Infinite dimensional Lie groups: Diffeomorphism groups

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Abstract: Groups of diffeomorphisms of a manifold M have many of the properties of finite dimensional Lie groups, but also differ in surprising ways. I review some (or all or more) of the following properties or I do something else:
No complexification. Exponential mappings are defined but are not locally surjective or injective. Right invariant Riemannian metrics might have vanishing geodesic distance. Many famous PDE's arise as geodesic equations on Diffeomorphism groups. There are topological groups of diffeomorphisms which are smooth manifolds but only right translations are smooth. There are diffeomorphism groups which are smooth in a certain sense (Some Denjoy-ultradifferentiable class) but not better (not real analytic).

Why Banach Lie groups are not enough

One of the important objects is the diffeomorphism group

$$\mathrm{Diff}(M) = \{\varphi \in C^\infty(M,M) \,:\, \varphi \text{ bijective}, \varphi^{-1} \in C^\infty(M,M)\},$$

of a compact smooth manifold M. We will see soon that $\mathrm{Diff}(M)$ is a smooth Fréchet–Lie group. What about a Banach manifold version of the diffeomorphism group? If $n \geq 1$, then one can consider

$$\mathrm{Diff}_{C^n}(M) = \{\varphi \in C^n(M,M) \,:\, \varphi \text{ bijective}, \varphi^{-1} \in C^n(M,M)\},$$

the group of C^n-diffeomorphisms. The space $\mathrm{Diff}_{C^n}(M)$ is a Banach manifold and a topological group, but not a Lie group. What went wrong? The group operations are continuous, but not differentiable.

Fix $\varphi \in \mathrm{Diff}_{C^n}(M)$ and consider left translation

$$L_\varphi : \mathrm{Diff}_{C^n}(M) \to \mathrm{Diff}_{C^n}(M) , \quad \psi \mapsto \varphi \circ \psi ;$$

its derivative should be

$$T_\psi L_\varphi . h = (D\varphi \circ \psi).h ,$$

with $T_\psi L_\varphi$ denoting the derivative of the map L_ψ and $D\varphi$ denotes the derivative of the diffeomorphism φ; the former is a map between infinite-dimensional manifolds, while the latter maps M to itself. To see this, consider a one-parameter variation $\psi(t,x)$, such that $\psi(0,x) = \psi(x)$ and $\partial_t \psi(t,x)|_{t=0} = h(x)$, and compute

$$\partial_t \varphi(\psi(t,x))|_{t=0} = D\varphi(\psi(x)).h(x) .$$

We see that in general $T_\psi L_\varphi .h$ lies only in C^{n-1}. However, if composition were to be a differentiable operator, $T_\varphi L_\psi$ would have to map into C^n-functions.

There seems to be a trade off involved: we can consider smooth functions, in which case the diffeomorphism group is a Lie group, but can be modelled only on a Fréchet space; or we look at functions with finite regularity, but then composition ceases to be differentiable. This choice cannot be avoided.

Theorem (Omori, 1978)

If a connected Banach–Lie group G acts effectively, transitively and smoothly on a compact manifold, then G must be a finite-dimensional Lie group.

A short introduction to convenient calculus in infinite dimensions.

Traditional differential calculus works well for finite dimensional vector spaces and for Banach spaces.
Beyond Banach spaces, the main difficulty is that composition of linear mappings stops to be jointly continuous at the level of Banach spaces, for any compatible topology.
For more general locally convex spaces we sketch here the convenient approach to C^∞ as explained in [Frölicher-Kriegl 1988] and [Kriegl-M 1997].

The c^∞-topology

Let E be a locally convex vector space. A curve $c : \mathbb{R} \to E$ is called *smooth* or C^∞ if all derivatives exist and are continuous. Let $C^\infty(\mathbb{R}, E)$ be the space of smooth curves. It can be shown that the set $C^\infty(\mathbb{R}, E)$ does not entirely depend on the locally convex topology of E, only on its associated bornology (system of bounded sets). The final topologies with respect to the following sets of mappings into E coincide:

1. $C^\infty(\mathbb{R}, E)$.
2. The set of all Lipschitz curves (so that $\{\frac{c(t)-c(s)}{t-s} : t \neq s, |t|, |s| \leq C\}$ is bounded in E, for each C).
3. The set of injections $E_B \to E$ where B runs through all bounded absolutely convex subsets in E, and where E_B is the linear span of B equipped with the Minkowski functional $\|x\|_B := \inf\{\lambda > 0 : x \in \lambda B\}$.
4. The set of all Mackey-convergent sequences $x_n \to x$ (there exists a sequence $0 < \lambda_n \nearrow \infty$ with $\lambda_n(x_n - x)$ bounded).

The c^∞-topology. II

This topology is called the c^∞-topology on E and we write $c^\infty E$ for the resulting topological space.
In general (on the space $\mathcal{D}$ of test functions for example) it is finer than the given locally convex topology, it is not a vector space topology, since addition is no longer jointly continuous. Namely, even $c^\infty(\mathcal{D} \times \mathcal{D}) \neq c^\infty\mathcal{D} \times c^\infty\mathcal{D}$.
The finest among all locally convex topologies on E which are coarser than $c^\infty E$ is the bornologification of the given locally convex topology. If E is a Fréchet space, then $c^\infty E = E$.

Convenient vector spaces

A locally convex vector space E is said to be a *convenient vector space* if one of the following holds (called c^∞-completeness):

1. For any $c \in C^\infty(\mathbb{R}, E)$ the (Riemann-) integral $\int_0^1 c(t)dt$ exists in E.
2. Any Lipschitz curve in E is locally Riemann integrable.
3. A curve $c : \mathbb{R} \to E$ is C^∞ if and only if $\lambda \circ c$ is C^∞ for all $\lambda \in E^*$, where E^* is the dual of all cont. lin. funct. on E.
    - Equiv., for all $\lambda \in E'$, the dual of all bounded lin. functionals.
    - Equiv., for all $\lambda \in \mathcal{V}$, where $\mathcal{V}$ is a subset of E' which recognizes bounded subsets in E.

   We call this *scalarwise* C^∞.
4. Any Mackey-Cauchy-sequence (i. e. $t_{nm}(x_n - x_m) \to 0$ for some $t_{nm} \to \infty$ in $\mathbb{R}$) converges in E. This is visibly a mild completeness requirement.

Convenient vector spaces. II

5. If B is bounded closed absolutely convex, then E_B is a Banach space.
6. If $f : \mathbb{R} \to E$ is scalarwise Lip^k, then f is Lip^k, for $k > 1$.
7. If $f : \mathbb{R} \to E$ is scalarwise C^∞ then f is differentiable at 0.

Here a mapping $f : \mathbb{R} \to E$ is called Lip^k if all derivatives up to order k exist and are Lipschitz, locally on $\mathbb{R}$. That f is scalarwise C^∞ means $\lambda \circ f$ is C^∞ for all continuous (equiv., bounded) linear functionals on E.

Smooth mappings

Let E, and F be convenient vector spaces, and let $U \subset E$ be c^∞-open. A mapping $f : U \to F$ is called smooth or C^∞, if $f \circ c \in C^\infty(\mathbb{R}, F)$ for all $c \in C^\infty(\mathbb{R}, U)$.
If E is a Fréchet space, then this notion coincides with all other reasonable notions of C^∞-mappings. Beyond Fréchet mappings, as a rule, there are more smooth mappings in the convenient setting than in other settings, e.g., C_c^∞.

Main properties of smooth calculus

1. For maps on Fréchet spaces this coincides with all other reasonable definitions. On $\mathbb{R}^2$ this is non-trivial [Boman,1967].
2. Multilinear mappings are smooth iff they are bounded.
3. If $E \supseteq U \xrightarrow{f} F$ is smooth then the derivative $df : U \times E \to F$ is smooth, and also $df : U \to L(E, F)$ is smooth where $L(E, F)$ denotes the space of all bounded linear mappings with the topology of uniform convergence on bounded subsets.
4. The chain rule holds.
5. The space $C^\infty(U, F)$ is again a convenient vector space where the structure is given by the obvious injection
$$C^\infty(U, F) \xrightarrow{C^\infty(c,\ell)} \prod_{c \in C^\infty(\mathbb{R},U), \ell \in F^*} C^\infty(\mathbb{R}, \mathbb{R}), \quad f \mapsto (\ell \circ f \circ c)_{c,\ell},$$
where $C^\infty(\mathbb{R}, \mathbb{R})$ carries the topology of compact convergence in each derivative separately.

Main properties of smooth calculus, II

6. The exponential law holds: For c^∞-open $V \subset F$,

$$C^\infty(U, C^\infty(V, G)) \cong C^\infty(U \times V, G)$$

   is a linear diffeomorphism of convenient vector spaces.
   Note that this is the main assumption of variational calculus. Here it is a theorem.

7. A linear mapping $f : E \to C^\infty(V, G)$ is smooth (by (2) equivalent to bounded) if and only if
   $E \xrightarrow{f} C^\infty(V, G) \xrightarrow{\mathrm{ev}_v} G$ is smooth for each $v \in V$.
   (*Smooth uniform boundedness theorem*, see [Kriegl M 1997], theorem 5.26).

Main properties of smooth calculus, III

8. The following canonical mappings are smooth.

$$\begin{aligned}
&\mathrm{ev} : C^\infty(E,F) \times E \to F, \quad \mathrm{ev}(f,x) = f(x) \\
&\mathrm{ins} : E \to C^\infty(F, E \times F), \quad \mathrm{ins}(x)(y) = (x,y) \\
&(\quad)^\wedge : C^\infty(E, C^\infty(F,G)) \to C^\infty(E \times F, G) \\
&(\quad)^\vee : C^\infty(E \times F, G) \to C^\infty(E, C^\infty(F,G)) \\
&\mathrm{comp} : C^\infty(F,G) \times C^\infty(E,F) \to C^\infty(E,G) \\
&C^\infty(\quad,\quad) : C^\infty(F,F_1) \times C^\infty(E_1,E) \to \\
&\qquad\qquad\qquad \to C^\infty(C^\infty(E,F), C^\infty(E_1,F_1)) \\
&\qquad (f,g) \mapsto (h \mapsto f \circ h \circ g) \\
&\prod : \prod C^\infty(E_i,F_i) \to C^\infty(\prod E_i, \prod F_i)
\end{aligned}$$

This ends our review of standard results of C^∞ convenient calculus. Convenient calculus (having properties 6 and 7) exists also for:

- Real analytic mappings [Kriegl,M,1990]. Mappings are smooth along smooth curves and C^ω along C^ω-curves.
- Holomorphic mappings [Kriegl,Nel,1985] (notion of [Fantappié, 1930-33]). Mappings are holomorphic along affine complex lines.
- Many classes of Denjoy Carleman ultradifferentible functions, both of Beurling type and of Roumieu-type [Kriegl,M,Rainer, 2009, 2011, 2013]. We will come back to this later, since it has surprising consequences for diffeomorphism groups.
- With some adaptations, Lip^k [Frölicher-Kriegl, 1988].
- With more adaptations, even $C^{k,\alpha}$ (k-th derivative Hölder-contin. with index α) [Faure,Frölicher 1989], [Faure, These Geneve, 1991].

Manifolds of mappings and diffeomorphism groups as convenient manifolds.

We do this for C^∞. It works for each real convenient calculus $\mathcal{S}$ (not for holomorphic). See later for DC ultradifferentiable calculus. Let M be a compact (for simplicity's sake) fin. dim. manifold and N a manifold. We use an auxiliary Riemann metric $\bar{g}$ on N. Then

$$\begin{tikzcd} & 0_N \arrow[dl, bend right, "\text{zero section}"'] \arrow[d, hook] & & N \arrow[d, hook] \arrow[dr, bend left, "\text{diagonal}"] & \\ TN & V^N \arrow[l, hook', "\text{open}"] \arrow[rr, "(\pi_N, \exp^{\bar{g}})", "\cong"'] & & V^{N\times N} \arrow[r, hook, "\text{open}"'] & N \times N \end{tikzcd}$$

$C^\infty(M,N)$, the space of smooth mappings $M \to N$, has the following manifold structure. Chart, centered at $f \in C^\infty(M,N)$, is:

$$C^\infty(M,N) \supset U_f = \{g : (f,g)(M) \subset V^{N\times N}\} \xrightarrow{u_f} \tilde{U}_f \subset \Gamma(f^*TN)$$

$$u_f(g) = (\pi_N, \exp^{\bar{g}})^{-1} \circ (f,g), \quad u_f(g)(x) = (\exp^{\bar{g}}_{f(x)})^{-1}(g(x))$$

$$(u_f)^{-1}(s) = \exp^{\bar{g}}_f \circ s, \qquad (u_f)^{-1}(s)(x) = \exp^{\bar{g}}_{f(x)}(s(x))$$

Manifolds of mappings II

Lemma: $C^\infty(\mathbb{R}, \Gamma(M; f^*TN)) = \Gamma(\mathbb{R} \times M; {\mathrm{pr}_2}^* f^*TN)$
By Cartesian Closedness (after trivializing the bundle f^*TN).
Lemma: Chart changes are smooth (C^∞)
$\tilde{U}_{f_1} \ni s \mapsto (\pi_N, \exp^{\bar{g}}) \circ s \mapsto (\pi_N, \exp^{\bar{g}})^{-1} \circ (f_2, \exp^{\bar{g}}_{f_1} \circ s)$
since they map smooth curves to smooth curves.
Lemma: $C^\infty(\mathbb{R}, C^\infty(M, N)) \cong C^\infty(\mathbb{R} \times M, N)$.
By the first lemma.
Lemma: Composition $C^\infty(P, M) \times C^\infty(M, N) \to C^\infty(P, N)$, $(f, g) \mapsto g \circ f$, is smooth, since it maps smooth curves to smooth curves
Corollary (of the chart structure):

$$TC^\infty(M, N) = C^\infty(M, TN) \xrightarrow{C^\infty(M, \pi_N)} C^\infty(M, N).$$

$$T_f C^\infty(M, N) = \left\{ h : \begin{tikzcd} & TN \arrow[d, "\pi"] \\ M \arrow[ur, "h"] \arrow[r, "f"] & N \end{tikzcd} \right\} \cong \Gamma(f^*TN)$$

Regular Lie groups

We consider a smooth Lie group G with Lie algebra $\mathfrak{g} = T_e G$ modelled on convenient vector spaces. The notion of a regular Lie group is originally due to Omori et al. for Fréchet Lie groups, was weakened and made more transparent by Milnor, and then carried over to convenient Lie groups; see [Kriegl, M 1997], 38.4.
A Lie group G is called *regular* if the following holds:

- For each smooth curve $X \in C^\infty(\mathbb{R}, \mathfrak{g})$ there exists a curve $g \in C^\infty(\mathbb{R}, G)$ whose right logarithmic derivative is X, i.e.,

$$\begin{cases} g(0) & = e \\ \partial_t g(t) & = T_e(\mu^{g(t)})X(t) = X(t).g(t) \end{cases}$$

  The curve g is uniquely determined by its initial value $g(0)$, if it exists.
- Put $\mathrm{evol}^r_G(X) = g(1)$ where g is the unique solution required above. Then $\mathrm{evol}^r_G : C^\infty(\mathbb{R}, \mathfrak{g}) \to G$ is required to be C^∞ also. We have $\mathrm{Evol}^X_t := g(t) = \mathrm{evol}_G(tX)$.

Diffeomorphism group of compact M

Theorem: *For each compact manifold M, the diffeomorphism group is a regular Lie group.*

Proof: $\mathrm{Diff}(M) \xrightarrow{open} C^\infty(M, M)$. Composition is smooth by restriction. Inversion is smooth: If $t \mapsto f(t, \quad)$ is a smooth curve in $\mathrm{Diff}(M)$, then $f(t, \quad)^{-1}$ satisfies the implicit equation $f(t, f(t, \quad)^{-1}(x)) = x$, so by the finite dimensional implicit function theorem, $(t, x) \mapsto f(t, \quad)^{-1}(x)$ is smooth. So inversion maps smooth curves to smooth curves, and is smooth.

Let $X(t, x)$ be a time dependent vector field on M (in $C^\infty(\mathbb{R}, \mathfrak{X}(M))$). Then $\mathrm{Fl}_s^{\partial_t \times X}(t, x) = (t + s, \mathrm{Evol}^X(t, x))$ satisfies the ODE $\quad \partial_t \mathrm{Evol}(t, x) = X(t, \mathrm{Evol}(t, x))$. If $X(s, t, x) \in C^\infty(\mathbb{R}^2, \mathfrak{X}(M))$ is a smooth curve of smooth curves in $\mathfrak{X}(M)$, then obviously the solution of the ODE depends smoothly also on the further variable s, thus evol maps smooth curves of time dependant vector fields to smooth curves of diffeomorphism. $\square$

Exponential mapping of $\mathrm{Diff}(M)$

The exponential mapping $\mathrm{Exp} : \mathfrak{X}_c(M) \to \mathrm{Diff}_c(M)$ satisfies $T_0 \, \mathrm{Exp} = \mathrm{Id}$, but it is not locally surjective near Id_M: This is due to [Freifeld67] and [Koppell70]. The strongest result in this direction is [Grabowski88], where it is shown, that $\mathrm{Diff}_c(M)$ contains a smooth curve through Id_M whose points (sauf Id_M) are free generators of an arcwise connected free subgroup which meets the image of Exp only at the identity.
The same is true for groups real-analytic diffemorphisms, and groups of Denjoy-Carleman ultradifferentiable diffeomorphisms (see below).

Proof

of a weak version of this result for $M = S^1$. For large $n \in \mathbb{N}$ we consider the diffeomorphism

$$f_n(\theta) = \theta + \tfrac{2\pi}{n} + \tfrac{1}{2^n}\sin^2(\frac{n\theta}{2}) \mod 2\pi;$$

the diffeomorphism f_n has just one periodic orbit and this is of period n, namely $\{\frac{2\pi k}{n} : k = 0, \ldots, n-1\}$. For even n the diffeomorphism f_n cannot be written as $g \circ g$ for a diffeomorphism g (so f_n is not contained in a flow), by the following argument: If g has a periodic orbit of odd period, then this is also a periodic orbit of the same period of $g \circ g$, whereas a periodic orbit of g of period $2n$ splits into two disjoint orbits of period n each, of $g \circ g$. Clearly, a periodic orbit of $g \circ g$ is a subset of a periodic orbit of g. So if $g \circ g$ has only finitely many periodic orbits of some even order, there must be an even number of them.

More on Diffeomorphism groups

- Let $f \in \mathrm{Diff}(S^1)$ be fixed point free and in the image of Exp. Then f is conjugate to some translation R_θ.
- A formula for the tangent mapping of the exponential of a Lie group in the case $G = \mathrm{Diff}(M)$ looks as follows:

$$T_X \operatorname{Exp} . Y = \int_0^1 (\mathrm{Fl}^X_{-t})^* Y \, dt \circ \mathrm{Fl}^X_1$$

- For each finite dimensional manifold M of dimension $m > 1$ and for $M = S^1$ the mapping $T_X \operatorname{Exp}$ is not injective for some X arbitrarily near to 0.
- The mapping

$$\mathrm{Ad} \circ \operatorname{Exp} : \mathfrak{X}_c(M) \to \mathrm{Diff}(M) \to L(\mathfrak{X}_c(M), \mathfrak{X}_c(M))$$

is not real analytic since
$\mathrm{Ad}(\operatorname{Exp}(sX))Y(x) = (\mathrm{Fl}^X_{-s})^* Y(x) = T_x(\mathrm{Fl}^X_s)(Y(\mathrm{Fl}^X_{-s}(x)))$ is not real analytic in s in general: choose Y constant in a chart and X not real analytic.

- The group $\mathrm{Diff}^\omega(M)$ of real analytic diffeomorphisms is a real analytic regular Lie group in the convenient sense (see below).
- But is is not real analytic in the sense of extendability to complexifications [Dahmen-Schmeding,2015]. Thus it has no complexification.

Lie subalgebras do not correspond to Lie subgroups
Let $\overline{g} \subset \mathfrak{X}_c(\mathbb{R}^2)$ be the closed Lie subalgebra of all vector fields with compact support on $\mathbb{R}^2$ of the form
$X(x,y) = f(x,y)x + g(x,y)y$ where g vanishes on the strip $0 \le x \le 1$.
Claim. *There is no Lie subgroup G of* $\mathrm{Diff}(\mathbb{R}^2)$ *corresponding to* $\overline{g}$.
If G exists there is a smooth curve $t \mapsto f_t \in G \subset \mathrm{Diff}_c(\mathbb{R}^2)$. Then $X_t := (tf_t) \circ f_t^{-1}$ is a smooth curve in $\overline{g}$, and we may assume that $X_0 = fx$ where $f = 1$ on a large ball. But then $\mathrm{Ad}^G(f_t) = f_t^* : \overline{g} \not\to \overline{g}$, a contradiction.

Denjoy-Carleman ultradifferentiable functions

Fix a sequence $M = (M_k)$ of positive reals. A C^∞-mapping f on an open set $U \subset \mathbb{R}$ is said to be of class $C^{\{M\}}$ if for each compact set K there $\exists \rho > 0$ such that the set

$$\left\{ \frac{f^{(k)}(x)}{\rho^k \, k! \, M_k} : x \in K, k \in \mathbb{N} \right\} \quad \text{is bounded.}$$

In this way we get the so-called *Denjoy–Carleman classes of Roumieu type* $C^{\{M\}}$.
If we replace $\exists \rho > 0$ by a $\forall \rho > 0$ we obtain the *Denjoy–Carleman classes of Beurling type* $C^{(M)}$.
We will denote by $C^{[M]}$ either of them, and write $\Box$ for $\exists$ or $\forall$.

Properties of M		**Properties of $C^{[M]}$**
M increasing	$\Rightarrow$	$C^\omega(U) \subseteq C^{\{M\}}(U) \subseteq C^\infty(U)$
M logarithmically convex i.e., $M_k^2 \le M_{k-1}\, M_{k+1}\ \forall\ k$. Then: $(\frac{M_k}{M_0})^{1/k}$ increasing, $M_l\, M_k \le M_0\, M_{l+k}\ \forall\ l, k$, $M_1^k\, M_k \ge M_j\, M_{\alpha_1} \cdots M_{\alpha_j}$, $\alpha_i \in \mathbb{N}_{>0}, \alpha_1 + \cdots + \alpha_j = k$	$\Rightarrow$	$C^{[M]}(U)$ is a ring. $C^{[M]}$ closed under compos. $C^{[M]}$ closed under appl. inverse function thm. $C^{[M]}$ is closed under solving ODEs.
M weakly log-convex i.e., $(k!\, M_k)$ log-convex		(always assumed below)
$\sup_{k\in\mathbb{N}_{>0}}(M_k/N_k)^{1/k} < \infty$	$\Leftrightarrow$	$C^{[M]}(U) \subseteq C^{[N]}(U)$
$\sup_{k\in\mathbb{N}_{>0}}(M_k)^{1/k} < \infty$	$\Leftrightarrow$	$C^\omega(U) = C^{\{M\}}(U)$
$\lim_{k\to\infty}(M_k)^{1/k} = \infty$	$\Leftrightarrow$	$C^\omega(U) \subsetneq C^{\{M\}}(U)$
$\sup_{k\in\mathbb{N}_{>0}}(M_{k+1}/M_k)^{1/k} < \infty$	$\Leftrightarrow$	$C^{[M]}$ closed under derivat.

$\sum_{k=0}^{\infty} \frac{M_k}{(k+1)M_{k+1}} = \infty$ or, equivalently, $\sum_{k=1}^{\infty} (\frac{1}{k!\,M_k})^{1/k} = \infty$	$\Leftrightarrow$	$C^{[M]}$ is quasianalytic, i.e., $T_a : C^{[M]}(U) \to \mathcal{F}_n^M$ is inject. not surj.if $C^\omega(U) \subsetneq C^{[M]}(U)$
$\sum_{k=0}^{\infty} \frac{M_k}{(k+1)M_{k+1}} < \infty$	$\Leftrightarrow$	$C^{[M]}$ is non-quasianalytic. $\Rightarrow C^{[M]}$ part. of unity exist.
$\lim_{k\to\infty}(M_k)^{1/k} = \infty$ and $\sum_{k=j}^{\infty} \frac{M_k}{(k+1)M_{k+1}} \le C\frac{M_j}{M_{j+1}}$ for all $j \in \mathbb{N}$ and some C	$\Leftrightarrow$	$C^\omega(U) \subsetneq C^{[M]}(U)$ and $T_a : C^{[M]}(U) \to \mathcal{F}_n^M$ is surj.: $C^{[M]}$ strongly non-quasianal.
M has moderate growth, $\sup_{j,k\in\mathbb{N}_{>0}} (\frac{M_{j+k}}{M_j\,M_k})^{1/(j+k)} < \infty$		necessary for cartesian closedness

In [Kriegl,M,Rainer, 2009, 2011, 2013] the class $C^{[M]}$ was extended to mappings between admissible convenient locally convex spaces. It was proved that $C^{[M]}$ gives rise to a convenient calculus in the following sense, then forms a cartesian closed category, provided that $M = (M_k)$ is log-convex and has moderate growth.

A differentiabilty class $\mathcal{S}$ is a convenient calculus if:

(1) For c^∞-open sets in convenient vector spaces $U \subseteq E, V \subseteq F$ we can define $\mathcal{S}$-mappings, and $\mathcal{S}(U, F)$ is again a convenient space in a suitable lcs structure. c^∞-open subsets in convenient vector spaces and $\mathcal{S}$-mappings form a category. Any $\mathcal{S}$-mapping is continuous for the c^∞-topologies. If E, F are of dim $< \infty$ (or even Banach spaces) then $\mathcal{S}(U, F)$ is the classically defined space (this is usually hard!).
(2) We have a linear $\mathcal{S}$-diffeomorphism (G a convenient vs) $\mathcal{S}(U \times V, G) = \mathcal{S}(U, \mathcal{S}(V, G))$. (*Cartesian Closedness*)
(3) A map $f : U \to F$ is $\mathcal{S}$ iff $\lambda \circ f$ is $\mathcal{S}$ for all bounded linear functionals λ in a set $\subseteq E'$ which describes the bornology. (*$\mathcal{S}$ is a bornological concept*). Linear $\mathcal{S}$-mappings are exactly the bounded ones, and $L(E, F)$ (with the lc-topology of bounded convergence) is bornologically embedded in $\mathcal{S}(E, F)$.
(4) A linear map $\ell : E \to \mathcal{S}(V, G)$ is $\mathcal{S}$ ($\Leftrightarrow$ bounded) iff $\mathrm{ev}_x \circ \ell : E \to G$ is $\mathcal{S}$ for each $x \in F$. (*$\mathcal{S}$-uniform boundedness theorem).*

Thm. *Let $\mathcal{S}$ be a real differentiability class as above which admits convenient calculus. Let A and B be finite dimensional $\mathcal{S}$ manifolds with A compact. Then the space $\mathcal{S}(A,B)$ of all $\mathcal{S}$-mappings $A \to B$ is a $\mathcal{S}$-manifold modelled on convenient vector spaces $\Gamma_{\mathcal{S}}(f^*TB)$ of $\mathcal{S}$ sections of pullback bundles along $f : A \to B$. Moreover, a curve $c : \mathbb{R} \to \mathcal{S}(A,B)$ is $\mathcal{S}$ if and only if $c^\wedge : \mathbb{R} \times A \to B$ is $\mathcal{S}$. Similarly for Banach-plots.*
Corollary. *Let A_1, A_2 and B be finite dimensional $\mathcal{S}$ manifolds with A_1 and A_2 compact. Then composition*

$$\mathcal{S}(A_2, B) \times \mathcal{S}(A_1, A_2) \to \mathcal{S}(A_1, B), \quad (f, g) \mapsto f \circ g$$

is $\mathcal{S}$. This is best possible. For example, if $\mathcal{S} = C^{[M]}$ for a weight sequence M which is logarithmically convex and of moderate growth, and if $N = (N_k)$ is another with $(N_k/M_k)^{1/k} \searrow 0$, then composition is **not** *C^N.*
Proof. Composition maps $\mathcal{S}$-curves to $\mathcal{S}$-curves, so it is $\mathcal{S}$. For $\mathcal{S} = C^Q$ (i.e., Q is quasianalytic) we need Banach-plots.

Theorem. *Let $\mathcal{S}$ be as above. Let A be a compact ($\Rightarrow$ finite dimensional) $\mathcal{S}$ manifold. Then the group $\mathrm{Diff}^{\mathcal{S}}(A)$ of all $\mathcal{S}$-diffeomorphisms of A is an open subset of the $\mathcal{S}$ manifold $\mathcal{S}(A, A)$. Moreover, it is a $\mathcal{S}$-regular $\mathcal{S}$ Lie group: Inversion and composition are $\mathcal{S}$. Its Lie algebra consists of all $\mathcal{S}$-vector fields on A, with the negative of the usual bracket as Lie bracket. The exponential mapping is $\mathcal{S}$. It is not surjective onto any neighborhood of Id_A.*

This is best possible, similarly as in the composition theorem.

A Zoo of diffeomorphism groups on $\mathbb{R}^n$

Theorem. *The following groups of diffeomorphisms on $\mathbb{R}^n$ are C^∞-regular Lie groups:*

- $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R}^n)$, *the group of all diffeomorphisms which differ from the identity by a function which is bounded together with all derivatives separately.*
- $\mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$, *the group of all diffeomorphisms which differ from the identity by a function in the intersection H^∞ of all Sobolev spaces H^k for $k \in \mathbb{N}_{\geq 0}$.*
- $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R}^n)$, *the group of all diffeomorphisms which fall rapidly to the identity.*
- $\mathrm{Diff}_c(\mathbb{R}^n)$ *of all diffeomorphisms which differ from the identity only on a compact subset. (well known since 1980)*

[M, Mumford,2013], partly [B.Walter,2012]; for Denjoy-Carleman ultradifferentiable diffeomorphisms [Kriegl, M, Rainer 2014].
In particular, $\mathrm{Diff}_{H^\infty}(\mathbb{R}^n)$ is essential if one wants to prove that the geodesic equation of a right Riemannian invariant metric is well-posed with the use of Sobolev space techniques.

An exotic zoo of diffeomorphisms on $\mathbb{R}^n$

Various sets of $C^{[M]}$-diffeomorphisms of $\mathbb{R}^n$ form $C^{[M]}$-regular Lie groups. We denote by $\mathrm{Diff}\mathcal{A}$ the set of all mappings $\mathrm{Id}+f : \mathbb{R}^n \to \mathbb{R}^n$, where $\inf_{x\in\mathbb{R}^n} \det({}_n + df(x)) > 0$ and $f \in \mathcal{A}$, for any of the following classes $\mathcal{A}$ of test functions:

Global Denjoy–Carleman classes

$$\mathcal{B}^{[M]}(\mathbb{R}^n) = \Big\{ f \in C^\infty(\mathbb{R}^n) : \square\rho > 0 \sup_{\alpha\in\mathbb{N}^n} \frac{\|\partial^\alpha f\|_{L^\infty(\mathbb{R}^n)}}{\rho^{|\alpha|}\ |\alpha|!\ M_{|\alpha|}} < \infty \Big\}.$$

Sobolev–Denjoy–Carleman classes

$$W^{[M],p}(\mathbb{R}^n) = \Big\{ f \in C^\infty(\mathbb{R}^n) : \square\rho > 0 \sup_{\alpha\in\mathbb{N}^n} \frac{\|\partial^\alpha f\|_{L^p(\mathbb{R}^n)}}{\rho^{|\alpha|}\ |\alpha|!\ M_{|\alpha|}} < \infty \Big\}, \quad 1 \le p < \infty.$$

Gelfand–Shilov classes

$$\mathcal{S}^{[M]}_{[L]}(\mathbb{R}^n) = \Big\{ f \in C^\infty(\mathbb{R}^n) : \square\rho > 0 \sup_{\substack{p\in\mathbb{N}\\ \alpha\in\mathbb{N}^n}} \frac{\|(1+|x|)^p \partial^\alpha f\|_{L^\infty(\mathbb{R}^n)}}{\rho^{p+|\alpha|}\ p!|\alpha|!\ L_p M_{|\alpha|}} < \infty \Big\}.$$

Denjoy–Carleman functions with compact support

$$\mathcal{D}^{[M]}(\mathbb{R}^n) = C^{[M]}(\mathbb{R}^n) \cap (\mathbb{R}^n) = \mathcal{B}^{[M]}(\mathbb{R}^n) \cap \mathcal{D}(\mathbb{R}^n).$$

Note that $\mathcal{D}^{[M]}(\mathbb{R}^n)$ is trivial unless $M = (M_k)$ is non-quasianalytic.
For the sequence $L = (L_k)$ we just assume $L_k \geq 1$ for all k. Note that $\mathcal{D}^{[M]} \subseteq \mathcal{S}^{[M]}_{[L]}$, and hence $\mathcal{S}^{[M]}_{[L]}$ is certainly non-trivial if $M = (M_k)$ is non-quasianalytic.

Theorem

Let $M = (M_k)$ be log-convex and have moderate growth; in the Beurling case we also assume $C^{(M)} \supseteq C^\omega$. Assume that $L = (L_k)$ satisfies $L_k \geq 1$ for all k. Let $1 \leq p < q \leq \infty$. Then $\operatorname{Diff}\mathcal{B}^{[M]}$, $\operatorname{Diff}W^{[M],p}$, $\operatorname{Diff}\mathcal{S}^{[M]}_{[L]}$, *and* $\operatorname{Diff}\mathcal{D}^{[M]}$ *are -regular Lie groups. We have the following $C^{[M]}$ injective group homomorphisms*

$$\operatorname{Diff}\mathcal{D}^{[M]} \rightarrowtail \operatorname{Diff}\mathcal{S}^{[M]}_{[L]} \rightarrowtail \operatorname{Diff}W^{[M],p} \rightarrowtail \operatorname{Diff}W^{[M],q}(\mathbb{R}^n) \rightarrowtail \operatorname{Diff}\mathcal{B}^{[M]}.$$

Each group in this diagram is normal in the groups on its right.

Surprising behavior of right invariant weak Riemannian metrics.

Groups related to $\mathrm{Diff}_c(\mathbb{R})$

The reflexive nuclear (LF) space $C_c^\infty(\mathbb{R})$ of smooth functions with compact support leads to the well-known regular Lie group $\mathrm{Diff}_c(\mathbb{R})$.
Define $C_{c,2}^\infty(\mathbb{R}) = \{f : f' \in C_c^\infty(\mathbb{R})\}$ to be the space of antiderivatives of smooth functions with compact support. It is a reflexive nuclear (LF) space. We also define the space $C_{c,1}^\infty(\mathbb{R}) = \left\{f \in C_{c,2}^\infty(\mathbb{R}) \ : \ f(-\infty) = 0\right\}$ of antiderivatives of the form $x \mapsto \int_{-\infty}^x g\, dy$ with $g \in C_c^\infty(\mathbb{R})$.
$\mathrm{Diff}_{c,2}(\mathbb{R}) = \left\{\varphi = \mathrm{Id} + f : f \in C_{c,2}^\infty(\mathbb{R}), f' > -1\right\}$ *is the corresponding group.*
Define the two functionals $\mathrm{Shift}_\ell, \mathrm{Shift}_r : \mathrm{Diff}_{c,2}(\mathbb{R}) \to \mathbb{R}$ by

$$\mathrm{Shift}_\ell(\varphi) = \mathrm{ev}_{-\infty}(f) = \lim_{x\to-\infty} f(x), \quad \mathrm{Shift}_r(\varphi) = \mathrm{ev}_\infty(f) = \lim_{x\to\infty} f(x)$$

for $\varphi(x) = x + f(x)$.

Then the short exact sequence of smooth homomorphisms of Lie groups

$$\mathrm{Diff}_c(\mathbb{R}) \rightarrowtail \mathrm{Diff}_{c,2}(\mathbb{R}) \xrightarrow{(\mathrm{Shift}_\ell, \mathrm{Shift}_r)} (\mathbb{R}^2, +)$$

describes a semidirect product, where a smooth homomorphic section $s : \mathbb{R}^2 \to \mathrm{Diff}_{c,2}(\mathbb{R})$ is given by the composition of flows $s(a,b) = \mathrm{Fl}_a^{X_\ell} \circ \mathrm{Fl}_b^{X_r}$ for the vectorfields $X_\ell = f_\ell \partial_x$, $X_r = f_r \partial_x$ with $[X_\ell, X_r] = 0$ where $f_\ell, f_r \in C^\infty(\mathbb{R}, [0,1])$ satisfy

$$f_\ell(x) = \begin{cases} 1 & \text{for } x \leq -1 \\ 0 & \text{for } x \geq 0, \end{cases} \qquad f_r(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ 1 & \text{for } x \geq 1. \end{cases} \tag{1}$$

The normal subgroup
$\mathrm{Diff}_{c,1}(\mathbb{R}) = \ker(\mathrm{Shift}_\ell) = \{\varphi = \mathrm{Id} + f : f \in C^\infty_{c,1}(\mathbb{R}), f' > -1\}$ of diffeomorphisms which have no shift at $-\infty$ will play an important role later on.

Some diffeomorphism groups on $\mathbb{R}$

We have the following smooth injective group homomorphisms:

$$\begin{tikzcd}
\mathrm{Diff}_c(\mathbb{R}) \arrow[r] \arrow[d] & \mathrm{Diff}_{\mathcal{S}}(\mathbb{R}) \arrow[r] \arrow[d] & \mathrm{Diff}_{W^{\infty,1}}(\mathbb{R}) \arrow[d] & \\
\mathrm{Diff}_{c,1}(\mathbb{R}) \arrow[r] \arrow[d] & \mathrm{Diff}_{\mathcal{S}_1}(\mathbb{R}) \arrow[r] \arrow[d] & \mathrm{Diff}_{W_1^{\infty,1}}(\mathbb{R}) \arrow[d] & \\
\mathrm{Diff}_{c,2}(\mathbb{R}) \arrow[r] & \mathrm{Diff}_{\mathcal{S}_2}(\mathbb{R}) \arrow[r] & \mathrm{Diff}_{W_2^{\infty,1}}(\mathbb{R}) \arrow[r] & \mathrm{Diff}_{\mathcal{B}}(\mathbb{R})
\end{tikzcd}$$

Each group is a normal subgroup in any other in which it is contained, in particular in $\mathrm{Diff}_{\mathcal{B}}(\mathbb{R})$.
For $\mathcal{S}$ and $W^{\infty,1}$ this works the same as for C_c^∞. For $H^\infty = W^{\infty,2}$ it is surprisingly more subtle.

Solving the Hunter-Saxton equation: The setting

We will denote by $\mathcal{A}(\mathbb{R})$ any of the spaces $C_c^\infty(\mathbb{R})$, $\mathcal{S}(\mathbb{R})$ or $W^{\infty,1}(\mathbb{R})$ and by $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ the corresponding groups $\mathrm{Diff}_c(\mathbb{R})$, $\mathrm{Diff}_{\mathcal{S}}(\mathbb{R})$ or $\mathrm{Diff}_{W^{\infty,1}}(\mathbb{R})$.
Similarly $\mathcal{A}_1(\mathbb{R})$ will denote any of the spaces $C_{c,1}^\infty(\mathbb{R})$, $\mathcal{S}_1(\mathbb{R})$ or $W_1^{\infty,1}(\mathbb{R})$ and $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ the corresponding groups $\mathrm{Diff}_{c,1}(\mathbb{R})$, $\mathrm{Diff}_{\mathcal{S}_1}(\mathbb{R})$ or $\mathrm{Diff}_{W_1^{\infty,1}}(\mathbb{R})$.
The $\dot{H}^1$-metric. For $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ and $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ the homogeneous H^1-metric is given by

$$G_\varphi(X\circ\varphi, Y\circ\varphi) = G_{\mathrm{Id}}(X,Y) = \int_{\mathbb{R}} X'(x)Y'(x)\,dx\,,$$

where X, Y are elements of the Lie algebra $\mathcal{A}(\mathbb{R})$ or $\mathcal{A}_1(\mathbb{R})$. We shall also use the notation

$$\langle\cdot,\cdot\rangle_{\dot{H}^1} := G(\cdot,\cdot)\,.$$

Theorem

On $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ *the geodesic equation is the Hunter-Saxton equation*

$$\boxed{(\varphi_t)\circ\varphi^{-1} = u \qquad u_t = -uu_x + \frac{1}{2}\int_{-\infty}^{x}(u_x(z))^2\,dz\ ,}$$

and the induced geodesic distance is positive.
On the other hand the geodesic equation does not exist on the subgroups $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$, *since the adjoint* $\mathrm{ad}(X)^*\check{G}_{\mathrm{Id}}(X)$ *does not lie in* $\check{G}_{\mathrm{Id}}(\mathcal{A}(\mathbb{R}))$ *for all* $X\in\mathcal{A}(\mathbb{R})$*.*
One obtains the classical form of the Hunter-Saxton equation by differentiating:

$$u_{tx} = -uu_{xx} - \frac{1}{2}u_x^2\ ,$$

Note that $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ is a natural example of a non-robust Riemannian manifold.

Proof

Note that $\check{G}_{\mathrm{Id}} : \mathcal{A}_1(\mathbb{R}) \to \mathcal{A}_1(\mathbb{R})^*$ is given by $\check{G}_{\mathrm{Id}}(X) = -X''$ if we use the L^2-pairing $X \mapsto (Y \mapsto \int XY dx)$ to embed functions into the space of distributions. We now compute the adjoint of $\mathrm{ad}(X)$:

$$\begin{aligned}\left\langle \mathrm{ad}(X)^* \check{G}_{\mathrm{Id}}(Y), Z \right\rangle &= \check{G}_{\mathrm{Id}}(Y, \mathrm{ad}(X)Z) = G_{\mathrm{Id}}(Y, -[X, Z]) \\ &= \int_{\mathbb{R}} Y'(x)\big(X'(x)Z(x) - X(x)Z'(x)\big)'\, dx \\ &= \int_{\mathbb{R}} Z(x)\big(X''(x)Y'(x) - (X(x)Y'(x))''\big)\, dx \,.\end{aligned}$$

Therefore the adjoint as an element of $\mathcal{A}_1^*$ is given by

$$\mathrm{ad}(X)^* \check{G}_{\mathrm{Id}}(Y) = X''Y' - (XY')'' \,.$$

For $X = Y$ we can rewrite this as

$$\mathrm{ad}(X)^* \check{G}_{\mathrm{Id}}(X) = \tfrac{1}{2}\big((X'^2)' - (X^2)'''\big) = \frac{1}{2}\Big(\int_{-\infty}^{x} X'(y)^2\, dy - (X^2)' \Big)''$$
$$= \frac{1}{2} \check{G}_{\mathrm{Id}} \Big(- \int_{-\infty}^{x} X'(y)^2\, dy + (X^2)' \Big)\, .$$

If $X \in \mathcal{A}_1(\mathbb{R})$ then the function $-\frac{1}{2}\int_{-\infty}^{x} X'(y)^2\, dy + \frac{1}{2}(X^2)'$ is again an element of $\mathcal{A}_1(\mathbb{R})$. This follows immediately from the definition of $\mathcal{A}_1(\mathbb{R})$. Therefore the geodesic equation exists on $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ and is as given.
However if $X \in \mathcal{A}(\mathbb{R})$, a neccessary condition for $\int_{-\infty}^{x} (X'(y))^2 dy \in \mathcal{A}(\mathbb{R})$ would be $\int_{-\infty}^{\infty} X'(y)^2 dy = 0$, which would imply $X' = 0$. Thus the geodesic equation does not exist on $\mathcal{A}(\mathbb{R})$.
The positivity of geodesic distance will follow from the explicit formula for geodesic distance below. QED.

Theorem.

[BBM2014] [A version for *Diff* (S^1) is by J.Lenells 2007,08,11]
We define the R-map by:

$$R : \begin{cases} \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R}) \to \mathcal{A}\big(\mathbb{R}, \mathbb{R}_{>-2}\big) \subset \mathcal{A}(\mathbb{R}, \mathbb{R}) \\ \varphi \mapsto 2\left((\varphi')^{1/2} - 1\right) . \end{cases}$$

The R-map is invertible with inverse

$$R^{-1} : \begin{cases} \mathcal{A}\big(\mathbb{R}, \mathbb{R}_{>-2}\big) \to \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R}) \\ \gamma \mapsto x + \dfrac{1}{4}\displaystyle\int_{-\infty}^{x} \gamma^2 + 4\gamma \, dx . \end{cases}$$

The pull-back of the flat L^2-metric via R is the $\dot{H}^1$-metric on $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$*, i.e.,*

$$R^* \langle \cdot, \cdot \rangle_{L^2} = \langle \cdot, \cdot \rangle_{\dot{H}^1} .$$

Thus the space $\left(\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R}), \dot{H}^1\right)$ *is a flat space in the sense of Riemannian geometry.*
Here $\langle \cdot, \cdot \rangle_{L^2}$ denotes the L^2-inner product on $\mathcal{A}(\mathbb{R})$ with constant volume dx.

Proof

To compute the pullback of the L^2-metric via the R-map we first need to calculate its tangent mapping. For this let $h = X \circ \varphi \in T_\varphi \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ and let $t \mapsto \psi(t)$ be a smooth curve in $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ with $\psi(0) = \mathrm{Id}$ and $\partial_t|_0 \psi(t) = X$. We have:

$$\begin{aligned}
T_\varphi R.h &= \partial_t|_0 R(\psi(t)\circ\varphi) = \partial_t|_0 2\Big(((\psi(t)\circ\varphi)_x)^{1/2} - 1\Big)\\
&= \partial_t|_0 2((\psi(t)_x \circ \varphi)\,\varphi_x)^{1/2}\\
&= 2(\varphi_x)^{1/2}\partial_t|_0((\psi(t)_x)^{1/2}\circ\varphi) = (\varphi_x)^{1/2}\Big(\frac{\psi_{tx}(0)}{(\psi(0)_x)^{-1/2}}\circ\varphi\Big)\\
&= (\varphi_x)^{1/2}(X'\circ\varphi) = (\varphi')^{1/2}(X'\circ\varphi)\,.
\end{aligned}$$

Using this formula we have for $h = X_1 \circ \varphi, k = X_2 \circ \varphi$:

$$R^*\langle h, k\rangle_{L^2} = \langle T_\varphi R.h, T_\varphi R.k\rangle_{L^2} = \int_{\mathbb{R}} X_1'(x) X_2'(x)\, dx = \langle h, k\rangle_{\dot{H}^1} \; QED$$

Corollary

Given $\varphi_0, \varphi_1 \in \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ *the geodesic* $\varphi(t,x)$ *connecting them is given by*

$$\varphi(t,x) = R^{-1}\Big((1-t)R(\varphi_0) + tR(\varphi_1)\Big)(x)$$

and their geodesic distance is

$$d(\varphi_0,\varphi_1)^2 = 4\int_{\mathbb{R}} \left((\varphi_1')^{1/2} - (\varphi_0')^{1/2}\right)^2\, dx\ .$$

But this construction shows much more: For $\mathcal{S}_1$, C_1^∞, and even for many kinds of Denjoy-Carleman ultradifferentiable model spaces as explained above. This shows that Sobolev space methods for treating nonlinear PDEs is not the only method.

Corollary: *The metric space* $\left(\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R}), \dot{H}^1\right)$ *is path-connected and geodesically convex but not geodesically complete. In particular, for every* $\varphi_0 \in \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ *and* $h \in T_{\varphi_0}\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$, $h \neq 0$ *there exists a time* $T \in \mathbb{R}$ *such that* $\varphi(t,\cdot)$ *is a geodesic for* $|t| < |T|$ *starting at* φ_0 *with* $\varphi_t(0) = h$, *but* $\varphi_x(T,x) = 0$ *for some* $x \in \mathbb{R}$.

Theorem: *The square root representation on the diffeomorphism group* $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ *is a bijective mapping, given by:*

$$R : \begin{cases} \mathrm{Diff}_{\mathcal{A}}(\mathbb{R}) \to \left(\mathrm{Im}(R), \|\cdot\|_{L^2}\right) \subset \left(\mathcal{A}\left(\mathbb{R}, \mathbb{R}_{>-2}\right), \|\cdot\|_{L^2}\right) \\ \qquad \varphi \mapsto 2\left((\varphi')^{1/2} - 1\right). \end{cases}$$

The pull-back of the restriction of the flat L^2*-metric to* $\mathrm{Im}(R)$ *via* R *is again the homogeneous Sobolev metric of order one. The image of the* R*-map is the splitting submanifold of* $\mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2})$ *given by:*

$$\mathrm{Im}(R) = \left\{\gamma \in \mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2}) : F(\gamma) := \int_{\mathbb{R}} \gamma(\gamma + 4)\, dx = 0\right\}.$$

On the space $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ the geodesic equation does not exist. Still:
Corollary: *The geodesic distance* $d^{\mathcal{A}}$ *on* $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ *coincides with the restriction of* $d^{\mathcal{A}_1}$ *to* $\mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$, *i.e., for* $\varphi_0, \varphi_1 \in \mathrm{Diff}_{\mathcal{A}}(\mathbb{R})$ *we have*

$$d^{\mathcal{A}}(\varphi_0, \varphi_1) = d^{\mathcal{A}_1}(\varphi_0, \varphi_1) .$$

Continuing Geodesics Beyond the Group, or How Solutions of the Hunter–Saxton Equation Blow Up

Consider a straight line $\gamma(t) = \gamma_0 + t\gamma_1$ in $\mathcal{A}(\mathbb{R}, \mathbb{R})$. Then $\gamma(t) \in \mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2})$ precisely for t in an open interval (t_0, t_1) which is finite at least on one side, say, at $t_1 < \infty$. Note that

$$\varphi(t)(x) := R^{-1}(\gamma(t))(x) = x + \frac{1}{4}\int_{-\infty}^{x} \gamma^2(t)(u) + 4\gamma(t)(u)\, du$$

makes sense for all t, that $\varphi(t) : \mathbb{R} \to \mathbb{R}$ is smooth and that $\varphi(t)'(x) \geq 0$ for all x and t; thus, $\varphi(t)$ is monotone non-decreasing. Moreover, $\varphi(t)$ is proper and surjective since $\gamma(t)$ vanishes at $-\infty$ and ∞. Let

$$\mathrm{Mon}_{\mathcal{A}_1}(\mathbb{R}) := \left\{ \mathrm{Id} + f : f \in \mathcal{A}_1(\mathbb{R}, \mathbb{R}), f' \geq -1 \right\}$$

be the monoid (under composition) of all such functions.

For $\gamma \in \mathcal{A}(\mathbb{R},\mathbb{R})$ let $x(\gamma) := \min\{x \in \mathbb{R} \cup \{\infty\} : \gamma(x) = -2\}$. Then for the line $\gamma(t)$ from above we see that $x(\gamma(t)) < \infty$ for all $t > t_1$. Thus, if the 'geodesic' $\varphi(t)$ leaves the diffeomorphism group at t_1, it never comes back but stays inside $\mathrm{Mon}_{\mathcal{A}_1}(\mathbb{R}) \setminus \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ for the rest of its life. In this sense, $\mathrm{Mon}_{\mathcal{A}_1}(\mathbb{R})$ is a *geodesic completion* of $\mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$, and $\mathrm{Mon}_{\mathcal{A}_1}(\mathbb{R}) \setminus \mathrm{Diff}_{\mathcal{A}_1}(\mathbb{R})$ is the *boundary*.
What happens to the corresponding solution $u(t,x) = \varphi_t(t, \varphi(t)^{-1}(x))$ of the HS equation? In certain points it has infinite derivative, it may be multivalued, or its graph can contain whole vertical intervals. If we replace an element $\varphi \in \mathrm{Mon}_{\mathcal{A}_1}(\mathbb{R})$ by its graph $\{(x, \varphi(x)) : x \in \mathbb{R}\} \subset \mathbb{R}$ we get a smooth 'monotone' submanifold, a smooth monotone relation. The inverse φ^{-1} is then also a smooth monotone relation. Then $t \mapsto \{(x, u(t,x)) : x \in \mathbb{R}\}$ is a (smooth) curve of relations. Checking that it satisfies the HS equation is an exercise left for the interested reader. What we have described here is the *flow completion* of the HS equation in the spirit of [Khesin M 2004].

Soliton-Like Solutions of the Hunter Saxton equation

For a right-invariant metric G on a diffeomorphism group one can ask whether (generalized) solutions $u(t) = \varphi_t(t) \circ \varphi(t)^{-1}$ exist such that the momenta $\check{G}(u(t)) =: p(t)$ are distributions with finite support. Here the geodesic $\varphi(t)$ may exist only in some suitable Sobolev completion of the diffeomorphism group. By the general theory, the momentum $\mathrm{Ad}(\varphi(t))^* p(t) = \varphi(t)^* p(t) = p(0)$ is constant. In other words,

$$p(t) = (\varphi(t)^{-1})^* p(0) = \varphi(t)_* p(0),$$

i.e., the momentum is carried forward by the flow and remains in the space of distributions with finite support. The infinitesimal version (take ∂_t of the last expression) is

$$p_t(t) = -\mathcal{L}_{u(t)} p(t) = -\operatorname{ad}_{u(t)}{}^* \, p(t).$$

The space of N-solitons of order 0 consists of momenta of the form $p_{y,a} = \sum_{i=1}^N a_i \delta_{y_i}$ with $(y,a) \in \mathbb{R}^{2N}$. Consider an initial soliton $p_0 = \check{G}(u_0) = -u_0'' = \sum_{i=1}^N a_i \, \delta_{y_i}$ with $y_1 < y_2 < \cdots < y_N$. Let H be the Heaviside function

$$H(x) = \begin{cases} 0, & x < 0, \\ \frac{1}{2}, & x = 0, \\ 1, & x > 0, \end{cases}$$

and $D(x) = 0$ for $x \leq 0$ and $D(x) = x$ for $x > 0$. We will see later why the choice $H(0) = \frac{1}{2}$ is the most natural one; note that the behavior is called the Gibbs phenomenon. With these functions we can write

$$u_0''(x) = -\sum_{i=1}^N a_i \delta_{y_i}(x)$$

$$u_0'(x) = -\sum_{i=1}^N a_i H(x - y_i)$$

$$u_0(x) = -\sum_{i=1}^N a_i D(x - y_i).$$

We will assume henceforth that $\sum_{i=1}^{N} a_i = 0$. Then $u_0(x)$ is constant for $x > y_N$ and thus $u_0 \in H_1^1(\mathbb{R})$; with a slight abuse of notation we assume that $H_1^1(\mathbb{R})$ is defined similarly to $H_1^\infty(\mathbb{R})$. Defining $S_i = \sum_{j=1}^{i} a_j$ we can write

$$u_0'(x) = -\sum_{i=1}^{N} S_i \left(H(x - y_i) - H(x - y_{i+1}) \right).$$

This formula will be useful because
$\operatorname{supp}(H(. - y_i) - H(. - y_{i+1})) = [y_i, y_{i+1}]$.
The evolution of the geodesic $u(t)$ with initial value $u(0) = u_0$ can be described by a system of ordinary differential equations (ODEs) for the variables (y, a).
Theorem *The map $(y, a) \mapsto \sum_{i=1}^{N} a_i \delta_{y_i}$ is a Poisson map between the canonical symplectic structure on $\mathbb{R}^{2N}$ and the Lie–Poisson structure on the dual $T^*_{\mathrm{Id}} \operatorname{Diff}_{\mathcal{A}}(\mathbb{R})$ of the Lie algebra.*

In particular, this means that the ODEs for (y, a) are Hamilton's equations for the pullback Hamiltonian

$$E(y, a) = \frac{1}{2} G_{\mathrm{Id}}(u_{(y,a)}, u_{(y,a)}),$$

with $u_{(y,a)} = \check{G}^{-1}(\sum_{i=1}^{N} a_i \delta_{y_i}) = -\sum_{i=1}^{N} a_i D(. - y_i)$. We can obtain the more explicit expression

$$\begin{aligned} E(y, a) &= \frac{1}{2} \int_{\mathbb{R}} \left(u_{(y,a)}(x)'\right)^2 \, dx = \frac{1}{2} \int_{\mathbb{R}} \left(\sum_{i=1}^{N} S_i \mathbb{1}_{[y_i, y_{i+1}]} \right)^2 \, dx \\ &= \frac{1}{2} \sum_{i=1}^{N} S_i^2 (y_{i+1} - y_i). \end{aligned}$$

Hamilton's equations $\dot{y}_i = \partial E / \partial a_i$, $\dot{a}_i = -\partial E / \partial y_i$ are in this case

$$\begin{aligned} \dot{y}_i(t) &= \sum_{j=i}^{N-1} S_i(t)(y_{i+1}(t) - y_i(t)), \\ \dot{a}_i(t) &= \frac{1}{2} \left(S_i(t)^2 - S_{i-1}(t)^2 \right). \end{aligned}$$

Using the R-map we can find explicit solutions for these equations as follows. Let us write $a_i(0) = a_i$ and $y_i(0) = y_i$. The geodesic with initial velocity u_0 is given by

$$\varphi(t,x) = x + \frac{1}{4}\int_{-\infty}^{x} t^2(u_0'(y))^2 + 4tu_0'(y)\,dy$$

$$u(t,x) = u_0(\varphi^{-1}(t,x)) + \frac{t}{2}\int_{-\infty}^{\varphi^{-1}(t,x)} u_0'(y)^2\,dy.$$

First note that

$$\varphi'(t,x) = \left(1 + \frac{t}{2}u_0'(x)\right)^2$$

$$u'(t,z) = \frac{u_0'\left(\varphi^{-1}(t,z)\right)}{1 + \frac{t}{2}u_0'\left(\varphi^{-1}(t,z)\right)}.$$

Using the identity $H(\varphi^{-1}(t,z)-y_i)=H(z-\varphi(t,y_i))$ we obtain

$$u_0'\left(\varphi^{-1}(t,z)\right)=-\sum_{i=1}^{N}a_iH\left(z-\varphi(t,y_i)\right),$$

and thus

$$\left(u_0'\left(\varphi^{-1}(t,z)\right)\right)'=-\sum_{i=1}^{N}a_i\delta_{\varphi(t,y_i)}(z).$$

Combining these we obtain

$$\begin{aligned}u''(t,z)&=\frac{1}{\left(1+\frac{t}{2}u_0'\left(\varphi^{-1}(t,z)\right)\right)^2}\left(-\sum_{i=1}^{N}a_i\delta_{\varphi(t,y_i)}(z)\right)\\&=\sum_{i=1}^{N}\frac{-a_i}{\left(1+\frac{t}{2}u_0'(y_i)\right)^2}\delta_{\varphi(t,y_i)}(z).\end{aligned}$$

From here we can read off the solution of Hamilton's equations

$$\begin{aligned}y_i(t)&=\varphi(t,y_i)\\a_i(t)&=-a_i\left(1+\tfrac{t}{2}u_0'(y_i)\right)^{-2}.\end{aligned}$$

When trying to evaluate $u_0'(y_i)$,

$$u_0'(y_i) = a_i H(0) - S_i,$$

we see that u_0' is discontinuous at y_i and it is here that we seem to have the freedom to choose the value $H(0)$. However, it turns out that we observe the Gibbs phenomenon, i.e., only the choice $H(0) = \frac{1}{2}$ leads to solutions of Hamilton's equations. Also, the regularized theory of multiplications of distributions (Colombeau, Kunzinger et.al.) leads to this choice. Thus we obtain

$$y_i(t) = y_i + \sum_{j=1}^{i-1} \left(\frac{t^2}{4} S_j^2 - t S_j \right) (y_{j+1} - y_j)$$
$$a_i(t) = \frac{-a_i}{\left(1 + \frac{t}{2}\left(\frac{a_i}{2} - S_i\right)\right)^2} = -\left(\frac{S_i}{1 - \frac{t}{2} S_i} - \frac{S_{i-1}}{1 - \frac{t}{2} S_{i-1}} \right).$$

It can be checked by direct computation that these functions indeed solve Hamilton's equations.