Combinatorial Interpretations for Lucas Analogues

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joint with Curtis Bennett, Juan Carrillo, and John Machacek

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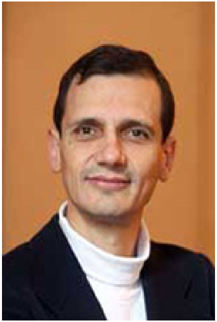

LATTICE PATHS, REFLECTIONS, & DIMENSION–CHANGING BIJECTIONS

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ABSTRACT. We enumerate various families of planar lattice paths consisting of unit steps in directions N, S, E, or W, which do not cross the x-axis or both x- and y-axes. The proofs are purely combinatorial throughout, using either reflections or bijections between these NSEW-paths and linear NS-paths. We also consider other dimension-changing bijections.

1. Introduction. Consider lattice paths in the plane consisting of unit steps, each in a direction N, S, E, or W. Such NSEW-paths were first investigated by DeTemple & Robertson [DR] and Csáki, Mohanty & Saran [CMS]. The basic result of these papers is the following.

Outline

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So when proving theorems about the Lucas sequence, one gets results about the Fibonacci numbers, the nonnegative integers, and q-analogues for free.

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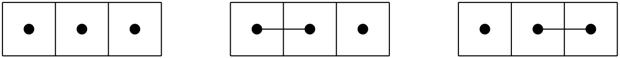

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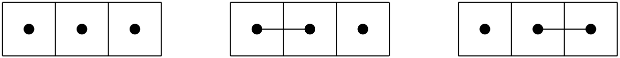

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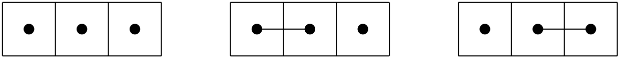

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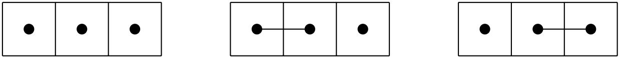

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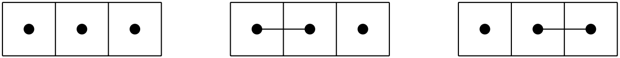

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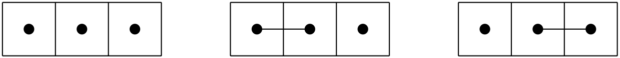

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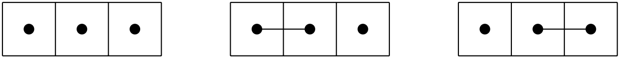

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Previous work on the Lucas analogue of the binomial coefficients was done by Gessel-Viennot, Benjamin-Plott, Savage-Sagan.

Outline

- The Lucas sequence
- Binomial coefficient analogue
- Catalan number analogue
- Coxeter groups
- Comments and open problems

Given $0 \leq k \leq n$ the corresponding *Lucasnomial* is

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \frac{\{n\}!}{\{k\}!\{n-k\}!}$$

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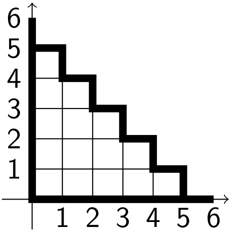

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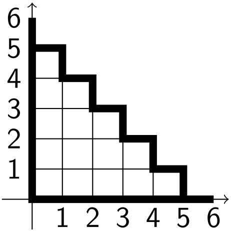
δ_6: 

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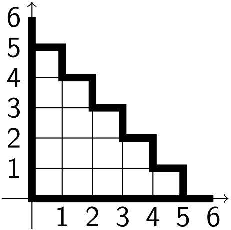

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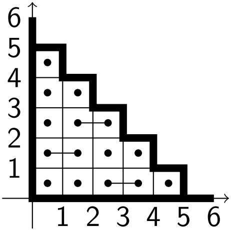

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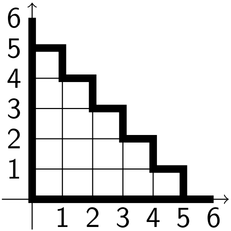

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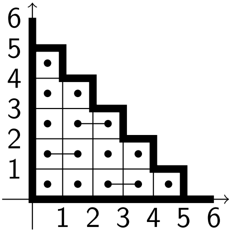

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$$\mathrm{wt}\, \mathcal{T}(\delta_n) = \{n\}!$$

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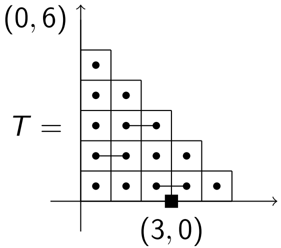

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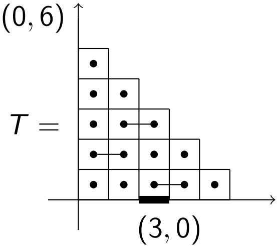

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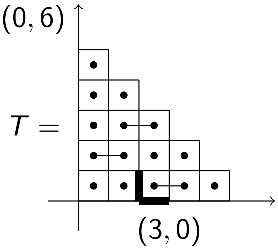

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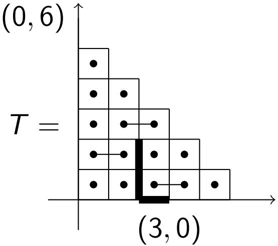

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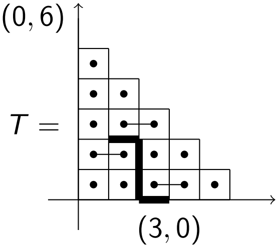

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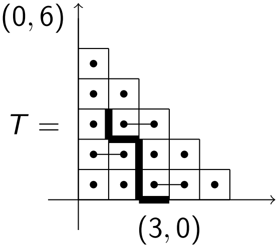
$T =$ 

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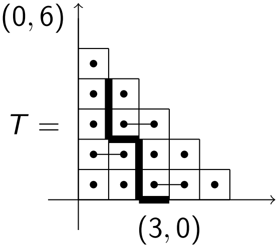
$T =$ 

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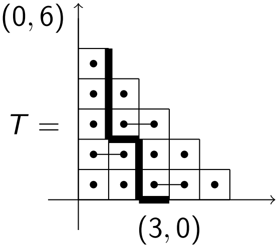
$T =$ 

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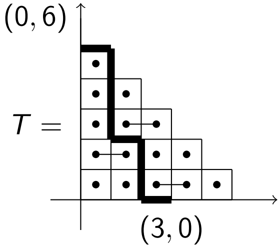
$T =$ 

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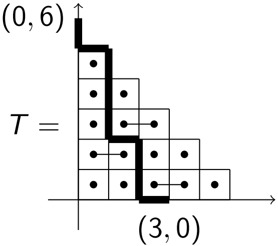

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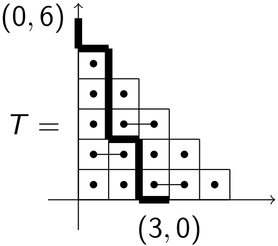

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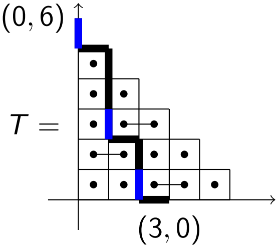
$T =$ 

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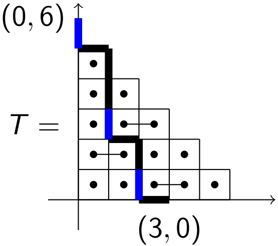
$T =$ 

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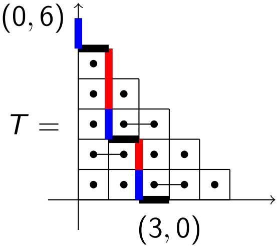
$T =$ 

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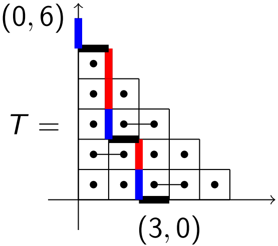
$T =$ 

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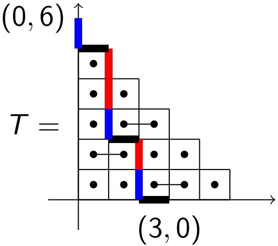
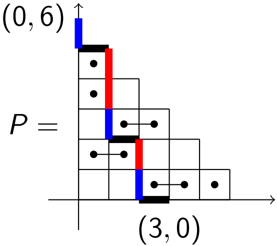
$T =$  $P =$ 

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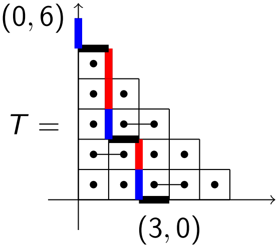

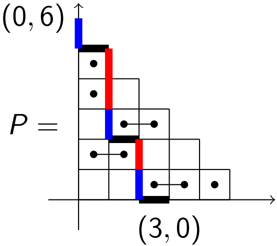

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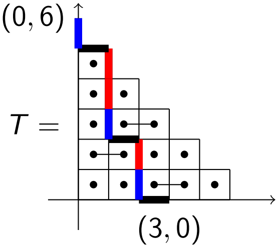
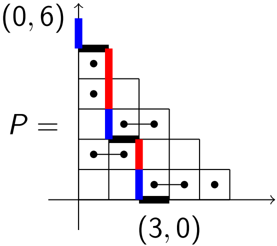
$T =$ 

$P =$ 

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Proposition $\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \{k+1\} \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} + t\{n-k-1\} \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\}.$

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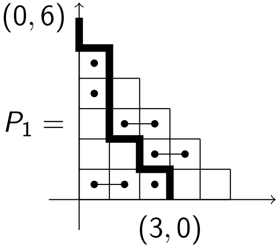
$P_1 =$ 

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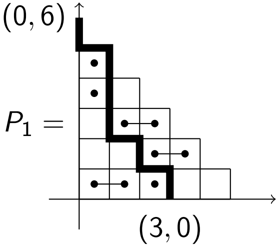
$P_1 =$ 

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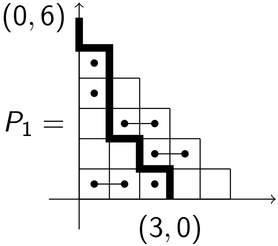
$P_1 =$ 

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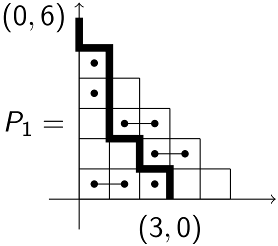

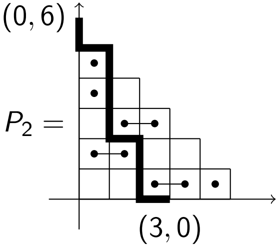

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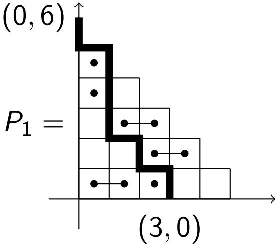
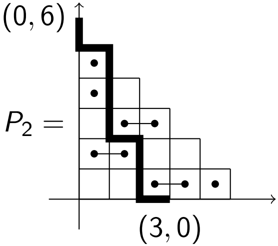
$P_1 =$ 

$P_2 =$ 

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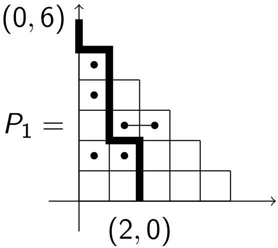

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B_n	$2, 4, 6, \ldots, 2n$	$2n$
D_n	$2, 4, 6, \ldots, 2(n-1), n$	$2(n-1)$ (for $n \geq 3$)
E_6	$2, 5, 6, 8, 9, 12$	12
E_7	$2, 6, 8, 10, 12, 14, 18$	18
E_8	$2, 8, 12, 14, 18, 20, 24, 30$	30
F_4	$2, 6, 8, 12$	12
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$$\mathrm{Cat}\, W = \prod_{i=1}^{n} \frac{h + d_i}{d_i}.$$

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$I_2(m)$	$2, m$	m (for $m \geq 2$)

The *finite Coxeter groups* W are those generated by reflections. Each irreducible W has *degree set* $D = \{d_1, \ldots, d_n\}$. The *Coxeter number* of W is $h = \max D$. The *Coxeter-Catalan number* of W is

$$\mathrm{Cat}\, W = \prod_{i=1}^{n} \frac{h + d_i}{d_i}.$$

$$\therefore \ \mathrm{Cat}\, A_n = \frac{(n+3)(n+4)\ldots(2n+2)}{(2)(3)\ldots(n+1)} = C_{n+1}.$$

W	$d_1, \ldots, d_n$	h
A_n	$2, 3, 4, \ldots, n+1$	$n+1$
B_n	$2, 4, 6, \ldots, 2n$	$2n$
D_n	$2, 4, 6, \ldots, 2(n-1), n$	$2(n-1)$ (for $n \geq 3$)
E_6	$2, 5, 6, 8, 9, 12$	12
E_7	$2, 6, 8, 10, 12, 14, 18$	18
E_8	$2, 8, 12, 14, 18, 20, 24, 30$	30
F_4	$2, 6, 8, 12$	12
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For $0 \le k \le n$ and $d \ge 1$ define the *d-divisible Lucasnomial*

$$\left\{ \begin{matrix} n : d \\ k : d \end{matrix} \right\} = \frac{\{n : d\}!}{\{k : d\}!\{n-k : d\}!}$$

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Outline

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What can be said about Fuss-Catalan Lucas analogues for other Coxeter goups?

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The number of lattice paths using steps N and E from $(0,0)$ to (a,b) and staying weakly above the line $y = (b/a)x$ is $\mathrm{Cat}(a,b)$.

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We have $C_n = \sum_{k=1}^{n} N_{n,k}$. The Lucas analogue of $N_{n,k}$ is a polynomial in s, t for $n \leq 100$.

FÜR CHRISTIAN!