\theta^6 e^{-\theta^2\frac{n}{2}}\mathbf{d}\theta\right).\end{aligned}$$

Here, the imaginary part is zero since θ^{2k-1} *is an* odd *function.*
With the integrals

$$\int_{-\infty}^{\infty} \theta^4 e^{-\theta^2 \frac{n}{2}} \mathbf{d}\theta = \frac{3\sqrt{2\pi}}{n^{5/2}}$$

$$\int_{-\infty}^{\infty} \theta^6 e^{-\theta^2 \frac{n}{2}} \mathbf{d}\theta = \frac{15\sqrt{2\pi}}{n^{7/2}},$$

we obtain

$$\frac{\sqrt{2\pi}}{\sqrt{n}} + \frac{1}{8}\frac{\sqrt{2\pi}}{n^{3/2}} + O\left(n^{-5/2}\right).$$

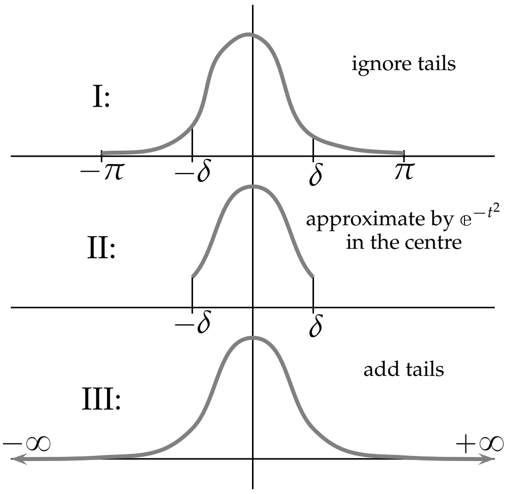
4.4.2. Hayman's theorem. Our typical situation is the following: let $f(z) = \sum_{n\geqslant 0} f_n z^n$ be analytic about $z = 0$; we want to express the coefficient f_n by using Cauchy's integral formula, where we integrate along a circle $z(\theta) = re^{i\theta}$:

$$f_n = \frac{r^{-n}}{2\pi} \int_{-\pi}^{\pi} \frac{f\left(re^{i\theta}\right)}{e^{ni\theta}} \mathbf{d}\theta.$$

The subsequent steps follow the general pattern:

- show that the "tails" of the integral (i.e., $\theta \notin (-\delta, \delta)$) are "asymptotically negligible in comparison to the main part";
- show that, for the main part of the integral (i.e., $\theta \in (-\delta, \delta)$), there exists an approximation of the integrand of the type $e^{-\theta^2}$;
- show that the tails of these "new, approximative integrals" (i.e., the integration domains $\theta < -\delta$ and $\theta > \delta$) are again asymptotically negligible;

see the following schematic illustration.



We now make situations in which this "general pattern" works more precise. For $r \geqslant 0$ in the circle of convergence with $f(r) > 0$ and small θ, we have the series expansion *in* θ

$$\log\left(f\left(re^{i\theta}\right)\right) = \log\left(f(r)\right) + \sum_{p=1}^{\infty} \alpha_p(r) \frac{(i\theta)^p}{p!}. \tag{4.46}$$

Of particular interest for our purposes are the first two terms, which, using the notation $h(r) := \log(f(r))$, we write

$$a(r) := \alpha_1(r) = rh'(r) = r\frac{f'(r)}{f(r)} \tag{4.47}$$

$$b(r) := \alpha_2(r) = r^2 h''(r) + rh'(r)$$
$$= r\frac{f'(r)}{f(r)} + r^2\left(\frac{f''(r)}{f(r)} - \left(\frac{f'(r)}{f(r)}\right)^2\right). \tag{4.48}$$

DEFINITION 4.4.5. *Let $f(z)$ be analytic about 0 with radius of convergence $0 < R \leqslant \infty$, and assume furthermore that $f(z) > 0$ for $z \in (R_0, R)$ for some $0 < R_0 < R$. The function $f(z)$ is called* Hayman-admissible, *if it satisfies the following three conditions:*

- **H1:** $\lim_{r\to R} a(r) = +\infty$ *and* $\lim_{r\to R} b(r) = +\infty$;
- **H2:** *there is a function $\delta(r)$ on (R_0, R) with $0 < \delta(r) < \pi$ such that we have* uniformly *for $|\theta| \leqslant \delta(r)$:*

$$f\left(r\mathrm{e}^{\mathrm{i}\theta}\right) = f(r)\,\mathrm{e}^{\mathrm{i}\theta a(r) - \left(\theta^2 b(r)\right)/2 + o(1)} \quad (r \to R);$$

- **H3:** *we have* uniformly *for $\delta(r) \leqslant |\theta| < \pi$:*

$$f\left(r\mathrm{e}^{\mathrm{i}\theta}\right) = o\left(\frac{f(r)}{\sqrt{b(r)}}\right) \quad (r \to R).$$

In practice, the first thing to do is to find the function $\delta(r)$: with the notation from (4.46), by **H2** and **H3** we should for $r \to R$ on the one hand have

$$\alpha_2(r)\,\delta(r)^2 \to \infty, \quad \leftarrow \text{(follows from the } \textit{combination} \text{ of } \mathbf{H2} \text{ and } \mathbf{H3}\text{)},$$

and on the other hand

$$\alpha_3(r)\,\delta(r)^3 \to 0. \quad \leftarrow \text{(follows from } \mathbf{H2} \text{ since the error term in the exponent is of the order } o(1)\text{)}$$

Altogether, this leads to the *necessary* condition

$$\lim_{r\to R} \frac{\alpha_3(r)^2}{\alpha_2(r)^3} = 0.$$

For the "saddle point Ansatz", we need

$$\alpha_2(r)^{-1/2} = o(\delta(r)) \text{ and } \delta(r) = o\left(\alpha_3(r)^{-1/3}\right), \tag{4.49}$$

that is, e.g.

$$\delta(r) = \alpha_2(r)^{-1/4} \cdot \alpha_3(r)^{-1/6}.$$

REMARK 4.4.6. *For example, e^z, $\mathrm{e}^{\mathrm{e}^z - 1}$, and $\mathrm{e}^{z + \frac{z^2}{2}}$ (with $R = \infty$) or $\mathrm{e}^{\frac{z}{1-z}}$ (with $R = 1$) are* Hayman-admissible *functions.* Not *Hayman-admissible is for example e^{z^2} (becomes "too large" about π and violates* **H3***).*

THEOREM 4.4.7 (Hayman). *Let $f(z) = \sum_{n \geqslant o} f_n z^n$ be Hayman-admissible. For $n \in \mathbb{N}$, let $\zeta = \zeta(n)$ be the largest (and, thus, unique) solution of the equation*

$$a(z) = z\frac{f'(z)}{f(z)} = n$$

in the interval (R_0, R). Then, for $n \to \infty$, we have

$$f_n \sim \frac{f(\zeta)}{\zeta^n \sqrt{2\pi b(\zeta)}},$$

where $b(z) = z^2 \frac{\mathbf{d}^2}{\mathbf{d}z^2} \log f(z) + z\frac{\mathbf{d}}{\mathbf{d}z} \log f(z)$.

PROOF. As always, we use Cauchy's integral formula in the "split" Version

$$f_n r^n = \underbrace{\frac{1}{2\pi}\int_{-\delta}^{\delta} f\left(r\mathbb{e}^{\mathbb{i}\theta}\right) e^{-n\mathbb{i}\theta}\mathbf{d}\theta}_{I_n^0} + \underbrace{\frac{1}{2\pi}\int_{\delta}^{2\pi-\delta} f\left(r\mathbb{e}^{\mathbb{i}\theta}\right) e^{-n\mathbb{i}\theta}\mathbf{d}\theta}_{I_n^1},$$

where we set $\delta = \delta(r)$. The part I_n^1 can be immediately approximated by assumption **H3**:

$$\left|I_n^1\right| \leqslant \frac{1}{2\pi} 2\pi \cdot o\left(\frac{f(r)}{\sqrt{b(r)}}\right).$$

For the main part I_n^0, we want to show:

$$I_n^0 = \frac{f(r)}{\sqrt{2\pi b(r)}}\left(\exp\left(-\frac{(a(r)-n)^2}{2b(r)}\right) + o(1)\right) \quad (r \to R) \tag{4.50}$$

(independent of n). To achieve this, we use assumption **H2**:

$$\begin{aligned} I_n^0 &= \frac{1}{2\pi}\int_{-\delta}^{\delta} f(r)\exp\left(\mathbb{i}(a(r)-n)\theta - \frac{1}{2}b(r)\theta^2 + o(1)\right)\mathbf{d}\theta \\ &= \frac{f(r)}{2\pi}\int_{-\delta}^{\delta} \mathbb{e}^{\mathbb{i}\theta(a(r)-n)-\frac{1}{2}b(r)\theta^2}(1+o(1))\,\mathbf{d}\theta \quad \leftarrow \mathbb{e}^{o(1)}=1+o(1) \\ &= \frac{f(r)}{2\pi}\left(\int_{-\delta}^{\delta} \mathbb{e}^{\mathbb{i}\theta(a(r)-n)-\frac{1}{2}b(r)\theta^2}\mathbf{d}\theta + o\left(\int_{-\delta}^{\delta}\mathbb{e}^{-\frac{1}{2}b(r)\theta^2}\mathbf{d}\theta\right)\right) \quad \leftarrow \left|\mathbb{e}^{\mathbb{i}\theta(a-n)}\right|=1 \\ &= \frac{f(r)}{2\pi}\left(\int_{-\delta}^{\delta} \mathbb{e}^{\mathbb{i}\theta(a(r)-n)-\frac{1}{2}b(r)\theta^2}\mathbf{d}\theta + o\left(b(r)^{-1/2}\right)\right). \quad \leftarrow \int_{-\infty}^{\infty}\mathbb{e}^{-\frac{t^2 b}{2}}\mathbf{d}t = \frac{\sqrt{2\pi}}{\sqrt{b}} \end{aligned}$$

The assumptions **H2** and **H3** *both* hold for $\theta = \delta(r)$. Together, they yield

$$\sqrt{b(r)}\mathbb{e}^{\mathbb{i}\delta(r)a(r)-\delta(r)^2 b(r)/2} \to 0 \implies \delta(r)^2 b(r) \to \infty \text{ for } r \to R.$$

We perform the substitution $t = \sqrt{\frac{b(r)}{2}}\theta$ (i.e., $\mathbf{d}\theta = \frac{\sqrt{2}}{\sqrt{b(r)}}\mathbf{d}t$), and we obtain for $r \to R$:

$$\begin{aligned}
I_n^0 &= \frac{f(r)}{2\pi} \cdot \left(\int_{-\delta\sqrt{\frac{b(r)}{2}}}^{\delta\sqrt{\frac{b(r)}{2}}} \sqrt{\frac{2}{b(r)}} e^{-t^2 + i t \sqrt{\frac{2}{b(r)}}(a(r)-n)} \mathbf{d}t + o\left(b(r)^{-1/2}\right) \right) \\
&= \frac{f(r)}{\pi\sqrt{2b(r)}} \left(\int_{-\delta\sqrt{\frac{b(r)}{2}}}^{\delta\sqrt{\frac{b(r)}{2}}} e^{-t^2 + i t \sqrt{\frac{2}{b(r)}}(a(r)-n)} \mathbf{d}t + o(1) \right) \quad \leftarrow \text{form a square} \\
&= \frac{f(r)}{\pi\sqrt{2b(r)}} \left(\int_{-\infty}^{\infty} e^{-\left(t - i\sqrt{\frac{1}{2b(r)}}(a(r)-n)\right)^2 - \frac{(a(r)-n)^2}{2b(r)}} \mathbf{d}t + o(1) \right) \quad \leftarrow \delta\sqrt{b(r)/2} \to \infty \\
&= \frac{f(r)}{\pi\sqrt{2b(r)}} \left(e^{-\frac{(a(r)-n)^2}{2b(r)}} \underbrace{\int_{-\infty - i\sqrt{\frac{1}{2b(r)}}(a(r)-n)}^{\infty - i\sqrt{\frac{1}{2b(r)}}(a(r)-n)} e^{-s^2} \mathbf{d}s}_{\sqrt{\pi}} + o(1) \right).
\end{aligned}$$

Thus, we established (4.50), and the claim of the theorem follows immediately: for, by **H1**, we have $\lim_{r\to R} a(r) = +\infty$, and hence

$$\lim_{n\to\infty} \zeta(n) = R;$$

we simply put $r = \zeta(n)$. ☐

The convenient feature of the concept of Hayman-admissibility is that there hold the following "closure properties": "Large classes of functions are Hayman-admissible".

THEOREM 4.4.8. *Let $f(z)$ and $g(z)$ be Hayman-admissible functions, and let $p(z)$ be a polynomial with real coefficients. Then:*

- *$f(z) \cdot g(z)$ and $e^{f(z)}$ are also Hayman-admissible;*
- *$f(z) + p(z)$ is Hayman-admissible; if the leading coefficient of $p(z)$ is positive, then also $f(z)\,p(z)$ and $p(f(z))$ are Hayman-admissible;*
- *If* almost all *Taylor coefficients of $e^{p(z)}$ are positive, then also $e^{p(z)}$ is Hayman-admissible.*

SKETCH. In the first place, one has to find the function $\delta(r)$: to this end, one uses (4.49). ☐

We give some examples in order to illustrate the concept of Hayman-admissibility.

EXAMPLE 4.4.9. *For $f(z) = e^z$, we have the expansion*

$$\begin{aligned}
f\left(re^{i\theta}\right) &= e^{r + ri\theta - r\frac{\theta^2}{2} + O\left(r\theta^3\right)} \\
&= f(r) \cdot e^{ri\theta - r\frac{\theta^2}{2} + O\left(r\theta^3\right)}.
\end{aligned}$$

*Hence, $a(r) = b(r) = r$, and thus **H1** is of course satisfied.*

For **H2** *we must choose* $\delta = \delta(r)$ *such that*

$$\lim_{r \to R} r\delta^3 = 0.$$

Thus, $\delta = o\left(r^{-1/3}\right)$*, and for* **H3** *to hold we must achieve*

$$\left|f\left(r\mathbb{e}^{\mathbb{i}\theta}\right)\right| = \left|\mathbb{e}^{r\mathbb{e}^{\mathbb{i}\theta}}\right| = \left|\mathbb{e}^{r\cos\theta}\right| \overset{!?!}{=} o\left(\frac{\mathbb{e}^r}{\sqrt{r}}\right).$$

Since $r\cos\theta = r - r\frac{\theta^2}{2} + O\left(r\theta^4\right)$ *and* $\sqrt{r} = \mathbb{e}^{\frac{1}{2}\log r}$*, these conditions will be satisfied for*

$$\log r = o\left(r\theta^2\right),$$

so, for example, for $\delta = r^{-2/5}$*.*

EXAMPLE 4.4.10. *For* $f(z) = \frac{1}{1-z}$*, we consider the Taylor series expansion of* $\log\frac{1}{1-r\mathbb{e}^{\mathbb{i}t}}$ *about* $t = 0$*:*

$$\log\frac{1}{1-r\mathbb{e}^{\mathbb{i}t}} = \log\frac{1}{1-r}\cdot t^0 + \frac{\mathbb{i}r}{1-r}\cdot t^1 + \frac{-r}{(1-r)^2}\cdot t^2 + O\left(\frac{r\cdot(r+1)}{(1-r)^3}\cdot t^3\right).$$

We obtain the expansion

$$f\left(r\mathbb{e}^{\mathbb{i}\theta}\right) = \frac{1}{1-r}\exp\left(\frac{\mathbb{i}r}{1-r}\cdot\theta - \frac{r}{(1-r)^2}\cdot\theta^2 + O\left(\frac{r\cdot(r+1)}{(1-r)^3}\cdot\theta^3\right)\right)$$

Clearly, here we have $a(r) = \frac{r}{1-r}$ *and* $b(r) = \frac{r}{(1-r)^2}$*, and* **H1** *is satisfied* ($R = 1$)*. In order to satisfy* **H2***, we should have* $\theta = o(1-r)$*. For* **H3** *to hold we must achieve*

$$\left|f\left(r\mathbb{e}^{\mathbb{i}\theta}\right)\right| = o\left(\frac{f(r)}{\sqrt{b(r)}}\right),$$

that is,

$$\left|\frac{1}{1-r\mathbb{e}^{\mathbb{i}\theta}}\right| \overset{!?!}{=} o\left(\frac{\frac{1}{1-r}}{\frac{\sqrt{r}}{1-r}}\right) = o\left(\frac{1}{\sqrt{r}}\right).$$

However,

$$\frac{1}{\left|1-r\mathbb{e}^{\mathbb{i}\theta}\right|} = \frac{1}{\sqrt{(1-r\cos\theta)^2 + (r\sin\theta)^2}}$$
$$= \frac{1}{\sqrt{1-2r\cos\theta+r^2}}, \quad \leftarrow \sin^2\theta + \cos^2\theta = 1$$

and for $r \to R = 1$ *we have* $\theta \to 0$ *(since* θ *must be of order* $\theta = o(1-r)$*); more precisely,*

$$\cos\theta = 1 - \frac{\theta^2}{2} + O\left(\theta^4\right) = 1 - o(1-r)^2.$$

Hence,

$$\frac{1}{\left|1-r\mathbb{e}^{\mathbb{i}\theta}\right|} = \frac{1}{\sqrt{(1-r)^2 + 2r\cdot o(1-r)}},$$

and for $r \to 1$ *this tends to* ∞*: consequently* $f(z)$ *is* not *Hayman-admissible.*

EXAMPLE 4.4.11. *For* $f(z) = e^{z^2}$*, we have*

$$\log f\left(re^{i\theta}\right) = r^2 e^{2i\theta} = r^2 + 2r^2 i\theta - 4r^2\frac{\theta^2}{2} + O\left(r^2\theta^3\right),$$

thus $a(r) = 2r^2$ *and* $b(r) = 4r^2$*. However,* **H3** *cannot be achieved. For, if we would assume that condition, then, for* $\delta \leqslant |\theta| \leqslant \pi$*, we would need*

$$\left|f\left(re^{i\theta}\right)\right| = e^{r^2\cos 2\theta} \overset{???}{=} o\left(\frac{e^{r^2}}{2r}\right) \quad (r \to R = \infty),$$

which is of course wrong for $\theta = \pi$*.*

EXAMPLE 4.4.12. *Let* f_n *be the number of permutations of* $[n]$*, which have only cycle lengths* 2 *and* 3 *in their disjoint cycle decomposition. In the language of species, we consider the composition* `sets` $\left(\texttt{cycles}_{2,3}\right)$*:*

$$f(z) = \sum_{n\geqslant 0}\frac{f_n}{n!}z^n = e^{\frac{z^2}{2}+\frac{z^3}{3}}.$$

We have

$$\begin{aligned}\log f\left(re^{i\theta}\right) &= \frac{1}{2}e^{2i\theta}r^2 + \frac{1}{3}e^{3i\theta}r^3\\ &= \left(\frac{r^2}{2}+\frac{r^3}{3}\right) + \underbrace{\left(r^2+r^3\right)}_{a(r)} i\theta - \underbrace{\left(2r^2+3r^3\right)}_{b(r)}\frac{\theta^2}{2} + O\left(\theta^3\right)\end{aligned}$$

This function is *Hayman-admissible, and hence*

$$f_n \sim n!\frac{e^{\frac{\zeta^2}{2}+\frac{\zeta^3}{3}}}{\zeta^n\sqrt{2\pi\zeta^2(3\zeta+2)}}, \quad \text{as } n \to \infty, \tag{4.51}$$

where $\zeta = \zeta(n)$ *is a solution of*

$$a(\zeta) = \zeta^2(1+\zeta) = n; \tag{4.52}$$

i.e., $\zeta(n) \sim n^{1/3}$*. However, since* ζ *appears in the exponential in* (4.51)*, this first approximation for* ζ *is not good enough for getting an asymptotic formula for* f_n *in terms of familiar functions; we need more terms in the asymptotic expansion for* ζ*. How this is done is explained below. It illustrates another important concept in asymptotic analysis, the so-called* "bootstrap method"*.*[10]

In (4.52)*, we make the Ansatz* $\zeta = n^{1/3} + \rho$*, with* ρ *of smaller asymptotic order than* $n^{1/3}$*. We expand the product and obtain*

$$n + n^{2/3}(1+3\rho) + n^{1/3}\left(2\rho + 3\rho^2\right) + \rho^2 + \rho^3 = n,$$

[10]The name comes from the idea to pull oneself out of one's boots, similar to Münchhausen's famous story where he pulls himself out of a swamp. This is somehow how the method works.

or, equivalently,

$$n^{2/3}\left(1+3\rho\right)+n^{1/3}\left(2\rho+3\rho^2\right)+\rho^2+\rho^3=0. \tag{4.53}$$

Now we suppose that ρ is of larger asymptotic order than a constant. In that case, the term $3n^{2/3}\rho$ is asymptotically larger than all other terms on the left-hand side of (4.53)*. This is a contradiction because then the left-hand side cannot be zero. Hence, ρ must be asymptotically constant. In that case, the asymptotically largest terms on the left-hand side of* (4.53) *are $n^{2/3}\left(1+3\rho\right)$. They must (asymptotically) "cancel" each other, by which we mean that ρ must be chosen such that they actually are of smaller order. We infer that $\rho=-\frac{1}{3}+o\left(1\right)$.*

We need to continue since, by our findings so far, the term ζ^3 (which appears in the exponential in (4.51)*) is still not approximated to an order that would tend to zero. So, we let $\zeta=n^{1/3}-\frac{1}{3}+\rho$, with ρ of smaller (asymptotic) order than a constant.*[11] *If we substitute this Ansatz in* (4.53)*, then we obtain*

$$3n^{2/3}\rho+3n^{1/3}\rho^2-\frac{1}{3}n^{1/3}+\rho^3-\frac{1}{3}\rho+\frac{2}{27}=0. \tag{4.54}$$

By arguments that are analogous to the previous ones, here we find that ρ must be of the order $O\left(n^{-1/3}\right)$. In that case, the asymptotically largest terms in (4.54) *are $3n^{2/3}\rho$ and $-\frac{1}{3}n^{1/3}$. They must (asymptotically) cancel each other, and therefore we see that $\rho=\frac{1}{9}n^{-1/3}+o\left(n^{-1/3}\right)$.*

This is still not good enough. Another round of "bootstrap" reveals that

$$\zeta=n^{1/3}-\frac{1}{3}+\frac{1}{9}n^{-1/3}-\frac{2}{81}n^{-2/3}+o\left(n^{-2/3}\right).$$

If this is substituted in (4.51)*, we obtain*

$$f_n\sim n!\frac{e^{\frac{1}{3}n+\frac{1}{2}n^{2/3}-\frac{1}{6}n^{1/3}+\frac{1}{18}}}{\sqrt{6\pi}n^{\frac{n}{3}+\frac{1}{2}}},\quad \text{as } n\to\infty.$$

Here we used that

$$\begin{aligned}\zeta^n&=n^{n/3}\left(\zeta n^{-1/3}\right)^n\\&=n^{n/3}\left(1-\frac{1}{3}n^{-1/3}+\frac{1}{9}n^{-2/3}-\frac{2}{81}n^{-1}+o\left(n^{-1}\right)\right)^n\\&=n^{n/3}\exp\left(n\log\left(1-\frac{1}{3}n^{-1/3}+\frac{1}{9}n^{-2/3}-\frac{2}{81}n^{-1}+o\left(n^{-1}\right)\right)\right)\\&=n^{n/3}\exp\left(-\tfrac{1}{3}n^{2/3}+\tfrac{1}{18}n^{1/3}+o\left(1\right)\right),\quad \text{as } n\to\infty.\end{aligned}$$

[11]We allow ourselves the liberty of using the same symbol again although the ρ here has nothing to do with the earlier ρ.

4.4.3. A saddle point theorem for large powers.

EXAMPLE 4.4.13. *If we consider power series F and f which are compositional inverses of each other, that is,*

$$F(f(z)) = z,$$

then the coefficients of $F(z)$ can be determined by the use of Lagrange inversion*:*

$$[\![z^n]\!]\, F(z) = \frac{1}{n}\left[\!\!\left[z^{-1}\right]\!\!\right] f(z)^{-n}.$$

We write f as

$$f(z) = \frac{z}{\varphi(z)}.$$

Then we obtain the following contour integral for the coefficients of $F(z)$:

$$[\![z^n]\!]\, F(z) = \frac{1}{n}\frac{1}{2\pi \mathbb{i}}\int_\Gamma \frac{\varphi^n(z)}{z^n}\mathbf{d}z.$$

EXAMPLE 4.4.14. *Let W_n denote the number of all paths in the lattice $\mathbb{Z}\times\mathbb{Z}$ that consist of right steps $(1,0)$, diagonal up-steps $(1,1)$, and diagonal down-steps $(1,-1)$, and which start at $(0,0)$ and end at $(n,0)$. We have*

$$W_n = \left[\!\!\left[z^0\right]\!\!\right]\left(\frac{1}{z}+1+z\right)^n = \frac{1}{2\pi\mathbb{i}}\int_\Gamma \frac{\left(1+z+z^2\right)^n}{z^{n+1}}\mathbf{d}z.$$

These two examples motivate to consider a more general problem: determine

$$\left[\!\!\left[z^N\right]\!\!\right] A(z)\,B(z)^n = \frac{1}{2\pi\mathbb{i}}\int_\Gamma \frac{A(z)\,B(z)^n}{z^{N+1}}\mathbf{d}z$$

for $n\to\infty$ *and* $N\to\infty$.

THEOREM 4.4.15. *Let $A(z) = \sum_{j\geqslant 0} a_j z^j$ and $B(z) = \sum_{j\geqslant 0} b_j z^j$ be two functions analytic about 0 and satisfying the following properties:*
L1: *A and B have non-negative coefficients and $B(0)\neq 0$.*
L2: *$B(z)$ is* aperiodic, *i.e.*, $\gcd\left(\{j : b_j > 0\}\right) = 1$.
L3: *The radius of convergence R of $B(z)$ is $<\infty$; the radius of convergence of $A(z)$ is $\geqslant R$.*
Let T be the (left) limit

$$T := \lim_{s\to R-} s\frac{B'(s)}{B(s)}.$$

Then, for $\lambda := \frac{N}{n}$, we have: if $\lambda\in(0,T)$, then

$$\left[\!\!\left[z^N\right]\!\!\right] A(z)\,B(z)^n = A(\zeta)\frac{B(\zeta)^n}{\zeta^{N+1}\sqrt{2\pi n\cdot\xi}}\left(1+o(1)\right), \tag{4.55}$$

where ζ is the largest solution of the equation

$$\alpha_1^B(\zeta) = \zeta\frac{B'(\zeta)}{B(\zeta)} = \lambda$$

(cf. (4.47)*), and*

$$\xi := \frac{\mathbf{d}^2}{\mathbf{d}s^2}\left(\log B(s) - \lambda\log s\right)\Big|_{s=\zeta}.$$

This holds uniformly *for λ in an arbitrary* compact *subinterval of* $(0, T)$.

For the proof we need the following simple lemma.

LEMMA 4.4.16. *Let* $f(z) = \sum_{n \geqslant 0} f_n z^n$ *be analytic about* 0 *for* $|z| < R$ *with* non-negative *coefficients, of which at least two are positive. If there is a* non-real z *with* $|z| < r$, *for which*

$$|f(z)| = f(|z|),$$

then

- *this z must be of the form*

$$z = r \cdot \mathbb{e}^{2\pi \mathbb{i} \frac{p}{q}},$$

  where $0 < p < q$ *and* $\gcd(p, q) = 1$,
- *and there must exist an* $a \in \mathbb{Z}$ *such that* $f_n \neq 0$ only *for* $n \equiv a \pmod{q}$.

PROOF OF THE LEMMA. Of course, we have

$$\left| \sum_{n \geqslant 0} f_n z^n \right| \leqslant \sum_{n \geqslant 0} |f_n z^n| = \sum_{n \geqslant 0} f_n |z^n| . \leftarrow f_n \geqslant 0$$

Equality for a $z = r\mathbb{e}^{\mathbb{i}\theta}$ can only hold if all powers $z^n = r^n \mathbb{e}^{n\mathbb{i}\theta}$ (interpreted as vectors in $\mathbb{R}^2$) point in the same direction. So let two arbitrary coefficients be non-zero, say $f_{n_1}, f_{n_2} \neq 0$. Then we must have

$$\mathbb{e}^{n_1 \mathbb{i}\theta} = \mathbb{e}^{n_2 \mathbb{i}\theta} \iff |n_1 - n_2| \, \theta = 2k\pi,$$

and thus $\theta = 2\pi \frac{k}{|n_1 - n_2|}$, where we may of course choose $k < |n_1 - n_2|$. Let $\frac{p}{q} = \frac{k}{|n_1 - n_2|}$ be the reduced fraction. Then we have

$$q \mid n_1 - n_2,$$

or, equivalently, $n_1 \equiv n_2 \pmod{q}$. □

PROOF OF THE THEOREM. For fixed r with $0 < r < R$, by assumption **L2** (aperiodicity of B) and Lemma 4.4.16, the function $\left|B\left(r\mathbb{e}^{\mathbb{i}\theta}\right)\right|$ attains its *unique* maximum at $\theta = 0$. Therefore there exists a (small) $\theta_1 \in (0, \pi)$ such that

- $\left|B\left(r\mathbb{e}^{\mathbb{i}\theta}\right)\right| < \left|B\left(r\mathbb{e}^{\mathbb{i}\theta_1}\right)\right|$ for $\theta \in [\theta_1, \pi]$;
- $\left|B\left(r\mathbb{e}^{\mathbb{i}\theta}\right)\right|$ is (strictly) monotone decreasing on $[0, \theta_1]$.

Consequently, if we integrate along the contour $z = \zeta\mathbb{e}^{\mathbb{i}\theta}$, then we obtain the desired coefficient as $J(\pi)$, where

$$J(\theta) := \frac{1}{\zeta^N 2\pi} \int_{-\theta}^{\theta} A\left(\zeta\mathbb{e}^{\mathbb{i}\theta}\right) B\left(\zeta\mathbb{e}^{\mathbb{i}\theta}\right)^n \mathbb{e}^{-N\mathbb{i}\theta} \mathbf{d}\theta,$$

and where for $n \to \infty$ the difference $J(\pi) - J(\theta_1)$ is exponentially small (because of the above estimation for $\left|B\left(r\mathbb{e}^{\mathbb{i}\theta}\right)\right|$). If we expand as in (4.46), we obtain for the integrand the approximation ($N = \lambda n$)

$$A(\zeta)\, B(\zeta)^n \cdot \exp(\mathbb{i}\theta \left(\alpha_1^A(\zeta) + n \left(\underbrace{\alpha_1^B(\zeta) - \lambda}_{=0 \text{ by assumpt.}} \right)\right) \\ - \frac{\theta^2}{2} \left(\alpha_2^A(\zeta) + n\alpha_2^B(\zeta) \right) + O\left(n \cdot \theta^3 \right)),$$

for $\alpha_3^A(\zeta)$ and $\alpha_3^B(\zeta)$ are $O(1)$ for $n \to \infty$ since ζ does not depend on n. Now let $\theta_0 := n^{-2/5}$. Then, for $\theta_0 \leqslant \theta \leqslant \theta_1$, we have (of course)

$$\mathbb{e}^{-\frac{\theta^2}{2}\left(\alpha_2^A(\zeta)+n\alpha_2^B(\zeta)\right)+O\left(n\cdot\theta^3\right)} = O\left(\mathbb{e}^{-n^{1/5}}\right) \cdot \mathbb{e}^{O\left(n^{-1/5}\right)} = O\left(\mathbb{e}^{-n^{1/5}}\right),$$

for $n \to \infty$. Hence also $J(\theta_1) - J(\theta_0) = O\left(\mathbb{e}^{-n^{1/5}}\right)$ for $n \to \infty$, that is, exponentially small. The desired coefficient is therefore asymptotically $J(\theta_0)$: for $\theta < \theta_0$ we have however (of course) $\frac{\theta^2}{2} \cdot \alpha_2^A(\zeta) = O\left(n^{-4/5}\right)$ $(n \to \infty)$ — that is, the desired coefficient is (after "appending" the asymptotically negligible integral ends) asymptotically equal to

$$\frac{A(\zeta)\, B(\zeta)^n}{2\pi\zeta^N} \int_{-\infty}^{\infty} \mathbb{e}^{-\frac{\theta^2}{2} n\alpha_2^B(\zeta)} \mathbf{d}\theta = \frac{A(\zeta)\, B(\zeta)^n}{\zeta^N \sqrt{2\pi n \cdot \alpha_2^B(\zeta)}}.$$

The coefficient α_2^B is the third Taylor coefficient in the expansion of $\log B\left(r\mathbb{e}^{\mathbb{i}\theta}\right)$, hence

$$\alpha_2^B(r) = -\left.\frac{\mathbf{d}^2}{\mathbf{d}\theta^2} \log B\left(r\mathbb{e}^{\mathbb{i}\theta}\right)\right|_{\theta=0} = r\frac{B'(r)}{B(r)} + r^2\frac{B''(r)}{B(r)} - r^2\left(\frac{B'(r)}{B(r)}\right)^2,$$

and we have

$$\xi = \left.\frac{\mathbf{d}^2}{\mathbf{d}s^2} (\log B(s) - \lambda \log s)\right|_{s=\zeta} = \frac{B''(\zeta)}{B(\zeta)} - \left(\frac{B'(\zeta)}{B(\zeta)}\right)^2 + \frac{\lambda}{\zeta^2} = \frac{\alpha_2^B(\zeta)}{\zeta^2}.$$

This establishes our claim. □

EXAMPLE 4.4.17. *Let $f(z)$ be given by the equation*

$$f(z) = z \cdot \mathbb{e}^{f(z)} \iff f(z)\, \mathbb{e}^{-f(z)} = z.$$

I.e., f is the compositional inverse of $z \cdot \mathbb{e}^{-z}$. By Lagrange inversion, we get

$$\begin{aligned} f_n &= \frac{1}{n} \left[\!\left[z^{-1} \right]\!\right] \left(z\mathbb{e}^{-z} \right)^{-n} \\ &= \frac{1}{n} \left[\!\left[z^{-1} \right]\!\right] \frac{\mathbb{e}^{nz}}{z^n} \\ &= \frac{1}{n} \left[\!\left[z^{n-1} \right]\!\right] \left(\mathbb{e}^{z} \right)^n \\ &= \frac{1}{n} [\![z^n]\!] \left(\mathbb{e}^{z} \right)^n \cdot z. \end{aligned}$$

In the notation of Theorem 4.4.15, we have $A(z) = z$ *and* $B(z) = \mathrm{e}^z$*, as well as* $n = N$ *(hence* $\lambda = 1$*). By the theorem, we obtain*

$$\frac{n^{n-1}}{n!} \sim \frac{1}{n}\zeta \frac{\mathrm{e}^{n\zeta}}{\zeta^{n+1}\sqrt{2\pi n\xi}},$$

where $\lambda = \xi = \zeta = 1$*, thus (not very surprisingly, see* (4.1)*):*

$$\frac{n^{n-1}}{n!} \sim \frac{1}{n}\frac{\mathrm{e}^{n}}{\sqrt{2\pi n}}.$$

EXAMPLE 4.4.18. *We consider the trinomial coefficients*

$$T_n := [\![z^n]\!] \left(1 + z + z^2\right)^n.$$

In the notation of Theorem 4.4.15, we have $A(z) = 1$ *and* $B(z) = \left(1 + z + z^2\right)$*, as well as* $n = N$ *(hence* $\lambda = 1$*). By the theorem, we get*

$$T_n \sim \frac{\left(1 + \zeta + \zeta^2\right)^n}{\zeta^n\sqrt{2\pi n\xi}},$$

where ζ *is determined by the equation*

$$\zeta\frac{1 + 2\zeta}{1 + \zeta + \zeta^2} = \lambda = 1,$$

that is, $\zeta = 1$*, and*

$$\begin{aligned}
\xi &= \frac{\mathbf{d}^2}{\mathbf{d}s^2}\left(\log B(s) - \lambda \log s\right)\Big|_{s=\zeta=1} \\
&= \frac{\mathbf{d}}{\mathbf{d}s}\left(\frac{1 + 2s}{1 + s + s^2} - \frac{1}{s}\right)\Bigg|_{s=1} \quad \leftarrow\lambda=1 \\
&= \frac{1 - 2s - 2s^2}{\left(1 + s + s^2\right)^2} + \frac{1}{s^2}\Bigg|_{s=1} = \frac{2}{3}.
\end{aligned}$$

Thus, we obtain

$$T_n \sim \frac{3^{n+1/2}}{2\sqrt{\pi n}}.$$

EXAMPLE 4.4.19 (Asymptotics for unary-binary trees). *In Example 4.3.34, we had already considered the number of unary-binary planar rooted trees with* $n > 0$ *vertices, which we denoted by* T_n*. The corresponding generating function* $T(z) = \sum T_n z^n$ *satisfies the equation*

$$T(z) = z\left(1 + T(z) + T(z)^2\right).$$

In other words, T *is the compositional inverse of* $\frac{z}{1+z+z^2}$*:* $\frac{T}{1+T+T^2} = z$*.*
By Lagrange inversion, we obtain

$$\begin{aligned}
M_n &= \frac{1}{n}\left[\!\left[z^{-1}\right]\!\right] \frac{\left(1 + z + z^2\right)^n}{z^n} \\
&= \frac{1}{n}[\![z^n]\!]\, z\left(1 + z + z^2\right)^n.
\end{aligned}$$

Completely analogously as in the preceding example, here we obtain

$$T_n \sim \frac{3^{n+1/2}}{2\sqrt{\pi n^3}},$$

in agreement with our finding in Example 4.3.34.

4.5. Asymptotics of combinatorial sums

The idea that we successfully applied in the saddle point method, can also be applied to sums (instead of to integrals). We illustrate this by considering an example.

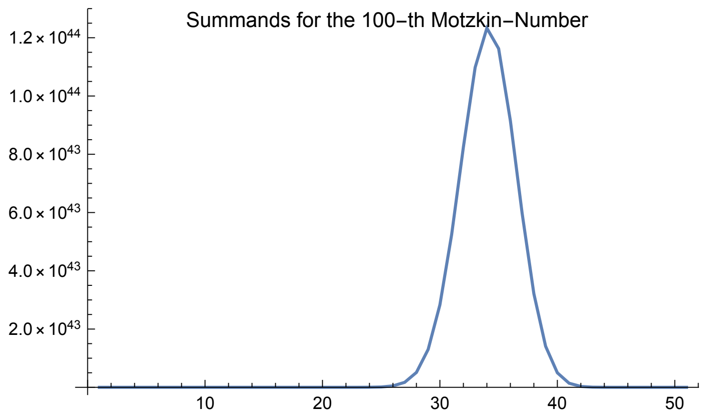
EXAMPLE 4.5.1 (Motzkin numbers). *The number of all paths in the lattice $\mathbb{Z} \times \mathbb{Z}$ which*

- *run from $(0,0)$ to $(n,0)$;*
- *consist of right steps $(1,0)$, diagonal up-steps $(1,1)$, and diagonal down-steps $(1,-1)$;*
- *never pass below the x-axis;*

is called n-th Motzkin number, *and is denoted by M_n. If, in such a* Motzkin path, *we delete the horizontal steps, then what remains is evidently a Dyck path that consists of $2k \leqslant n$ diagonal steps. This leads us straightforwardly to the formula*

$$M_n = \sum_{k=0}^{n/2} \underbrace{\frac{1}{k+1}\binom{2k}{k}\binom{n}{n-2k}}_{=:s(n,k)}. \tag{4.56}$$

The summands $s(n,k)$ in this sum "optically" show a similar behaviour as the function e^{-t^2} that strongly decreases away from zero, see the following plot for the summands $s(100,k)$:



For the determination of the place of the "peak" of these summands, we look for the k, for which $s(n,k+1)/s(n,k) = 1$. This leads to the equation

$$\frac{k+1}{k+2}\frac{(n-2k)(n-2k-1)}{(k+1)^2} = 1,$$

respectively equivalently

$$(n-2k)(n-2k-1) = (k+1)(k+2).$$

Again, we are not necessarily interested in the "exact" location of the peak. With the "approximate" Ansatz $k \sim \lambda n$, we are "approximatively" led to the quadratic equation

$$(1-2\lambda)^2 = \lambda^2,$$

with obvious solutions $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{3}$. Therefore, we set $k_0 = \frac{n}{3}$, and approximate $s\,(n, n/3 - k_1)$ with the help of Stirling's formula (4.1)*, in logarithmic form:*

$$\log n! \sim -n + \left(n + \frac{1}{2}\right)\log n + \frac{1}{2}\left(\log 2\pi\right).$$

We have

$$\begin{aligned} s\,(n,k) &= \frac{n!}{(k+1)!\,(k)!\,(n-2k)!} \\ &= \exp\Bigg(\left(n+\frac{1}{2}\right)\log n + 1 - \log 2\pi \\ &\qquad - \left(n - 2k + \frac{1}{2}\right)\log\left(n-2k\right) \\ &\qquad - \left(k+\frac{1}{2}\right)\log(k) - \left(k+\frac{3}{2}\right)\log(k+1)\Bigg). \end{aligned}$$

Now we substitute $k = n/3 - k_1$ and use

$$\log\left(-z+\frac{n}{3}\right) = \log n - \log 3 + \log\left(1 - \frac{3z}{n}\right)$$

(for $z = k_1$ and $z = k_1 - 1$) together with

$$\log\left(1-z\right) = z + \frac{z^2}{2} + O\left(z^3\right)$$

to obtain (after some computation):

$$\begin{aligned} s\,(n,k) &= \exp\Bigg(\frac{9k_1^2 - 3k_1 + 3}{n} - 2\log(n) + n\log(3) + 2 \\ &\qquad - \log(2\pi) + \frac{5\log(3)}{2} + O\left(\frac{k_1^3}{n^2}\right)\Bigg) \\ &= \frac{3^{n+\frac{5}{2}}}{2\pi n^2} \mathrm{e}^{2-\frac{1}{n}\left(9k_1^2 - 3k_1 + 3\right)}\left(1 + O\left(\frac{k_1^3}{n^2}\right)\right). \end{aligned}$$

(The expression becomes so simple because many terms can be "subsumed" in the error term $O\left(\frac{k_1^3}{n^2}\right)$.) computations ok, fix this??? Consequently, we have

$$M_n = \mathrm{e}^2 \sum_k \frac{3^{n+\frac{5}{2}}}{2\pi n^2} \mathrm{e}^{-\frac{1}{n}\left(9k^2 - 3k + 3\right)}\left(1 + O\left(\frac{k^3}{n^2}\right)\right). \tag{4.57}$$

We split this sum as follows:

$$M_n = \sum_{|k| \leqslant n^{3/5}} \left(\cdots\right) + \sum_{|k| > n^{3/5}} \left(\cdots\right).$$

For $|k| > n^{3/5}$ (i.e., $k = \pm\left(\left\lceil n^{3/5}\right\rceil + k_1\right)$ with $k_1 > 0$), the summand for $|k| = n^{3/5}$ dominates the other summands, provided n is large enough:

$$\mathrm{e}^{-\frac{1}{n}\left(9\left(n^{3/5}+k_1\right)^2 \pm 3\left(n^{3/5}+k_1\right)+3\right)} = \mathrm{e}^{-9\left(n^{3/5-1/2}+k_1/\sqrt{n}\right)^2+O\left(n^{-2/5}\right)}.$$

Consequently, we can bound that part of the sum by

$$\left|\sum_{|k|>n^{3/5}} (\cdots)\right| = O\left(n \cdot 3^n n^{-2} \mathrm{e}^{-9n^{1/5}}\right).$$

This is exponentially small in comparison with $3^n/n^{3/2}$. The other part of the sum is

$$\sum_{|k|\leqslant n^{3/5}} \frac{3^{n+\frac{5}{2}}}{2\pi n^2} \mathrm{e}^{2-9\left(\frac{k}{\sqrt{n}}\right)^2} \cdot \left(1 + O\left(n^{-1/5}\right)\right).$$

If we extend the domain of summation of this sum to all of $\mathbb{Z}$, then we introduce an error, which however is exponentially small. Thus we arrive at

$$\frac{3^{n+5/2}}{2\pi n^2}\sqrt{n} \sum_{k\in\mathbb{Z}} \frac{1}{\sqrt{n}} \mathrm{e}^{-9\left(\frac{k}{\sqrt{n}}\right)^2},$$

and the sum can be identified as Riemann sum *for $\int_{-\infty}^{\infty} \mathrm{e}^{-9t^2}\mathbf{d}t = \frac{1}{3}\sqrt{\pi}$. Hence, we get*

$$M_n \sim \mathrm{e}^2 \frac{3^{n+5/2}}{2\pi n^{3/2}} \frac{1}{3}\sqrt{\pi} = \mathrm{e}^2 \frac{3^{n+3/2}}{2\sqrt{\pi n^3}}.$$

Exercise 38: *Let w_n be the number of possibilities of paying an amount of n Euro using 1–Euro–coins, 2–Euro–coins and 5–Euro–notes (the* order *of the coins and notes is irrelevant).*

(1) *Determine the generating function $\sum_{n\geqslant 0} w_n z^n$.*
(2) *Determine the asymptotic behaviour of w_n for $n \to \infty$.*

Exercise 39: *Let $D_{n,k}$ be the number of permutations of $[n]$, whose disjoint cycle decomposition does not contain any cycle of length $\leqslant k$. (So $D_{n,1}$ is the number of fixed–point–free permutations of $[n]$.)*

(1) *Show:*

$$\sum_{n\geqslant 0} \frac{D_{n,k}}{n!} z^n = \frac{\mathrm{e}^{-z-\frac{z^2}{2}-\cdots-\frac{z^k}{k}}}{1-z}.$$

(2) *For k fixed, what is the asymptotic behaviour of $D_{n,k}$ for $n \to \infty$?*

Exercise 40: *Let $w_{n,k}$ be the number of possibilities of paying an amount of n Euro using 1–Euro–coins, 2–Euro–coins and 5–Euro–notes, where exactly k coins or notes are used (again, the* order *of the coins and notes is irrelevant).*

(1) *Determine the generating function $\sum_{n,k\geqslant 0} w_{n,k} z^n t^k$.*
(2) *If we assume that all possibilities which are enumerated by $w_n = \sum_k w_{n,k}$ have the same probability: What is the asymptotic behaviour of the* expected value *for the number of coins and notes which are used to pay an amount of n Euro, for $n \to \infty$?*

Exercise 41: *Let R_n be the number of possibilities of (completely) tiling a $2 \times n$ rectangle by 1×1 squares and 1×2 rectangles (dominoes).*

(1) *Determine the generating function $\sum_{n \geqslant 0} R_n z^n$.*
(2) *Determine the asymptotic behaviour of R_n for $n \to \infty$?*

Hint: The fact that the dominant singularity can not be calculated explicitly is not an obstacle. One has to continue to calculate with the dominant singularity symbolically.

Exercise 42: *Let $p(n,k)$ be the number of all (integer) partitions of n with at most k summands. Show:*

$$\sum_{n \geqslant 0} p(n,k) z^n = \frac{1}{(1-z)(1-z^2)\cdots(1-z^k)}.$$

Determine the asymptotic behaviour of $p(n,k)$ for fixed k and $n \to \infty$.

Exercise 43: *The exponential generating function of the Bernoulli numbers b_n is*

$$\sum_{n \geqslant 0} b_n z^n = \frac{z}{\mathrm{e}^z - 1}.$$

Determine the asymptotic behaviour of b_n for $\to \infty$.

Exercise 44: *The fraction $\frac{1}{\Gamma(z)}$ is an entire function with zeroes $0, -1, -2, \ldots$. Show Weierstraß' product representation:*

$$\frac{1}{\Gamma(s)} = s\mathrm{e}^{\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) \mathrm{e}^{-s/n},$$

where γ denotes Eulers constant

$$\gamma = \lim_{n \to \infty} (H_n - \log n)$$

and

$$H_n = \sum_{j=1}^{n} \frac{1}{j}$$

denotes the n–th harmonic number.

Exercise 45: *Show the* reflection formula *for the gamma function:*

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{4.58}$$

Hint: Use the product representation of the sine:

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2 \pi^2}\right). \tag{4.59}$$

Exercise 46: *Show the* duplication formula *for the gamma function:*

$$\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z} \pi^{\frac{1}{2}} \Gamma(2z)$$

and its generalisation

$$\prod_{j=0}^{m} \Gamma\left(z + \frac{j}{m}\right) = m^{\frac{1}{2} - mz} (2\pi)^{\frac{m-1}{2}} \Gamma(mz).$$

Exercise 47: *Show using Stirling's Formula for the* Γ*–Function*

$$\binom{n+\alpha-1}{n} = \frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1+O\left(\frac{1}{n}\right)\right)$$

Hint: Here is Stirling's Formula:

$$\Gamma(s+1) \sim s^s e^{-s}\sqrt{2\pi s}\left(1+O\left(\frac{1}{s}\right)\right).$$

Exercise 48: *A* Motzkin path *is a lattice path, where every step is of the form* $(1,0)$, $(1,1)$, $(1,1)$ *(horizontal, up– and down–steps), which starts at the origin, returns to the* x*–axis and never goes below the* x*–axis. Let* M_n *be the number of all Motzkin paths of length (i.e., number of steps)* n*. Show that the generating function for Motzkin paths is given by*

$$\sum_{n\geqslant 0} M_n z^n = \frac{1-z-\sqrt{1-2z-3z^2}}{2z^2}.$$

Derive an explizit formula for M_n*. Use the generating function to determine the asymptotic behaviour of* M_n *for* $n \to \infty$.

Exercise 49: *A* Schröder path *of length* n *is a lattice path consisting of steps* $(2,0)$, $(1,1)$ *and* $(1,-1)$ *(i.e., double horizontal, upward and downward steps) which starts at the origin, ends in* $(n,0)$ *but never falls below the* x*–axis. If we assume that all Schröder paths of length* n *have the same probability: What is the asymptotics of the expected value of the number of steps for a Schröder path of length* n *for* $n \to \infty$*?*

Exercise 50: *Consider the number of cycles in the disjoint cycle decomposition of permutations of* $[n]$ on average*: What is the asymptotics for this average for* $n \to \infty$*?*

Exercise 51: *Let* $H_n = \sum_{j=1}^{n} j^{-1}$ *be the* n*–th* harmonic number*. Show that*

$$\sum_{n\geqslant 0} H_n z^n = \frac{1}{1-z}\log\frac{1}{1-z},$$

and use this result together with singularity analysis to obtain an asymptotic expansion of H_n *for* $n \to \infty$.

Exercise 52: *Let* u_n *be the number of permutations of* $[n]$*, which only have cycles of* odd *length in their decomposition into disjoint cycles. Determine the asymptotic behaviour of* u_n *for* $n \to \infty$.

Hint: Observe that the generating function is analytic in a "Double–Delta–Domain" (i.e., in a disk with two "dents" at the two singularities), so we have two contributions according to the Transfer Theorem.

Exercise 53: *Determine the asymptotic behaviour of the sum*

$$f_n = \sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}$$

for $n \to \infty$.

Hint: Compute the generating function of these sums, i.e., multiply the above expression by z^n and sum over all $n \geqslant 0$; apply the binomial theorem for simplifying the double sum thus obtained.

Exercise 54: *Denote by I_n the number of all involutions (an* involution *is a self–inverse bijection) on $[n]$.*

(1) *Show*
$$\sum_{n\geqslant 0} I_n \frac{z^n}{n!} = e^{z+\frac{z^2}{2}}.$$
(2) *Use the saddle point method to determine the asymptotic behaviour of I_n for $n \to \infty$.*

Exercise 55: *The exponential generating function of the Bell–numbers B_n (B_n is the number of all partitions of $[n]$) is*
$$\sum_{n\geqslant 0} B_n \frac{z^n}{n!} = e^{e^z-1}.$$
Use the saddle point method to determine the asymptotic behaviour of B_n for $n \to \infty$.

Exercise 56: *Determine the asymptotic behaviour of the sum*
$$\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k!}$$
for $n \to \infty$.

Hint: Determine the generating function for this sum!

Exercise 57: *The saddle point method can also be used for the asymptotics of the Motzkin numbers M_n (see exercise 48) for $n \to \infty$:*
Show
$$M_n = [\![z^0]\!] \left(z+1+z^{-1}\right)^n - [\![z^2]\!] \left(z+1+z^{-1}\right)^n$$
and obtain a complex contour integral for M_n, which can be dealt with using the saddle point method.

4.6. Exp–log scheme

As another application, we mention the following (without proof):

DEFINITION 4.6.1. *A function $G(z)$ which is analytic at 0, has only non–negative coefficients and finite radius of convergence ρ, is said to be of logarithmic type with parameters (κ, λ), where $\kappa, \lambda \in \mathbb{R}$, $\kappa \neq 0$, if the following conditions hold:*

(1) *the number ρ is the unique singularity of $G(z)$ on $|z| = \rho$,*
(2) *$G(z)$ is continuable to a Δ–domain at ρ,*
(3) *$G(z)$ satisfies*
$$G(z) = \kappa \cdot \log \frac{1}{1-z} + \lambda + O\left(\frac{1}{(\log(1-z/\rho))^2}\right) \text{ as } z \to \rho \text{ in } \Delta. \tag{4.60}$$

DEFINITION 4.6.2. *The labelled construction*

$$\mathcal{F} = \mathtt{Sets}\left(\mathcal{G}\right)$$

is called a (labelled) exp–log–scheme if the exponential generating function $G(z)$ *of* $\mathcal{G}$ *is of logarithmic type.*
The unlabelled construction

$$\mathcal{F} = \mathtt{Multisets}\left(\mathcal{G}\right)$$

is called an (unlabelled) exp–log–scheme if the ordinary generating function $G(z)$ *of* $\mathcal{G}$ *is of logarithmic type, with* $\rho < 1$.
In both cases (labelled and unlabelled), the quantities (κ, λ) *from* (4.60) *are called the* parameters *of the scheme.*

THEOREM 4.6.3 (Exp–log scheme). *Consider an exp–log scheme with parameters* (κ, λ).
Then we have

$$[\![z^n]\!]\, G(z) = \frac{\kappa}{n \cdot \rho^n} \cdot \left(1 + O\left((\log n)^{-2}\right)\right),$$

$$[\![z^n]\!]\, F(z) = \frac{e^{\lambda + r_0}}{\Gamma(\kappa)} \cdot n^{\kappa - 1} \cdot \rho^{-n} \cdot \left(1 + O\left((\log n)^{-2}\right)\right),$$

where $r_0 = 0$ *in the labelled case and* $r_0 = \sum_{j \geqslant 2} \frac{G(\rho^j)}{j}$ *in the unlabelled case.*
If we consider the number X *of* $\mathcal{G}$*–components in a (randomly chosen)* $\mathcal{F}$*–object of size* n*, then the* expected value *of* X *is*

$$\kappa \cdot (\log n - \Psi(\kappa)) + \lambda + r_1 + O\left((\log n)^{-1}\right) \text{ (where } \Psi(s) = \frac{d}{ds}\Gamma(s)\text{),}$$

where $r_1 = 0$ *in the labelled case and* $r_0 = \sum_{j \geqslant 2} G\left(\rho^j\right)$ *in the unlabelled case. The variance of* X *is* $O(\log n)$.

Exercise 58: *Determine the asymptotic behaviour of expected value and variance of the number of connected components of* 2*–regular labelled graphs with* n *vertices for* $n \to \infty$.

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