Remarks on Infinite dimensional symplectic and Poisson geometry

Peter W. Michor

University of Vienna, Austria
www.mat.univie.ac.at/~michor

Unendlich-dimensionale Analysis und Geometrie
University of Paderborn
March 26, 2025

Slides will be available at:
https://www.mat.univie.ac.at/~michor/lectures.html

Based on:

[KM97] Andreas Kriegl, Peter W. Michor: The Convenient Setting of Global Analysis. Mathematical Surveys and Monographs, Volume: 53, Amer. Math. Soc., 1997.

See also:
Wikipedia [https://en.wikipedia.org/wiki/Convenient_vector_space]

W. M. Tulczyjew. The graded Lie algebra of multivector fields and the generalized Lie deriv- ative of forms. Bull. Acad. Polon. Sci., 22, 9:937–942, 1974.

[BIM24] Martin Bauer, Sadashige Ishida, Peter W. Michor. Symplectic structures on the space of space curves. arXiv:2407.19908.

Review

For a finite dimensional symplectic manifold (M, ω) we have the following exact sequence of Lie algebras:

$$0 \to H^0(M) \to C^\infty(M, \mathbb{R}) \xrightarrow{\text{grad}^\omega} \mathfrak{X}(M, \omega) \to H^1(M) \to 0.$$

$H^*(M)$ De Rham cohomology of M with 0 bracket.
$C^\infty(M, \mathbb{R})$ is equipped with the Poisson bracket $\{\quad,\quad\}$,
$\mathfrak{X}(M, \omega)$ all vector fields ξ with $\mathcal{L}_\xi \omega = 0$ with usual Lie bracket.

Furthermore, $\text{grad}^\omega f$ is the Hamiltonian vector field for $f \in C^\infty(M, \mathbb{R})$ given by $i(\text{grad}^\omega f)\omega = df$ and $\gamma(\xi) = [i_\xi \omega]$.

Consider a symplectic right action $r : M \times G \to M$ of a connected Lie group G on M; we use the notation $r(x, g) = r^g(x) = r_x(g) = x.g$. By $\zeta_X(x) = T_e(r_x)X$ we get a mapping $\zeta : \mathfrak{g} \to \mathfrak{X}(M, \omega)$ which sends each element X of the Lie algebra $\mathfrak{g}$ of G to the fundamental vector field ζ_X. This is a Lie algebra homomorphism (for right actions!).

$$H^0(M) \xrightarrow{\;i\;} C^\infty(M,\mathbb{R}) \xrightarrow{\;\operatorname{grad}^\omega\;} \mathfrak{X}(M,\omega) \xrightarrow{\;\gamma\;} H^1(M)$$

$$j : \mathfrak{g} \to C^\infty(M,\mathbb{R}), \qquad \zeta : \mathfrak{g} \to \mathfrak{X}(M,\omega)$$

A linear lift $j : \mathfrak{g} \to C^\infty(M,\mathbb{R})$ of ζ with $\operatorname{grad}^\omega \circ j = \zeta$ exists if and only if $\gamma \circ \zeta = 0$ in $H^1(M)$. This lift j may be changed to a Lie algebra homomorphism if and only if the 2-cocycle $\bar{\jmath} : \mathfrak{g} \times \mathfrak{g} \to H^0(M)$, given by $(i \circ \bar{\jmath})(X,Y) = \{j(X), j(Y)\} - j([X,Y])$, vanishes in the Lie algebra cohomology $H^2(\mathfrak{g}, H^0(M))$, for if $\bar{\jmath} = \delta\alpha$ then $j - i \circ \alpha$ is a Lie algebra homomorphism.

If $j : \mathfrak{g} \to C^\infty(M,\mathbb{R})$ is a Lie algebra homomorphism, we may associate the *momentum mapping* $J : M \to \mathfrak{g}' = L(\mathfrak{g},\mathbb{R})$ to it, which is given by $J(x)(X) = \chi(X)(x)$ for $x \in M$ and $X \in \mathfrak{g}$. It is G-equivariant for a suitably chosen (in general affine) action of G on $\mathfrak{g}'$.

Infinite dimensional weak symplectic manifolds

Let M be a manifold, in general is infinite dimensional, Hausdorff, in the sense of convenient calculus.
A 2-form $\omega \in \Omega^2(M)$ is called a *weak symplectic structure* on M if the following three conditions holds:

1. ω is closed, $d\omega = 0$.
2. The associated vector bundle homomorphism $\check{\omega} : TM \to T^*M$ is injective.
3. The gradient of ω with respect to itself exists and is smooth; this can be expressed most easily in charts, so let M be open in a convenient vector space E. Then for $x \in M$ and $X, Y, Z \in T_xM = E$ we have $d\omega(x)(X)(Y,Z) = \omega(\Omega_x(Y,Z),X) = \omega(\tilde{\Omega}_x(X,Y),Z)$ for smooth $\Omega, \tilde{\Omega} : M \times E \times E \to E$ which are bilinear in $E \times E$.

A 2-form $\omega \in \Omega^2(M)$ is called a *strong symplectic structure* on M if it is closed ($d\omega = 0$) and if its associated vector bundle homomorphism $\check{\omega} : TM \to T^*M$ is invertible with smooth inverse.

In this case, the vector bundle TM has reflexive fibers T_xM: Let $i : T_xM \to (T_xM)''$ be the canonical mapping onto the bidual. Skew symmetry of ω is equivalent to the fact that the transposed $(\check{\omega})^t = (\check{\omega})^* \circ i : T_xM \to (T_xM)'$ satisfies $(\check{\omega})^t = -\check{\omega}$. Thus, $i = -((\check{\omega})^{-1})^* \circ \check{\omega}$ is an isomorphism.

Cotangent bundles

Every cotangent bundle T^*Q, viewed as a manifold, carries a canonical weak symplectic structure $\omega_Q \in \Omega^2(T^*Q)$, which is defined as follows. Let $\pi_Q^* : T^*Q \to Q$ be the projection. Then the *Liouville form* $\theta_Q \in \Omega^1(T^*Q)$ is given by $\theta_Q(X) = \langle \pi_{T^*Q}(X), T(\pi_Q^*)(X) \rangle$ for $X \in T(T^*Q)$, where $\langle \quad , \quad \rangle$ denotes the duality pairing $T^*Q \times_Q TQ \to \mathbb{R}$. Then the symplectic structure on T^*Q is given by $\omega_Q = -d\theta_Q$, which of course in a local chart looks like $\omega_E((v, v'), (w, w')) = \langle w', v \rangle_E - \langle v', w \rangle_E$. The associated mapping $\check{\omega} : T_{(0,0)}(E \times E') = E \times E' \to E' \times E''$ is given by $(v, v') \mapsto (-v', i_E(v))$, where $i_E : E \to E''$ is the embedding into the bidual. So the canonical symplectic structure on T^*Q is strong if and only if all model spaces of the manifold Q are reflexive.

Towards the Hamiltonian mapping

Let M be a weak symplectic manifold. The first thing to note is that the Hamiltonian mapping $\operatorname{grad}^\omega : C^\infty(M,\mathbb{R}) \to \mathfrak{X}(M,\omega)$ does not make sense in general, since $\check{\omega} : TM \to T^*M$ is not invertible. Namely, $\operatorname{grad}^\omega f = (\check{\omega})^{-1} \circ df$ is defined only for those $f \in C^\infty(M,\mathbb{R})$ with $df(x)$ in the image of $\check{\omega}$ for all $x \in M$. A similar difficulty arises for the definition of the Poisson bracket on $C^\infty(M,\mathbb{R})$.

For a weak symplectic manifold (M,ω) let $T_x^\omega M$ denote the real linear subspace $T_x^\omega M = \check{\omega}_x(T_xM) \subset T_x^*M = L(T_xM,\mathbb{R})$, and let us call it the *ω-smooth cotangent space* with respect to ω of M at x. The convenient structure on $T_x^\omega M$ is the one from T_xM. All $T_x^\omega M$ together form a subbundle of T^*M isomorphic to TM via $\check{\omega} : TM \to T^\omega M \subseteq T^*M$. It is in general not a splitting subbundle.

Note that only for strong symplectic structures the mapping $\check{\omega}_x : T_xM \to T_x^*M$ is a diffeomorphism onto $T_x^\omega M$ with the structure induces from T_x^*M.

Definition of $C^\infty_\omega(E,\mathbb{R}) \subset C^\infty(E,\mathbb{R})$.

For a weak symplectic vector space (E,ω) we consider linear subspace $C^\infty_\omega(E,\mathbb{R}) \subset C^\infty(E,\mathbb{R})$ consisting of all smooth functions $f : E \to \mathbb{R}$ such that

- each iterated derivative $d^k f(x) \in L^k_{\text{sym}}(E;\mathbb{R})$ has the property that
  $$d^k f(x)(\quad, y_2, \dots, y_k) \in E^\omega$$
  is actually in the smooth dual $E^\omega \subset E'$ for all $x, y_2, \dots, y_k \in E$,
- and that the mapping $\prod^k E \to E$
  $$(x, y_2, \dots, y_k) \mapsto (\check\omega)^{-1}(df(x)(\quad, y_2, \dots, y_k))$$
  is smooth. By the symmetry of higher derivatives, this is then true for all entries of $d^k f(x)$, for all x.

This makes sense even if (E,ω) is a weak symplectic manifold which happens to be a convenient vector space since $T^\omega E \cong TE = E \times E =: E \times E^\omega \subset T^*E = E \times E'$.

Lemma. [KM97, 48.6] *For $f \in C^\infty(E, \mathbb{R})$ the following assertions are equivalent:*

1. *$df : E \to E'$ factors to a smooth mapping $E \to E^\omega$.*
2. *f has a smooth ω-gradient $\operatorname{grad}^\omega f \in \mathfrak{X}(E) = C^\infty(E, E)$ which satisfies $df(x)y = \omega(\operatorname{grad}^\omega f(x), y)$.*
3. *$f \in C^\infty_\omega(E, \mathbb{R})$.*

Definition of $C^\infty_\omega(M, \mathbb{R}) \subset C^\infty(M, \mathbb{R})$:
For a weak symplectic manifold (M, ω) the space $C^\infty_\omega(M, \mathbb{R})$ is the linear subspace consisting of all smooth functions $f : M \to \mathbb{R}$ such that the differential $df : M \to T^*M$ factors to a smooth mapping $M \to T^\omega M$. It follows that these are exactly those smooth functions on M which admit a smooth ω-gradient $\operatorname{grad}^\omega f \in \mathfrak{X}(M)$.

Theorem [KM97, Thm 48.8] with gap closed in [BIM24, appendix]

Let (M, ω) *be a weak symplectic manifold. The Hamiltonian mapping* $\operatorname{grad}^\omega : C^\infty_\omega(M, \mathbb{R}) \to \mathfrak{X}(M, \omega)$*, which is given by*

$$i_{\operatorname{grad}^\omega f}\omega = df \quad \text{or} \quad \operatorname{grad}^\omega f := (\check{\omega})^{-1} \circ df$$

is well defined. Also the Poisson bracket

$$\begin{aligned} \{\ \ ,\ \ \} &: C^\infty_\omega(M, \mathbb{R}) \times C^\infty_\omega(M, \mathbb{R}) \to C^\infty_\omega(M, \mathbb{R}) \\ \{f, g\} &:= i_{\operatorname{grad}^\omega f} i_{\operatorname{grad}^\omega g}\omega = \omega(\operatorname{grad}^\omega g, \operatorname{grad}^\omega f) = \\ &= dg(\operatorname{grad}^\omega f) = (\operatorname{grad}^\omega f)(g) \end{aligned}$$

is well defined and gives a Lie algebra structure to the space $C^\infty_\omega(M, \mathbb{R})$*, which also fulfills*

$$\{f, gh\} = \{f, g\}h + g\{f, h\}.$$

Theorem, continued.

We equip $C^\infty_\omega(M,\mathbb{R})$ *with the initial structure with respect to the the two following mappings:*

$$C^\infty_\omega(M,\mathbb{R}) \xrightarrow{\subset} C^\infty(M,\mathbb{R}), \qquad C^\infty_\omega(M,\mathbb{R}) \xrightarrow{\text{grad}^\omega} \mathfrak{X}(M).$$

Then the Poisson bracket is bounded bilinear on $C^\infty_\omega(M,\mathbb{R})$*.*

We have the following long exact sequence of Lie algebras and Lie algebra homomorphisms:

$$0 \to H^0(M) \to C^\infty_\omega(M,\mathbb{R}) \xrightarrow{\text{grad}^\omega} \mathfrak{X}(M,\omega) \xrightarrow{\gamma} H^1_\omega(M) \to 0,$$

where $H^0(M)$ *is the space of locally constant functions, and*

$$H^1_\omega(M) = \frac{\{\varphi \in C^\infty(M \leftarrow T^\omega M) : d\varphi = 0\}}{\{df : f \in C^\infty_\omega(M,\mathbb{R})\}}$$

is the first symplectic cohomology space of (M,ω)*, a linear subspace of the De Rham cohomology space* $H^1(M)$*.*

The Diez-Rudolph topology

In [DR24, 5.3: T.Diez, G.Rudolph: Symplectic Reduction in Infinite Dimensions, arXiv:2409.05829], for a weak symplectic vector space (E, ω), a locally convex topology τ on E is called *compatible with* ω if the dual $(E, \tau)' = \check{\omega}(E) = E^\omega \subset E'$.

Proposition. [DR24,5.4] *For a convenient weak symplectic vector space the bornological topology on E is compatible with ω*

- *in the Bastiani setting: iff E is a reflexive Banach space and ω is strong.*
- *here: iff E is reflexive and ω is strong.*

Note that $L^p \times L^{p'}$ is symplectic, Banach, but i,g, not Hilbert. Namely: If we take $E' \times E \to \mathbb{R}$ is given by $(x', x) \mapsto \omega(\check{\omega}^{-1}(x'), x)$ as duality reflexivity follows.

How does this notion fit into the convenient framework?

Example: Let $E = \ell^2 \times \ell^2$ with the weak symplectic structure $\omega((x,y),(x',y')) = \sum_n c_n(x_n y'_n - y_n x'_n)$ for a sequence $0 < c_n \searrow 0$ sufficiently fast.

Then any l.c. topology on E compatible with ω is NOT convenient: Namely, let $0 < b_n \nearrow \infty$ with $b_n c_n \searrow 0$. Then for suitable $x \in \ell^2$ the sequence $X_k := (b_n x_n)_{n=1}^k \in \ell^2$ is a Mackey-Cauchy sequence for the weak $\sigma(E, E^\omega)$-topology but its limit $X = (b_n x_n)$ is i.g. not in ℓ^2.

Smooth Curves into (E, τ). [KM97, Section 1] Since (E, τ) is not Mackey complete in general, we define $c : \mathbb{R} \to (E, \tau)$ to be smooth if $\lambda \circ c : \mathbb{R} \to \mathbb{R}$ is smooth **and** each iterated derivative $c^{(n)}(t)$ lies in E (a priori only in the c^∞-completion of E). We denote this space by $C^\infty(\mathbb{R}, (E, \tau))$, and by $c^\infty(\tau)$ we denote the final topology on E with respect to $C^\infty(\mathbb{R}, (E, \tau))$.

Question. Let (E, ω) be a convenient weak symplectic vector space and let τ be any l.c. topology compatible with ω. Under which conditions do we have $C^\infty(\mathbb{R}, (E, \tau)) = C^\infty(\mathbb{R}, E)$?

Proposition. *Let (E, ω) be a convenient weak symplectic vector space and let τ be any l.c. topology compatible with ω. Suppose that the bornology of E has a basis of $\sigma(E, \check{\omega}(E))$-closed sets (i.e., each bounded set is contained in a $\sigma(E, \omega(E))$-closed bounded set). This is he case if (E, ω) is a convenient weak symplectic vector space which is a dual space $E = F'$ such that $\check{\omega}(E) \subseteq F \subseteq E' = E''$.*
Then we have $C^\infty(\mathbb{R}, (E, \tau)) = C^\infty(\mathbb{R}, E)$.

This includes the the $\ell^2 \times \ell^2$ example from above.

In the convenient spirit, under this condition we then have $C^\infty_\omega(E, \mathbb{R}) = C^\infty((E, \tau), \mathbb{R})$, although (E, τ) is NOT a convenient space.

Proof. This is a special case of the following theorem.

Theorem[KF88, Theorem 4.1.19] *Let $c : \mathbb{R} \to E$ be a curve in a convenient vector space E. Let $\mathcal{F} \subseteq E'$ be a subset of bounded linear functionals such that the bornology of E has a basis of $\sigma(E, \mathcal{F})$-closed sets. Then the following are equivalent:*

1. *c is smooth*
2. *There exist locally bounded curves $c^k : \mathbb{R} \to E$ such that $\lambda \circ c$ is smooth $\mathbb{R} \to \mathbb{R}$ with $(\lambda \circ c)^{(k)} = \lambda \circ c^k$, for each $\lambda \in \mathcal{F}$ and each k.*

If $E = F'$ is the dual of a convenient vector space F, then for any point separating subset $\mathcal{F} \subseteq F$ the bornology of E has a basis of $\sigma(E, \mathcal{F})$-closed subsets, by [FK88 4.1.22]*.*

[FK88] Frölicher, A.; Kriegl, A., Linear spaces and differentiation theory, Pure Appl. Math., J. Wiley, Chichester, 1988.

Weakly symplectic group actions.

An infinite dimensional regular Lie group G with Lie algebra $\mathfrak{g}$ acts from the right on a weak symplectic manifold (M, ω) by $r : M \times G \to M$ (notation $r(x, g) = r^g(x) = r_x(g)$), so that each r^g is a symplectomorphism. Some immediate consequences:

(1) *The space $C^\infty_\omega(M)^G$ of G-invariant smooth functions with ω-gradients is a Lie subalgebra for the Poisson bracket*, since for each $g \in G$ and $f, h \in C^\infty(M)^G$ we have $(r^g)^*\{f, h\} = \{(r^g)^* f, (r^g)^* h\} = \{f, h\}$.

(2) *For $x \in M$ the pullback of ω to the orbit $x.G$ is a 2-form, invariant under the action of G on the orbit.* In finite dimensions the orbit is an initial submanifold. Here this has to be checked directly in each example. There is a tangent bundle $T_x(x.G) = T(r_x)\mathfrak{g}$. If $i : x.G \to M$ is the embedding of the orbit then $r^g \circ i = i \circ r^g$, so that $i^*\omega = i^*(r^g)^*\omega = (r^g)^* i^*\omega$ holds for each $g \in G$ and thus $i^*\omega$ is invariant.

(3) The infinitesimal action $\zeta : \mathfrak{g} \to \mathfrak{X}(M, \omega)$, given by $\zeta_X(x) = T_e(r_x)X$ for $X \in \mathfrak{g}$ and $x \in M$, is a homomorphism of Lie algebras (for a left action we get an anti homomorphism of Lie algebras). We have the exact sequence of Lie algebra homomorphisms

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & H^0(M) & \xrightarrow{\alpha} & C^\infty_\omega(M) & \xrightarrow{\operatorname{grad}^\omega} & \mathfrak{X}(M,\omega) & \xrightarrow{\gamma} & H^1_\omega(M) & \longrightarrow & 0 \\
 & & & & & \overset{j}{\nwarrow} & \uparrow{\scriptstyle \zeta} & & & & \\
 & & & & & & \mathfrak{g} & & & &
\end{array}
$$

(4) If $H^1_\omega(M) = 0$ then any symplectic action on (M, ω) is a Hamiltonian action.

(5) *If the Lie algebra* $\mathfrak{g}$ *is equal to its commutator subalgebra* $[\mathfrak{g}, \mathfrak{g}]$, *the linear span of all* $[X, Y]$ *for* $X, Y \in \mathfrak{g}$ *(true for all full diffeomorphism groups), then any infinitesimal symplectic action* $\zeta : \mathfrak{g} \to \mathfrak{X}(M, \omega)$ *is a Hamiltonian action,* since then any $Z \in \mathfrak{g}$ can be written as $Z = \sum_i [X_i, Y_i]$ so that $\zeta_Z = \sum [\zeta_{X_i}, \zeta_{Y_i}] \in \operatorname{im}(\operatorname{grad}^\omega)$ since $\gamma : \mathfrak{X}(M, \omega) \to H^1_\omega(M)$ is a homom.into the zero Lie bracket.

(6) If $j : \mathfrak{g} \to (C^\infty_\omega(M), \{\ \ ,\ \ \})$ happens to be not a homomorphism of Lie algebras then $c(X, Y) = \{j(X), j(Y)\} - j([X, Y])$ lies in $H^0(M)$, and indeed $c : \mathfrak{g} \times \mathfrak{g} \to H^0(M)$ is a cocycle for the Lie algebra cohomology: $c([X, Y], Z) + c([Y, Z], X) + c([Z, X], Y) = 0$. If c is a coboundary, i.e., $c(X, Y) = -b([X, Y])$, then $j + \alpha \circ b$ is a Lie algebra homomorphism. If the cocycle c is non-trivial we can use the central extension $H^0(M) \times_c \mathfrak{g}$ with bracket $[(a, X), (b, Y)] = (c(X, Y), [X, Y])$ in the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(M) & \xrightarrow{\alpha} & C^\infty_\omega(M) & \xrightarrow{\text{grad}^\omega} & \mathfrak{X}(M, \omega) & \xrightarrow{\gamma} & H^1_\omega(M) \longrightarrow 0 \\
 & & & & \uparrow{\scriptstyle \bar{\jmath}} & & \uparrow{\scriptstyle \zeta} & & \\
 & & & & H^1(M) \times_c \mathfrak{g} & \xrightarrow{\text{pr}_2} & \mathfrak{g} & &
\end{array}$$

where $\bar{\jmath}(a, X) = j(X) + \alpha(a)$. Then $\bar{\jmath}$ is a homomorphism of Lie algebras.

Momentum mapping

For an infinitesimal symplectic action $\zeta : \mathfrak{g} \to \mathfrak{X}(M, \omega)$ we can find a linear lift $j : \mathfrak{g} \to C^\infty_\omega(M, \mathbb{R})$ iff there exists $J \in C^\infty_\omega(M, \mathfrak{g}^*) := \{f \in C^\infty(M, \mathfrak{g}^*) : \langle f(\quad), X\rangle \in C^\infty_\omega(M)$ for all $X \in \mathfrak{g}\}$ such that

$$\operatorname{grad}^\omega(\langle J, X\rangle) = \zeta_X \quad \text{for all } X \in \mathfrak{g}.$$

$J \in C^\infty_\omega(M, \mathfrak{g}^*)$ is called the *momentum mapping*for the infinitesimal action $\zeta : \mathfrak{g} \to \mathfrak{X}(M, \omega)$.

Basic properties of the momentum mapping

(1) *For $x \in M$, the transposed mapping of the linear mapping $dJ(x) : T_xM \to \mathfrak{g}^*$ is*

$$dJ(x)^\top : \mathfrak{g} \to T^*_xM, \qquad dJ(x)^\top = \check{\omega}_x \circ \zeta$$

(2) *The closure of the image $dJ(T_xM)$ of $dJ(x) : T_xM \to \mathfrak{g}*$ is the annihilator $\mathfrak{g}_x^\circ$ of the isotropy Lie algeba $\mathfrak{g}_x := \{X \in \mathfrak{g} : \zeta_X(x) = 0\}$ in $\mathfrak{g}^*$*, since the annihilator of the image is the kernel of the transposed mapping,

(3) *The kernel of* $dJ(x)$ *is the symplectic orthogonal*

$$(T(r_x)\mathfrak{g})^{\perp,\omega} = (T_x(x.G))^{\perp,\omega} \subseteq T_x M.$$

(4) *If* G *is connected,* $x \in M$ *is a fixed point for the* G*-action if and only if* x *is a critical point of* J*, i.e.* $dJ(x) = 0$*.*

(5) (Emmy Noether's theorem) *Let* $h \in C^\infty_\omega(M)$ *be a Hamiltonian function which is invariant under the Hamiltonian* G *action. Then* $dJ(\mathrm{grad}^\omega(h)) = 0$*. Thus the momentum mapping* $J : M \to \mathfrak{g}^*$ *is constant on each trajectory (if it exists) of the Hamiltonian vector field* $\mathrm{grad}^\omega(h)$*.*

Towards the Schouten-Nijenhuis bracket

Let M be a convenient smooth manifold. We shall use the graded differential algebra of differential forms consisting of smooth sections of the bundle of bounded skew symmetric multilinear forms $L^*_{\text{skew}}(TM,\mathbb{R})$ on the the tangent bundle:

$$\Omega(M) = \bigoplus_{k=0}^{\infty} \Omega^k(M) = \bigoplus_{k=0}^{\infty} C^\infty(M \leftarrow L^k_{\text{skew}}(TM,\mathbb{R})).$$

Later we shall use only manifolds M having the following property: *For each covector $\alpha \in T^*M$ there exists a function $f \in C^\infty(M)$ with $df_{\pi(\alpha)} = \alpha$.* The following classes of manifolds have this property: Smoothly paracompact manifolds (having smoothly paracompact modelling spaces). Each manifold M such that $C^\infty(M,\mathbb{R})$ separates points on TM: Then $\text{ev} : M \ni x \mapsto \text{ev}_x \in C^\infty(M,\mathbb{R})'$ is a smooth injective immersion and linear functionals in $C^\infty(M,\mathbb{R})''$ restricted to $T\,\text{ev}\,.TM \subset C^\infty(M,\mathbb{R})'$ suffice.

The bornological tensor product

For a convenient vector space E, let $E\bar{\otimes}_\beta E$ be the c^∞-completed bornological tensor product which linearizes bibounded bilinear mappings. If E is a Banach or Fréchet or (DF) space then each bibounded bilinear mapping is jointly continuous and thus $E\bar{\otimes}_\beta E$ agrees with the completed projective tensor product of Grothendieck.

Let $\bigwedge^n E$ be the (Mackey-) closed linear subspace of all *alternating tensors* in $\bar{\bigotimes}^n_\beta E$. It is the universal solution for convenient vector spaces F of the linearization problem $L(\bigwedge^n E, F) \cong L^n_{\text{alt}}(E; F)$, where $L^n_{\text{alt}}(E; F)$ is the space of all bounded n-linear alternating mappings $E \times \ldots \times E \to F$, a direct summand of $L^n(E; F) := L(E, \ldots, E; F)$. The mapping $\bigwedge^n : L(E, F) \to L(\bigwedge^n E, \bigwedge^n F)$ is bounded multilinear and thus smooth.

Summable skew multi vector fields

We apply the smooth mapping

$$\bigwedge^n : L(E, F) \to L(\bigwedge^n E, \bigwedge^n F)$$

to the chart change mappings for the tangent bundle $TM \to M$ to obtain the smooth vector bundle $\pi_M : \bigwedge^n TM \to M$ of *summable n-multi vectors* on M. Note that the space linearly generated by $X_1 \wedge \cdots \wedge X_n$ for $X_i \in T_x M$ is dense in the fiber $\bigwedge^n T_x M$. The space $\Gamma(\bigwedge^n TM)$ of smooth sections of this bundle is the space of *summable multi vector fields* on M. We write $\Gamma(\bigwedge^0 TM) = C^\infty(M, \mathbb{R})$ and $\Gamma(\bigwedge TM) = \bigoplus_{n \geq 0} \Gamma(\bigwedge^n TM)$ which is a graded commutative algebra for the usual wedge-product of multi vector fields for the grading $(\Gamma(\bigwedge TM), \wedge)_n = \Gamma(\bigwedge^n TM)$. The wedge product is a bounded bilinear operation on the convenient space $\Gamma(\bigwedge TM)$, by the universal property of the bornological tensor product.

Easy Theorem.

Schouten-Nijenhuis bracket for summable multi vector fields. *Let M be a smooth manifold. We consider the space $\Gamma(\bigwedge_{sum} TM)$ of multivector fields on M. This space carries a graded Lie bracket for the grading $\Gamma(\bigwedge_{sum}^{*+1} TM), * = -1, 0, 1, 2, \dots$, called the Schouten-Nijenhuis bracket, which is given by*

$$[X_1 \wedge \cdots \wedge X_p, Y_1 \wedge \cdots \wedge Y_q] = \sum_{i,j} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \cdots \widehat{X}_i \cdots \wedge X_p \wedge Y_1 \wedge \cdots \widehat{Y}_j \cdots \wedge Y_q,$$

$$[f, U] = -\bar{\imath}(df)U,$$

where $\bar{\imath}(df)$ is the insertion operator $\bigwedge_{sum}^{k} TM \to \bigwedge_{sum}^{k-1} TM$, the adjoint of $df \wedge (\quad) : \bigwedge_{sum}^{l} T^*M \to \bigwedge_{sum}^{l+1} T^*M$.

Easy Theorem continued

Let $U \in \Gamma(\bigwedge_{sum}^{u} TM)$*,* $V \in \Gamma(\bigwedge_{sum}^{v} TM)$*,* $W \in \Gamma(\bigwedge_{sum}^{w} TM)$*, and* $f \in C^{\infty}(M, \mathbb{R})$*. Then we have:*

$$[X, V] = \mathcal{L}_X V \quad \textit{is the Lie derivation along a vector field } X \; [U, V] \quad =$$

$$[U, [V, W]] = [[U, V], W] + (-1)^{(u-1)(v-1)}[V, [U, W]].$$

$$[U, V \wedge W] = [U, V] \wedge W + (-1)^{(u-1)v} V \wedge [U, W].$$

$$[X, U] = \mathcal{L}_X U.$$

The last 4 properties determine the Schouten bracket uniquely on $\Gamma(\bigwedge_{sum} TM)$
Let $P \in \Gamma(\bigwedge_{sum}^{2} TM)$*. Then the product* $\{f, g\} := \frac{1}{2}\langle df \wedge dg, P\rangle$ *on* $C^{\infty}(M)$ *satisfies the Jacobi identity if and only if* $[P, P] = 0$*.*

Duality between multivector fields and differential forms

Let M be a smooth manifold modeled on a convenient vector space E. By the universal property of the bornological tensor product described in !1.1, the dual space of $\bigwedge^n E$ is the space $L^n_{\text{skew}}(E;\mathbb{R})$. Using and extending the conventions of *Greub78*, we start from the duality

$$\langle\ ,\ \rangle : \bigwedge^n E^* \times \bigwedge^n E \to \mathbb{R}$$
$$\langle \varphi_1 \wedge \cdots \wedge \varphi_n, X_1 \wedge \cdots \wedge X_n \rangle = \det(\langle \varphi_i, X_j \rangle_{i,j})$$

we get the complete fiberwise duality

$$\langle\ ,\ \rangle : \Omega^n(M) \times \Gamma(\bigwedge^n TM) \to C^\infty(M)\,,$$
$$\langle \omega, X_1 \wedge \cdots \wedge X_n \rangle = \omega(X_1, \ldots, X_n)$$

We have the following dual pairs of operators: For $\omega \in \Omega^p(M)$ the linear map $\mu(\omega) : \Omega^k(M) \to \Omega^{k+p}(M)$ given by $\mu(\omega)\psi := \omega \wedge \psi$ is the fiberwise dual operator to $\bar{\iota}(\omega) : \Gamma(\bigwedge^{k+p} TM) \to \Gamma(\bigwedge^k TM)$, where

$$\bar{\iota}(\omega)(X_1 \wedge \dots \wedge X_{k+p}) =$$
$$= \frac{1}{p!k!} \sum_{\sigma \in \mathfrak{S}_{k+p}} \operatorname{sign}(\sigma)\, \omega(X_{\sigma(1)}, \dots, X_{\sigma(p)}) X_{\sigma(p+1)} \wedge \dots \wedge X_{\sigma(p+k)} .$$

Likewise, for $U \in \Gamma(\bigwedge^p TM)$ the fiberwise linear mapping $\bar{\mu}(U) : \Gamma(\bigwedge^k TM) \to \Gamma(\bigwedge^{k+p} TM)$ given by $\bar{\mu}(U)V = U \wedge V$ is the fiberwise dual of the 'insertion operator' $i(U) : \Omega^{k+p}(M) \to \Omega^k(M)$.

Lemma.

Let U be in $\Gamma(\bigwedge^u TM)$. Then we have:

- $i(U) : \Omega(M) \to \Omega(M)$ *is a homogeneous bounded module homomorphism of degree* $-u$*. It is a graded derivation of* $\Omega(M)$ *if and only if* $p = 1$*. For* $f \in C^\infty(M)$ *we have* $i(f)\omega = f.\omega$*.*
- $i(U \wedge V) = i(V) \circ i(U)$*, thus the graded commutator vanishes:*

$$[i(U), i(V)] = i(U)i(V) - (-1)^{uv} i(V)i(U) = 0.$$

- $i(U)(\omega \wedge \psi) = i(\bar{\iota}(\omega)U)\psi + (-1)^u \omega \wedge i(U)\psi$ *for* $\omega \in \Omega^1(M)$ *and* $\psi \in \Omega(M)$*, that is:* $[i(U), \mu(\omega)] = i(\bar{\iota}(\omega)U)$

The Lie differential operator

For $U \in \Gamma(\bigwedge^u TM)$ we define the *Lie derivation* $\mathcal{L}(U) : \Omega^k(M) \to \Omega^{k-u+1}(M)$ by

$$\mathcal{L}(U) := [i(U), d] = i(U) \circ d - (-1)^p d \circ i(U)$$

which is homogeneous of degree $1 - u$ and is called the Lie differential operator. It is a derivation if and only if U is a vector field. We have $[\mathcal{L}(U), d] = 0$ by the graded Jacobi identity of the graded commutator.

Theorem

Let $U \in \Gamma(\bigwedge^u TM)$, $V \in \Gamma(\bigwedge^v TM)$, and $f \in C^\infty(M)$. Then we have:

$$\mathcal{L}(U \wedge V) = i(V) \circ \mathcal{L}(U) + (-1)^u \mathcal{L}(V) \circ i(U) \tag{1}$$

$$\mathcal{L}(X_1 \wedge \cdots \wedge X_u) = \\ = \sum_j (-1)^{j-1} i(X_u) \cdots i(X_{j+1}) \mathcal{L}(X_j) i(X_{j-1}) \cdots i(X_1) \tag{2}$$

$$\mathcal{L}(f) = [i(f), d] = [\mu(f), d] = -\mu(df) \tag{3}$$

$$[\mathcal{L}(U), i(V)] = (-1)^{(u-1)(v-1)} i([U, V]) = -i([V, U]) \tag{4}$$

$$[\mathcal{L}(U), \mathcal{L}(V)] = (-1)^{(u-1)(v-1)} \mathcal{L}([U, V]) = -\mathcal{L}([V, U]) \tag{5}$$

$$\langle d\omega, -[V, U] \rangle = \langle di(V) d\omega, U \rangle \\ - (-1)^{(u-1)(v-1)} \langle di(U) d\omega, V \rangle \tag{6}$$

Formula (6) was the starting point of the treatment of the Schouten-Nijenhuis bracket in *Tulczyjew74*.

The general Schouten bracket

For a convenient manifold the *general multivector fields* of order k are the smooth sections of the vector bundle $L^k_{\text{skew}}(T^*M, \mathbb{R}) \to M$. We could call these

$$\text{MV}(M) = \sum_{k=0}^{\infty} \text{MV}^k(M) := \sum_{k=0}^{\infty} \Gamma(L^k_{\text{skew}}(T^*M, \mathbb{R})).$$

A *summable differential form* ω on M is a smooth section of the bundle of skew symmetric tensors $\bigwedge^k_{\text{sum}} T^*M \subset \bar{\otimes}^k_\beta T^*M = T^*M \bar{\otimes}_\beta T^*M \bar{\otimes}_\beta \dots \bar{\otimes}_\beta T^*M \to M$, where $\bar{\otimes}_\beta$ denotes the c^∞-completed bornological tensor product which linearizes bounded bilinear mappings.

Let us denote by $\Omega^k_{\text{sum}}(M)$ the graded algebra of all summable differential forms. Note that exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$ does not map $\Omega^k_{\text{sum}}(M)$ into $\Omega^{k+1}_{\text{sum}}(M)$; summability of a form is destroyed by the exterior derivative.

The graded algebra $\Omega_{\text{sum},d}(M)$

Therefore we let $\Omega^k_{\text{sum},d}(M)$ be the graded differential subalgebra of all summable forms ω such that $d\omega$ is again summable. Note that the latter condition is a linear partial differential relation. Here we assume that $C^\infty(M)$ separates points on TM: *For each* $\alpha \in T^*M$ *there exists* $f \in C^\infty(M)$ *with* $df_{\pi(\alpha)} = \alpha$. Consequently, $\text{ev}_x \circ d : \Omega^k_{\text{sum},d}(M) \to \bigwedge^{k+1} T^*_x M$ is surjective for all $x \in M$

The vector bundle $L^k_{\text{skew}}(T^*M, \mathbb{R}) \to M$ is the dual bundle of $\bigwedge^k_{\text{sum}} T^*M \to M$; we will denote the duality by (the dual space is always on the feft hand side)

$$\langle \quad , \quad \rangle : L^k_{\text{skew}}(T^*M, \mathbb{R}) \times_M \textstyle\bigwedge^k_{\text{sum},\beta} T^*M \to \mathbb{R}$$

$$\langle U, \varphi_1 \wedge \cdots \wedge \varphi_k \rangle = U(\varphi_1, \ldots, \varphi_k)$$

as well as its extension to spaces of sections.

For $\omega \in \Omega^k_{\mathrm{sum}}(M)$ we consider the pointwise linear (i.e., vector bundle push-forward) mapping

$$\mu(\omega) : \Omega^\ell_{\mathrm{sum}}(M) \to \Omega^{\ell+k}_{\mathrm{sum}}(M), \quad \mu(\omega)\varphi = \omega \wedge \varphi$$

and its pointwise dual

$$\begin{aligned} \bar{\iota}(\omega) = \mu(\omega)^* : \mathrm{MV}^{\ell+k}(M) \to \mathrm{MV}^\ell(M), \\ \langle U, \mu(\omega)\varphi\rangle = \langle U, \omega \wedge \varphi\rangle = \langle \bar{\iota}(\omega)U, \varphi\rangle \end{aligned}$$

For a decomposable k-form $\omega = \varphi_1 \wedge \cdots \wedge \varphi_k$ we have

$$\begin{aligned} \langle \bar{\iota}(\varphi_1 \wedge \cdots \wedge \varphi_k)U, \varphi_{k+1} \wedge \cdots \wedge \varphi_{k+\ell}\rangle = \\ &= \langle U, \varphi_1 \wedge \cdots \wedge \varphi_k \wedge \varphi_{k+1}, \ldots, \varphi_{k+\ell}) \\ &= U(\varphi_1, \ldots, \varphi_{k+\ell}) \end{aligned}$$

Similarly, for $U \in \mathrm{MV}^u(M)$ we consider

$$\bar{\mu}(U) : \mathrm{MV}^\ell(M) \to \mathrm{MV}^{u+\ell}(M), \quad \bar{\mu}(U)V = U \wedge V$$

which is the dual of $i(U) : \Omega^{\ell+u}_{\mathrm{sum}}(M) \to \Omega^{\ell}_{\mathrm{sum}}(M)$ which on decomposable $u+\ell$-forms is given by

$$i(U)(\varphi_1 \wedge \cdots \wedge \varphi_{u+\ell}) = \\ = \frac{1}{u!\ell!} \sum_{\sigma \in \mathcal{S}_{k+\ell}} \mathrm{sign}(\sigma) U(\varphi_{\sigma(1)}, \ldots, \varphi_{\sigma(u)}) \varphi_{\sigma(u+1)} \wedge \cdots \wedge \varphi_{\sigma(k+\ell)}$$

Thus $i(U)$ respects the d-stable subalgebra $\Omega^k_{\mathrm{sum},d}(M)$ so that

$$i(U) : \Omega^{\ell+u}_{\mathrm{sum},d}(M) \to \Omega^{\ell}_{\mathrm{sum},d}(M)$$

Lemma

Let U *be in* $\mathrm{MV}^u(M)$ *and* $V \in \mathrm{MV}^v(M)$. *Then we have:*

1. $i(U) : \Omega_{sum}(M) \to \Omega_{sum}(M)$ *is a homogeneous bounded module homomorphism of degree* $-u$. *It is a graded derivation of* $\Omega_{sum}(M)$ *if and only if* $p = 1$. *For* $f \in C^\infty(M)$ *we have* $i(f)\omega = f.\omega$.
2. $i(U \wedge V) = i(V) \circ i(U)$, *thus the graded commutator vanishes:*
$$[i(U), i(V)] = i(U)i(V) - (-1)^{uv} i(V)i(U) = 0.$$
3. $i(U)(\omega \wedge \psi) = i(\bar{\iota}(\omega)U)\psi + (-1)^u \omega \wedge i(U)\psi$, *i.e.,* $[i(U), \mu(\omega)] = i(\bar{\iota}(\omega)U)$, *for* $\omega \in \Omega^1(M)$ *and* $\psi \in \Omega_{sum}(M)$.

The Schouten-Nijenhuis bracket: Tulczyjew's Approach

We turn the above Theorem around and use (6) as definition: For $\omega \in \Omega^{u+v-2}_{\text{sum},d}(M)$ we put

$$\langle [U, V], d\omega\rangle = -\langle V, di(U)d\omega\rangle + (-1)^{(u-1)(v-1)}\langle U, di(V)d\omega\rangle$$

We can also prove

$$\langle [U, V], fd\omega\rangle = -f\langle V, di(U)d\omega\rangle + (-1)^{(u-1)(v-1)}f\langle U, di(V)d\omega\rangle \,.$$

So $[U, V]$ is a multivector field of order $u + v - 1$: To see this, note first that $[\ ,\ ]$ respects f-dependence of mutlivector field; then restrict U and V to chart, and compute (1) where $d\omega = \varphi_1 \wedge \dots \varphi_{u+v-1}$ for constant 1-forms φ_i. Then

$$[\ ,\ ] : \mathrm{MV}^u(M) \times \mathrm{MV}^v(M) \to \mathrm{MV}^{u+v-1}(M)$$

is a smooth (bounded) bilinear operator satisfying $[U, V] = -(-1)^{(u-1)(v-1)}[V, U]$. It also satisfies

$$\bar\iota(df)[U, V] = [\bar\iota(df)U, V] + (-1)^{u-1}[U, \bar\iota(df)V]$$

We also have

$$i([U,V]) = -i(V)di(U) + (-1)^{(u-1)(v-1)}i(U)di(V) \\ + (-1)^v di(U\wedge V) + (-1)^u i(U\wedge V)d$$

Definition of Lie differentials: For $U \in \mathrm{MV}^u(M)$ the *Lie differential operator*

$$\mathcal{L}(U) := [i(U), d] = i(U) \circ d - (-1)^p d \circ i(U) \\ : \Omega^k_{\mathrm{sum},d}(M) \to \Omega^{k-u+1}_{\mathrm{sum},d}(M)$$

is well defined. It is a derivation if and only if U is a vector field. We have $[\mathcal{L}(U), d] = 0$ by the graded Jacobi identity of the graded commutator. We now generalise the above Theorem to this new situation:

Theorem

Let $U \in \mathrm{MV}^u(M)$*,* $V \in \mathrm{MV}^v(M)$*, and* $f \in C^\infty(M)$*. Then we have:*

$$\mathcal{L}(U \wedge V) = i(V) \circ \mathcal{L}(U) + (-1)^u \mathcal{L}(V) \circ i(U) \tag{1}$$

$$\mathcal{L}(X_1 \wedge \cdots \wedge X_u) = $$

$$= \sum_j (-1)^{j-1} i(X_u) \cdots i(X_{j+1}) \mathcal{L}(X_j) i(X_{j-1}) \cdots i(X_1) \tag{2}$$

$$\mathcal{L}(f) = [i(f), d] = [\mu(f), d] = -\mu(df) \tag{3}$$

$$[\mathcal{L}(U), i(V)] = (-1)^{(u-1)(v-1)} i([U, V]) = -i([V, U]) \tag{4}$$

$$[\mathcal{L}(U), \mathcal{L}(V)] = (-1)^{(u-1)(v-1)} \mathcal{L}([U, V]) = -\mathcal{L}([V, U]) \tag{5}$$

Thank you for listening.

Thank you for listening.

Thank you for listening.

Thank you for listening.