Overview on analysis and geometries of shape spaces and diffeomorphism groups.

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Based on collaborations with: M. Bauer, M. Bruveris, P. Harms, D. Mumford

This talk falls fully into the infinite dimensional differential geometry part of the workshop. Some or all of the following topics will be covered:

- A short introduction to convenient calculus in infinite dimensions, with an application to Sobolev spaces.
- Manifolds of mappings (with compact source) and diffeomorphism groups as convenient manifolds.
- A diagram of actions of diffeomorphism groups.
- The manifold of immersions and its orbifold quotient under the reparameterization group.
- Riemannian geometries of spaces of immersions, diffeomorphism groups, shape spaces, Riemannian metrics, their geodesic equations with well posedness results and vanishing geodesic distance.
- Robust Infinite Dimensional Riemannian manifolds, and Riemannian homogeneous spaces of diffeomorphism groups.

Some words on smooth convenient calculus

Traditional differential calculus works well for finite dimensional vector spaces and for Banach spaces.
Beyond Banach spaces, the main difficulty is that composition of linear mappings stops to be jointly continuous at the level of Banach spaces, for any compatible topology.

For more general locally convex spaces we sketch here the convenient approach as explained in [Frölicher-Kriegl 1988] and [Kriegl-Michor 1997].

I explain this to show how simple differential calculus can be!

The c^∞-topology

Let E be a locally convex vector space. A curve $c : \mathbb{R} \to E$ is called *smooth* or C^∞ if all derivatives exist and are continuous. Let $C^\infty(\mathbb{R}, E)$ be the space of smooth functions. It can be shown that the set $C^\infty(\mathbb{R}, E)$ does not depend on the locally convex topology of E, only on its associated bornology (system of bounded sets). The final topologies with respect to the following sets of mappings into E coincide:

1. $C^\infty(\mathbb{R}, E)$.
2. The set of all Lipschitz curves (so that $\{\frac{c(t)-c(s)}{t-s} : t \neq s, |t|, |s| \leq C\}$ is bounded in E, for each C).
3. The set of injections $E_B \to E$ where B runs through all bounded absolutely convex subsets in E, and where E_B is the linear span of B equipped with the Minkowski functional $\|x\|_B := \inf\{\lambda > 0 : x \in \lambda B\}$.
4. The set of all Mackey-convergent sequences $x_n \to x$ (there exists a sequence $0 < \lambda_n \nearrow \infty$ with $\lambda_n(x_n - x)$ bounded).

The c^∞-topology. II

This topology is called the c^∞-topology on E and we write $c^\infty E$ for the resulting topological space.

In general (on the space $\mathcal{D}$ of test functions for example) it is finer than the given locally convex topology, it is not a vector space topology, since addition is no longer jointly continuous. Namely, $c^\infty(\mathcal{D} \times \mathcal{D})$ is strictly finer than $c^\infty\mathcal{D} \times c^\infty\mathcal{D}$.

The finest among all locally convex topologies on E which are coarser than $c^\infty E$ is the bornologification of the given locally convex topology. If E is a Fréchet space, then $c^\infty E = E$.

Convenient vector spaces

A locally convex vector space E is said to be a *convenient vector space* if one of the following holds (called c^∞-completeness):

1. For any $c \in C^\infty(\mathbb{R}, E)$ the (Riemann-) integral $\int_0^1 c(t)dt$ exists in E.
2. Any Lipschitz curve in E is locally Riemann integrable.
3. A curve $c : \mathbb{R} \to E$ is C^∞ if and only if $\lambda \circ c$ is C^∞ for all $\lambda \in E^*$, where E^* is the dual of all cont. lin. funct. on E.
   - Equiv., for all $\lambda \in E'$, the dual of all bounded lin. functionals.
   - Equiv., for all $\lambda \in \mathcal{V}$, where $\mathcal{V}$ is a subset of E' which recognizes bounded subsets in E.

   We call this *scalarwise* C^∞.
4. Any Mackey-Cauchy-sequence (i. e. $t_{nm}(x_n - x_m) \to 0$ for some $t_{nm} \to \infty$ in $\mathbb{R}$) converges in E. This is visibly a mild completeness requirement.

Convenient vector spaces. II

5. If B is bounded closed absolutely convex, then E_B is a Banach space.
6. If $f : \mathbb{R} \to E$ is scalarwise Lip^k, then f is Lip^k, for $k > 1$.
7. If $f : \mathbb{R} \to E$ is scalarwise C^∞ then f is differentiable at 0.

Here a mapping $f : \mathbb{R} \to E$ is called Lip^k if all derivatives up to order k exist and are Lipschitz, locally on $\mathbb{R}$. That f is scalarwise C^∞ means $\lambda \circ f$ is C^∞ for all continuous (equiv., bounded) linear functionals on E.

Smooth mappings

Let E, and F be convenient vector spaces, and let $U \subset E$ be c^∞-open. A mapping $f : U \to F$ is called smooth or C^∞, if $f \circ c \in C^\infty(\mathbb{R}, F)$ for all $c \in C^\infty(\mathbb{R}, U)$.

If E is a Fréchet space, then this notion coincides with all other reasonable notions of C^∞-mappings. Beyond Fréchet mappings, as a rule, there are more smooth mappings in the convenient setting than in other settings, e.g., C^∞_c.

Main properties of smooth calculus

1. For maps on Fréchet spaces this coincides with all other reasonable definitions. On $\mathbb{R}^2$ this is non-trivial [Boman,1967].
2. Multilinear mappings are smooth iff they are bounded.
3. If $E \supseteq U \xrightarrow{f} F$ is smooth then the derivative $df : U \times E \to F$ is smooth, and also $df : U \to L(E, F)$ is smooth where $L(E, F)$ denotes the space of all bounded linear mappings with the topology of uniform convergence on bounded subsets.
4. The chain rule holds.
5. The space $C^\infty(U, F)$ is again a convenient vector space where the structure is given by the obvious injection

$$C^\infty(U, F) \xrightarrow{C^\infty(c,\ell)} \prod_{c \in C^\infty(\mathbb{R}, U), \ell \in F^*} C^\infty(\mathbb{R}, \mathbb{R}), \quad f \mapsto (\ell \circ f \circ c)_{c,\ell},$$

where $C^\infty(\mathbb{R}, \mathbb{R})$ carries the topology of compact convergence in each derivative separately.

Main properties of smooth calculus, II

6. The exponential law holds: For c^∞-open $V \subset F$,

$$C^\infty(U, C^\infty(V, G)) \cong C^\infty(U \times V, G)$$

   is a linear diffeomorphism of convenient vector spaces. **Note that this is the main assumption of variational calculus. Here it is a theorem.**

7. A linear mapping $f : E \to C^\infty(V, G)$ is smooth (by (2) equivalent to bounded) if and only if $E \xrightarrow{f} C^\infty(V, G) \xrightarrow{\mathrm{ev}_v} G$ is smooth for each $v \in V$. (*Smooth uniform boundedness theorem*, [KM97], theorem 5.26).
   A mapping $f : U \to L(F, G)$ is smooth iff $U \xrightarrow{f} L(F, G) \xrightarrow{\mathrm{ev}_x} G$ is smooth for all $x \in F$.

Main properties of smooth calculus, III

8. The following canonical mappings are smooth.

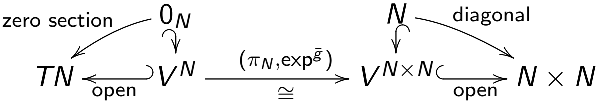
$$\begin{aligned}
&\mathrm{ev}: C^\infty(E,F)\times E \to F, \quad \mathrm{ev}(f,x)=f(x)\\
&\mathrm{ins}: E \to C^\infty(F, E\times F), \quad \mathrm{ins}(x)(y)=(x,y)\\
&(\quad)^\wedge : C^\infty(E, C^\infty(F,G)) \to C^\infty(E\times F, G)\\
&(\quad)^\vee : C^\infty(E\times F, G) \to C^\infty(E, C^\infty(F,G))\\
&\mathrm{comp}: C^\infty(F,G)\times C^\infty(E,F)\to C^\infty(E,G)\\
&C^\infty(\quad,\quad): C^\infty(F,F_1)\times C^\infty(E_1,E)\to\\
&\qquad\qquad \to C^\infty(C^\infty(E,F), C^\infty(E_1,F_1))\\
&\qquad (f,g)\mapsto (h\mapsto f\circ h\circ g)\\
&\prod : \prod C^\infty(E_i,F_i)\to C^\infty(\prod E_i, \prod F_i)
\end{aligned}$$

This ends our review of the standard results of convenient calculus. Convenient calculus (having properties 6 and 7) exists also for:

- Real analytic mappings [Kriegl,M,1990]
- Holomorphic mappings [Kriegl,Nel,1985] (notion of [Fantappié, 1930-33])
- Many classes of Denjoy Carleman ultradifferentible functions, both of Beurling type and of Roumieu-type [Kriegl,M,Rainer, 2009, 2011, 2013]

Manifolds of mappings

Let M be a compact (for simplicity's sake) fin. dim. manifold and N a manifold. We use an auxiliary Riemann metric $\bar{g}$ on N. Then



$C^\infty(M,N)$, the space of smooth mappings $M \to N$, has the following manifold structure. Chart, centered at $f \in C^\infty(M,N)$, is:

$$C^\infty(M,N) \supset U_f = \{g : (f,g)(M) \subset V^{N\times N}\} \xrightarrow{u_f} \tilde{U}_f \subset \Gamma(f^* TN)$$

$$u_f(g) = (\pi_N, \exp^{\bar{g}})^{-1} \circ (f,g), \quad u_f(g)(x) = (\exp^{\bar{g}}_{f(x)})^{-1}(g(x))$$

$$(u_f)^{-1}(s) = \exp^{\bar{g}}_f \circ s, \qquad (u_f)^{-1}(s)(x) = \exp^{\bar{g}}_{f(x)}(s(x))$$

Manifolds of mappings II

Lemma: $C^\infty(\mathbb{R}, \Gamma(M; f^*TN)) = \Gamma(\mathbb{R} \times M; \mathrm{pr}_2{}^* f^*TN)$
By Cartesian Closedness (after handling local trivializations).
Lemma: Chart changes are smooth (C^∞)
$\tilde{U}_{f_1} \ni s \mapsto (\pi_N, \exp^{\bar{g}}) \circ s \mapsto (\pi_N, \exp^{\bar{g}})^{-1} \circ (f_2, \exp^{\bar{g}}_{f_1} \circ s)$
since they map smooth curves to smooth curves.
Lemma: $C^\infty(\mathbb{R}, C^\infty(M, N)) \cong C^\infty(\mathbb{R} \times M, N)$.
By Cartesian closedness.
Lemma: Composition $C^\infty(P, M) \times C^\infty(M, N) \to C^\infty(P, N)$, $(f, g) \mapsto g \circ f$, is smooth, since it maps smooth curves to smooth curves
Corollary (of the chart structure):

$$TC^\infty(M, N) = C^\infty(M, TN) \xrightarrow{C^\infty(M, \pi_N)} C^\infty(M, N).$$

Regular Lie groups

We consider a smooth Lie group G with Lie algebra $\mathfrak{g} = T_eG$ modelled on convenient vector spaces. The notion of a regular Lie group is originally due to Omori et al. for Fréchet Lie groups, was weakened and made more transparent by Milnor, and then carried over to convenient Lie groups; see [KM97], 38.4.

A Lie group G is called *regular* if the following holds:

- For each smooth curve $X \in C^\infty(\mathbb{R}, \mathfrak{g})$ there exists a curve $g \in C^\infty(\mathbb{R}, G)$ whose right logarithmic derivative is X, i.e.,

$$\begin{cases} g(0) & = e \\ \partial_t g(t) & = T_e(\mu^{g(t)})X(t) = X(t).g(t) \end{cases}$$

  The curve g is uniquely determined by its initial value $g(0)$, if it exists.

- Put $\mathrm{evol}^r_G(X) = g(1)$ where g is the unique solution required above. Then $\mathrm{evol}^r_G : C^\infty(\mathbb{R}, \mathfrak{g}) \to G$ is required to be C^∞ also. We have $\mathrm{Evol}^X_t := g(t) = \mathrm{evol}_G(tX)$.

Diffeomorphism group of compact M

Theorem: *For each compact manifold M, the diffeomorphism group is a regular Lie group.*
Proof: $\text{Diff}(M) \xrightarrow{open} C^\infty(M, M)$. Composition is smooth by restriction. Inversion is smooth: If $t \mapsto f(t, \quad)$ is a smooth curve in $\text{Diff}(M)$, then $f(t, \quad)^{-1}$ satisfies the implicit equation $f(t, f(t, \quad)^{-1}(x)) = x$, so by the finite dimensional implicit function theorem, $(t, x) \mapsto f(t, \quad)^{-1}(x)$ is smooth. So inversion maps smooth curves to smooth curves, and is smooth.
Let $X(t, x)$ be a time dependent vector field on M (in $C^\infty(\mathbb{R}, \mathfrak{X}(M))$). Then $\text{Fl}_s^{\partial_t \times X}(t, x) = (t + s, \text{Evol}^X(t, x))$ satisfies the ODE $\quad \partial_t \text{Evol}(t, x) = X(t, \text{Evol}(t, x))$. If $X(s, t, x) \in C^\infty(\mathbb{R}^2, \mathfrak{X}(M))$ is a smooth curve of smooth curves in $\mathfrak{X}(M)$, then obviously the solution of the ODE depends smoothly also on the further variable s, thus evol maps smooth curves of time dependant vector fields to smooth curves of diffeomorphism. QED.

The principal bundle of embeddings

For finite dimensional manifolds M, N with M compact, $\mathrm{Emb}(M,N)$, the space of embeddings of M into N, is open in $C^\infty(M,N)$, so it is a smooth manifold. $\mathrm{Diff}(M)$ acts freely and smoothly from the right on $\mathrm{Emb}(M,N)$.

Theorem: $\mathrm{Emb}(M,N) \to \mathrm{Emb}(M,N)/\mathrm{Diff}(M)$ *is a principal fiber bundle with structure group* $\mathrm{Diff}(M)$.

Proof: Auxiliary Riem. metric $\bar g$ on N. Given $f \in \mathrm{Emb}(M,N)$, view $f(M)$ as submanifold of N. $TN|_{f(M)} = \mathrm{Nor}(f(M)) \oplus Tf(M)$.

$\mathrm{Nor}(f(M)) : \xrightarrow[\cong]{\exp^{\bar g}} W_{f(M)} \xrightarrow{p_{f(M)}} f(M)$ tubular nbhd of $f(M)$.

If $g : M \to N$ is C^1-near to f, then
$\varphi(g) := f^{-1} \circ p_{f(M)} \circ g \in \mathrm{Diff}(M)$ and
$g \circ \varphi(g)^{-1} \in \Gamma(f^* W_{f(M)}) \subset \Gamma(f^* \mathrm{Nor}(f(M)))$.
This is the required local splitting. QED

The orbifold bundle of immersions

$\text{Imm}(M,N)$, the space of immersions $M \to N$, is open in $C^\infty(M,N)$, and is thus a smooth manifold. The regular Lie group $\text{Diff}(M)$ acts smoothly from the right, but no longer freely.
Theorem: [Cervera,Mascaro,M,1991] *For an immersion $f : M \to N$, the isotropy group $\text{Diff}(M)_f = \{\varphi \in \text{Diff}(M) : f \circ \phi = f\}$ is always a finite group, acting freely on M; so $M \xrightarrow{p} M/\text{Diff}(M)_f$ is a convering of manifold and f factors to $f = \bar{f} \circ p$.*
Thus $\text{Imm}(M,N) \to \text{Imm}(M,N)/Diff(M)$ is a projection onto an honest infinite dimensional orbifold.

A message from convenient analysis

Theorem. [4.1.19 and 4.1.22 of Frölicher Kriegl: Linear spaces and differentiation theory, 1988] *Let $c : \mathbb{R} \to E$ be a curve in a convenient vector space E. Let $\mathcal{V} \subset E'$ be a subset of bounded linear functionals such that the bornology of E has a basis of $\sigma(E, \mathcal{V})$-closed sets. Then the following are equivalent:*

(i) *c is smooth*

(ii) *For each $k \in \mathbb{N}$ there exists a locally bounded curve $c^k : \mathbb{R} \to E$ such that for each $\ell \in \mathcal{V}$ the function $\ell \circ c$ is smooth $\mathbb{R} \to \mathbb{R}$ with $(\ell \circ c)^{(k)} = \ell \circ c^k$.*

If $E = F'$ is the dual of convenient vector space F, then for any point separating subset $\mathcal{V} \subset F$ the bornology of E has a basis of $\sigma(E, \mathcal{V})$-closed subsets.

Corollary. Let E be a vector bundle over M. Then for each $s \in (\dim(M)/2, \infty)$ the space $C^\infty(\mathbb{R}, \Gamma_{H^s(\hat{g})}(E))$ of smooth curves in $\Gamma_{H^s(\hat{g})}(E)$ consists of all continuous mappings $c : \mathbb{R} \times M \to E$ with $p \circ c = \mathrm{pr}_2 : \mathbb{R} \times M \to M$ such that:

- For each $x \in M$ the curve $t \mapsto c(t, x) \in E_x$ is smooth; let $(\partial_t^p c)(t, x) = \partial_t^p (c(t, x))$, and
- For each $p \in \mathbb{N}_{\geq 0}$, the curve $\partial_t^p c$ has values in $\Gamma_{H^s(\hat{g})}(E)$ so that $\partial_t^p c : \mathbb{R} \to \Gamma_{H^s(\hat{g})}(E)$, and $t \mapsto \|\partial_t c(t, \quad)\|_{H^s(\hat{g})}$ is bounded, locally in t.

Corollary *Let E_1, E_2 be vector bundles over M, let $U \subset E_1$ be an open neighborhood of the image of a smooth section, let $F : U \to E_2$ be a fiber preserving smooth mapping, and let $s \in (m/2, \infty)$. Then the set $\Gamma_{H^s}(U) := \{h \in \Gamma_{H^s}(E_1) : h(M) \subset U\}$ is open in $\Gamma_{H^s}(E_1)$, and the mapping $F_* : \Gamma_{H^s}(U) \to \Gamma_{H^s}(E_2)$ given by $h \mapsto F \circ h$, is smooth. If the restriction of F to each fiber of E_1 is real analytic, then F_* is real analytic.*

The Laplacian depends smoothly on the metric

Theorem. *Let* $\alpha \in (\frac{\dim(M)}{2}, \infty)$ *and let* $E \to M$ *be a natural bundle of first order. Then* $g \mapsto \nabla^g$ *is a smooth mapping:*

$$\nabla : \mathrm{Met}_{H^\alpha}(M) \to L^2(\Gamma_{H^\alpha}(TM), \Gamma_{H^s}(E); \Gamma_{H^{s-1}}(E)),$$
$$\nabla : \mathrm{Met}_{H^\alpha}(M) \to L(\Gamma_{H^s}(E); \Gamma_{H^{s-1}}(T^*M \otimes E)),$$

for $1 \leq s \leq \alpha$. *Consequently,* $g \mapsto \Delta^g$ *is a real analytic mapping*

$$\mathrm{Met}_{H^\alpha}(M) \to L(\Gamma_{H^s}(E), \Gamma_{H^{s-2}}(E)),$$

for $2 \leq s \leq \alpha$. *If* $E = \mathbb{R}$ *then* $g \mapsto \Delta^g$ *is a real analytic mapping*

$$\mathrm{Met}_{H^\alpha}(M) \to L(H^s(M, \mathbb{R}), H^{s-2}(M, \mathbb{R})),$$

for $2 \leq s \leq \alpha + 1$.

The operator $F(1+\Delta^g)$

Let $g \in \mathrm{Met}_{H^\alpha}(M)$ for $\alpha \in (\frac{m}{2}, \infty)$, and let E be a natural first order vector bundle over M. Let $(e_i)_{i\in\mathbb{N}}$ be an $L^2(g)$-orthonormal basis of $\Gamma_{H^0}(E)$ of eigenvectors of $1+\Delta^g$ with eigenvalues $(\lambda_i)_{i\in\mathbb{N}} \subset \mathbb{R}_{>0}$.
In general the eigenvalues cannot be chosen smoothly, the eigenfunctions not even continuously, as functions of g. By

[A. Kriegl, P. W. Michor, and A. Rainer. Many parameter Hölder perturbation of unbounded operators. Math. Ann., 353:519â522, 2012],

the increasingly ordered eigenvalues are Lipschitz in g. However, along any real analytic curve $t \mapsto g(t)$ in $\mathrm{Met}_{H^\alpha}(M)$ the eigenvalues and the eigenfunctions can be parameterized real analytically in t. This follows from a result due to Rellich.

The global resolvent set

$$\{(g,\lambda) \in \mathrm{Met}_{H^\alpha}(M)\times\mathbb{C} : (1+\Delta^g-\lambda) : \Gamma_{H^2}(E) \to \Gamma_{H^0(\hat{g})}(E) \text{ invertible}\}$$

is open in $\mathrm{Met}_{H^\alpha}(M) \times \mathbb{C}$ and contains $\mathrm{Met}_{H^\alpha}(M) \times (\mathbb{C} \setminus \mathbb{R}_{>0})$. For any simple closed positively oriented C^1-curve γ in $\mathbb{C}$ which does not meet any eigenvalue of $1 + \Delta^g$ the operator

$$P(g,\gamma) = -\frac{1}{2\pi i}\int_\gamma (1+\Delta^g-\lambda)^{-1} d\lambda : \Gamma_{H^0}(E) \to \Gamma_{H^2}(E)$$

is the orthogonal projection onto the finite dimensional direct sum of all eigenspaces for those eigenvalues of $1 + \Delta^g$ which lie in the interior of γ. For fixed γ the operator $P(g,\gamma)$ is defined for all g in the open set of those g such that no eigenvalue of $1 + \Delta^g$ lies on γ. It depends smoothly, even C^ω, on those g, since inversion

$$GL(\Gamma_{H^2}(E), \Gamma_{H^0}(E)) \to L(\Gamma_{H^0}(E), \Gamma_{H^2}(E))$$

is real analytic, and since $\Gamma_{H^2}(E) \to \Gamma_{H^0}(E)$ is a compact operator.

Let $\mathbb{R}_{>}0 \subset U \xrightarrow{F} \mathbb{C}$ be a holomorphic function with $F(\mathbb{R}_{>0}) = \mathbb{R}_{>0}$ where U is an open neighborhood of $\mathbb{R}_{>0}$ in $\mathbb{C}$.

$$\Gamma_{H^0}(E) \supset D(f(1+\Delta^g)) \xrightarrow{F(1+\Delta^g)} \Gamma_{H^0}(E), \quad h \mapsto \sum_{i\in\mathbb{N}} \langle h_i, e_i \rangle f(\lambda_i) e_i$$

domain $D(F(1+\Delta^g)) = \left\{ h \in \Gamma_{H^0}(E); \sum_{i\in\mathbb{N}} \langle h_i, e_i \rangle^2 F(\lambda_i)^2 < \infty \right\}$ is densely defined and self-adjoint with respect to $L^2(g)$. The domain $D(F(1+\Delta^g))$ is a Hilbert space.

Theorem *Let $\alpha \in$ The mapping*

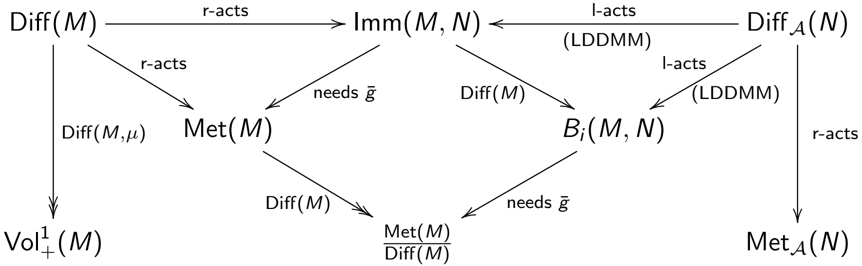
$$\begin{aligned} & g \mapsto F(1+\Delta^g) \\ & \mathrm{Met}_{H^\alpha}(M) \to L(D(F(1+\Delta^g)), \Gamma_{H^0}(E)) \end{aligned}$$

is smooth. Real analytic? Still lacking proof.
The proof uses the message from convenient calculus. Namely, The elements $h \otimes k \in V \otimes \Gamma_{H^0}(E)$, where V is a dense subspace in $D(F(1+\Delta^g))$, separate points in $L(D(F(1+\Delta^g)), \Gamma_{H^0}(E))$ and the latter space has a basis of bounded sets which closed with respect to it.
Then we use a smooth curve $g(t) \in \mathrm{Met}_{H^\alpha}(M)$, a curve γ enclosing the first N eigenvalues of $1+\Delta^{g(0)}$ in its interior, $h = \sum_{i=1}^{N} h_i e_i$,

$$-\frac{1}{2\pi i}\int_\gamma F(\lambda)\Big\langle (1+\Delta^g-\lambda)^{-1}h, k\Big\rangle d\lambda$$

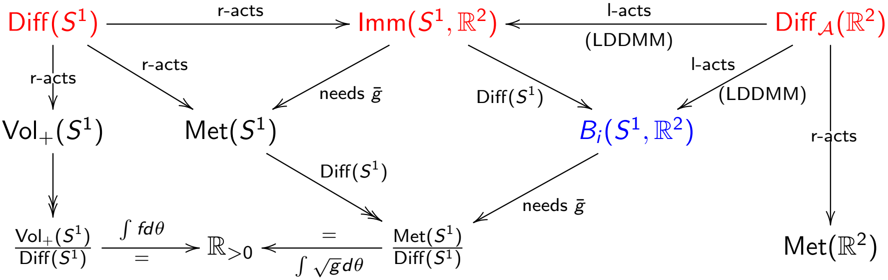
and their derivatives in t as canditates.

The diagram



M compact , N possibly non-compact manifold

$\mathrm{Met}(N) = \Gamma(S^2_+ T^* N)$	space of all Riemann metrics on N
$\bar{g}$	one Riemann metric on N
$\mathrm{Diff}(M)$	Lie group of all diffeos on compact mf M
$\mathrm{Diff}_{\mathcal{A}}(N),\ \mathcal{A} \in \{H^\infty, \mathcal{S}, c\}$	Lie group of diffeos of decay $\mathcal{A}$ to Id_N
$\mathrm{Imm}(M, N)$	mf of all immersions $M \to N$
$B_i(M, N) = \mathrm{Imm}/\mathrm{Diff}(M)$	shape space
$\mathrm{Vol}^1_+(M) \subset \Gamma(\mathrm{vol}(M))$	space of positive smooth probability densities



$\mathrm{Diff}(S^1)$	Lie group of all diffeos on compact mf S^1
$\mathrm{Diff}_{\mathcal{A}}(\mathbb{R}^2)$, $\mathcal{A} \in \{\mathcal{B}, H^\infty, \mathcal{S}, c\}$	Lie group of diffeos of decay $\mathcal{A}$ to $\mathrm{Id}_{\mathbb{R}^2}$
$\mathrm{Imm}(S^1, \mathbb{R}^2)$	mf of all immersions $S^1 \to \mathbb{R}^2$
$B_i(S^1, \mathbb{R}^2) = \mathrm{Imm}/\mathrm{Diff}(S^1)$	shape space
$\mathrm{Vol}_+(S^1) = \left\{ f\, d\theta : f \in C^\infty(S^1, \mathbb{R}_{>0}) \right\}$	space of positive smooth probability densities
$\mathrm{Met}(S^1) = \left\{ g\, d\theta^2 : g \in C^\infty(S^1, \mathbb{R}_{>0}) \right\}$	space of metrics on S^1

Riemannian metrics on shape space

$$\begin{array}{c} \mathrm{Imm}(M,N) \\ \big\downarrow \pi \\ B_i(M,N) \\ \| \\ \mathrm{Imm}(M,N)/\mathrm{Diff}(M) \end{array}$$

Given a Diff(M)-invariant metric G on Imm(M, N), the the horizontal space (G-perpendicular to the Diff(M)-orbit) is or is not a complement to the orbit (it might even be 0). Nevertheless G induces a Riemannian metric on the quotient $B_i(M, N)$ (at least off the singularities) such that π is a *Riemannian submersion*. Then:

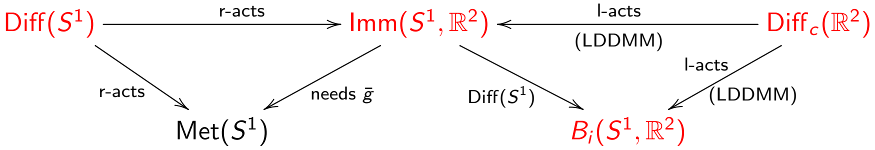
If a geodesics on Imm(M, N) is horizontal at one time, then for all times and it projects down to geodesic on shape space.
O'Neill's formula connects sectional curvature on Imm(M, N) and on B_i.

[Micheli, M, Mumford, Izvestija 2013]

L^2 metric

$$G^0_c(h,k) = \int_M \langle h(\theta), k(\theta) \rangle ds.$$

Problem: The induced geodesic distance vanishes.



[MichorMumford2005a,2005b], [BauerBruverisHarmsMichor2011,2012]

Weak Riem. metrics on $\mathrm{Emb}(M,N) \subset \mathrm{Imm}(M,N)$.

Metrics on the space of immersions of the form:

$$G^P_f(h,k) = \int_M \bar{g}(P^f h, k)\, \mathrm{vol}(f^*\bar{g})$$

where $\bar{g}$ is some fixed metric on N, $g = f^*\bar{g}$ is the induced metric on M, $h,k \in \Gamma(f^*TN)$ are tangent vectors at f to $\mathrm{Imm}(M,N)$, and P^f is a positive, selfadjoint, bijective (scalar) pseudo differential operator of order $2p$ depending smoothly on f. Also P has to be $\mathrm{Diff}(M)$-invariant: $\varphi^* \circ P_f = P_{f\circ\varphi} \circ \varphi^*$.
Good example:

- $P^f = 1 + A(\Delta^g)^p$ (p need not be an integer!), where Δ^g is the Bochner-Laplacian on M induced by the metric $g = f^*\bar{g}$.
- Or even $P^f = f(1+\Delta^g)$ for a suitable resolvent function f.

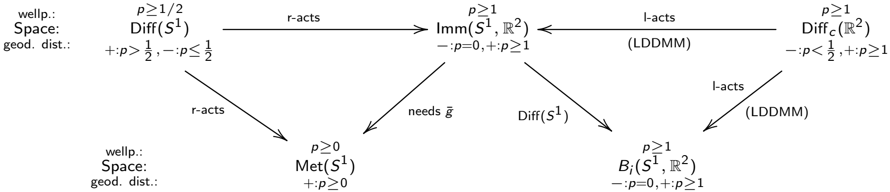
Sobolev type metrics

Advantages of Sobolev type metrics:

1. Positive geodesic distance
2. Geodesic equations are well posed
3. Spaces are geodesically complete for $p > \frac{\dim(M)}{2} + 1$.

Problems:

1. Analytic solutions to the geodesic equation?
2. Curvature of shape space with respect to these metrics?
3. Numerics are in general computational expensive



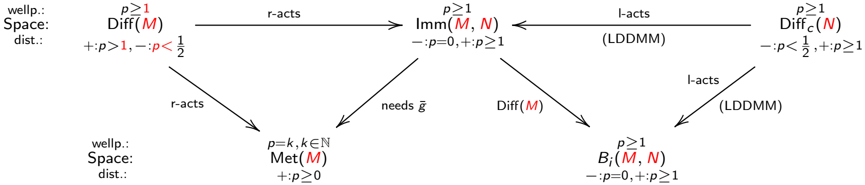
Sobolev type metrics

Advantages of Sobolev type metrics:

1. Positive geodesic distance
2. Geodesic equations are well posed
3. Spaces are conjectured to be geodesically complete for $p > \frac{\dim(M)}{2} + 1$.

Problems:

1. Analytic solutions to the geodesic equation?
2. Curvature of shape space with respect to these metrics?
3. Numerics are in general computational expensive



Geodesic equation.

The geodesic equation for a Sobolev-type metric G^P on immersions is given by

$$\begin{aligned}\nabla_{\partial_t} f_t =& \frac{1}{2} P^{-1}\Big(\operatorname{Adj}(\nabla P)(f_t, f_t)^\perp - 2.Tf.\bar{g}(Pf_t, \nabla f_t)^\sharp \\ & \qquad - \bar{g}(Pf_t, f_t).\operatorname{Tr}^g(S)\Big) \\ & - P^{-1}\Big((\nabla_{f_t} P) f_t + \operatorname{Tr}^g\big(\bar{g}(\nabla f_t, Tf)\big) Pf_t \Big).\end{aligned}$$

The geodesic equation written in terms of the momentum for a Sobolev-type metric G^P on Imm is given by:

$$\begin{cases} p = Pf_t \otimes \operatorname{vol}(f^*\bar{g}) \\ \nabla_{\partial_t} p = \frac{1}{2}\big(\operatorname{Adj}(\nabla P)(f_t, f_t)^\perp - 2Tf.\bar{g}(Pf_t, \nabla f_t)^\sharp \\ \qquad - \bar{g}(Pf_t, f_t) \operatorname{Tr}^{f^*\bar{g}}(S)\big) \otimes \operatorname{vol}(f^*\bar{g}) \end{cases}$$

Assumptions for Wellposedness

Assumption 1: $P, \nabla P$ and $\mathrm{Adj}(\nabla P)^\perp$ are smooth sections of the bundles

$$\begin{array}{ccc} L(T\mathrm{Imm}; T\mathrm{Imm}) & L^2(T\mathrm{Imm}; T\mathrm{Imm}) & L^2(T\mathrm{Imm}; T\mathrm{Imm}) \\ \downarrow & \downarrow & \downarrow \\ \mathrm{Imm} & \mathrm{Imm} & \mathrm{Imm}, \end{array}$$

and of the Sobolev H^s-completions for large s, of all spaces, respectively.

Assumption 2: For each $f \in \mathrm{Imm}(M, N)$, the operator P_f is an elliptic pseudo-differential operator of order $2p$ for $p > 0$ which is positive and symmetric with respect to the H^0-metric on Imm, i.e.

$$\int_M \bar{g}(P_f h, k)\,\mathrm{vol}(g) = \int_M \bar{g}(h, P_f k)\,\mathrm{vol}(g) \qquad \text{for } h, k \in T_f\mathrm{Imm}.$$

Wellposedness result

Theorem [Bauer, Harms, M, 2011, with a small gap] *Let* $p \in [1, \infty)$ *and* $k \in (\dim(M)/2 + 1, \infty)$, *and let* P *satisfy the assumptions. Then the geodesic equation has unique local solutions in the Sobolev manifold* Imm^{k+2p} *of* H^{k+2p}*-immersions. The solutions depend smoothly on* t *and on the initial conditions* $f(0, \,.\,)$ *and* $f_t(0, \,.\,)$. *The domain of existence (in* t*) is uniform in* k *and thus this also holds in* $\mathrm{Imm}(M, N)$. *Moreover, in each Sobolev completion* Imm^{k+2p}, *the Riemannian exponential mapping* $\exp^P$ *exists and is smooth on a neighborhood of the zero section in the tangent bundle, and* $(\pi, \exp^P)$ *is a diffeomorphism from a (smaller) neigbourhood of the zero section to a neighborhood of the diagonal in* $\mathrm{Imm}^{k+2p} \times \mathrm{Imm}^{k+2p}$. *All these neighborhoods are uniform in* $k > \dim(M)/2 + 1$ *and can be chosen* H^{k_0+2p}*-open, for* $k_0 > \dim(M)/2 + 1$. *Thus both properties of the exponential mapping continue to hold in* $\mathrm{Imm}(M, N)$.

Theorem. [O. Müller. Applying the index theorem to non-smooth operators. J. Geometry and Physics, 116:140-145, 2017], for integer p and k. A forthcoming paper for the general situation.

The operators $f \mapsto P_f = (1 + A\Delta^{f^*\bar{g}})^p$ *and* $f \mapsto F(1 + \Delta^{f^*\bar{g}})$ *for a suitable resolvent function* F*, satisfy both assumptions for wellposedness.*

The wellposedness result carries over to the case of diffeomorphism groups.

A variant of the proof furnishes a similar wellposedness result for metrics on the space $\mathrm{Met}(M)$ of all Riemannian metrics of the form

$$G_g(h,k) = \int_M \Big(C_1 \operatorname{Trace}(g^{-1}\cdot P^1_g(h)\cdot g^{-1}\cdot k) \\ + C_2 \operatorname{Trace}(g^{-1}\cdot P^2_g(h)) \operatorname{Trace}(g^{-1}\cdot k)\Big) \operatorname{vol}(g)\,,$$

where P^i_g is any of $(1+A\Delta^g)^p$ or $F(1+A\Delta^g)$ for a suitable resolvent function F.

Thank you for your attention