# Overview on Convenient Calculus and Differential Geometry in Infinite dimensions, with Applications to Diffeomorphism Groups and Shape Spaces

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Based on collaborations with: M. Bauer, M. Bruveris, P. Harms, D. Mumford

- 1. A short introduction to convenient calculus in infinite dimensions.
- 2. Manifolds of mappings (with compact source) and diffeomorphism groups as convenient manifolds
- 3. A diagram of actions of diffeomorphism groups
- 4. Riemannian geometries of spaces of immersions, diffeomorphism groups, and shape spaces, their geodesic equations with well posedness results and vanishing geodesic distance.
- 5. Riemannian geometries on spaces of Riemannian metrics and pulling them back to diffeomorphism groups.
- 6. Robust Infinite Dimensional Riemannian manifolds, and Riemannian homogeneous spaces of diffeomorphism groups.

We will discuss geodesic equations of many different metrics on these spaces and make contact to many well known equations (Cammassa-Holm, KdV, Hunter-Saxton, Euler for ideal fluids).

# Some words on smooth convenient calculus

Traditional differential calculus works well for finite dimensional vector spaces and for Banach spaces.
Beyond Banach spaces, the main difficulty is that composition of linear mappings stops to be jointly continuous at the level of Banach spaces, for any compatible topology.
For more general locally convex spaces we sketch here the convenient approach as explained in [Frölicher-Kriegl 1988] and [Kriegl-Michor 1997].

# The $c^\infty$-topology

Let $E$ be a locally convex vector space. A curve $c : \mathbb{R} \to E$ is called *smooth* or $C^\infty$ if all derivatives exist and are continuous. Let $C^\infty(\mathbb{R}, E)$ be the space of smooth functions. It can be shown that the set $C^\infty(\mathbb{R}, E)$ does not depend on the locally convex topology of $E$, only on its associated bornology (system of bounded sets). The final topologies with respect to the following sets of mappings into E coincide:

1. $C^\infty(\mathbb{R}, E)$.
2. The set of all Lipschitz curves (so that $\{\frac{c(t)-c(s)}{t-s} : t \neq s, |t|, |s| \leq C\}$ is bounded in $E$, for each $C$).
3. The set of injections $E_B \to E$ where $B$ runs through all bounded absolutely convex subsets in $E$, and where $E_B$ is the linear span of $B$ equipped with the Minkowski functional $\|x\|_B := \inf\{\lambda > 0 : x \in \lambda B\}$.
4. The set of all Mackey-convergent sequences $x_n \to x$ (there exists a sequence $0 < \lambda_n \nearrow \infty$ with $\lambda_n(x_n - x)$ bounded).

# The $c^\infty$-topology. II

*This topology is called the $c^\infty$-topology on $E$ and we write $c^\infty E$ for the resulting topological space.*

In general (on the space $\mathcal{D}$ of test functions for example) it is finer than the given locally convex topology, it is not a vector space topology, since scalar multiplication is no longer jointly continuous. The finest among all locally convex topologies on $E$ which are coarser than $c^\infty E$ is the bornologification of the given locally convex topology. If $E$ is a Fréchet space, then $c^\infty E = E$.

# Convenient vector spaces

A locally convex vector space $E$ is said to be a *convenient vector space* if one of the following holds (called $c^\infty$-completeness):

1. For any $c \in C^\infty(\mathbb{R}, E)$ the (Riemann-) integral $\int_0^1 c(t)dt$ exists in $E$.
2. Any Lipschitz curve in $E$ is locally Riemann integrable.
3. A curve $c : \mathbb{R} \to E$ is $C^\infty$ if and only if $\lambda \circ c$ is $C^\infty$ for all $\lambda \in E^*$, where $E^*$ is the dual of all cont. lin. funct. on $E$.
   - Equiv., for all $\lambda \in E'$, the dual of all bounded lin. functionals.
   - Equiv., for all $\lambda \in \mathcal{V}$, where $\mathcal{V}$ is a subset of $E'$ which recognizes bounded subsets in $E$.

   We call this *scalarwise* $C^\infty$.
4. Any Mackey-Cauchy-sequence (i. e. $t_{nm}(x_n - x_m) \to 0$ for some $t_{nm} \to \infty$ in $\mathbb{R}$) converges in $E$. This is visibly a mild completeness requirement.

# Convenient vector spaces. II

5. If $B$ is bounded closed absolutely convex, then $E_B$ is a Banach space.
6. If $f : \mathbb{R} \to E$ is scalarwise $\mathrm{Lip}^k$, then $f$ is $\mathrm{Lip}^k$, for $k > 1$.
7. If $f : \mathbb{R} \to E$ is scalarwise $C^\infty$ then $f$ is differentiable at 0.

Here a mapping $f : \mathbb{R} \to E$ is called $\mathrm{Lip}^k$ if all derivatives up to order $k$ exist and are Lipschitz, locally on $\mathbb{R}$. That $f$ is scalarwise $C^\infty$ means $\lambda \circ f$ is $C^\infty$ for all continuous (equiv., bounded) linear functionals on $E$.

# Smooth mappings

*Let $E$, and $F$ be convenient vector spaces, and let $U \subset E$ be $c^\infty$-open. A mapping $f : U \to F$ is called smooth or $C^\infty$, if $f \circ c \in C^\infty(\mathbb{R}, F)$ for all $c \in C^\infty(\mathbb{R}, U)$.*

If $E$ is a Fréchet space, then this notion coincides with all other reasonable notions of $C^\infty$-mappings. Beyond Fréchet mappings, as a rule, there are more smooth mappings in the convenient setting than in other settings, e.g., $C_c^\infty$.

# Main properties of smooth calculus

1. For maps on Fréchet spaces this coincides with all other reasonable definitions. On $\mathbb{R}^2$ this is non-trivial [Boman,1967].
2. Multilinear mappings are smooth iff they are bounded.
3. If $E \supseteq U \xrightarrow{f} F$ is smooth then the derivative $df : U \times E \to F$ is smooth, and also $df : U \to L(E, F)$ is smooth where $L(E, F)$ denotes the space of all bounded linear mappings with the topology of uniform convergence on bounded subsets.
4. The chain rule holds.
5. The space $C^\infty(U, F)$ is again a convenient vector space where the structure is given by the obvious injection

$$C^\infty(U, F) \xrightarrow{C^\infty(c,\ell)} \prod_{c \in C^\infty(\mathbb{R}, U), \ell \in F^*} C^\infty(\mathbb{R}, \mathbb{R}), \quad f \mapsto (\ell \circ f \circ c)_{c,\ell},$$

where $C^\infty(\mathbb{R}, \mathbb{R})$ carries the topology of compact convergence in each derivative separately.

# Main properties of smooth calculus, II

6. The exponential law holds: For $c^\infty$-open $V \subset F$,

$$C^\infty(U, C^\infty(V, G)) \cong C^\infty(U \times V, G)$$

is a linear diffeomorphism of convenient vector spaces.
*Note that this is the main assumption of variational calculus. Here it is a theorem.*

7. A linear mapping $f : E \to C^\infty(V, G)$ is smooth (by (2) equivalent to bounded) if and only if $E \xrightarrow{f} C^\infty(V, G) \xrightarrow{\mathrm{ev}_v} G$ is smooth for each $v \in V$.
(*Smooth uniform boundedness theorem*, see [KM97], theorem 5.26).

# Main properties of smooth calculus, III

8. The following canonical mappings are smooth.

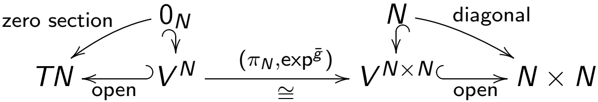
$$
\begin{aligned}
&\mathrm{ev}: C^\infty(E,F)\times E \to F, \quad \mathrm{ev}(f,x)=f(x)\\
&\mathrm{ins}: E \to C^\infty(F, E\times F), \quad \mathrm{ins}(x)(y)=(x,y)\\
&(\quad)^\wedge : C^\infty(E, C^\infty(F,G)) \to C^\infty(E\times F, G)\\
&(\quad)^\vee : C^\infty(E\times F, G) \to C^\infty(E, C^\infty(F,G))\\
&\mathrm{comp}: C^\infty(F,G)\times C^\infty(E,F) \to C^\infty(E,G)\\
&C^\infty(\quad,\quad): C^\infty(F,F_1)\times C^\infty(E_1,E)\to\\
&\qquad\qquad\to C^\infty(C^\infty(E,F), C^\infty(E_1,F_1))\\
&\qquad (f,g)\mapsto (h\mapsto f\circ h\circ g)\\
&\prod : \prod C^\infty(E_i,F_i) \to C^\infty(\prod E_i, \prod F_i)
\end{aligned}
$$

This ends our review of the standard results of convenient calculus. Convenient calculus (having properties 6 and 7) exists also for:

- Real analytic mappings [Kriegl,M,1990]
- Holomorphic mappings [Kriegl,Nel,1985] (notion of [Fantappié, 1930-33])
- Many classes of Denjoy Carleman ultradifferentible functions, both of Beurling type and of Roumieu-type [Kriegl,M,Rainer, 2009, 2011, 2013]

# Manifolds of mappings

Let $M$ be a compact (for simplicity's sake) fin. dim. manifold and $N$ a manifold. We use an auxiliary Riemann metric $\bar{g}$ on $N$. Then




$C^\infty(M,N)$, the space of smooth mappings $M \to N$, has the following manifold structure. Chart, centered at $f \in C^\infty(M,N)$, is:

$$C^\infty(M,N) \supset U_f = \{g : (f,g)(M) \subset V^{N\times N}\} \xrightarrow{u_f} \tilde{U}_f \subset \Gamma(f^* TN)$$

$$u_f(g) = (\pi_N, \exp^{\bar{g}})^{-1} \circ (f,g), \quad u_f(g)(x) = (\exp^{\bar{g}}_{f(x)})^{-1}(g(x))$$

$$(u_f)^{-1}(s) = \exp^{\bar{g}}_f \circ s, \qquad (u_f)^{-1}(s)(x) = \exp^{\bar{g}}_{f(x)}(s(x))$$

# Manifolds of mappings II

**Lemma:** $C^\infty(\mathbb{R}, \Gamma(M; f^*TN)) = \Gamma(\mathbb{R} \times M; \mathrm{pr}_2{}^* f^*TN)$
By Cartesian Closedness (I am lying a little).
**Lemma:** Chart changes are smooth ($C^\infty$)
$\tilde{U}_{f_1} \ni s \mapsto (\pi_N, \exp^{\bar{g}}) \circ s \mapsto (\pi_N, \exp^{\bar{g}})^{-1} \circ (f_2, \exp^{\bar{g}}_{f_1} \circ s)$
since they map smooth curves to smooth curves.
**Lemma:** $C^\infty(\mathbb{R}, C^\infty(M, N)) \cong C^\infty(\mathbb{R} \times M, N)$.
By Cartesian closedness.
**Lemma:** Composition $C^\infty(P, M) \times C^\infty(M, N) \to C^\infty(P, N)$, $(f, g) \mapsto g \circ f$, is smooth, since it maps smooth curves to smooth curves
**Corollary** (of the chart structure):

$$TC^\infty(M, N) = C^\infty(M, TN) \xrightarrow{C^\infty(M, \pi_N)} C^\infty(M, N).$$

# Regular Lie groups

We consider a smooth Lie group $G$ with Lie algebra $\mathfrak{g} = T_e G$ modelled on convenient vector spaces. The notion of a regular Lie group is originally due to Omori et al. for Fréchet Lie groups, was weakened and made more transparent by Milnor, and then carried over to convenient Lie groups; see [KM97], 38.4.

A Lie group $G$ is called *regular* if the following holds:

- For each smooth curve $X \in C^\infty(\mathbb{R}, \mathfrak{g})$ there exists a curve $g \in C^\infty(\mathbb{R}, G)$ whose right logarithmic derivative is $X$, i.e.,

$$\begin{cases} g(0) & = e \\ \partial_t g(t) & = T_e(\mu^{g(t)})X(t) = X(t).g(t) \end{cases}$$

  The curve $g$ is uniquely determined by its initial value $g(0)$, if it exists.
- Put $\operatorname{evol}^r_G(X) = g(1)$ where $g$ is the unique solution required above. Then $\operatorname{evol}^r_G : C^\infty(\mathbb{R}, \mathfrak{g}) \to G$ is required to be $C^\infty$ also. We have $\operatorname{Evol}^X_t := g(t) = \operatorname{evol}_G(tX)$.

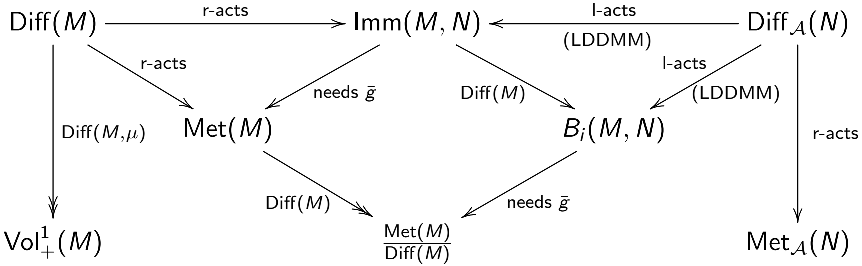
# Diffeomorphism group of compact $M$

**Theorem:** *For each compact manifold $M$, the diffeomorphism group is a regular Lie group.*

**Proof:** $\text{Diff}(M) \xrightarrow{\text{open}} C^\infty(M, M)$. Composition is smooth by restriction. Inversion is smooth: If $t \mapsto f(t, \quad)$ is a smooth curve in $\text{Diff}(M)$, then $f(t, \quad)^{-1}$ satisfies the implicit equation $f(t, f(t, \quad)^{-1}(x)) = x$, so by the finite dimensional implicit function theorem, $(t, x) \mapsto f(t, \quad)^{-1}(x)$ is smooth. So inversion maps smooth curves to smooth curves, and is smooth.

Let $X(t, x)$ be a time dependent vector field on $M$ (in $C^\infty(\mathbb{R}, \mathfrak{X}(M))$). Then $\text{Fl}_s^{\partial_t \times X}(t, x) = (t + s, \text{Evol}^X(t, x))$ satisfies the ODE $\quad \partial_t \text{Evol}(t, x) = X(t, \text{Evol}(t, x))$. If $X(s, t, x) \in C^\infty(\mathbb{R}^2, \mathfrak{X}(M))$ is a smooth curve of smooth curves in $\mathfrak{X}(M)$, then obviously the solution of the ODE depends smoothly also on the further variable $s$, thus evol maps smooth curves of time dependant vector fields to smooth curves of diffeomorphism. QED.

# The diagram




$M$ compact , $N$ pssibly non-compact manifold

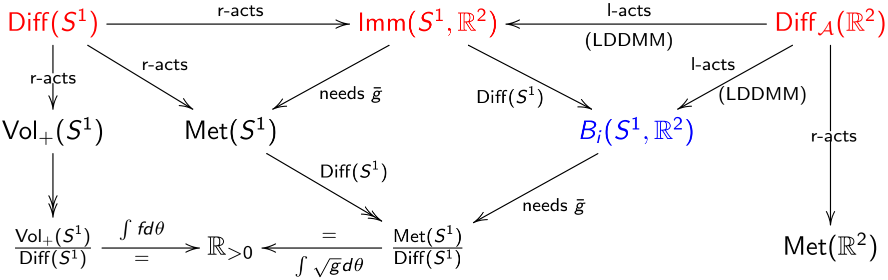
| | |
|---|---|
| $\mathrm{Met}(N) = \Gamma(S^2_+ T^* N)$ | space of all Riemann metrics on $N$ |
| $\bar{g}$ | one Riemann metric on $N$ |
| $\mathrm{Diff}(M)$ | Lie group of all diffeos on compact mf $M$ |
| $\mathrm{Diff}_{\mathcal{A}}(N),\ \mathcal{A} \in \{H^\infty, \mathcal{S}, c\}$ | Lie group of diffeos of decay $\mathcal{A}$ to $\mathrm{Id}_N$ |
| $\mathrm{Imm}(M,N)$ | mf of all immersions $M \to N$ |
| $B_i(M,N) = \mathrm{Imm}/\mathrm{Diff}(M)$ | shape space |
| $\mathrm{Vol}^1_+(M) \subset \Gamma(\mathrm{vol}(M))$ | space of positive smooth probability densities |

$$\mathrm{Emb}(M,N) \to \mathrm{Emb}(M,N)/\mathrm{Diff}(M) = B(M,N)$$

is a smooth principal fiber bundle with structure group $\mathrm{Diff}(M)$; $B(M,N)$ is the smooth manifold of submanifolds of $N$ of type $M$.

The right action of $\mathrm{Diff}(M)$ on $\mathrm{Imm}(M,N)$ is not free; each isotropy group is a finite group [Cervera, Mascaro, M, 1991] and thus the quotient $\mathrm{Imm}(M,N)/\mathrm{Diff}(M) = B_i(M,N)$ is an honest orbifold, albeit infinite dimensional.

The right action of $\mathrm{Diff}(M)$ on $\mathrm{Met}(M)$ is not free; each isotropy group is a finite dimensional Lie group (compact for compact $M$). The quotient $\mathrm{Met}(M)/\mathrm{Diff}(M)$ is a stratified space, the 'true' phace space for Einstein's equation, sometimes called 'superspace', [Ebin] [Ebin,Marsden].




| | |
|---|---|
| $\mathrm{Diff}(S^1)$ | Lie group of all diffeos on compact mf $S^1$ |
| $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^2),\ \mathcal{A} \in \{\mathcal{B}, H^\infty, \mathcal{S}, c\}$ | Lie group of diffeos of decay $\mathcal{A}$ to $\mathrm{Id}_{\mathbb{R}^2}$ |
| $\mathrm{Imm}(S^1, \mathbb{R}^2)$ | mf of all immersions $S^1 \to \mathbb{R}^2$ |
| $B_i(S^1, \mathbb{R}^2) = \mathrm{Imm}/\mathrm{Diff}(S^1)$ | shape space |
| $\mathrm{Vol}_+(S^1) = \left\{ f\, d\theta : f \in C^\infty(S^1, \mathbb{R}_{>0}) \right\}$ | space of positive smooth probability densities |
| $\mathrm{Met}(S^1) = \left\{ g\, d\theta^2 : g \in C^\infty(S^1, \mathbb{R}_{>0}) \right\}$ | space of metrics on $S^1$ |

# The manifold of immersions

Let $M$ be either $S^1$ or $[0, 2\pi]$.

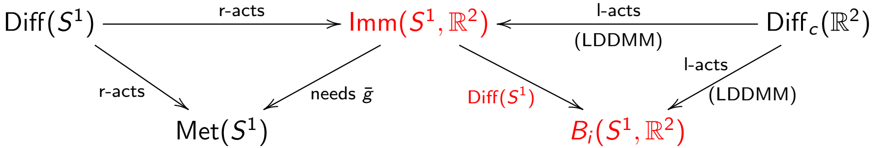
$$\operatorname{Imm}(M, \mathbb{R}^2) := \{c \in C^\infty(M, \mathbb{R}^2) : c'(\theta) \neq 0\} \subset C^\infty(M, \mathbb{R}^2).$$

The tangent space of $\operatorname{Imm}(M, \mathbb{R}^2)$ at a curve $c$ is the set of all vector fields along $c$:

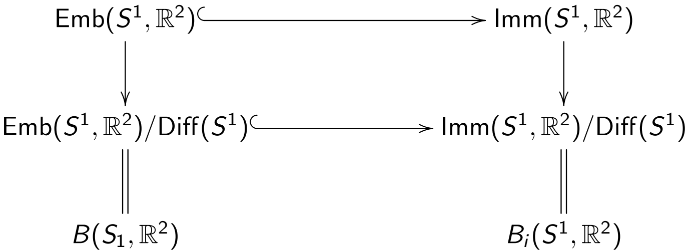
$$T_c \operatorname{Imm}(M, \mathbb{R}^2) = \left\{ h : \begin{array}{ccc} & & T\mathbb{R}^2 \\ & \overset{h}{\nearrow} & \downarrow \pi \\ M & \xrightarrow{c} & \mathbb{R}^2 \end{array} \right\} \cong \{h \in C^\infty(M, \mathbb{R}^2)\}$$

Some Notation:

$$v(\theta) = \frac{c'(\theta)}{|c'(\theta)|}, \quad n(\theta) = iv(\theta), \quad ds = |c'(\theta)| d\theta, \quad D_s = \frac{1}{|c'(\theta)|}\partial_\theta$$

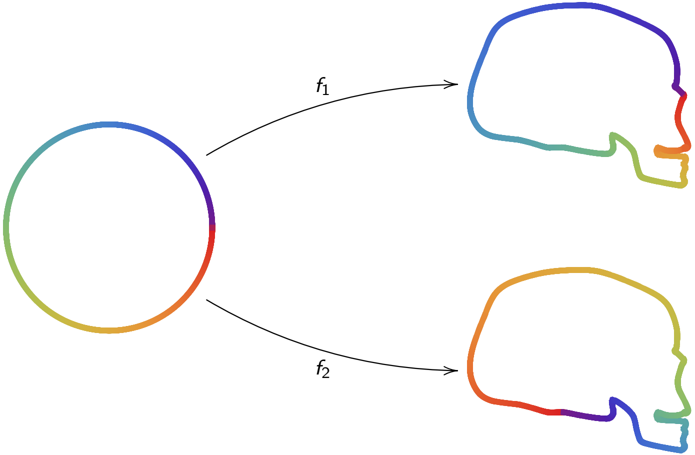


Diff($S^1$)
r-acts
Imm($S^1, \mathbb{R}^2$)
l-acts
(LDDMM)
Diff$_c$($\mathbb{R}^2$)
r-acts
needs $\bar{g}$
Diff($S^1$)
l-acts
(LDDMM)
Met($S^1$)
$B_i(S^1, \mathbb{R}^2)$

Emb($S^1, \mathbb{R}^2$)
Imm($S^1, \mathbb{R}^2$)
Emb($S^1, \mathbb{R}^2$)/Diff($S^1$)
Imm($S^1, \mathbb{R}^2$)/Diff($S^1$)
$B(S_1, \mathbb{R}^2)$
$B_i(S^1, \mathbb{R}^2)$

# Different parameterizations




$$f_1, f_2 : S^1 \to \mathbb{R}^2, \qquad f_1 = f_2 \circ \varphi, \qquad \varphi \in \mathrm{Diff}(S^1)$$

# Inducing a metric on shape space

$$\begin{array}{c} \mathrm{Imm}(M, \mathbb{R}^2) \\ \big\downarrow \pi \\ B_i := \mathrm{Imm}(M, \mathbb{R}^2)/\,\mathrm{Diff}(M) \end{array}$$

Every Diff($M$)-invariant metric "above" induces a unique metric "below" such that $\pi$ is a Riemannian submersion.

# The vertical and horizontal bundle

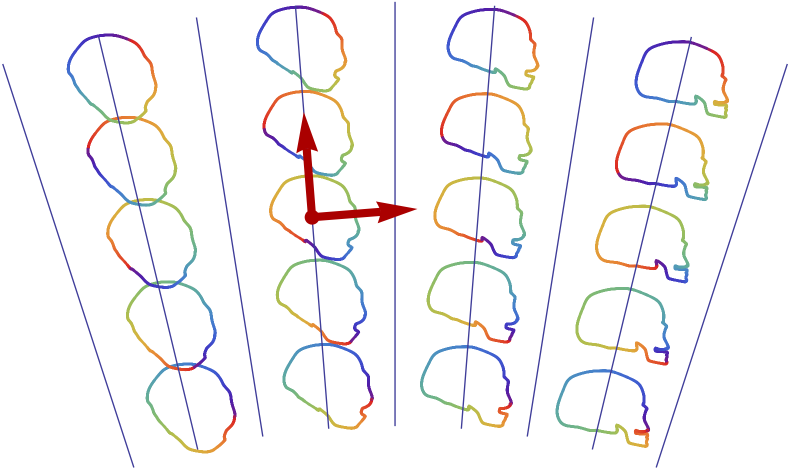
- $T\,\mathrm{Imm} = \mathrm{Vert} \bigoplus \mathrm{Hor}$.
- The vertical bundle is

$$\mathrm{Vert} := \ker T\pi \subset T\,\mathrm{Imm}\,.$$

- The horizontal bundle is

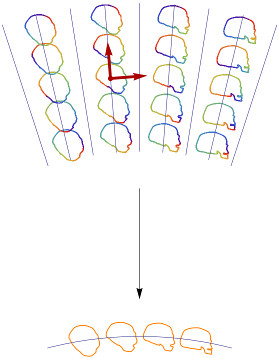
$$\mathrm{Hor} := (\ker T\pi)^{\perp,G} \subset T\,\mathrm{Imm}\,.$$

# The vertical and horizontal bundle



# Definition of a Riemannian metric

$$\mathrm{Imm}(M,N) \xrightarrow{\ \pi\ } B_i(M,N)$$



1. Define a $\mathrm{Diff}(M)$-invariant metric $G$ on Imm.
2. Then $T\pi$ restricted to the horizontal space yields an isomorphism

$$(\ker T_f\pi)^{\perp,G} \cong T_{\pi(f)}B_i.$$

3. Define a metric on $B_i$ such that

$$(\ker T_f\pi)^{\perp,G} \cong T_{\pi(f)}B_i$$

is an isometry.

# Riemannian submersions

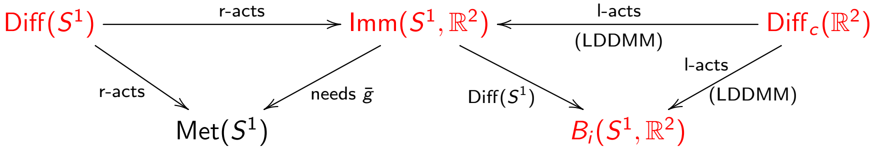
$$\begin{array}{c} \operatorname{Imm}(M, \mathbb{R}^2) \\ \Big\downarrow \pi \\ B_i := \operatorname{Imm}(M, \mathbb{R}^2) / \operatorname{Diff}(M) \end{array}$$

- Horizontal geodesics on $\operatorname{Imm}(M, \mathbb{R}^2)$ project down to geodesics in shape space.
- O'Neill's formula connects sectional curvature on $\operatorname{Imm}(M, \mathbb{R}^2)$ and on $B_i$.

# $L^2$ metric

$$G^0_c(h,k) = \int_M \langle h(\theta), k(\theta) \rangle ds.$$

Problem: The induced geodesic distance vanishes.




[MichorMumford2005a,2005b], [BauerBruverisHarmsMichor2011,2012]

# Weak Riem. metrics on $\mathrm{Emb}(M,N) \subset \mathrm{Imm}(M,N)$.

Metrics on the space of immersions of the form:

$$G^P_f(h,k) = \int_M \bar g(P^f h, k)\, \mathrm{vol}(f^*\bar g)$$

where $\bar g$ is some fixed metric on $N$, $g = f^*\bar g$ is the induced metric on $M$, $h,k \in \Gamma(f^*TN)$ are tangent vectors at $f$ to $\mathrm{Imm}(M,N)$, and $P^f$ is a positive, selfadjoint, bijective (scalar) pseudo differential operator of order $2p$ depending smoothly on $f$. Good example: $P^f = 1 + A(\Delta^g)^p$, where $\Delta^g$ is the Bochner-Laplacian on $M$ induced by the metric $g = f^*\bar g$. Also $P$ has to be $\mathrm{Diff}(M)$-invariant: $\varphi^* \circ P_f = P_{f\circ\varphi} \circ \varphi^*$.

# Elastic metrics on plane curves

Here $M = S^1$ or $[0, 1\pi]$, $N = \mathbb{R}^2$. The elastic metrics on $\operatorname{Imm}(M, \mathbb{R}^2)$ is

$$G_c^{a,b}(h,k) = \int_0^{2\pi} a^2 \langle D_s h, n\rangle \langle D_s k, n\rangle + b^2 \langle D_s h, v\rangle \langle D_s k, v\rangle \, ds,$$

with

$$\begin{aligned} P_c^{a,b}(h) = &- a^2 \langle D_s^2 h, n\rangle n - b^2 \langle D_s^2 h, v\rangle v \\ &+ (a^2 - b^2)\kappa\big(\langle D_s h, v\rangle n + \langle D_s h, n\rangle v\big) \\ &+ (\delta_{2\pi} - \delta_0)\big(a^2 \langle n, D_s h\rangle n + b^2 \langle v, D_s h\rangle v\big). \end{aligned}$$

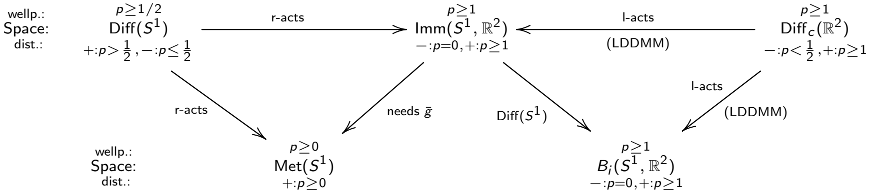
# Sobolev type metrics

Advantages of Sobolev type metrics:

1. Positive geodesic distance
2. Geodesic equations are well posed
3. Generalization to higher dimension

Problems:

1. Analytic solutions to the geodesic equation?
2. Curvature of shape space with respect to these metrics?
3. Numerics are in general computational expensive



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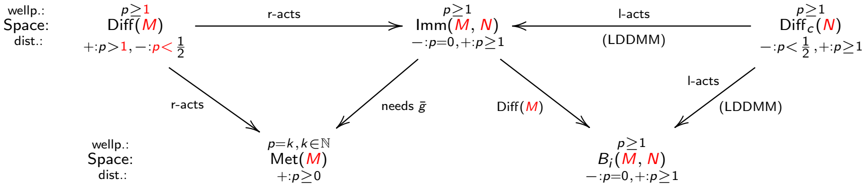


# Geodesic equation.

The geodesic equation for a Sobolev-type metric $G^P$ on immersions is given by

$$\begin{aligned}\nabla_{\partial_t} f_t =& \frac{1}{2} P^{-1}\Big( \operatorname{Adj}(\nabla P)(f_t, f_t)^\perp - 2.Tf.\bar{g}(Pf_t, \nabla f_t)^\sharp \\ & \qquad - \bar{g}(Pf_t, f_t).\operatorname{Tr}^g(S)\Big) \\ & - P^{-1}\Big((\nabla_{f_t} P)f_t + \operatorname{Tr}^g\big(\bar{g}(\nabla f_t, Tf)\big)Pf_t\Big).\end{aligned}$$

The geodesic equation written in terms of the momentum for a Sobolev-type metric $G^P$ on Imm is given by:

$$\begin{cases} p = Pf_t \otimes \operatorname{vol}(f^*\bar{g}) \\ \nabla_{\partial_t} p = \frac{1}{2}\big( \operatorname{Adj}(\nabla P)(f_t, f_t)^\perp - 2Tf.\bar{g}(Pf_t, \nabla f_t)^\sharp \\ \qquad\qquad - \bar{g}(Pf_t, f_t)\operatorname{Tr}^{f^*\bar{g}}(S)\big) \otimes \operatorname{vol}(f^*\bar{g}) \end{cases}$$

## Wellposedness

**Assumption 1:** $P, \nabla P$ and $\mathrm{Adj}(\nabla P)^\perp$ are smooth sections of the bundles

$$\begin{array}{ccccc} L(T\mathrm{Imm}; T\mathrm{Imm}) & \quad & L^2(T\mathrm{Imm}; T\mathrm{Imm}) & \quad & L^2(T\mathrm{Imm}; T\mathrm{Imm}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Imm} & & \mathrm{Imm} & & \mathrm{Imm}, \end{array}$$

respectively. Viewed locally in trivializ. of these bundles,

$P_f h, \quad (\nabla P)_f(h,k), \quad \big(\mathrm{Adj}(\nabla P)_f(h,k)\big)^\perp$ are pseudo-differential operators of order $2p$ in $h, k$ separately. As mappings in $f$ they are non-linear, and we assume they are a composition of operators of the following type:

(a) Local operators of order $l \le 2p$, i.e., nonlinear differential operators $A(f)(x) = A(x, \hat{\nabla}^l f(x), \hat{\nabla}^{l-1} f(x), \ldots, \hat{\nabla} f(x), f(x))$

(b) Linear pseudo-differential operators of degrees $l_i$,

such that the total (top) order of the composition is $\le 2p$.

**Assumption 2:** For each $f \in \mathrm{Imm}(M,N)$, the operator $P_f$ is an elliptic pseudo-differential operator of order $2p$ for $p > 0$ which is positive and symmetric with respect to the $H^0$-metric on Imm, i.e.

$$\int_M \bar{g}(P_f h, k)\,\mathrm{vol}(g) = \int_M \bar{g}(h, P_f k)\,\mathrm{vol}(g) \qquad \text{for } h, k \in T_f\mathrm{Imm}.$$

**Theorem** [Bauer, Harms, M, 2011] *Let* $p \geq 1$ *and* $k > \dim(M)/2 + 1$, *and let* $P$ *satisfy the assumptions.*
*Then the geodesic equation has unique local solutions in the Sobolev manifold* $\mathrm{Imm}^{k+2p}$ *of* $H^{k+2p}$*-immersions. The solutions depend smoothly on* $t$ *and on the initial conditions* $f(0, \,.\,)$ *and* $f_t(0, \,.\,)$*. The domain of existence (in* $t$*) is uniform in* $k$ *and thus this also holds in* $\mathrm{Imm}(M, N)$*.*
*Moreover, in each Sobolev completion* $\mathrm{Imm}^{k+2p}$*, the Riemannian exponential mapping* $\exp^P$ *exists and is smooth on a neighborhood of the zero section in the tangent bundle, and* $(\pi, \exp^P)$ *is a diffeomorphism from a (smaller) neigbourhood of the zero section to a neighborhood of the diagonal in* $\mathrm{Imm}^{k+2p} \times \mathrm{Imm}^{k+2p}$*. All these neighborhoods are uniform in* $k > \dim(M)/2 + 1$ *and can be chosen* $H^{k_0+2p}$*-open, for* $k_0 > \dim(M)/2 + 1$*. Thus both properties of the exponential mapping continue to hold in* $\mathrm{Imm}(M, N)$*.*

# Sobolev completions of $\Gamma(E)$, where $E \to M$ is a VB

Fix (background) Riemannian metric $\hat{g}$ on $M$ and its covariant derivative $\nabla^M$. Equip $E$ with a (background) fiber Riemannian metric $\hat{g}^E$ and a compatible covariant derivative $\hat{\nabla}^E$. Then *Sobolev space* $H^k(E)$ is the completion of $\Gamma(E)$ for the Sobolev norm

$$\|h\|_k^2 = \sum_{j=0}^{k} \int_M (\hat{g}^E \otimes \hat{g}_j^0)\big((\hat{\nabla}^E)^j h, (\hat{\nabla}^E)^j h\big) \operatorname{vol}(\hat{g}).$$

This Sobolev space is independ of choices of $\hat{g}$, $\nabla^M$, $\hat{g}^E$ and $\hat{\nabla}^E$ since $M$ is compact: The resulting norms are equivalent.

**Sobolev lemma:** *If $k > \dim(M)/2$ then the identy on $\Gamma(E)$ extends to a injective bounded linear map $H^{k+p}(E) \to C^p(E)$ where $C^p(E)$ carries the supremum norm of all derivatives up to order $p$.*

**Module property of Sobolev spaces:** *If $k > \dim(M)/2$ then pointwise evaluation $H^k(L(E,E)) \times H^k(E) \to H^k(E)$ is bounded bilinear. Likewise all other pointwise contraction operations are multilinear bounded operations.*

# Proof of well-posedness

By assumption 1 the mapping $P_f h$ is of order $2p$ in $f$ and in $h$ where $f$ is the footpoint of $h$. Therefore $f \mapsto P_f$ extends to a smooth section of the smooth Sobolev bundle

$$L\big(T\mathrm{Imm}^{k+2p}; T\mathrm{Imm}^{k} \mid \mathrm{Imm}^{k+2p}\big) \to \mathrm{Imm}^{k+2p},$$

where $T\mathrm{Imm}^k \mid \mathrm{Imm}^{k+2p}$ denotes the space of all $H^k$ tangent vectors with foot point a $H^{k+2p}$ immersion, i.e., the restriction of the bundle $T\mathrm{Imm}^k \to \mathrm{Imm}^k$ to $\mathrm{Imm}^{k+2p} \subset \mathrm{Imm}^k$.
This means that $P_f$ is a bounded linear operator

$$P_f \in L\big(H^{k+2p}(f^*TN), H^k(f^*TN)\big) \quad \text{for} \quad f \in \mathrm{Imm}^{k+2p}.$$

It is injective since it is positive. As an elliptic operator, it is an unbounded operator on the Hilbert completion of $T_f\mathrm{Imm}$ with respect to the $H^0$-metric, and a Fredholm operator $H^{k+2p} \to H^k$ for each $k$. It is selfadjoint elliptic, so the index $=0$. Since it is injective, it is thus also surjective.

By the implicit function theorem on Banach spaces, $f \mapsto P_f^{-1}$ is then a smooth section of the smooth Sobolev bundle

$$L\big(T\mathrm{Imm}^k \mid \mathrm{Imm}^{k+2p}; T\mathrm{Imm}^{k+2p}\big) \to \mathrm{Imm}^{k+2p}$$

As an inverse of an elliptic pseudodifferential operator, $P_f^{-1}$ is also an elliptic pseudo-differential operator of order $-2p$.

By assumption 1 again, $(\nabla P)_f(m,h)$ and $\big(\mathrm{Adj}(\nabla P)_f(m,h)\big)^\perp$ are of order $2p$ in $f, m, h$ (locally). Therefore $f \mapsto P_f$ and $f \mapsto \mathrm{Adj}(\nabla P)^\perp$ extend to smooth sections of the Sobolev bundle

$$L^2\big(T\mathrm{Imm}^{k+2p}; T\mathrm{Imm}^k \mid \mathrm{Imm}^{k+2p}\big) \to \mathrm{Imm}^{k+2p}$$

Using the module property of Sobolev spaces, one obtains that the "Christoffel symbols"

$$\begin{aligned}\Gamma_f(h,h) = \frac{1}{2}P^{-1}\Big(&\mathrm{Adj}(\nabla P)(h,h)^\perp - 2.Tf.\overline{g}(Ph,\nabla h)^\sharp \\ &- \overline{g}(Ph,h).\mathrm{Tr}^g(S) - (\nabla_h P)h - \mathrm{Tr}^g\big(\overline{g}(\nabla h, Tf)\big)Ph\Big)\end{aligned}$$

extend to a smooth ($C^\infty$) section of the smooth Sobolev bundle

$$L^2_{\mathrm{sym}}\big(T\mathrm{Imm}^{k+2p}; T\mathrm{Imm}^{k+2p}\big) \to \mathrm{Imm}^{k+2p}$$

Thus $h \mapsto \Gamma_f(h,h)$ is a smooth quadratic mapping $T\mathrm{Imm} \to T\mathrm{Imm}$ which extends to smooth quadratic mappings $T\mathrm{Imm}^{k+2p} \to T\mathrm{Imm}^{k+2p}$ for each $k \geq \frac{\dim(2)}{2}+1$. The geodesic equation $\boxed{\nabla^{\overline{g}}_{\partial_t} f_t = \Gamma_f(f_t,f_t)}$ can be reformulated using the linear connection $C^g : TN \times_N TN \to TTN$ (horizontal lift mapping) of $\nabla^{\overline{g}}$:

$$\partial_t f_t = C\Big(\frac{1}{2}H_f(f_t,f_t) - K_f(f_t,f_t), f_t\Big).$$

The right-hand side is a smooth vector field on $T\mathrm{Imm}^{k+2p}$, the geodesic spray. Note that the restriction to $T\mathrm{Imm}^{k+1+2p}$ of the geodesic spray on $T\mathrm{Imm}^{k+2p}$ equals the geodesic spray there. By the theory of smooth ODE's on Banach spaces, the flow of this vector field exists in $T\mathrm{Imm}^{k+2p}$ and is smooth in time and in the initial condition, for all $k \geq \frac{\dim(2)}{2}+1$.
It remains to show that the domain of existence is independent of $k$. I omit this. QED

# The elastic metric

$$G_c^{a,b}(h,k) = \int_0^{2\pi} a^2 \langle D_s h, n\rangle \langle D_s k, n\rangle + b^2 \langle D_s h, v\rangle \langle D_s k, v\rangle \, ds,$$

$$\boxed{\begin{aligned} c_t &= u \in C^\infty(\mathbb{R}_{>0} \times M, \mathbb{R}^2) \\ L(u_t) &= L(\frac{1}{2} H_c(u,u) - K_c(u,u)) \\ &= \frac{1}{2}(\delta_0 - \delta_{2\pi})\Big(\langle D_s u, D_s u\rangle v + \tfrac{3}{4}\langle v, D_s u\rangle^2 v \\ &\qquad - 2\langle D_s u, v\rangle D_s u - \tfrac{3}{2}\langle n, D_s u\rangle^2 v\Big) \\ &\qquad + D_s\Big(\langle D_s u, D_s u\rangle v + \tfrac{3}{4}\langle v, D_s u\rangle^2 v \\ &\qquad - 2\langle D_s u, v\rangle D_s u - \tfrac{3}{2}\langle n, D_s u\rangle^2 v\Big) \end{aligned}}$$

Note: Only a metric on Imm/transl.

# Representation of the elastic metrics

Aim: Represent the class of elastic metrics as the pullback metric of a flat metric on $C^\infty(M, \mathbb{R}^2)$, i.e.: find a map

$$R : \mathrm{Imm}(M, \mathbb{R}^2) \mapsto C^\infty(M, \mathbb{R}^n)$$

such that

$$G_c^{a,b}(h, k) = R^*\langle h, k\rangle_{L^2} = \langle T_c R.h, T_c R.k\rangle_{L^2}.$$

[YounesMichorShahMumford2008] [SrivastavaKlassenJoshiJermyn2011]

# The $R$ transform on open curves

**Theorem**

*The metric $G^{a,b}$ is the pullback of the flat $L^2$ metric via the transform $R$:*

$$
\begin{aligned}
R^{a,b}:\ \operatorname{Imm}([0,2\pi],\mathbb{R}^2) &\to C^\infty([0,2\pi],\mathbb{R}^3) \\
R^{a,b}(c) &= |c'|^{1/2}\left(a\begin{pmatrix} v \\ 0\end{pmatrix} + \sqrt{4b^2-a^2}\begin{pmatrix} 0 \\ 1\end{pmatrix}\right) .
\end{aligned}
$$

*The metric $G^{a,b}$ is flat on open curves, geodesics are the preimages under the $R$-transform of geodesics on the flat space* $\operatorname{im} R$ *and the geodesic distance between $c, \overline{c} \in \operatorname{Imm}([0,2\pi],\mathbb{R}^2)/\operatorname{trans}$ is given by the integral over the pointwise distance in the image* $\operatorname{Im}(R)$*.*
*The curvature on $B([0,2\pi],\mathbb{R}^2)$ is non-negative.*

[BauerBruverisCotterMarslandMichor2012]

# The $R$ transform on open curves II

Image of $R$ is characterized by the condition:

$$(4b^2 - a^2)(R_1^2(c) + R_2^2(c)) = a^2 R_3^2(c)$$

Define the flat cone

$$C^{a,b} = \{q \in \mathbb{R}^3 : (4b^2 - a^2)(q_1^2 + q_2^2) = a^2 q_3^2, q_3 > 0\}.$$

Then $\operatorname{Im} R = C^\infty(S^1, C^{a,b})$. The inverse of $R$ is given by:

$$R^{-1} : \operatorname{im} R \to \operatorname{Imm}([0, 2\pi], \mathbb{R}^2)/\operatorname{trans}$$
$$R^{-1}(q)(\theta) = p_0 + \frac{1}{2ab} \int_0^\theta |q(\theta)| \begin{pmatrix} q_1(\theta) \\ q_2(\theta) \end{pmatrix} d\theta\,.$$

# The $R$ transform on closed curves I

Characterize image using the inverse:

$$R^{-1}(q)(\theta) = p_0 + \frac{1}{2ab} \int_0^\theta |q(\theta)| \begin{pmatrix} q_1(\theta) \\ q_2(\theta) \end{pmatrix} d\theta \,.$$

$R^{-1}(q)(\theta)$ is closed iff

$$F(q) = \int_0^{2\pi} |q(\theta)| \begin{pmatrix} q_1(\theta) \\ q_2(\theta) \end{pmatrix} d\theta = 0$$

A basis of the orthogonal complement $\left(T_q \mathscr{C}^{a,b}\right)^\perp$ is given by the two gradients $\operatorname{grad}^{L^2} F_i(q)$

# The $R$ transform on closed curves II

**Theorem**

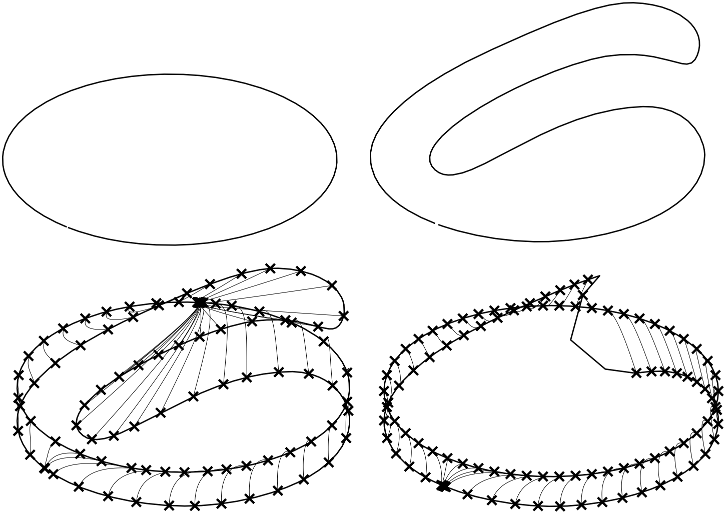
*The image $\mathscr{C}^{a,b}$ of the manifold of closed curves under the $R$-transform is a codimension 2 submanifold of the flat space $\mathrm{Im}(R)_{open}$. A basis of the orthogonal complement $\left(T_q\mathscr{C}^{a,b}\right)^{\perp}$ is given by the two vectors*

$$U_1(q) = \frac{1}{\sqrt{q_1^2+q_2^2}}\begin{pmatrix} 2q_1^2+q_2^2 \\ q_1 q_2 \\ 0 \end{pmatrix} + \frac{2}{a}\sqrt{4b^2-a^2}\begin{pmatrix} 0 \\ 0 \\ q_1 \end{pmatrix},$$

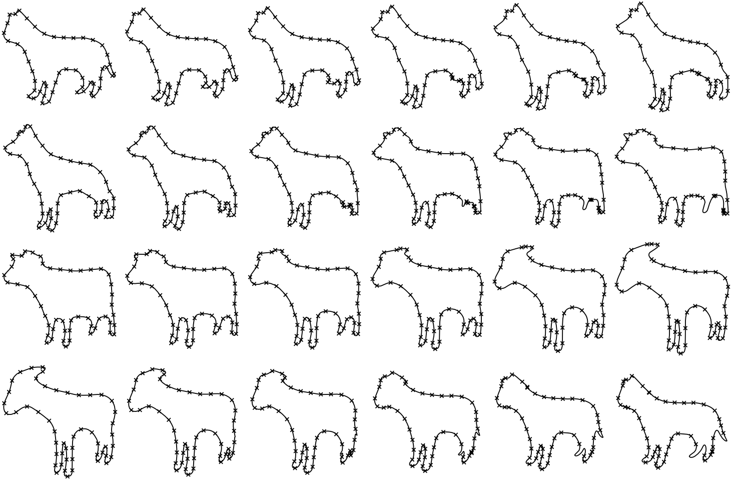
$$U_2(q) = \frac{1}{\sqrt{q_1^2+q_2^2}}\begin{pmatrix} q_1 q_2 \\ q_1^2+2q_2^2 \\ 0 \end{pmatrix} + \frac{2}{a}\sqrt{4b^2-a^2}\begin{pmatrix} 0 \\ 0 \\ q_2 \end{pmatrix}.$$

[BauerBruverisCotterMarslandMichor2012]

# compress and stretch

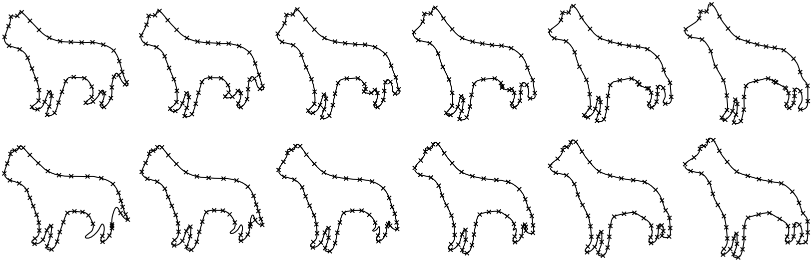


# A geodesic Rectangle

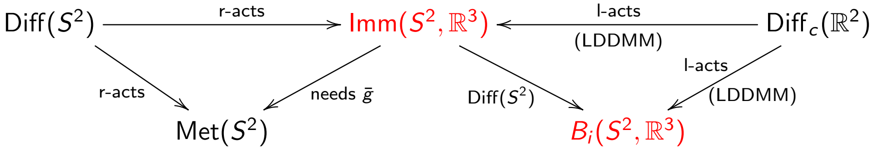


# Non-symmetric distances

| $I_1$ | $I_2$ | $I_1 \to I_2$ # iterations | $I_1 \to I_2$ # points | $I_1 \to I_2$ distance | $I_2 \to I_1$ # iterations | $I_2 \to I_1$ # points | $I_2 \to I_1$ distance | %diff |
|---|---|---|---|---|---|---|---|---|
| cat | cow | 28 | 456 | 7.339 | 33 | 462 | 8.729 | 15.9 |
| cat | dog | 36 | 475 | 8.027 | 102 | 455 | 10.060 | 20.2 |
| cat | donkey | 73 | 476 | 12.620 | 102 | 482 | 12.010 | 4.8 |
| cow | donkey | 32 | 452 | 7.959 | 26 | 511 | 7.915 | 0.6 |
| dog | donkey | 15 | 457 | 8.299 | 10 | 476 | 8.901 | 6.8 |
| shark | airplane | 63 | 491 | 13.741 | 40 | 487 | 13.453 | 2.1 |



# An example of a metric space with strongly negatively curved regions




$$G_f^{\Phi}(h,k) = \int_M \Phi(f)\overline{g}(h,k)\,\mathrm{vol}(g)$$

[BauerHarmsMichor2012]

# Non-vanishing geodesic distance

The pathlength metric on shape space induced by $G^\Phi$ separates points if one of the following holds:

- $\Phi \geq C_1 + C_2 \|\mathrm{Tr}^g(S)\|^2$ with $C_1, C_2 > 0$ or
- $\Phi \geq C_3 \,\mathrm{Vol}$

This leads us to consider $\Phi = \Phi(\mathrm{Vol}, \|\mathrm{Tr}^g(S)\|^2)$.
Special cases:

- $G^A$-metric: $\Phi = 1 + A\|\mathrm{Tr}^g(S)\|^2$
- Conformal metrics: $\Phi = \Phi(\mathrm{Vol})$

# Geodesic equation on shape space $B_i(M, \mathbb{R}^n)$, with $\Phi = \Phi(\mathrm{Vol}, \mathrm{Tr}(L))$

$$
\begin{aligned}
f_t &= a.\nu, \\
a_t &= \frac{1}{\Phi}\Big[\frac{\Phi}{2} a^2 \,\mathrm{Tr}(L) - \frac{1}{2}\mathrm{Tr}(L)\int_M (\partial_1\Phi) a^2 \,\mathrm{vol}(g) - \frac{1}{2} a^2 \Delta(\partial_2\Phi) \\
&\quad + 2a g^{-1}(d(\partial_2\Phi), da) + (\partial_2\Phi)\|da\|^2_{g^{-1}} \\
&\quad + (\partial_1\Phi) a \int_M \mathrm{Tr}(L).a \,\mathrm{vol}(g) - \frac{1}{2}(\partial_2\Phi)\,\mathrm{Tr}(L^2) a^2\Big]
\end{aligned}
$$

## Sectional curvature on $B_i$

Chart for $B_i$ centered at $\pi(f_0)$ so that $\pi(f_0) = 0$ in this chart:

$$a \in C^\infty(M) \longleftrightarrow \pi(f_0 + a.\nu^{f_0}).$$

For a linear 2-dim. subspace $P \subset T_{\pi(f_0)}B_i$ spanned by $a_1, a_1$, the sectional curvature is defined as:

$$k(P) = -\frac{G^\Phi_{\pi(f_0)}\big(\mathcal{R}_{\pi(f_0)}(a_1, a_2)a_1, a_2\big)}{\|a_1\|^2\|a_2\|^2 - G^\Phi_{\pi(f_0)}(a_1, a_2)^2}\ , \text{ where}$$

$$\begin{aligned}
R_0(a_1, a_2, a_1, a_2) = G_0^\Phi(R_0(a_1, a_2)a_1, a_2) = \\
\frac{1}{2}d^2 G_0^\Phi(a_1, a_1)(a_2, a_2) + \frac{1}{2}d^2 G_0^\Phi(a_2, a_2)(a_1, a_1) \\
- d^2 G_0^\Phi(a_1, a_2)(a_1, a_2) \\
+ G_0^\Phi(\Gamma_0(a_1, a_1), \Gamma_0(a_2, a_2)) - G_0^\Phi(\Gamma_0(a_1, a_2), \Gamma_0(a_1, a_2)).
\end{aligned}$$

# Sectional curvature on $B_i$ for $\Phi = \mathrm{Vol}$

$$k(P) = -\frac{\mathcal{R}_0(a_1, a_2, a_1, a_2)}{\|a_1\|^2\|a_2\|^2 - G^{\Phi}_{\pi(f_0)}(a_1, a_2)^2}\ , \text{ where}$$

$$\begin{aligned}
R_0(a_1, a_2, a_1, a_2) &= -\frac{1}{2}\mathrm{Vol}\int_M \|a_1 da_2 - a_2 da_1\|^2_{g^{-1}}\,\mathrm{vol}(g) \\
&+ \frac{1}{4\,\mathrm{Vol}}\overline{\mathrm{Tr}(L)^2}\Big(\overline{a_1^2}.\overline{a_2^2} - \overline{a_1.a_2}^2\Big) \\
&+ \frac{1}{4}\Big(\overline{a_1^2}.\overline{\mathrm{Tr}(L)^2 a_2^2} - 2\overline{a_1.a_2}.\overline{\mathrm{Tr}(L)^2 a_1.a_2} + \overline{a_2^2}.\overline{\mathrm{Tr}(L)^2 a_1^2}\Big) \\
&- \frac{3}{4\,\mathrm{Vol}}\Big(\overline{a_1^2}.\overline{\mathrm{Tr}(L)a_2}^2 - 2\overline{a_1.a_2}.\overline{\mathrm{Tr}(L)a_1}.\overline{\mathrm{Tr}(L)a_2} + \overline{a_2^2}.\overline{\mathrm{Tr}(L)a_1}^2\Big) \\
&+ \frac{1}{2}\Big(\overline{a_1^2}.\overline{\mathrm{Tr}^g((da_2)^2)} - 2\overline{a_1.a_2}.\overline{\mathrm{Tr}^g(da_1.da_2)} + \overline{a_2^2}\overline{\mathrm{Tr}^g((da_1)^2)}\Big) \\
&- \frac{1}{2}\Big(\overline{a_1^2}.\overline{a_2^2.\,\mathrm{Tr}(L^2)} - 2.\overline{a_1.a_2}.\overline{a_1.a_2.\,\mathrm{Tr}(L^2)} + \overline{a_2^2}.\overline{a_1^2.\,\mathrm{Tr}(L^2)}\Big).
\end{aligned}$$

## Sectional curvature on $B_i$ for $\Phi = 1 + A\,\mathrm{Tr}(L)^2$

$$k(P) = -\frac{\mathcal{R}_0(a_1, a_2, a_1, a_2)}{\|a_1\|^2\|a_2\|^2 - G^{\Phi}_{\pi(f_0)}(a_1, a_2)^2}\ , \text{ where}$$

$$R_0(a_1, a_2, a_1, a_2) = \int_M A(a_1\Delta a_2 - a_2\Delta a_1)^2\,\mathrm{vol}(g)$$

$$+ \int_M 2A\,\mathrm{Tr}(L) g_2^0\big((a_1 da_2 - a_2 da_1) \otimes (a_1 da_2 - a_2 da_1), s\big)\,\mathrm{vol}(g)$$

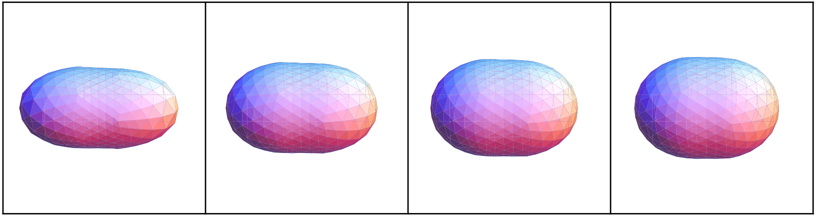
$$+ \int_M \frac{1}{1 + A\,\mathrm{Tr}(L)^2}\Big[ -4A^2 g^{-1}\big(d\,\mathrm{Tr}(L), a_1 da_2 - a_2 da_1\big)^2$$

$$- \Big(\frac{1}{2}\big(1 + A\,\mathrm{Tr}(L)^2\big)^2 + 2A^2\,\mathrm{Tr}(L)\Delta(\mathrm{Tr}(L)) + 2A^2\,\mathrm{Tr}(L^2)\,\mathrm{Tr}(L)^2\Big)\cdot$$

$$\cdot \|a_1 da_2 - a_2 da_1\|^2_{g^{-1}} + (2A^2\,\mathrm{Tr}(L)^2)\|da_1 \wedge da_2\|^2_{g_0^2}$$

$$+ (8A^2\,\mathrm{Tr}(L)) g_2^0\big(d\,\mathrm{Tr}(L) \otimes (a_1 da_2 - a_2 da_1), da_1 \wedge da_2\big)\Big]\,\mathrm{vol}(g)$$

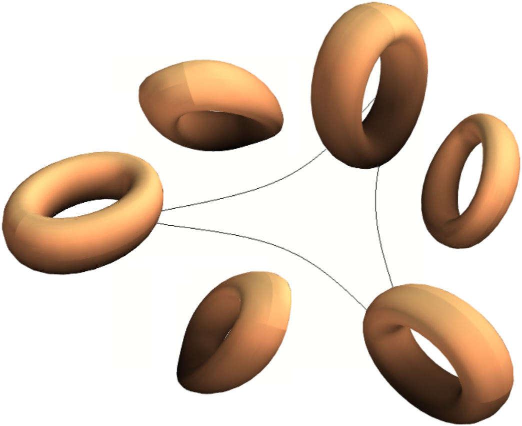
# Negative Curvature: A toy example



Ex1: $\Phi = 1 + .4\,\mathrm{Tr}(L)^2$ Ex2: $\Phi = e^{Vol}$ Ex3: $\Phi = e^{Vol}$

# Another toy example

$G_f^\Phi(h,k) = \int_{\mathbb{T}^2} \overline{g}((1+\Delta)h,k)\,\mathrm{vol}(g)$ on $\mathrm{Imm}(\mathbb{T}^2,\mathbb{R}^3)$:



[BauerBruveris2011]

# Special case: Diffeomorphism groups.

For $M = N$ the space $\text{Emb}(M, M)$ equals the *diffeomorphism group of* $M$. An operator $P \in \Gamma\big(L(T\,\text{Emb}; T\,\text{Emb})\big)$ that is invariant under reparametrizations induces a right-invariant Riemannian metric on this space. Thus one gets the geodesic equation for right-invariant Sobolev metrics on diffeomorphism groups and well-posedness of this equation. The geodesic equation on $\text{Diff}(M)$ in terms of the momentum $p$ is given by

$$\begin{cases} p = Pf_t \otimes \text{vol}(g), \\ \nabla_{\partial_t} p = -Tf.\bar{g}(Pf_t, \nabla f_t)^\sharp \otimes \text{vol}(g). \end{cases}$$

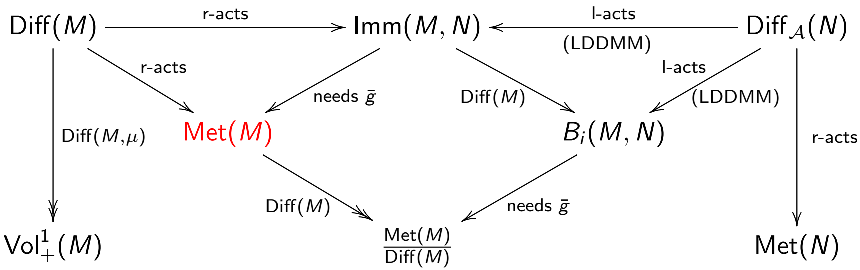
Note that this equation is not right-trivialized, in contrast to the equation given in [Arnold 1966]. The special case of theorem now reads as follows:

**Theorem.** [Bauer, Harms, M, 2011] *Let* $p \geq 1$ *and* $k > \frac{\dim(M)}{2} + 1$ *and let* $P$ *satisfy the assumptions.*

*The initial value problem for the geodesic equation has unique local solutions in the Sobolev manifold* $\mathrm{Diff}^{k+2p}$ *of* $H^{k+2p}$*-diffeomorphisms. The solutions depend smoothly on* $t$ *and on the initial conditions* $f(0, \,.\,)$ *and* $f_t(0, \,.\,)$*. The domain of existence (in* $t$*) is uniform in* $k$ *and thus this also holds in* $\mathrm{Diff}(M)$*.*

*Moreover, in each Sobolev completion* $\mathrm{Diff}^{k+2p}$*, the Riemannian exponential mapping* $\exp^P$ *exists and is smooth on a neighborhood of the zero section in the tangent bundle, and* $(\pi, \exp^P)$ *is a diffeomorphism from a (smaller) neigbourhood of the zero section to a neighborhood of the diagonal in* $\mathrm{Diff}^{k+2p} \times \mathrm{Diff}^{k+2p}$*. All these neighborhoods are uniform in* $k > \dim(M)/2 + 1$ *and can be chosen* $H^{k_0+2p}$*-open, for* $k_0 > \dim(M)/2 + 1$*. Thus both properties of the exponential mapping continue to hold in* $\mathrm{Diff}(M)$*.*

# About Met($M$)



Diff(M)
r-acts
Imm(M, N)
l-acts
(LDDMM)
Diff_A(N)
r-acts
needs ḡ
Diff(M)
l-acts
(LDDMM)
Diff(M,μ)
Met(M)
B_i(M, N)
r-acts
Diff(M)
needs ḡ
Vol^1_+(M)
Met(M)/Diff(M)
Met(N)

## Weak Riemann metrics on Met($M$)

All of them are Diff($M$)-invariant; natural, tautological.

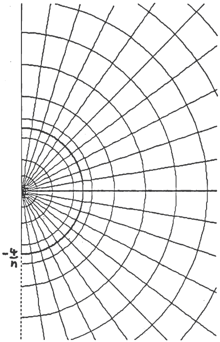
$$
\begin{aligned}
G_g(h,k) &= \int_M g_2^0(h,k)\,\mathrm{vol}(g) = \int \mathrm{Tr}(g^{-1}hg^{-1}k)\,\mathrm{vol}(g), \quad L^2\text{-metr.}\\
\text{or } &= \Phi(\mathrm{Vol(g)}) \int_M g_2^0(h,k)\,\mathrm{vol}(g) \qquad \text{conformal}\\
\text{or } &= \int_M \Phi(\mathrm{Scal}^g).g_2^0(h,k)\,\mathrm{vol}(g) \qquad \text{curvature modified}\\
\text{or } &= \int_M g_2^0((1+\Delta^g)^p h,k)\,\mathrm{vol}(g) \qquad \text{Sobolev order } p\\
\text{or } &= \int_M \Big(g_2^0(h,k) + g_3^0(\nabla^g h, \nabla^g k) + \dots\\
&\qquad + g_p^0((\nabla^g)^p h, (\nabla^g)^p k)\Big)\,\mathrm{vol}(g)
\end{aligned}
$$

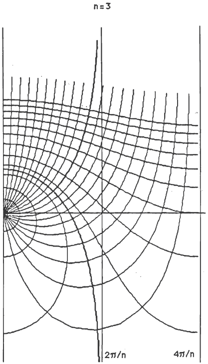
where $\Phi$ is a suitable real-valued function, $\mathrm{Vol} = \int_M \mathrm{vol}(g)$ is the total volume of $(M,g)$, Scal is the scalar curvature of $(M,g)$, and where $g_2^0$ is the induced metric on $\binom{0}{2}$-tensors.

# The $L^2$-metric on the space of all Riemann metrics

[Ebin 1970]. Geodesics and curvature [Freed Groisser 1989]. [Gil-Medrano Michor 1991] for non-compact $M$. [Clarke 2009] showed that geodesic distance for the $L^2$-metric is positive, and he determined the metric completion of $\mathrm{Met}(M)$.
The geodesic equation is completely decoupled from space, it is an ODE:

$$g_{tt} = g_t g^{-1} g_t + \tfrac{1}{4}\operatorname{Tr}(g^{-1} g_t g^{-1} g_t)\, g - \tfrac{1}{2}\operatorname{Tr}(g^{-1} g_t)\, g_t$$






$$\exp_0(A) = \frac{2}{n}\log\left((1+\tfrac{1}{4}\operatorname{Tr}(A))^2 + \tfrac{n}{16}\operatorname{Tr}(A_0^2)\right) Id \\ + \frac{4}{\sqrt{n\operatorname{Tr}(A_0^2)}}\arctan\left(\frac{\sqrt{n\operatorname{Tr}(A_0^2)}}{4+\operatorname{Tr}(A)}\right) A_0.$$

# Back to the the general metric on Met($M$).

We describe all these metrics uniformly as

$$G_g^P(h,k) = \int_M g_2^0(P_g h, k)\,\mathrm{vol}(g)$$
$$= \int_M \mathrm{Tr}(g^{-1}.P_g(h).g^{-1}.k)\,\mathrm{vol}(g),$$

where

$$P_g : \Gamma(S^2T^*M) \to \Gamma(S^2T^*M)$$

is a positive, symmetric, bijective pseudo-differential operator of order $2p$, $p \geq 0$, depending smoothly on the metric $g$, and also Diff($M$)-equivariantly:

$$\varphi^* \circ P_g = P_{\varphi^* g} \circ \varphi^*$$

The geodesic equation in this notation:

$$g_{tt} = P^{-1}\Big[(D_{(g,.)}Pg_t)^*(g_t) + \frac{1}{4}.g.\operatorname{Tr}(g^{-1}.Pg_t.g^{-1}.g_t) \\ + \frac{1}{2}g_t.g^{-1}.Pg_t + \frac{1}{2}Pg_t.g^{-1}.g_t - (D_{(g,g_t)}P)g_t \\ - \frac{1}{2}\operatorname{Tr}(g^{-1}.g_t).Pg_t\Big]$$

We can rewrite this equation to get it in a slightly more compact form:

$$(Pg_t)_t = (D_{(g,g_t)}P)g_t + Pg_{tt} \\ = (D_{(g,.)}Pg_t)^*(g_t) + \frac{1}{4}.g.\operatorname{Tr}(g^{-1}.Pg_t.g^{-1}.g_t) \\ + \frac{1}{2}g_t.g^{-1}.Pg_t + \frac{1}{2}Pg_t.g^{-1}.g_t - \frac{1}{2}\operatorname{Tr}(g^{-1}.g_t).Pg_t$$

# Well posedness of geodesic equation.

**Assumptions** *Let $P_g(h)$, $P_g^{-1}(k)$ and $(D_{(g,.)}Ph)^*(m)$ be linear pseudo-differential operators of order $2p$ in $m, h$ and of order $-2p$ in $k$ for some $p \geq 0$.*
*As mappings in the foot point $g$, we assume that all mappings are non-linear, and that they are a composition of operators of the following type:*
*(a) Non-linear differential operators of order $l \leq 2p$, i.e.,*

$$A(g)(x) = A\big(x, g(x), (\hat{\nabla} g)(x), \dots, (\hat{\nabla}^l g)(x)\big),$$

*(b) Linear pseudo-differential operators of order $\leq 2p$,*
*such that the total (top) order of the composition is $\leq 2p$.*
Since $h \mapsto P_g h$ induces a weak inner product, it is a symmetric and injective pseudodifferential operator. We *assume* that it is elliptic and selfadjoint. Then it is Fredholm and has vanishing index. Thus it is invertible and $g \mapsto P_g^{-1}$ is smooth
$H^k(S^2_+ T^*M) \to L(H^k(S^2 T^*M), H^{k+2p}(S^2 T^*M))$ by the implicit function theorem on Banach spaces.

**Theorem.** [Bauer, Harms, M. 2011] *Let the assumptions above hold. Then for* $k > \frac{\dim(M)}{2}$*, the initial value problem for the geodesic equation has unique local solutions in the Sobolev manifold* $\mathrm{Met}^{k+2p}(M)$ *of* $H^{k+2p}$*-metrics. The solutions depend* $C^\infty$ *on* $t$ *and on the initial conditions* $g(0, \,.\,) \in \mathrm{Met}^{k+2p}(M)$ *and* $g_t(0, \,.\,) \in H^{k+2p}(S^2 T^* M)$*. The domain of existence (in* $t$*) is uniform in* $k$ *and thus this also holds in* $\mathrm{Met}(M)$*.*
*Moreover, in each Sobolev completion* $\mathrm{Met}^{k+2p}(M)$*, the Riemannian exponential mapping* $\exp^P$ *exists and is smooth on a neighborhood of the zero section in the tangent bundle, and* $(\pi, \exp^P)$ *is a diffeomorphism from a (smaller) neighborhood of the zero section to a neighborhood of the diagonal in* $\mathrm{Met}^{k+2p}(M) \times \mathrm{Met}^{k+2p}(M)$*. All these neighborhoods are uniform in* $k > \frac{\dim(M)}{2}$ *and can be chosen* $H^{k_0+2p}$*-open, where* $k_0 > \frac{\dim(M)}{2}$*. Thus all properties of the exponential mapping continue to hold in* $\mathrm{Met}(M)$*.*

# Conserved Quantities on $\mathrm{Met}(M)$.

Right action of $\mathrm{Diff}(M)$ on $\mathrm{Met}(M)$ given by

$$(g, \phi) \mapsto \phi^* g.$$

Fundamental vector field (infinitesimal action):

$$\zeta_X(g) = \mathcal{L}_X g = -2 \operatorname{Sym} \nabla(g(X)).$$

If metric $G^P$ is invariant, we have the following conserved quantities

$$\begin{aligned} \text{const} &= G^P(g_t, \zeta_X(g)) \\ &= -2 \int_M g_1^0 \big( \nabla^* \operatorname{Sym} P g_t, g(X) \big) \operatorname{vol}(g) \\ &= -2 \int_M g \big( g^{-1} \nabla^* P g_t, X \big) \operatorname{vol}(g) \end{aligned}$$

Since this holds for all vector fields $X$,

$$\boxed{(\nabla^* P g_t) \operatorname{vol}(g) \in \Gamma(T^*M \otimes_M \operatorname{vol}(M)) \text{ is const. in } t.}$$

Thank you for your attention