Higher order symplectic structures on shape spaces of space curves

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Abstract: For $c \in \mathrm{Imm}(S^1, \mathbb{R}^3)$ the 2- form

$$\Omega_c^{MW}(h,k) = \int_{S^1} \det(D_s c, h, k) ds,$$

where $ds = |c'(\theta)| d\theta$ and $D_s = \frac{1}{|c'(\theta)|}\partial_\theta$, induces the Marsden-Weinstein symplectic structure[1] on the shape space $\mathrm{Imm}(S^1, \mathbb{R}^3)/\,\mathrm{Diff}(S^1)$, corresponding to a Kähler structure. The Hamiltonian flow for the length functional is the binormal flow. In this talk I will present other natural symplectic structures related to this one.

[1]Marsden, J., and Weinstein, A. Coadjoint orbits, vortices, and Clebsch variables for incompressible fluids. Physica D: Nonlinear Phenomena 7, 1 (1983), 305-323.

Infinite dim. symplectic manifolds

M be a manifold, infinite dimensional.
A 2-form $\omega \in \Omega^2(M)$ is called a *weak symplectic structure* on M if the following three conditions holds:

1. ω is closed, $d\omega = 0$.
2. The associated vector bundle homomorphism $\check{\omega} : TM \to T^*M$ is injective.
3. The gradient of ω with respect to itself exists and is smooth; in charts: so let M be open in a convenient vector space E. Then for $x \in M$ and $X, Y, Z \in T_xM = E$ we have $d\omega(x)(X)(Y,Z) = \omega(\Omega_x(Y,Z), X) = \omega(\tilde{\Omega}_x(X,Y), Z)$ for smooth $\Omega, \tilde{\Omega} : M \times E \times E \to E$ which are bilinear in $E \times E$.

$\omega \in \Omega^2(M)$ is called a *strong symplectic structure* on M if it is closed ($d\omega = 0$) and if $\check{\omega} : TM \to T^*M$ is invertible with smooth inverse. Then M turns out to be a Hilbert manifold.

Cotangent bundle (needs convenient calculus)

T^*Q carries a canonical weak symplectic structure $\omega_Q \in \Omega^2(T^*Q)$, which is defined as follows. Let $\pi_Q^* : T^*Q \to Q$ be the projection. Then the *Liouville form* $\theta_Q \in \Omega^1(T^*Q)$ is given by $\theta_Q(X) = \langle \pi_{T^*Q}(X), T(\pi_Q^*)(X) \rangle$ for $X \in T(T^*Q)$, where $\langle \quad , \quad \rangle$ denotes the duality pairing $T^*Q \times_Q TQ \to \mathbb{R}$. Then the symplectic structure on T^*Q is given by $\omega_Q = -d\theta_Q$, which of course in a local chart looks like $\omega_E((v, v'), (w, w')) = \langle w', v \rangle_E - \langle v', w \rangle_E$. The associated mapping $\check{\omega} : T_{(0,0)}(E \times E') = E \times E' \to E' \times E''$ is given by $(v, v') \mapsto (-v', i_E(v))$, where $i_E : E \to E''$ is the embedding into the bidual. So the canonical symplectic structure on T^*Q is strong if and only if all model spaces of the manifold Q are reflexive and Hilbert spaces.

Definition

For a weak symplectic manifold (M, ω) let
$T_x^\omega M = T_x^\omega M = \check{\omega}_x(T_x M) \subset T_x^* M = L(T_x M, \mathbb{R})$, called the
ω-smooth cotangent space The convenient structure on $T_x^\omega M$ is the one from $T_x M$. These vector spaces fit together to form a subbundle of $T^* M$ which is isomorphic to the tangent bundle TM via $\check{\omega} : TM \to T^\omega M \subseteq T^* M$. It is in general not a splitting subbundle.
For strong ω the mapping $\check{\omega}_x : T_x M \to T_x^* M$ is a diffeomorphism onto $T_x^\omega M$ with the structure induced from $T_x^* M$.
For a weak symplectic manifold (M, ω) let

$$C_\omega^\infty(M, \mathbb{R}) \subset C^\infty(M, \mathbb{R})$$

denote the subalgebra consisting of all smooth functions $f : M \to \mathbb{R}$ such that the differential $df : M \to T^* M$ factors to a smooth mapping $M \to T^\omega M$. These are exactly those smooth functions on M which admit a smooth ω-gradient $\operatorname{grad}^\omega f \in \mathfrak{X}(M)$.

Theorem *Let (M,ω) be a weak symplectic manifold. The Hamiltonian mapping* $\operatorname{grad}^\omega : C^\infty_\omega(M,\mathbb{R}) \to \mathfrak{X}(M,\omega)$*, which is given by* $i_{\operatorname{grad}^\omega f}\omega = df$ *or* $\operatorname{grad}^\omega f := (\check{\omega})^{-1} \circ df$ *is well defined. Also the Poisson bracket*

$$\{\ ,\ \} : C^\infty_\omega(M,\mathbb{R}) \times C^\infty_\omega(M,\mathbb{R}) \to C^\infty_\omega(M,\mathbb{R})$$

$$\{f,g\} = \omega(\operatorname{grad}^\omega g, \operatorname{grad}^\omega f) = dg(\operatorname{grad}^\omega f) = (\operatorname{grad}^\omega f)(g)$$

is well defined and gives a Lie algebra structure to the space $C^\infty_\omega(M,\mathbb{R})$*, which also fulfills* $\{f,gh\} = \{f,g\}h + g\{f,h\}$*. We equip* $C^\infty_\omega(M,\mathbb{R})$ *with the initial structure with respect to:*

$$C^\infty_\omega(M,\mathbb{R}) \xrightarrow{\subset} C^\infty(M,\mathbb{R}), \qquad C^\infty_\omega(M,\mathbb{R}) \xrightarrow{\operatorname{grad}^\omega} \mathfrak{X}(M).$$

Then the Poisson bracket is bounded bilinear on $C^\infty_\omega(M,\mathbb{R})$*.*
We have the following long exact sequence of Lie algebras and Lie algebra homomorphisms:

$$0 \to H^0(M) \to C^\infty_\omega(M,\mathbb{R}) \xrightarrow{\operatorname{grad}^\omega} \mathfrak{X}(M,\omega) \xrightarrow{\gamma} H^1_\omega(M) \to 0,$$

where $H^0(M)$ *is the space of locally constant functions, and* $H^1_\omega(M) = \frac{\{\varphi \in C^\infty(M \leftarrow T^\omega M) : d\varphi = 0\}}{\{df : f \in C^\infty_\omega(M,\mathbb{R})\}}$ *is the first symplectic cohomology space of* (M,ω)*, a linear subspace of the De Rham cohomology space* $H^1(M)$*.*

The space of regular space curves

$$\mathrm{Imm} = \mathrm{Imm}(S^1, \mathbb{R}^3) := \left\{ c \in C^\infty(S^1, \mathbb{R}^3) : |c'| \neq 0 \right\}.$$

is open subset in $C^\infty(S^1, \mathbb{R}^3)$, a manifold with tangent space

$$T_c\mathrm{Imm} = T_c\mathrm{Imm}(S^1, \mathbb{R}^3) = C^\infty(S^1, \mathbb{R}^3).$$

Consider the action of the reparametrization group
$\mathrm{Diff} = \mathrm{Diff}^+(S^1)$ or $= \mathrm{Diff}(S^1)$ by composition from the right and the quotient (shape) space

$$B_i = B_i(S^1, \mathbb{R}^3) := \mathrm{Imm}(S^1, \mathbb{R}^3)/\mathrm{Diff}(S^1),$$

which is an infinite dimensional orbifold; the isotropy groups are finite cyclic groups.[2]
The vertical fibers consist exactly of all fields h that are tangent to it's foot point c, i.e., $h = a.c'$ with $a \in C^\infty(S^1)$.

[2] V.Cervera, F.Mascaro, PM: The action of the diffeomorphism group on the space of immersions. Diff. Geom. Appl. 1 (1991), 391–401

Reparametrization invariant Riemannian metrics

on Imm are Riemannian metrics of the form:

$$G_c^L(h,k) = \int_{S^1} \langle L_c h, k\rangle ds = \int_{S^1} \langle h, L_c k\rangle ds, \quad \text{where}$$

$$ds = |c_\theta| d\theta, \quad D_s = (1/|c_\theta|)\partial_\theta, \quad \text{arc-lenght derivative,}$$

$$L \in \Gamma(\mathrm{End}(T\mathrm{Imm}), \quad L_c : T_c\mathrm{Imm} = C^\infty(S^1, \mathbb{R}^3) \to T_c\mathrm{Imm}\,,$$

called *inertia operator*, is an elliptic, pseudo differential operator that is equivariant under the right action of $\mathrm{Diff}(S^1)$ and also under left action of $SO(3)$ (or even the motion group $\mathbb{R}^3 \rtimes SO(3)$), and which is also selfadjoint with respect to the L^2-metric, i.e.,

$$L_{c\circ\varphi}(h\circ\varphi) = (L_c(h))\circ\varphi \quad \text{and} \quad \int \langle L_c h, k\rangle ds = \int \langle h, L_c k\rangle ds\,.$$

The class of Sobolev H^k-metrics, where $L = (1-(-1)^k D_s^{2k})$. For $k=0$ we have $G_c^{\mathrm{id}}(h,k) = \int \langle h,k\rangle ds$.

The induced Liouville one form

corresponding to the metric G^L is the one-form:

$$\Theta^L_c(h) := G^L_c(c\times D_s c, h) = \int \langle c\times D_s c, L_c h\rangle ds = \int \det(c, D_s c, L_c h) ds$$

for $h \in T_c\mathrm{Imm}$, where $\times$ denotes the vector cross product on $\mathbb{R}^3$.
We have:
For any inertia operator L, that is equivariant under the right action of Diff *and the left action of $SO(3)$, the induced Liouville one-form Θ^L is invariant under the right action of* Diff *and the left action of $SO(3)$, i.e., for any $c \in \mathrm{Imm}$, $h \in T_c\mathrm{Imm}$, $\varphi \in \mathrm{Diff}$ and $O \in SO(3)$ we have*

$$\Theta^L_{O\circ c\circ\varphi)}(O \circ h \circ \varphi)) = \Theta^L_c(h).$$

The induced pre-symplectic form on Imm is

$$\Omega^L_c(h,k) := -d\Theta^L_c(h,k) = -D_{c,h}\Theta^L_c(k) + D_{c,k}\Theta^L_c(h) + \Theta^L_c([h,k])\,,$$

$$\boxed{\begin{aligned}\Omega^L_c(h,k) = \int\Big(&\langle D_s c, L_c h \times k + h \times L_c k\rangle - \langle c, D_s h \times L_c k - D_s k \times L_c h\rangle \\ &- \langle c \times D_s c, (D_{c,h}L_c)k - (D_{c,k}L_c)h\rangle\Big)\,ds\,.\end{aligned}}$$

Ω^L is invariant under the actions of Diff and $SO(3)$. For the invariant L^2-metric, i.e., $L = \mathrm{id}$, one obtains three times the Marsden-Weinstein pre-symplectic structure, i.e.,

$$\Omega^{\mathrm{id}}_c(h,k) = 3\int_{S^1}\langle D_s c \times h, k\rangle ds = 3\int \det(D_s c, h, k) ds =: 3\Omega^{\mathrm{MW}}_c(h,k).$$

Its kernel consists exactly of all vector fields along c which are tangent to c, so by reduction it induces a pre-symplectic structure on shape space $\mathrm{Imm}(S^1,\mathbb{R}^3)/\mathrm{Diff}$ which is easily seen to be non-degenerate and thus is a symplectic structure there.

Theorem

The form Ω^L factors to a (pre)-symplectic form $\bar{\Omega}^L$ on B_i if the inertia operator L maps vertical tangent vectors to $\operatorname{span}\{c, c'\}$, *i.e., if for all $c \in \mathrm{Imm}$ and $a \in C^\infty(S^1)$ we have $L_c(a.c') = a_1 c' + a_2 c$ for some functions $a_i \in C^\infty(S^1)$.*

Proof. The 1-form Θ^L on Imm factors to a 1-form $\bar{\Theta}^L$ on shape space B_i with $\Theta^L = \pi^* \bar{\Theta}^L$ if and only if Θ^L is invariant under under Diff and is *horizontal* in the sense that it vanishes on each vertical tangent vector $h = a.c'$ for a in $C^\infty(S^1, \mathbb{R})$.

The condition on L such that Θ^L vanishes on all vertical h is

$$\Theta^L_c(ac') = \int \langle c \times D_s c, L_c(ac') \rangle ds = 0.$$

From here it is clear that this holds if $L_c(a.c') = a_1 c' + a_2 c$ for some functions $a_i \in C^\infty(S^1)$.

In that case also its exterior derivative satisfies

$$\Omega^L = -d\Theta^L = -d\pi^* \bar{\Theta}^L = -\pi^* d\bar{\Theta}^L =: \pi^* \bar{\Omega}^L$$

for the presymplectic form $\bar{\Omega}^L = -d\bar{\Theta}^L$ on B_i.

Inertia operators that satisfy these conditions are

$$\begin{aligned}
&L_c(h) = F(c).h \text{ for } F \in C^\infty(\mathrm{Imm}, \mathbb{R}_{>0}), \text{ for example}\\
&L_c(h) = \lambda(c) \quad \text{ for } \lambda : \mathrm{Imm} \to \mathbb{R}_{>0}\\
&L_c(h) = \Phi(\ell_c)h, \quad \text{or } L_c(h) = (1+\kappa_c^2)h, \quad \text{or } L_c(h) = \varphi(|c|)h,
\end{aligned}$$

Sobolev metrics do not satisfy the conditions. By using a projection in the definiton one can, however, modify them to still respect the vertical bundle:

$$L_c h = \Big(\mathrm{pr}_c(1-(-1)^k D_s^{2k}) \,\mathrm{pr}_c + (1-\mathrm{pr}_c)(1-(-1)^k D_s^{2k})(1-\mathrm{pr}_c)\Big) h,$$

where $\mathrm{pr}_c\, h = \langle D_s c, h\rangle D_s c$ is the L^2-orthogonal projection to the vertical bundle; see [Bauer, Harms, 2015], where metrics of this form were studied.

Horizontal Ω^L-Hamiltonian vector fields.

If the inertia operator $L \in \Gamma(\mathrm{End}(T\mathrm{Imm}(S^1, \mathbb{R}^3))$ induces a weak symplectic structure on B_i, and is weakly non-degenerate in the sense that $\bar{\Omega}^L : TB_i \to T^*B_i$ is injective then the kernel of $\Omega^L_c : T_c\mathrm{Imm} \to T^*_c\mathrm{Imm}$ equals $T_c(c \circ \mathrm{Diff})$ for all c. Thus Ω^L_c restricted to the G^L-orthogonal complement (if it is a complement) of $T_c(c \circ \mathrm{Diff})$ is injective. Let us suppose that H is a Diff-invariant smooth function on Imm. The 2-form Ω^L on Imm is still only presymplectic, but if each dH_c lies in the image of $\Omega^L : T\mathrm{Imm} \to T^*\mathrm{Imm}$, then a unique smooth *horizontal Hamiltonian vector field* $X \in \mathfrak{X}(\mathrm{Imm})$ is determined by

$$dH = i_X\Omega^L = \Omega^L(X,\ ), \quad G^L_c(X_c, Y_c) = 0 \quad \forall Y_c \in \ker(\Omega^L_c)$$

which we will denote by $\mathrm{hgrad}^{\Omega^L}(H)$. Obviously we then have

$$\mathrm{grad}^{\bar{\Omega}^L}(\bar{H}) \circ \pi = T\pi \circ \mathrm{hgrad}^{\Omega^L}(H), \quad \text{where } \bar{H} \circ \pi = H\,.$$

Momentum mappings

Since Θ^L is invariant for the action of group G with infinites. action $\zeta : \mathfrak{g} \to \mathfrak{X}(\mathrm{Imm}(S^1, \mathbb{R}^3))$, the *momentum mapping* is given, for $Y \in \mathfrak{g}$, by

$$\langle J(c), Y\rangle = \Theta^L(\zeta_Y)_c = \int \langle c \times D_s c, L_c(Y \circ c)\rangle ds,$$

where we denote the duality as $\langle \quad , \quad \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$. Namely,

$$d\Theta^L(\zeta_Y) = di_{\zeta_Y}\Theta^L = \mathcal{L}_{\zeta_Y}\Theta^L - i_{\zeta_Y} d\Theta^L = 0 - i_{\zeta_Y}\Omega^L.$$

Thus we have for $X = a.\partial_s \in \mathfrak{X}(S^1) = C^\infty(S^1)\partial_s$ and $Y \in \mathfrak{so}(3)$

$$\begin{aligned}
\zeta_{a.\partial_\theta}(c) &= D_{c,a.c_\theta} \quad \text{as derivation at } c \text{ on } C^\infty(\mathrm{Imm}, \mathbb{R}) \\
&= a.c_\theta = a.|c_\theta| D_s c \in T_c\mathrm{Imm} = C^\infty(S^1, \mathbb{R}^3) \\
L_{c\circ\varphi}(h \circ \varphi) &= (L_c h) \circ \varphi \implies (D_{c,a.c_\theta} L_c)(h) + L_c(a.h_\theta) = a.(L_c h)_\theta \\
\langle J^{\mathrm{Diff}}(c), a.\partial_\theta\rangle &= \Theta^L_c(\zeta_{a.\partial_\theta}(c)) = \Theta^L_c(a.c_\theta) = \int \langle c \times D_s c, L_c(a.c_\theta)\rangle ds \\
&= \int \langle c \times D_s c, a.(L_c c)_\theta - (D_{c,a.c_\theta} L_c)(c)\rangle ds
\end{aligned}$$

For $Y \in \mathfrak{so}(3)$ the *angular momentum* is

$$\begin{aligned}\langle J^{SO(3)}(c), Y\rangle = \Theta^L(Y\circ c) &= \int \langle c \times D_s c, L_c(Y \circ c)\rangle ds \\ &= \int \langle c \times D_s c, y \circ L_c(c) - (D_{c,Y\circ c} L_c(c)\rangle ds\end{aligned}$$

For a correct interpretation of the angular momentum recall that the action of $Y \in \mathbb{R}^3 \cong \mathfrak{so}(3) \cong L_{\text{skew}}(\mathbb{R}^3, \mathbb{R}^3)$ on $\mathbb{R}^3$ is given by $X \mapsto 2Y \times X$.

If L is also invariant under translations, then the *linear momentum*, for $y \in \mathbb{R}^3$, is

$$\langle J^{\mathbb{R}^3}(c), y\rangle = \Theta^L_c(y) = \int \langle c \times D_s c, L_c(y)\rangle ds \,.$$

Note that the above also furnishes conserved quantities on B_i, if $\bar{\Omega}^L$ is non-degenerate.

Theorem [Conformal factors] *Let $L_c = \lambda(c)\,\mathrm{id}$. Then the induced (pre)symplectic structure on* $\mathrm{Imm}(S^1, \mathbb{R}^3)$ *is given by*

$$\Omega^\lambda = \lambda\Omega^{\mathrm{id}} + \Theta^{\mathrm{id}} \wedge d\lambda.$$

Furthermore we have
(a) If $3\lambda_c + i_c d\lambda_c = 3\lambda(c) + D_{c,c}\lambda \neq 0$ on any open subset of Imm, *then Ω^λ induces a non-degenerate two-form on $B_i(S^1, \mathbb{R}^3)$, which is thus symplectic.*
(b) Assume in addition, that $X := \mathrm{hgrad}^{\Omega^{\mathrm{id}}} \lambda$ exists and that $3\lambda_c + i_c d\lambda_c = 0$ for all c. Denote by $\mathcal{F}$ the involutive 2-dimensional vector sub-bundle spanned by $I_c = c$ and $\mathrm{grad}^{\bar{\Omega}^{\mathrm{id}}} \bar{\lambda}$. Then Ω^λ induces a non-degenerate two-form on $\mathrm{Imm}/(\mathrm{Diff} \times \mathcal{F}) \simeq \{\bar{c} \in B_i(S^1, \mathbb{R}^3) : \bar{\lambda}_{\bar{c}} = 1\}/\,\mathrm{span}(\mathrm{grad}^{\bar{\Omega}^{\mathrm{id}}} \bar{\lambda})$, *where it agrees with a multiple of the Marsden-Weinstein symplectic structure. It is also non-degenerate on* $\{\bar{c} \in B_i(S^1, \mathbb{R}^3) : \bar{\ell}_{\bar{c}} = 1\}/\,\mathrm{span}(\mathrm{grad}^{\bar{\Omega}^{\mathrm{id}}} \bar{\lambda})$.

Theorem *(a)) Consider a -invariant Hamiltonian* $H\colon \mathrm{Imm}(S^1,\mathbb{R}^3) \to \mathbb{R}^3$*. If* $3\lambda_c + i_c d\lambda_c = 3\lambda(c) + D_{c,c}\lambda \neq 0$ *on any open subset of* Imm *then*

$$\mathrm{hgrad}^{\Omega^\lambda} H = -\frac{1}{3\lambda_c}\Big\{D_s c \times \mathrm{grad}^{G^{\mathrm{id}}} H + \frac{1}{3\lambda_c + D_{c,c}\lambda}\Big[\langle \mathrm{grad}_c^{G^{\mathrm{id}}} \lambda, D_s c \times \mathrm{grad}^{G^{\mathrm{id}}} H\rangle_{L^2_{ds}(S^1)} D_s c \times (D_s c \times c) - \langle c, \mathrm{grad}^{G^{\mathrm{id}}} H\rangle_{L^2_{ds}(S^1)} D_s c \times \mathrm{grad}_c^{G^{\mathrm{id}}} \lambda\Big]\Big\}.$$

(b) Consider a Hamiltonian $H\colon \mathrm{Imm}(S^1,\mathbb{R}^3) \to \mathbb{R}^3$ *invariant under and the flows of* $I_c = c$ *and* $\mathrm{hgrad}_c^{\Omega^{\mathrm{id}}} \lambda = -D_s c \times \mathrm{grad}^{G^{\mathrm{id}}} \lambda$*. If* $3\lambda_c + i_c d\lambda_c = 0$ *for all* c *then* $\mathrm{hgrad}_c^{\Omega^\lambda} H$ *is the orthonormal projection of*

$$Y_c^H = -\frac{1}{3\lambda_c} D_s c \times \mathrm{grad}_c^{G^{\mathrm{id}}} H$$

to the G_c^{id}*-orthogonal complement of the kernel of* Ω^λ*, which is spanned by* $I_c = c$*,* $\mathrm{hgrad}_c^{\Omega^{\mathrm{id}}} \lambda$*, and* $\{a.D_s c \mid a \in C^\infty(S^1)\}$*.*

Example: Length (here $\lambda(c) = \Phi(\ell_c)$)

Suppose $H(c) = f \circ \ell$ for some function f. Then:

$$dH_c(k) = f'(\ell_c) \int \langle D_s k, D_s c \rangle ds = -f'(\ell_c) \int \langle D_s^2 c, k \rangle ds,$$

$$\operatorname{grad}^{G^{\mathrm{id}}} H = -f'(\ell_c) D_s^2 c.$$

$$\operatorname{hgrad}^{\Omega^{\Phi(\ell)}} H = \frac{f'(\ell_c)}{3\Phi(\ell_c)} \left(1 + \frac{\Phi'(\ell_c)\ell_c}{3\Phi(\ell_c) + \Phi'(\ell_c)\ell_c} \right) D_s c \times D_s^2 c.$$

If $f'(\ell_c) = 0$ then c is a fixed point of the Hamitonian flow. If $f'(\ell_c) \neq 0$, then the length ℓ_c is conserved along the flow as $H = f \circ \ell$ is conserved. Note that $\operatorname{hgrad}^{\Omega^{\Phi(\ell)}} H$ is a constant multiple of the binormal equation (also known as the vortex filament equation),

$$\operatorname{hgrad}^{\Omega^{\mathrm{MW}}} \ell = D_s c \times D_s^2 c$$

using the Marsden-Weinstein symplectic structure.

Example: Kinetic energy as a Hamiltonian function:

$E(c) = \frac{1}{2}\int |c|^2 ds$, Using $\mathrm{grad}^{G^{\mathrm{id}}} E = c$, the horizontal Hamiltonian field is:

$$\begin{aligned}
\mathrm{hgrad}^{\Omega^{\Phi(\ell)}} E = \frac{1}{3\Phi(\ell_c)} \Big\{ &- D_s c \times c + \frac{1}{2}|c|^2 D_s c \times D_s^2 c \\
&+ \frac{\Phi'(\ell_c)}{3\Phi(\ell_c) + \Phi'(\ell_c)\ell_c} \Big[- \Theta_c^{\mathrm{id}}(D_s^2 c) D_s c \times (D_s c \times c) \\
&- \left(\|D_s c \times c\|^2_{L^2_{ds}(S^1)} - \frac{1}{2}\langle c, |c|^2 D_s^2 c\rangle_{L^2_{ds}(S^1)} \right) D_s c \times D_s^2 c \Big] \Big\} \\
= \frac{1}{3\Phi(\ell_c)} \Big\{ &- D_s c \times c + \frac{1}{2}|c|^2 D_s c \times D_s^2 c \\
&+ \frac{\Phi'(\ell_c)}{3\Phi(\ell_c) + \Phi'(\ell_c)\ell_c} \Big[\Theta_c^{\mathrm{id}}(D_s^2 c)(1 - \mathrm{pr}_c) c \\
&- \left(\|D_s c \times c\|^2_{L^2_{ds}(S^1)} + E(c) \right) D_s c \times D_s^2 c \Big] \Big\}.
\end{aligned}$$

Squared curvature as Hamiltonian function

$$H(c) = \frac{1}{2}\int \kappa^2 ds.$$

We have

$$\begin{aligned}
\operatorname{hgrad}^{\Omega^{\Phi(\ell)}} H &= \frac{1}{3\Phi(\ell_c)}\Big\{ - D_s c \times D_s^4 c - \frac{3}{2}\kappa^2 D_s c \times D_s^2 c \\
&\qquad + \frac{H\Phi'(\ell_c)}{3\Phi(\ell_c) + \Phi'(\ell_c)\ell_c} D_s c \times D_s^2 c\Big\} \\
&= \frac{1}{3\Phi(\ell_c)}\left\{ \operatorname{hgrad}^{\Omega^{\mathrm{id}}} H + \frac{H\Phi'(\ell_c)}{3\Phi(\ell_c) + \Phi'(\ell_c)\ell_c} \operatorname{hgrad}^{\Omega^{\mathrm{id}}} \ell \right\}.
\end{aligned}$$

Since both H and ℓ are again constants in motion along both $\operatorname{hgrad}^{\Omega^{\mathrm{id}}} \ell$ and $\operatorname{hgrad}^{\Omega^{\mathrm{id}}} H$, $\operatorname{hgrad}^{\Omega^{\Phi(\ell)}} H$ is also realized as a Hamiltonian vector field on Ω^{id}.

Thank you for your attention