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Duality for Contravariant Functors on Banach Spaces

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Dedicated to Prof. Dr. E. Hlawka on the occasion of his 60th birthday

Abstract

The following notion of duality is studied: If G is a contravariant functor on Ban, then $G^{\times}(X) = \mathrm{Nat}(G, (X \hat{\hat{\otimes}} .)')$. We derive the following results: The $^{\times}$-reflexive functors are exactly the maximal subfunctors of $H(.,A)$ with reflexive A, $G^{\times}$ is $^{\times}$-reflexive if and only if $G(I)$ is a reflexive Banach space, $G^{\times} = (G_e)^{\times}$, and $G^{\times}(X') = G_e(X)'$ if X' has the metric approximation property. The last result has the consequence that for the tensor product of functors $G \underset{\mathrm{Ban}}{\otimes} (X' \hat{\hat{\otimes}} .) = G_e(X)$ holds, if X' has the m. A. P.

In this article a notion of duality for contravariant functors on Banach spaces is studied, which is based on the class of integral operators in the sense of Buchwalter ([1], [2]): $\mathfrak{L}^1(X, Y') = (X \hat{\hat{\otimes}} Y)'$. We employ the technique of constructing certain normed right ideals to functors, here called subfunctors; this technique was developed in [12].

We are able to derive the following conclusions on the duality $G \mapsto G^{\times}$: The $^{\times}$-reflexive functors are exactly the maximal subfunctors of $H(.,A)$ with reflexive A, $G^{\times}$ is $^{\times}$-reflexive if and only if $G(I)$ is a reflexive Banach space, $G^{\times} = (G_e)^{\times}$ and $G^{\times}(X') = G_e(X)'$ if X' satisfies the metric approximation condition. This last result implies that under the same restrictions we have $G \underset{\mathrm{Ban}}{\hat{\otimes}} (X' \hat{\hat{\otimes}} .) = G_e(X)$, a result that was known until now only for reflexive X with the metric approximation property. There is some connection of this notion of duality to the concept of the adjoint ideal in the theory of Banach operator ideals, which can be found in Pietsch [14].

In [11] we studied the notion of Δ-duality, formulated in the language of Waelbroeck spaces ([1], [2]) and their projective tensor product. If we suppose, that all Banach spaces in our category fulfil the metric approximation condition, then the Δ-duality coincides with our notion by duality of categories. Here we need neither this hypothesis nor the theory of Waelbroeck spaces.

A similar concept of duality for covariant functors was studied by CIGLER [4]; our methods could be used to get rid of the metric approximation property, which he uses heavily throughout the paper.

We use the notation and the basic result of [3], [10], [9] with the only exception that we write $\mathrm{Nat}(G_1, G_2)$ for the space of all natural transformations between admissible functors. For information on tensorproducts see [6], [7]. We write $\|.\|_I$ for the norm on all spaces of the form $(X \hat{\hat{\otimes}} Y)'$; I for integral.

All functors $G, G_1, \ldots$ are admissible contravariant functors defined on a small full subcategory $\underline{B}$ of the category Ban of all Banach spaces, which contains all finite dimensional spaces.

1. Definition: If G is a functor then its *dual functor* $G^\times$ is defined by $G^\times(X) = \mathrm{Nat}(G, (X \hat{\hat{\otimes}} .)')$ with the obvious action on morphisms.

The equation $\mathrm{Nat}(G_1, G_2^\times) = \mathrm{Nat}(G_2, G_1^\times)$ holds and is natural in G_1 and G_2 and is thus an adjointness relation for the contravariant functor $G \mapsto G^\times$. This follows from

$$\begin{aligned}
\mathrm{Nat}(G_1, G_2^\times) &= \mathrm{Nat}_{(.)}\left(G_1(.), \mathrm{Nat}_{(..)}\left(G_2(..), (. \hat{\hat{\otimes}} ..)'\right)\right) = \\
&= \mathrm{Nat}_{(.)}\left(G_1(.), \mathrm{Nat}_{(..)}\left((. \hat{\hat{\otimes}} ..), G_2(..)'\right)\right) = \\
&= \mathrm{Nat}_{(..)}\left(G_1(.) \hat{\otimes}_{(.)\in \underline{B}} (. \hat{\hat{\otimes}} ..), G_2(..)'\right) = \\
&= \mathrm{Nat}_{(..)}\left(G_2(..), \left(G_1(.) \hat{\otimes}_{(.)\in \underline{B}} (. \hat{\hat{\otimes}} ..)\right)'\right) = \\
&= \mathrm{Nat}_{(..)}\left(G_2(..), \mathrm{Nat}_{(.)}\left(G_1(.), (. \hat{\hat{\otimes}} ..)'\right)\right) = \\
&= \mathrm{Nat}(G_2, G_1^\times).
\end{aligned}$$

Specializing we have for every G: $\mathrm{Nat}(G, G^{\times\times}) = \mathrm{Nat}(G^\times, G^\times)$ and the natural map $\iota^G: G \to G^{\times\times}$ corresponding to the identity on $G^\times$ is called the canonical injection. G is called $^\times$-reflexive, if ι^G is a natural equivalence.

Let us consider the map $\varphi_X^G: G(X) \to H(X, G(I))$, defined by $\varphi_X^G(g)(x) = G(\hat{x})g$, $x \in X$, $g \in G(X)$, where $\hat{x}: I \to X$, $\hat{x}(1) = x$ (I is the one dimensional Banach space). It is easily seen that φ_X^G is a linear contraction and natural in G and X. The quotient functor of G by φ^G happens to be in the following class of functors:

2. Definition: Let A be a Banach space. A *subfunctor* $\Lambda(.,A)$ of $H(.,A)$ is a functor $X \mapsto \Lambda(X,A)$ together with an injective contractive natural transformation $\Lambda(.,A) \to H(.,A)$—we will consider $\Lambda(.,A)$ to be an algebraic subfunctor of $H(.,A)$—such that $\Lambda(I,A) = A$ via $\hat{a} \leftrightarrow a$. We can derive the following properties:

a) $\|f \circ g\|_\Lambda \leqslant \|f\|_\Lambda \|g\|$, $f \in \Lambda(X,A)$, $g \in H(Y,X)$, because $\Lambda(.,A)$ is an admissible functor.

b) $\Lambda(X,A) \supseteq X' \otimes A$, i. e. the space of all finite-dimensional maps $X \to A$, because

$$\sum_i \hat{a}_i \circ x_i' = \sum_i \Lambda(x_i', A)\hat{a} \quad \text{and} \quad A = \Lambda(I,A).$$

c) $\|\hat{a} \circ x'\|_\Lambda = \|a\| \|x'\|$ by
$\|a\| \|x'\| = \|\hat{a} \circ x'\| \leqslant \|\hat{a} \circ x'\|_\Lambda \leqslant \|\hat{a}\|_\Lambda \|x'\| = \|a\| \|x'\|$.

3. Proposition: *The quotient functor* $G/\ker \varphi^G$ *of* G *by* φ^G *is a subfunctor* $\Lambda^G(., G(I))$ *of* $H(., G(I))$.

Proof: Let $\Lambda(X, G(I))$ be the image of φ_X^G with quotient norm. As φ_X^G is natural in X and contractive and φ_I^G is the map $g \mapsto \hat{g}$, $G(I) \to H(I, G(I))$, all properties of a subfunctor are trivially satisfied.

4. Definition: If $\Lambda(.,A)$ is a subfunctor we define $\Lambda^\times(X,A')$ to be the space of all $h \in H(X,A')$ such that $h^t \circ f \in (X \hat{\hat{\otimes}} Y)'$ for all $f \in \Lambda(Y,A)$, $Y \in \underline{B}$ and moreover

$$\|h\|_{\Lambda^\times} = \sup\{\|h^t \circ f\|_I : f \in \Lambda(Y,A),\ Y \in \underline{B},\ \|f\|_\Lambda \leqslant 1\} < \infty$$

and we equip $\Lambda^\times(X,A')$ with this norm. We shall show that $\Lambda^\times(.,A')$ is again a subfunctor and that it is the dual functor of $\Lambda(.,A)$ via some identifications.

5. Lemma: $G^\times(I) = G(I)'$.

Proof: $G^\times(I) = \mathrm{Nat}(G, (.)') = \mathrm{Nat}(\mathrm{Id}, G') = \mathrm{Nat}(H_I, G') = G(I)'$ and the correspondence is given by transposition, followed by the Yoneda map, i. e.:

$$\beta \in G^\times(I) \mapsto (\beta_I)^t(1) \in G(I)'$$

$$g' \in G(I)' \mapsto \beta(g') \in G^\times(I),\ \langle x, \beta(g')_X(g_X)\rangle = \langle G(\hat{x})g_X, g'\rangle,$$

where $x \in X$, $g_X \in G(X)$ and ${}^t: H(Z, Y') \to H(Y, Z')$ is the natural isomorphism $f \to (f' \circ \iota_Y : Y \mapsto Y'' \to Z')$.

6. Lemma: *For $\beta \in G^{\times}(X)$, $x \in X$ and $g \in G(I)$ we have*

$$\langle g, (\varphi_X^{G^\times}(\beta)(x))_I^t(1)\rangle = \langle x, \beta_I(g)\rangle$$

Proof:

$$\langle g, (\varphi_X^{G^\times}(\beta)(x))_I^t(1)\rangle = \langle 1, (\varphi_X^{G^\times}(\beta)(x))_I(g)\rangle = \langle 1, (G^\times(\hat{x})\beta)_I(g)\rangle =$$

$$= \langle 1, (\hat{x} \hat{\hat{\otimes}} I)' \beta_I(g)\rangle = \langle (\hat{x} \hat{\hat{\otimes}} I)(1), \beta_I(g)\rangle = \langle x, \beta_I(g)\rangle .$$

This result shows that $\varphi_X^{G^\times}: G^\times(X) \to H(X, G^\times(I))$ has the form $\beta \to (\beta_I)^t$ modulo the identification $G^\times(I) = G(I)'$ of Lemma 4. It is easily seen that $\beta \to (\beta_I)^t$ is injective, therefore $G^\times$ coincides with the subfunctor $\Lambda^{G^\times}(., G^\times(I)) = \Lambda^{G^\times}(., G(I)')$.

7. Proposition: $\Lambda^{G^\times}(., G(I)') = (\Lambda^G)^\times(., G(I)')$.

For $G = \Lambda(., A)$ it follows that $\Lambda^\times(., A')$ is again a subfunctor.

Proof. By 5. every $h \in \Lambda^{G^\times}(X, G(I)')$ has the form $(\beta_I)^t$ for an unique $\beta \in G^\times(X)$. If $f \in \Lambda^G(Y, G(I))$ for an arbitrary $Y \in \underline{B}$, then $f = \varphi_Y^G(g)$ for $g \in G(Y)$ with $\|g\| \leqslant \|f\|_{\Lambda^G} + \varepsilon$. Let be $y \in Y$:

$$h^t \circ f(y) = \beta_I^{tt} \circ \varphi_Y^G(g)(y) = \beta_I G(\hat{y}) g =$$

$$= (X \hat{\hat{\otimes}} \hat{y})' \beta_Y(g) = (\beta_Y(g))(y),$$

thus $h^t \circ f = \beta_Y(g) \in (X \hat{\hat{\otimes}} Y)'$ and

$$\|h^t \circ f\|_I = \|\beta_Y(g)\|_I \leqslant \|\beta\|\,\|g\| \leqslant \|h\|_{\Lambda^{G^\times}}(\|f\|_{\Lambda^G} + \varepsilon),$$

i. e. $h \in (\Lambda^G)^\times(X, G(I)')$ and $\|h\|_{(\Lambda^G)^\times} \leqslant \|h\|_{\Lambda^{G^\times}}$.

If on the other hand we have $h \in (\Lambda^G)^\times(X, G(I)')$, then we define $\beta \in \mathrm{Nat}(G, (X \hat{\hat{\otimes}} .)')$ by

$$\beta_Y(g_Y) = h^t \circ (\varphi_Y^G(g_Y)) \in (X \hat{\hat{\otimes}} Y') \text{ for } g_Y \in G(Y).$$

$$\|\beta_Y(g_Y)\|_I = \|h^t \circ (\varphi_Y^G(g_Y))\|_I \leqslant \|h\|_{(\Lambda^G)^\times} \|\varphi_Y^G(g_Y)\|_\Lambda \leqslant$$

$$\leqslant \|h\|_{(\Lambda^G)^\times} \|g_Y\|, \text{ thus } \|\beta\| \leqslant \|h\|_{(\Lambda^G)^\times}.$$

It is easily seen that β is natural and therefore $\beta \in G^\times(X)$. For $g \in G(I)$ and $x \in X$ we have

$$\langle g, (\beta_I)^t(x)\rangle = \langle x, \beta_I(g)\rangle = \langle x, h^t \circ \varphi_I^G(g)\rangle = \langle x, h^t(\hat{g})\rangle = \langle g, h(x)\rangle,$$

i. e. $h = (\beta_I)^t$ is indeed of the form of lemma 5 and therefore $h \in (\Lambda^G)^\times(X, G(I)')$ and $\|h\|_{\Lambda^{G^\times}} = \|\beta\| \leqslant \|h\|_{(\Lambda^G)^\times}$. This ends the proof.

8. Lemma: *Let $\Lambda(.,A)$ be a subfunctor and $h \in \Lambda^{\times}(X,A')$. Then $\|h\|_{\Lambda^{\times}} = \sup|\operatorname{tr}(h^t \circ v \circ u)|$, where $u: X' \to Z$ runs through all weak*-continuous quotient maps onto finite dimensional Banach spaces Z and $v \in \Lambda(Z,A), \|v\|_{\Lambda} \leqslant 1$. Every $h: X \to A'$, for which that supremum is finite, belongs to $\Lambda^{\times}(X,A')$.*

Proof: $\|h\|_{\Lambda^{\times}} = \sup\{\|h^t \circ f\|_I, f \in \Lambda(Y,A), Y \in \underline{B}, \|f\|_{\Lambda} \leqslant 1\} =$

$$= \sup\{|\operatorname{tr}(h^t \circ f \circ w)| : f \in \Lambda(Y,A), Y \in \underline{B},$$
$$\|f\|_{\Lambda} \leqslant 1, w \in X \otimes Y, \lambda(w) \leqslant 1\}.$$

We now fix f and w and consider the canonical factorization of w: $w = \operatorname{im} w \circ \tilde{w} \circ \operatorname{coim} w: X' \to X'/w^{-1}(0) \to w(X') \to Y$. $Z = X'/w^{-1}(0)$ is then a finite dimensional Banach space, $u = \operatorname{coim} w: X' \to Z$ a weak*-continuous quotient map and $v = f \circ \operatorname{im} w \circ \tilde{w} \in \Lambda(Z,A)$, $\|v\|_{\Lambda} \leqslant 1$ and we have $\operatorname{tr}(h^t \circ f \circ w) = \operatorname{tr}(h^t \circ v \circ u)$. The rest is clear.

This result shows that $\Lambda^{\times}(X,A')$ depends only on the spaces $\Lambda(Z,A)$ for finite dimensional Z, thus by 7 $G^{\times}(X) = G_e^{\times}(X)$.

9. Theorem: *If G is a functor and X' satisfies the metric approximation condition, then $G^{\times}(X') = G_e(X)'$.*

Proof: Since $G^{\times}(X) = G_e^{\times}(X)$ we may suppose that G is essential.

$G_e(X)$ is the completion of $G(I) \otimes X'$ in a reasonable norm $\alpha: \lambda \leqslant \alpha \leqslant \gamma$ (see [10]). In [6] it was shown, that $G_e(X)'$ is a linear subspace of $H(X', G(I)')$, consisting of all $f: X' \to G(I)'$, which fulfil

$$\|f\|_{G_e(X)'} = \sup\{|\textstyle\sum \langle g_i, f(x_i')\rangle| : \sum g_i \otimes x_i' \in G(I) \otimes X',$$
$$\alpha(\textstyle\sum g_i \otimes x_i') \leqslant 1\} < \infty.$$

On the other hand we have $G^{\times}(X') = \Lambda^{G^{\times}}(X', G(I)') = (\Lambda^G)^{\times}(X', G(I)')$ by 6 and 7. It remains to show that $(\Lambda^G)^{\times}(X', G(I)') = (G(I) \otimes_{\alpha} X')'$ and for that it is enough to show $\|f\|_{G_e(X)'} = \|f\|_{(\Lambda^G)^{\times}}$ for $f \in H(X', G(I)')$, because f is an element of the respective space, if the respective norm of f is finite.

Let us suppose $f: X' \to G(I)'$ and $\|f\|_{G(X)'} < \infty$. If $h \in \Lambda^G(Y, G(I))$, then there exists $g \in G(Y)$ with

$$h = \varphi_Y^G(g), \ \|g\| \leqslant \|h\|_{\Lambda^G} + \varepsilon,$$

$$\|f^t \circ h\|_I = \sup\{|\textstyle\sum \langle x_i', f^t \circ h(y_i)\rangle| : \sum x_i' \otimes y_i \in X' \otimes Y,$$
$$\lambda(\textstyle\sum x_i' \otimes y_i) \leqslant 1\}.$$

But we have

$$\sum \langle x_i', f^t \circ h(y_i)\rangle = \sum \langle h(y_i), f(x_i')\rangle = \sum \langle \varphi_Y^G(g)(y_i), f(x_i')\rangle =$$
$$= \sum \langle G(\hat{y}_i) g, f(x_i')\rangle$$

and $\sum G(\hat{y}_i)(g) \otimes x_i' \in G(I) \otimes X'$,

$$\alpha(\sum G(\hat{y}_i)(g) \otimes x_i') = \| \sum G(x_i')\, G(\hat{y}_i) g \|_{G(X)} = \| G(\sum \hat{y}_i \circ x_i')(g) \|_{G(X)} \leqslant$$
$$\leqslant \lambda(\sum \hat{y}_i \circ x_i') \|g\|_{G(Y)} \leqslant 1 \,.\, (\|h\|_{A^G} + \varepsilon)\,.$$

Therefore we can compute as follows:

$$\|f^t \circ h\|_I \leqslant \sup \{ |\sum \langle g_i, f(x_i')\rangle| : \sum g_i \otimes x_i' \in G(I) \otimes X',$$
$$\alpha(\sum g_i \otimes x_i') \leqslant \|h\|_{A^G} + \varepsilon\} \leqslant$$
$$\leqslant \|f\|_{G_e(X)'} (\|h\|_{A^G} + \varepsilon),$$

$$\|f\|_{(A^G)^\times} = \sup \{ \|f^t \circ h\|_I : h \in A(Y, G(I)),\ Y \in \underline{B},\ \|h\|_A \leqslant 1\} \leqslant$$
$$\leqslant \|f\|_{G_e(X)'} (1 + \varepsilon)\,.$$

Since $\varepsilon > 0$ is arbitrary, we have $\|f\|_{(A^G)^\times} \leqslant \|f\|_{G(X)'}$.

If on the other hand $f \in (A^G)^\times (X', G(I)')$, we proceed as follows:

$$\|f\|_{G_e(X)'} = \sup \{ |\sum \langle g_i, f(x_i')\rangle| : \sum g_i \otimes x_i' \in G(I) \otimes X',$$
$$\alpha(\sum g_i \otimes x_i') \leqslant 1\} =$$
$$= \sup \{ |\mathrm{Trace}(f^t \circ h)| : h \in X' \otimes G(I) \subseteq A^G(X, G(I)),\ \|h\|_{A^G} \leqslant 1\},$$

because G is essential and thus $\varphi_X^G : G(X) = G_e(X) \to A^G(X, G(I))$ is a quotient map and maps the dense subset $G(I) \otimes X' \cap \{g \in G_e(X) : \|g\| \leqslant 1\}$ of the unit ball of $G_e(X) = G(X)$ injectively onto a dense subset of the unit ball of $A^G(X, G(I))$. It is easily seen that φ_X^G induces the identity on $G(I) \otimes X'$, if we consider $G(I) \otimes X'$ in $G_e(X)$ and $A^G(X, G(I))$ respectively. Thus the above equation holds. Now $f^t \circ h \in X' \otimes X''$ and since X' has the metric approximation property we have $\|f^t \circ h\|_{X' \hat{\otimes} X''} = \|f^t \circ h\|_I$ and we can proceed as follows:

$$\|f\|_{G_e(X)'} \leqslant \sup \{ \|f^t \circ h\|_I : h \in X' \otimes G(I) \subseteq A^G(X, G(I)),\ \ \|h\|_{A^G} \leqslant 1\} \leqslant$$
$$\leqslant \sup \{ \|f^t \circ h\|_I : h \in A^G(Y, G(I)),\ Y \in \underline{B}\} = \|f\|_{(A^G)^\times}\,.$$

This ends the proof.

10. Lemma: *Let G be a functor and let $\iota^G: G \to G^{\times\times}$ be the canonical injection* (1.). *Then*

$$\varphi^{G^{\times\times}} \circ i^G = k \circ \varphi^G : G \to \Lambda^{G^{\times\times}}(., G(I)''),$$

where $$k_X : \Lambda^G(X, G(I)) \to \Lambda^{G^{\times\times}}(X, G(I)'')$$
is given by

$$k_X(f) = i \circ f, \quad i : G(I) \to G(I)''$$

the canonical embedding.

Proof: $i_X^G(g)_Y(\beta) = \beta_X(g) \in (X \hat{\hat{\otimes}} Y)'$, $g \in G(X)$, $\beta \in G^\times(Y)$ by definition.

$$\varphi_X^{G^{\times\times}}(\alpha) = (\alpha_I)^t, \; \alpha \in G^{\times\times}(X)$$

modulo the identification

$$G^{\times\times}(I) = \big(G^\times(I)\big)'.$$

Let be $x \in X$, $g' \in G(I)'$, $g \in G(X)$; then $g' = (\gamma_I)^t(1)$ for a uniquely determined $\gamma \in G^\times(I)$.

$$\langle g', [\varphi_X^{G^{\times\times}} \circ \iota_X^G(g)](x)\rangle = \langle g', \big(\iota_X^G(g)_I\big)^t(x)\rangle = \langle x, \iota_X^G(g)_I(\gamma)\rangle =$$

$$= \langle (\hat{x} \hat{\hat{\otimes}} I)'(1), \gamma_X(g)\rangle = \langle 1, (\hat{x} \hat{\hat{\otimes}} I)' \circ \gamma_X(g)\rangle = \langle 1, \gamma_I \circ G(\hat{x})(g)\rangle =$$

$$= \langle G(\hat{x})g, (\gamma_I)^t(1)\rangle = \langle \varphi_X^G(g)(x), g'\rangle = \langle g', i \circ \big(\varphi_X^G(g)\big)(x)\rangle =$$

$$= \langle g', [k_X \varphi_X^G(g)](x)\rangle .$$

This Lemma shows that $\Lambda^\times(., A')$ is indeed the dual functor of $\Lambda(., A)$ modulo some natural identifications—the notions of reflexivity coincide.

11. Definition: Let $\Lambda(., A)$ be a subfunctor and let $\Lambda^m(X, A)$ be the space of all $f \in H(X, A)$, for which

$$\|f\|_{\Lambda^m} = \sup\{\|f \circ h\|_\Lambda : h \in Y \otimes X, \lambda(h) \leqslant 1, Y \in B\} < \infty,$$

equipped with that norm. It is a routine matter to prove that $\Lambda^m(., A)$ is again a subfunctor, which contains $\Lambda(., A)$, but not necessarily isometrically. We call $\Lambda(., A)$ *maximal*, if $\Lambda(., A) = \Lambda^m(., A)$. It is easily seen that $\Lambda^m(., A)$ and $\Lambda^\times(., A')$ are maximal.

12. Lemma: *If $\Lambda(., A)$ is a subfunctor, then*

$$\Lambda^m(X, A) = \{f \in H(X, A) : i \circ f \in \Lambda^{\times\times}(X, A'')\},$$

where $i : A \to A''$ is the canonical embedding, and

$$\|f\|_{\Lambda^m} = \|i \circ f\|_{\Lambda^{\times\times}}.$$

Proof: If $f \in A^m(X, A)$ then $\|i \circ f\|_{A^{\times\times}} = \sup |\operatorname{tr}((i \circ f)^t \circ v \circ u)|$ by Lemma 8. We fix $v \in A^\times(Z, A')$, $\|v\|_{A^\times} \leqslant 1$ and $u = \sum z_i \otimes x_i$.

$$\begin{aligned}|\operatorname{tr}((i \circ f)^t \circ v \circ u)| &= |\operatorname{tr}((i \circ f)^t \circ v \circ (\textstyle\sum \hat{z}_i \circ \langle x_i, .\rangle))| = \\ &= |\textstyle\sum \langle x_i, (i \circ f)^t \circ v(z_i)\rangle| = |\textstyle\sum \langle f(x_i), v(z_i)\rangle| = \\ &= |\operatorname{tr}(v^t \circ f \circ (\textstyle\sum \hat{x}_i \langle z_i, .\rangle))| \leqslant \\ &\leqslant \|v^t \circ f \circ (\textstyle\sum \hat{x}_i \langle z_i, .\rangle)\|_{(Z' \hat{\otimes} Z)'},\end{aligned}$$

since Z is finite dimensional and therefore $(Z' \hat{\hat{\otimes}} Z)' = Z \hat{\otimes} Z'$ and trace is a contractive linear functional on $Z \hat{\otimes} Z'$. But then we can continue

$$\begin{aligned}&\leqslant \|v\|_{A^\times} \|f \circ (\textstyle\sum \hat{x}_i \langle z_i, .\rangle)\|_A \leqslant \\ &\leqslant 1 . \sup \{\|f \circ h\|_A, h \in Y \otimes X, \lambda(h) \leqslant 1\} = \|f\|_{A^m}.\end{aligned}$$

That is true for the supremum too and so we have

$$\|i \circ f\|_{A^{\times\times}} \leqslant \|f\|_{A^m}.$$

To derive the inverse inequality, we need first the fact that $k_Z \colon A(Z, A) \to A^{\times\times}(Z, A'')$ (Lemma 10) is an isometry for finite dimensional Z. Since trivially $\|k_Z\| \leqslant 1$ we have to show $\|k_Z(f)\|_{A^{\times\times}} \geqslant \|f\|_A$ for all $f \in A(Z, A)$. As Z fulfils the condition in Theorem 9, we have

$$A_e(Z, A)' = A(Z, A)' = A^\times(Z', A')$$

and for $f \in A(Z, A)$ we can find $h \in A^\times(Z', A')$ such that $|\operatorname{tr}(h^t \circ f)| = \|f\|_A$ and $\|h\|_{A^\times} = 1$. Now we have:

$$\|f\|_A = |\operatorname{tr}(h^t \circ f)| \leqslant \|h^t \circ f\|_{(Z \hat{\otimes} Z')'} \leqslant \|h\|_{A^\times} \|f\|_A = \|f\|_A,$$

because Z is finite dimensional (see the same argument used in the first part of the proof), and

$$\begin{aligned}\|f\|_A = \|h^t \circ f\|_I = \|(h^t \circ f)^t\|_I = \|(i \circ f)^t \circ h\|_I \leqslant \\ \leqslant \|i \circ f\|_{A^{\times\times}} \|h\|_{A^\times} = \|i \circ f\|_{A^{\times\times}}.\end{aligned}$$

We consider now $f \in H(X, A)$ with $i \circ f \in A^{\times\times}(X, A'')$. We factor $h \in Y \otimes X$, $\lambda(h) \leqslant 1$ as $h = \operatorname{im} h \circ \tilde{h} \colon Y' \to h(Y) \to X$. Then

$$\|f \circ h\|_A = \|f \circ \operatorname{im} h \circ \tilde{h}\|_A \leqslant \|f \circ \operatorname{im} h\|_A \|\tilde{h}\| \leqslant \|f \circ \operatorname{im} h\|_A.$$

But since $f \circ \operatorname{im} h \in A(h(Y'), A)$ and $h(Y')$ is finite dimensional we can apply the above result:

$$\|f \circ \operatorname{im} h\|_A = \|i \circ f \circ \operatorname{im} h\|_{A^{\times\times}} \leqslant \|i \circ f\|_{A^{\times\times}} \|\operatorname{im} h\| \leqslant \|i \circ f\|_{A^{\times\times}},$$

thus $\|f\|_{A^m} \leqslant \|i \circ f\|_{A^{\times\times}}$ and we are done.

Now the dirty work is done and we can harvest theorems, the first being Theorem 9.

13. Theorem: *The* $^{\times}$*-reflexive contravariant functors* $\underline{B} \to \mathrm{Ban}$ *are exactly the maximal subfunctors of* $H(.,A)$ *with* A *a reflexive Banach space. Especially* $G^{\times}$ *is reflexive if and only if* $G(I)$ *is reflexive.*

Proof: Use Lemmas 5 and 6 for the reflexivity of $G(I)$ (or A) and Lemma 12 and the last sentence in 11 for the rest.

14. Theorem: *If there exists a natural transformation* $\eta: G_1 \to G_2$ *such that* η_z *is an isometry onto for all finite dimensional* Z, *then* $G_1^{\times} = G_2^{\times}$.

Especially $G_e^{\times} = G^{\times}$, *the remark after* 8.

Proof: By Lemma 8 $G^{\times} = (A^G)^{\times}(., G(I)')$ depends only on the spaces $A^G(Z, G(I))$ for finite dimensional Z.

15. Theorem: *Let* $G: \underline{B} \to \mathrm{Ban}$ *be a functor. If* X' *has the metric approximation property, then*

$$G \underset{B}{\hat{\otimes}} (X' \hat{\hat{\otimes}} .) = G_e(X).$$

Proof: We have a linear contraction with dense image τ:

$$G \underset{B}{\hat{\otimes}} (X' \hat{\hat{\otimes}} .) \to G_e(X),$$

defined by

$$\tau(g_Z \otimes f_Z) = G(f_Z) g_Z, \quad g_Z \otimes f_Z \in G(Z) \hat{\otimes} (X' \hat{\hat{\otimes}} Z)$$

(see [3], [4], [10]). We consider

$$\tau': G_e(X)' \to \big(G \underset{B}{\hat{\otimes}} (X' \hat{\hat{\otimes}} .)\big)' = \mathrm{Nat}\big(G, (X' \hat{\hat{\otimes}} .)'\big) = G^{\times}(X'),$$

given by

$$\langle f_Z, \tau'(g')_Z(g_Z)\rangle = \langle G(f_Z) g_Z, g'\rangle, \quad g_Z \in G(Z), f_Z \in X' \hat{\hat{\otimes}} Z, g' \in G_e(X)'.$$

To reach the space $G^{\times}(X') = A^{\times}(X', G(I)')$ we have to investigate $(\tau'(g')_I)^t$ (see Lemma 5). Let be

$$g \in G(I), x' \in X', g' \in G_e(X)':$$

$$\langle g, (\tau'(g')_I)^t(x')\rangle = \langle x', \tau'(g')_I(g)\rangle = \langle G(x') g, g'\rangle.$$

Since the right-hand-side is the duality pairing $\langle G_e(X), G_e(X)'\rangle$ and $G_e(X)'$ can be considered as a linear subspace of $H(X', G(I)')$, we arrived at the identification $G_e(X)' = G^\times(X')$ of theorem 9.

Thus τ' is an isometry onto and so is τ.

Added in Proof: V. LOSERT and H. SPITZER have both given more direct proofs of theorem 15.

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