

DENJOY-CARLEMAN DIFFERENTIABLE PERTURBATION OF POLYNOMIALS AND UNBOUNDED OPERATORS

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ABSTRACT. Let $t \mapsto A(t)$ for $t \in T$ be a C^M -mapping with values unbounded operators with compact resolvents and common domain of definition which are self-adjoint or normal. Here C^M stands for C^ω (real analytic), a quasianalytic or non-quasianalytic Denjoy-Carleman class, C^∞ , or a Hölder continuity class $C^{0,\alpha}$. The parameter domain T is either \mathbb{R} or \mathbb{R}^n or an infinite dimensional convenient vector space. We prove and review results on C^M -dependence on t of the eigenvalues and eigenvectors of $A(t)$.

Theorem. *Let $t \mapsto A(t)$ for $t \in T$ be a parameterized family of unbounded operators in a Hilbert space H with common domain of definition and with compact resolvent. If $t \in T = \mathbb{R}$ and all $A(t)$ are self-adjoint then the following holds:*

- (A) *If $A(t)$ is real analytic in $t \in \mathbb{R}$, then the eigenvalues and the eigenvectors of $A(t)$ can be parameterized real analytically in t .*
- (B) *If $A(t)$ is quasianalytic of class C^Q in $t \in \mathbb{R}$, then the eigenvalues and the eigenvectors of $A(t)$ can be parameterized C^Q in t .*
- (C) *If $A(t)$ is non-quasianalytic of class C^L in $t \in \mathbb{R}$ and if no two different continuously parameterized eigenvalues (e.g., ordered by size) meet of infinite order at any $t \in \mathbb{R}$, then the eigenvalues and the eigenvectors of $A(t)$ can be parameterized C^L in t .*
- (D) *If $A(t)$ is C^∞ in $t \in \mathbb{R}$ and if no two different continuously parameterized eigenvalues meet of infinite order at any $t \in \mathbb{R}$, then the eigenvalues and the eigenvectors of $A(t)$ can be parameterized C^∞ in t .*
- (E) *If $A(t)$ is C^∞ in $t \in \mathbb{R}$, then the eigenvalues of $A(t)$ can be parameterized twice differentiably in t .*
- (F) *If $A(t)$ is $C^{1,\alpha}$ in $t \in \mathbb{R}$ for some $\alpha > 0$, then the eigenvalues of $A(t)$ can be parameterized C^1 in t .*

If $t \in T = \mathbb{R}$ and all $A(t)$ are normal then the following holds:

- (G) *If $A(t)$ is real analytic in $t \in \mathbb{R}$, then for each $t_0 \in \mathbb{R}$ and for each eigenvalue z_0 of $A(t_0)$ there exists $N \in \mathbb{N}_{>0}$ such that the eigenvalues near z_0 of $A(t_0 \pm s^N)$ and their eigenvectors can be parameterized real analytically in s near $s = 0$.*
- (H) *If $A(t)$ is quasianalytic of class C^Q in $t \in \mathbb{R}$, then for each $t_0 \in \mathbb{R}$ and for each eigenvalue z_0 of $A(t_0)$ there exists $N \in \mathbb{N}_{>0}$ such that the eigenvalues near z_0 of $A(t_0 \pm s^N)$ and their eigenvectors can be parameterized C^Q in s near $s = 0$.*
- (I) *If $A(t)$ is non-quasianalytic of class C^L in $t \in \mathbb{R}$, then for each $t_0 \in \mathbb{R}$ and for each eigenvalue z_0 of $A(t_0)$ at which no two of the different continuously*

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parameterized eigenvalues (which is always possible by [12, II 5.2]) meet of infinite order, there exists $N \in \mathbb{N}_{>0}$ such that the eigenvalues near z_0 of $A(t_0 \pm s^N)$ and their eigenvectors can be parameterized C^L in s near $s = 0$.

- (J) If $A(t)$ is C^∞ in $t \in \mathbb{R}$, then for each $t_0 \in \mathbb{R}$ and for each eigenvalue z_0 of $A(t_0)$ at which no two of the different continuously parameterized eigenvalues meet of infinite order, there exists $N \in \mathbb{N}_{>0}$ such that the eigenvalues near z_0 of $A(t_0 \pm s^N)$ and their eigenvectors can be parameterized C^∞ in s near $s = 0$.
- (K) If $A(t)$ is C^∞ in $t \in \mathbb{R}$, then for each $t_0 \in \mathbb{R}$ and for each eigenvalue z_0 of $A(t_0)$ at which no two of the different continuously parameterized eigenvalues meet of infinite order, the eigenvalues near z_0 of $A(t)$ and their eigenvectors can be parameterized by absolutely continuous functions in t near $t = t_0$.

If $t \in T = \mathbb{R}^n$ and all $A(t)$ are normal then the following holds:

- (L) If $A(t)$ is real analytic or quasianalytic of class C^Q in $t \in \mathbb{R}^n$, then for each $t_0 \in \mathbb{R}^n$ and for each eigenvalue z_0 of $A(t_0)$, there exist a neighborhood D of z_0 in \mathbb{C} , a neighborhood W of t_0 in \mathbb{R}^n , and a finite covering $\{\pi_k : U_k \rightarrow W\}$ of W , where each π_k is a composite of finitely many mappings each of which is either a local blow-up along a real analytic or C^Q submanifold or a local power substitution, such that the eigenvalues of $A(\pi_k(s))$, $s \in U_k$, in D and the corresponding eigenvectors can be parameterized real analytically or C^Q in s . If A is self-adjoint, then we do not need power substitutions.
- (M) If $A(t)$ is real analytic or quasianalytic of class C^Q in $t \in \mathbb{R}^n$, then for each $t_0 \in \mathbb{R}^n$ and for each eigenvalue z_0 of $A(t_0)$, there exist a neighborhood D of z_0 in \mathbb{C} and a neighborhood W of t_0 in \mathbb{R}^n such that the eigenvalues of $A(t)$, $t \in W$, in D and the corresponding eigenvectors can be parameterized by functions which are special functions of bounded variation (SBV), see [9] or [3], in t .

If $t \in T \subseteq E$, a c^∞ -open subset in a finite or infinite dimensional convenient vector space then the following holds:

- (N) For $0 < \alpha \leq 1$, if $A(t)$ is $C^{0,\alpha}$ (Hölder continuous of exponent α) in $t \in T$ and all $A(t)$ are self-adjoint, then the eigenvalues of $A(t)$ can be parameterized $C^{0,\alpha}$ in t .
- (O) For $0 < \alpha \leq 1$, if $A(t)$ is $C^{0,\alpha}$ in $t \in T$ and all $A(t)$ are normal, then we have: For each $t_0 \in T$ and each eigenvalue z_0 of $A(t_0)$ consider a simple closed C^1 -curve γ in the resolvent set of $A(t_0)$ enclosing only z_0 among all eigenvalues of $A(t_0)$. Then for t near t_0 in the c^∞ -topology on T , no eigenvalue of $A(t)$ lies on γ . Let $\lambda(t) = (\lambda_1(t), \dots, \lambda_N(t))$ be the N -tuple of all eigenvalues (repeated according to their multiplicity) of $A(t)$ inside of γ . Then $t \mapsto \lambda(t)$ is $C^{0,\alpha}$ for t near t_0 with respect to the non-separating metric

$$d(\lambda, \mu) = \min_{\sigma \in S_N} \max_{1 \leq i \leq N} |\lambda_i - \mu_{\sigma(i)}|$$

on the space of N -tuples.

Part (A) is due to Rellich [22] in 1942, see also [4] and [12, VII 3.9]. Part (D) has been proved in [2, 7.8], see also [13, 50.16], in 1997, which contains also a different proof of (A). (E) and (F) have been proved in [14] in 2003. (G) was proved in [19, 7.1]; it can be proved as (H) with some obvious changes, but it is not a special case since C^ω does not correspond to a sequence which is an \mathcal{L} -intersection (see ‘definitions and remarks’ below and [16]). (J) and (K) were proved in [19, 7.1]. (N) was proved in [17].

The purpose of this paper is to prove the remaining parts (B), (C), (H), (I), (L), (M), and (O).

Definitions and remarks. Let $M = (M_k)_{k \in \mathbb{N} = \mathbb{N}_{\geq 0}}$ be an increasing sequence ($M_{k+1} \geq M_k$) of positive real numbers with $M_0 = 1$. Let $U \subseteq \mathbb{R}^n$ be open. We denote by $C^M(U)$ the set of all $f \in C^\infty(U)$ such that, for each compact $K \subseteq U$, there exist positive constants C and ρ such that

$$|\partial^\alpha f(x)| \leq C \rho^{|\alpha|} |\alpha|! M_{|\alpha|} \quad \text{for all } \alpha \in \mathbb{N}^n \text{ and } x \in K.$$

The set $C^M(U)$ is a *Denjoy–Carleman class* of functions on U . If $M_k = 1$, for all k , then $C^M(U)$ coincides with the ring $C^\omega(U)$ of real analytic functions on U . In general, $C^\omega(U) \subseteq C^M(U) \subseteq C^\infty(U)$.

Throughout this paper $Q = (Q_k)_{k \in \mathbb{N}}$ is a sequence as above which is log-convex (i.e., $Q_k^2 \leq Q_{k-1}Q_{k+1}$ for all k), derivation closed (i.e., $\sup_k (Q_{k+1}/Q_k)^{1/k} < \infty$), quasianalytic (i.e., $\sum_k (k! Q_k)^{-1/k} = \infty$), and which is also an \mathcal{L} -intersection. We say that Q is an \mathcal{L} -intersection if $C^Q = \bigcap \{C^N : N \text{ non-quasianalytic, log-convex, } N \geq Q\}$. Moreover, $L = (L_k)_{k \in \mathbb{N}}$ is a sequence as above which is log-convex, derivation closed, and non-quasianalytic. Then C^Q and C^L are closed under composition and allow for the implicit function theorem. See [16] or [15] and references therein.

That $A(t)$ is a real analytic, C^M (where M is either Q or L), C^∞ , or $C^{k,\alpha}$ family of unbounded operators means the following: There is a dense subspace V of the Hilbert space H such that V is the domain of definition of each $A(t)$, and such that $A(t)^* = A(t)$ in the self-adjoint case, or $A(t)$ has closed graph and $A(t)A(t)^* = A(t)^*A(t)$ wherever defined in the normal case. Moreover, we require that $t \mapsto \langle A(t)u, v \rangle$ is of the respective differentiability class for each $u \in V$ and $v \in H$. From now on we treat only $C^M = C^\omega$, C^M for $M = Q$, $M = L$, and $C^M = C^{0,\alpha}$.

This implies that $t \mapsto A(t)u$ is of the same class $C^M(T, H)$ (where T is either \mathbb{R} or \mathbb{R}^n) or is in $C^{0,\alpha}(T, H)$ (if T is a convenient vector space) for each $u \in V$ by [13, 2.14.4, 10.3] for C^ω , by [15, 3.1, 3.3, 3.5] for $M = L$, by [16, 1.10, 2.1, 2.3] for $M = Q$, and by [13, 2.3], [11, 2.6.2] or [10, 4.14.4] for $C^{0,\alpha}$ because $C^{0,\alpha}$ can be described by boundedness conditions only and for these the uniform boundedness principle is valid.

A sequence of functions λ_i is said to *parameterize the eigenvalues*, if for each $z \in \mathbb{C}$ the cardinality $|\{i : \lambda_i(t) = z\}|$ equals the multiplicity of z as eigenvalue of $A(t)$.

Let X be a C^ω or C^Q manifold. A *local blow-up* Φ over an open subset U of X means the composition $\Phi = \iota \circ \varphi$ of a blow-up $\varphi : U' \rightarrow U$ with center a C^ω or C^Q submanifold and of the inclusion $\iota : U \rightarrow X$. A *local power substitution* is a mapping $\Psi : V \rightarrow X$ of the form $\Psi = \iota \circ \psi$, where $\iota : W \rightarrow X$ is the inclusion of a coordinate chart W of X and $\psi : V \rightarrow W$ is given by

$$(y_1, \dots, y_q) = ((-1)^{\epsilon_1} x_1^{\gamma_1}, \dots, (-1)^{\epsilon_q} x_q^{\gamma_q}),$$

for some $\gamma = (\gamma_1, \dots, \gamma_q) \in (\mathbb{N}_{>0})^q$ and all $\epsilon = (\epsilon_1, \dots, \epsilon_q) \in \{0, 1\}^q$, where y_1, \dots, y_q denote the coordinates of W (and $q = \dim X$).

This paper became possible only after some of the results of [15] and [16] were proved, in particular the uniform boundedness principles. The wish to prove the results of this paper was the main motivation for us to work on [15] and [16].

Applications. For brevity we confine ourselves to C^Q ; the same applies to C^ω . Let X be a compact C^Q manifold and let $t \mapsto g_t$ be a C^Q -curve of C^Q Riemannian metrics on X . Then we get the corresponding C^Q curve $t \mapsto \Delta(g_t)$ of Laplace–Beltrami operators on $L^2(X)$. By theorem (B) the eigenvalues and eigenvectors

can be arranged C^Q in t . By [1], the eigenfunctions are also C^Q as functions on X (at least for those C^Q which can be described by a weight function, see [7]). Question: Are the eigenvectors viewed as eigenfunctions then also in $C^Q(X \times \mathbb{R})$?

Let Ω be a bounded region in \mathbb{R}^n with C^Q boundary, and let $H(t) = -\Delta + V(t)$ be a C^Q -curve of Schrödinger operators with varying C^Q potential and Dirichlet boundary conditions. Then the eigenvalues and eigenvectors can be arranged C^Q in t . Question: Are the eigenvectors viewed as eigenfunctions then also in $C^Q(\Omega \times \mathbb{R})$?

Example. This is an elaboration of [2, 7.4] and [14, Example]. Let $S(2)$ be the vector space of all symmetric real (2×2) -matrices. We use the C^L -curve lemma [15, 3.6] or [16, 2.5]: *For each L , there exist sequences $\mu_n \rightarrow \infty$, $t_n \rightarrow t_\infty$, $s_n > 0$ in \mathbb{R} with the following property: For μ_n -converging sequences $A_n, B_n \in S(2)$, i.e., $\mu_n A_n$ and $\mu_n B_n$ are bounded in $S(2)$, there exists a curve $A \in C^L(\mathbb{R}, S(2))$ such that $A(t_n + t) = A_n + tB_n$ for $|t| \leq s_n$.*

Choose a sequence ν_n of reals satisfying $\mu_n \nu_n \rightarrow 0$ and $(\nu_n)^n \leq s_n$ for all n and use the C^L -curve lemma for

$$A_n := (\nu_n)^{n+1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_n := \nu_n \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues of $A_n + tB_n$ and their derivatives are

$$\lambda_n(t) = \pm \nu_n \sqrt{(\nu_n)^{2n} + t^2}, \quad \lambda'_n(t) = \pm \frac{\nu_n t}{\sqrt{(\nu_n)^{2n} + t^2}}.$$

Then

$$\begin{aligned} \frac{\lambda'(t_n + (\nu_n)^n) - \lambda'(t_n)}{((\nu_n)^n)^\alpha} &= \frac{\lambda'_n((\nu_n)^n) - \lambda'_n(0)}{(\nu_n)^{n\alpha}} = \pm \frac{\nu_n}{(\nu_n)^{n\alpha} \sqrt{2}} \\ &= \pm \frac{(\nu_n)^{1-n\alpha}}{\sqrt{2}} \rightarrow \infty \text{ for } \alpha > 0. \end{aligned}$$

So the condition (in (C), (D), (I), (J), and (K)) that no two different continuously parameterized eigenvalues meet of infinite order cannot be dropped. By [2, 2.1], we may always find a twice differentiable square root of a non-negative smooth function, so that the eigenvalues λ are functions which are twice differentiable but not $C^{1,\alpha}$ for any $\alpha > 0$.

Note that the normed eigenvectors cannot be chosen continuously in this example (see also example [21, §2]). Namely, we have

$$A(t_n) = (\nu_n)^{n+1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A(t_n + (\nu_n)^n) = (\nu_n)^{n+1} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Resolvent Lemma. *Let C^M be any of C^ω , C^Q , C^L , C^∞ , or $C^{0,\alpha}$, and let $A(t)$ be normal. If A is C^M then the resolvent $(t, z) \mapsto (A(t) - z)^{-1} \in L(H, H)$ is C^M on its natural domain, the global resolvent set*

$$\{(t, z) \in T \times \mathbb{C} : (A(t) - z) : V \rightarrow H \text{ is invertible}\}$$

which is open (and even connected).

Proof. By definition the function $t \mapsto \langle A(t)v, u \rangle$ is of class C^M for each $v \in V$ and $u \in H$. We may conclude that the mapping $t \mapsto A(t)v$ is of class C^M into H as follows: For $C^M = C^\infty$ we use [13, 2.14.4]. For $C^M = C^\omega$ we use in addition [13, 10.3]. For $C^M = C^Q$ or $C^M = C^L$ we use [16, 2.1] and/or [15, 3.3] where we replace \mathbb{R} by \mathbb{R}^n . For $C^M = C^{0,\alpha}$ we use [13, 2.3], [11, 2.6.2], or [10, 4.1.14] because $C^{0,\alpha}$ can be described by boundedness conditions only and for these the uniform boundedness principle is valid.

For each t consider the norm $\|u\|_t^2 := \|u\|^2 + \|A(t)u\|^2$ on V . Since $A(t)$ is closed, $(V, \|\cdot\|_t)$ is again a Hilbert space with inner product $\langle u, v \rangle_t := \langle u, v \rangle + \langle A(t)u, A(t)v \rangle$.

(1) *Claim (see [2, in the proof of 7.8], [13, in the proof of 50.16], or [14, Claim 1]). All these norms $\|\cdot\|_t$ on V are equivalent, locally uniformly in t . We then equip V with one of the equivalent Hilbert norms, say $\|\cdot\|_0$.*

We reduce this to $C^{0,\alpha}$. Namely, note first that $A(t) : (V, \|\cdot\|_s) \rightarrow H$ is bounded since the graph of $A(t)$ is closed in $H \times H$, contained in $V \times H$ and thus also closed in $(V, \|\cdot\|_s) \times H$. For fixed $u, v \in V$, the function $t \mapsto \langle u, v \rangle_t = \langle u, v \rangle + \langle A(t)u, A(t)v \rangle$ is $C^{0,\alpha}$ since so is $t \mapsto A(t)u$. By the multilinear uniform boundedness principle ([13, 5.18] or [11, 3.7.4]) the mapping $t \mapsto \langle \cdot, \cdot \rangle_t$ is $C^{0,\alpha}$ into the space of bounded sesquilinear forms on $(V, \|\cdot\|_s)$ for each fixed s . Thus the inverse image of $\langle \cdot, \cdot \rangle_s + \frac{1}{2}(\text{unit ball})$ in $L(\overline{(V, \|\cdot\|_s)} \oplus (V, \|\cdot\|_s); \mathbb{C})$ is a c^∞ -open neighborhood U of s in T . Thus $\sqrt{1/2}\|u\|_s \leq \|u\|_t \leq \sqrt{3/2}\|u\|_s$ for all $t \in U$, i.e., all Hilbert norms $\|\cdot\|_t$ are locally uniformly equivalent, and claim (1) follows.

By the linear uniform boundedness theorem we see that $t \mapsto A(t)$ is in $C^M(T, L(V, H))$ as follows (here it suffices to use a set of linear functionals which together recognize bounded sets instead of the whole dual): For $C^M = C^\infty$ we use [13, 1.7, 2.14.3]. For $C^M = C^\omega$ we use in addition [13, 9.4]. For $C^M = C^Q$ or $C^M = C^L$ we use [16, 2.2, 2.3] and/or [15, 3.5] where we replace \mathbb{R} by \mathbb{R}^n . For $C^M = C^{0,\alpha}$ see above.

If for some $(t, z) \in T \times \mathbb{C}$ the bounded operator $A(t) - z : V \rightarrow H$ is invertible, then this is true locally with respect to the c^∞ -topology on the product which is the product topology by [13, 4.16], and $(t, z) \mapsto (A(t) - z)^{-1} : H \rightarrow V$ is C^M , by the chain rule, since inversion is real analytic on the Banach space $L(V, H)$. \square

Note that $(A(t) - z)^{-1} : H \rightarrow H$ is a compact operator for some (equivalently any) (t, z) if and only if the inclusion $i : V \rightarrow H$ is compact, since $i = (A(t) - z)^{-1} \circ (A(t) - z) : V \rightarrow H \rightarrow H$.

Polynomial proposition. *Let P be a curve of polynomials*

$$P(t)(x) = x^n - a_1(t)x^{n-1} + \cdots + (-1)^n a_n(t), \quad t \in \mathbb{R}.$$

- (a) *If P is hyperbolic (i.e., all roots of $P(t)$ are real for each fixed t) and if the coefficient functions a_i are all C^Q then there exist C^Q functions λ_i which parameterize all roots.*
- (b) *If P is hyperbolic, the coefficient functions a_i are C^L , and no two of the different continuously arranged roots (e.g., ordered by size) meet of infinite order, then there exist C^L functions λ_i which parameterize all roots.*
- (c) *If the coefficient functions a_i are C^Q , then for each t_0 there exists $N \in \mathbb{N}_{>0}$ such that the roots of $s \mapsto P(t_0 \pm s^N)$ can be parameterized C^Q in s for s near 0.*
- (d) *If the coefficient functions a_i are C^L and no two of the different continuously arranged roots (by [12, II 5.2]) meet of infinite order, then for each t_0 there exists $N \in \mathbb{N}_{>0}$ such that the roots of $s \mapsto P(t_0 \pm s^N)$ can be parameterized C^L in s for s near 0.*

All C^Q or C^L solutions differ by permutations.

The proof of parts (a) and (b) is exactly as in [2] where the corresponding results were proven for C^∞ instead of C^L , and for C^ω instead of C^Q . For this we need only the following properties of C^Q and C^L :

- They allow for the implicit function theorem (for [2, 3.3]).
- They contain C^ω and are closed under composition (for [2, 3.4]).
- They are derivation closed (for [2, 3.7]).

Part (a) is also in [8, 7.6] which follows [2]. It also follows from the multidimensional version [20, 6.10] since blow-ups in dimension 1 are trivial. The proofs of parts (c) and (d) are exactly as in [19, 3.2] where the corresponding result was proven for C^ω instead of C^Q , and for C^∞ instead of C^L , if none of the different roots meet of infinite order. For these we need the properties of C^Q and C^L listed above.

Matrix proposition. *Let $A(t)$ for $t \in T$ be a family of $(N \times N)$ -matrices.*

- (e) *If $T = \mathbb{R} \ni t \mapsto A(t)$ is a C^Q -curve of Hermitian matrices, then the eigenvalues and the eigenvectors can be chosen C^Q .*
- (f) *If $T = \mathbb{R} \ni t \mapsto A(t)$ is a C^L -curve of Hermitian matrices such that no two different continuously arranged eigenvalues meet of infinite order, then the eigenvalues and the eigenvectors can be chosen C^L .*
- (g) *If $T = \mathbb{R} \ni t \mapsto A(t)$ is a C^L -curve of normal matrices such that no two different continuously arranged eigenvalues meet of infinite order, then for each t_0 there exists $N_1 \in \mathbb{N}_{>0}$ such that the eigenvalues and eigenvectors of $s \mapsto A(t_0 \pm s^{N_1})$ can be parameterized C^L in s for s near 0.*
- (h) *Let $T \subseteq \mathbb{R}^n$ be open and let $T \ni t \mapsto A(t)$ be a C^ω or C^Q -mapping of normal matrices. Let $K \subseteq T$ be compact. Then there exist a neighborhood W of K , and a finite covering $\{\pi_k : U_k \rightarrow W\}$ of W , where each π_k is a composite of finitely many mappings each of which is either a local blow-up along a C^ω or C^Q submanifold or a local power substitution, such that the eigenvalues and the eigenvectors of $A(\pi_k(s))$ can be chosen C^ω or C^Q in s . Consequently, the eigenvalues and eigenvectors of $A(t)$ are locally special functions of bounded variation (SBV). If A is a family of Hermitian matrices, then we do not need power substitutions.*

The proof of the matrix proposition in case (e) and (f) is exactly as in [2, 7.6], using the polynomial proposition and properties of C^Q and C^L . Item (g) is exactly as in [19, 6.2], using the polynomial proposition and properties of C^L . Item (h) is proved in [20, 9.1, 9.6], see also [18].

Proof of the theorem. We have to prove parts (B), (C), (H), (I), (L), (M), and (O). So let C^M be any of C^ω , C^Q , C^L , or $C^{0,\alpha}$, and let $A(t)$ be normal. Let z be an eigenvalue of $A(t_0)$ of multiplicity N . We choose a simple closed C^1 curve γ in the resolvent set of $A(t_0)$ for fixed t_0 enclosing only z among all eigenvalues of $A(t_0)$. Since the global resolvent set is open, see the resolvent lemma, no eigenvalue of $A(t)$ lies on γ , for t near t_0 . By the resolvent lemma, $A : T \rightarrow L((V, \|\cdot\|_0), H)$ is C^M , thus also

$$t \mapsto -\frac{1}{2\pi i} \int_\gamma (A(t) - z)^{-1} dz =: P(t, \gamma) = P(t)$$

is a C^M mapping. Each $P(t)$ is a projection, namely onto the direct sum of all eigenspaces corresponding to eigenvalues of $A(t)$ in the interior of γ , with finite rank. Thus the rank must be constant: It is easy to see that the (finite) rank cannot fall locally, and it cannot increase, since the distance in $L(H, H)$ of $P(t)$ to the subset of operators of rank $\leq N = \text{rank}(P(t_0))$ is continuous in t and is either 0 or 1.

So for t in a neighborhood U of t_0 there are equally many eigenvalues in the interior of γ , and we may call them $\lambda_i(t)$ for $1 \leq i \leq N$ (repeated with multiplicity).

Now we consider the family of N -dimensional complex vector spaces $t \mapsto P(t)H \subseteq H$, for $t \in U$. They form a C^M Hermitian vector subbundle over U of $U \times H \rightarrow U$: For given t , choose $v_1, \dots, v_N \in H$ such that the $P(t)v_i$ are linearly independent and thus span $P(t)H$. This remains true locally in t . Now we use the Gram Schmidt orthonormalization procedure (which is C^ω) for the $P(t)v_i$ to obtain a local orthonormal C^M frame of the bundle.

Now $A(t)$ maps $P(t)H$ to itself; in a C^M local frame it is given by a normal $(N \times N)$ -matrix parameterized C^M by $t \in U$.

Now all local assertions of the theorem follow:

- (B) Use the matrix proposition, part (e).
- (C) Use the matrix proposition, part (f).
- (H) Use the matrix proposition, part (h), and note that in dimension 1 blowups are trivial.
- (I) Use the matrix proposition, part (g).
- (L,M) Use the matrix proposition, part (h), for \mathbb{R}^n .
- (O) We use the following

Result ([6], [5, VII.4.1]) *Let A, B be normal $(N \times N)$ -matrices and let $\lambda_i(A)$ and $\lambda_i(B)$ for $i = 1, \dots, N$ denote the respective eigenvalues. Then*

$$\min_{\sigma \in \mathcal{S}_N} \max_j |\lambda_j(A) - \lambda_{\sigma(j)}(B)| \leq C \|A - B\|$$

for a universal constant C with $1 < C < 3$. Here $\| \cdot \|$ is the operator norm.

Finally, it remains to extend the local choices to global ones for the cases (B) and (C) only. There $t \mapsto A(t)$ is C^Q or C^L , respectively, which imply both C^∞ , and no two different eigenvalues meet of infinite order. So we may apply [2, 7.8] (in fact we need only the end of the proof) to conclude that the eigenvalues can be chosen C^∞ on $T = \mathbb{R}$, uniquely up to a global permutation. By the local result above they are then C^Q or C^L . The same proof then gives us, for each eigenvalue $\lambda_i : T \rightarrow \mathbb{R}$ with generic multiplicity N , a unique N -dimensional smooth vector subbundle of $\mathbb{R} \times H$ whose fiber over t consists of eigenvectors for the eigenvalue $\lambda_i(t)$. In fact this vector bundle is C^Q or C^L by the local result above, namely the matrix proposition, part (e) or (f), respectively. \square

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