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THE CONVENIENT SETTING FOR REAL ANALYTIC MAPPINGS

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Abstract. We present here "the" cartesian closed theory for real analytic mappings. It is based on the concept of real analytic curves in locally convex vector spaces. A mapping is real analytic, if it maps smooth curves to smooth curves and real analytic curves to real analytic curves. Under mild completeness conditions the second requirement can be replaced by: real analytic along affine lines. Enclosed and necessary is a careful study of locally convex topologies on spaces of real analytic mappings.

As an application we also present the theory of manifolds of real analytic mappings: the group of real analytic diffeomorphisms of a compact real analytic manifold is a real analytic Lie group.

Contents

0. Introduction 2
1. Real analytic curves 3
2. Real analytic mappings 11
3. Function spaces in finite dimensions 18
4. A uniform boundedness principle 26
5. Cartesian closedness 30
6. Consequences of cartesian closedness 36
7. Spaces of sections of vector bundles 41
8. Manifolds of analytic mappings 47

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0. Introduction

We always wanted to know whether the group of real analytic diffeomorphisms of a real analytic manifold is itself a real analytic manifold in some sense. The paper [16] contains the theorem, that this group for a compact real analytic manifold is a smooth Lie group modeled on locally convex vector spaces. (The proof, however, contains a gap, which goes back to Smale in [1]: in canonical charts, no partial mapping of the composition is linear off 0). The construction there relies on ad hoc descriptions of the topology on the space of real analytic functions. Also the literature dealing with the duals of these spaces like [9] does not really try to describe the topologies on spaces of real analytic functions. There are, however, some older papers on this subject, see [29], [25], [26], [27], [31], [12], [8].

For some other instances where real analytic mappings in infinite dimensions make their appearance, see the survey article [28].

In this article, we present a careful study of real analytic mappings in infinite (and finite) dimensions combined with a thorough treatment of locally convex topologies on spaces of real analytic functions. From the beginning our aim is cartesian closedness: a mapping $f : E \times F \to G$ should be real analytic if and only if the canonically associated mapping $\check{f} : E \to C^\omega(F, G)$ is it. Very simple examples, see 1.1, show that real analytic in the sense of having a locally converging Taylor series is too restrictive.

The right notion turns out to be scalarwise real analytic: A curve in a locally convex space is called (scalarwise) real analytic if and only if composed with each continuous linear functional it gives a real analytic function. Later we show, that the space of real analytic curves does not depend on the topology, only on the bornology described by the dual.

A mapping will be called real analytic if it maps smooth curves to smooth curves and real analytic curves to real analytic curves. This definition is in spirit very near to the original ideas of variational calculus and it leads to a simple and powerful theory. We will show the surprising result, that under some mild completeness conditions (i.e. for convenient vector spaces), the second condition can be replaced by: the mapping should be real analytic along affine lines, see 2.7. This is a version of Hartogs' theorem, which for Banach spaces is due to [2].

It is a very satisfying result, that the right realm of spaces for real analytic analysis is the category of convenient vector spaces, which is also the good setting in infinite dimensions for smooth analysis, see [5], and for holomorphic analysis, see [15].

The power of the cartesian closed calculus for real analytic mappings developed here is seen in [21], where it is used to construct, for any unitary representation of any Lie group, a real analytic moment mapping from the space of analytic vectors into the dual of the Lie algebra.

We do not give any hard implicit function theorem in this paper, because our setting is too weak to obtain one — but we do not think that this is a disadvantage. Let us make a programmatic statement here:

An eminent mathematician once said, that for infinite dimensional calculus each serious application needs its own foundation. By a serious application one obviously means some application of a hard inverse function theorem. These theorems can be proved, if by assuming enough a priori estimates one creates enough Banach space situation for some modified iteration procedure to converge. Many authors try to build their platonic idea of an a priori estimate into their differential calculus. We think that this makes the calculus inapplicable and hides the origin of the a priori estimates. We believe, that the calculus itself should be as easy to use as possible, and that all further assumptions (which most often come from ellipticity of some nonlinear partial differential equation of geometric origin) should be treated separately, in a setting depending on the specific problem. We are sure that in this sense the setting presented here (and the setting in [5]) is universally usable for most applications.

The later parts of this paper are devoted to the study of manifolds of real analytic mappings. We show indeed, that the set of real analytic mappings from a compact manifold to another one is a real analytic manifold, that composition is real analytic and that the group of real analytic diffeomorphisms is a real analytic Lie group. The exponential mapping of it (integration of vector fields) is real analytic, but as in the smooth case it is still not surjective on any neighborhood of the identity. We would like to stress the fact that the group of smooth diffeomorphisms of a manifold is a smooth but *not* a real analytic Lie group. We also show that the space of smooth mappings between real analytic manifolds is a real analytic manifold, but the composition is only smooth.

Throughout this paper our basic guiding line is the cartesian closed calculus for smooth mappings as exposed in [5]. The reader is assumed to be familiar with at least the rudiments of it; but section 1 contains a short summary of the essential parts.

We want to thank Janusz Grabowski for hints and discussions. This should have been a joint work with him, but distance prevented it.

1. Real analytic curves

1.1. As for smoothness and holomorphy we would like to obtain cartesian closedness for real analytic mappings. Thus one should have at least the following:

$f : \mathbb{R}^2 \to \mathbb{R}$ is real analytic in the classical sense if and only if $f^\vee : \mathbb{R} \to C^\omega(\mathbb{R}, \mathbb{R})$ is real analytic in some appropriate sense.

The following example shows that there are some subtleties involved.

Example. *The mapping*

$$f : \mathbb{R}^2 \ni (s,t) \mapsto \frac{1}{(st)^2+1} \in \mathbb{R}$$

is real analytic, whereas there is no reasonable topology on $C^\omega(\mathbb{R},\mathbb{R})$, such that the mapping $f^\vee : \mathbb{R} \to C^\omega(\mathbb{R},\mathbb{R})$ is locally given by its convergent Taylor series.

Proof. For a topology on $C^\omega(\mathbb{R},\mathbb{R})$ to be reasonable we require only that all evaluations $\mathrm{ev}_t : C^\omega(\mathbb{R},\mathbb{R}) \to \mathbb{R}$ are bounded linear functionals. Now suppose that $f^\vee(s) = \sum_{k=0}^\infty f_k s^k$ converges in $C^\omega(\mathbb{R},\mathbb{R})$ for small t, where $f_k \in C^\omega(\mathbb{R},\mathbb{R})$. Then the series converges even bornologically, see 1.7, so $f(s,t) = \mathrm{ev}_t(f^\vee(s)) = \sum f_k(t)\, s^k$ for all t and small s. On the other hand $f(s,t) = \sum_{k=0}^\infty (-1)^k (st)^{2k}$ for $|s| < 1/|t|$. So for all t we have $f_k(t) = (-1)^m t^k$ for $k = 2m$, and 0 otherwise, since for fixed t we have a real analytic function in one variable. Moreover, the series $\left(\sum f_k z^k\right)(t) = \sum (-1)^k t^{2k} z^{2k}$ has to converge in $C^\omega(\mathbb{R},\mathbb{R}) \otimes \mathbb{C}$ for $|z| \le \delta$ and all t, see 1.7. This is not the case: use $z = \sqrt{-1}\,\delta$, $t = 1/\delta$. □

1.2. There is, however, another notion of real analytic curves.
Example. Let $f : \mathbb{R} \to \mathbb{R}$ be a real analytic function with finite radius of convergence at 0. Now consider the curve $c : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$ defined by $c(t) := (f(k \cdot t))_{k\in\mathbb{N}}$. Clearly the composite of c with any continuous linear functional is real analytic, since these functionals depend only on finitely many coordinates. But the Taylor series of c at 0 has radius of convergence 0, since the radii of the coordinate functions go to 0. For an even more natural example see 5.2.

The natural setting for this notion of real analyticity is that of dual pairs:

Definition (Real analytic curves). Let a *dual pair* (E, E') be a real vector space E with prescribed point separating dual E'. A curve $c : \mathbb{R} \to E$ is called *real analytic* if $\lambda \circ c : \mathbb{R} \to \mathbb{R}$ is real analytic for all $\lambda \in E'$.

A subset $B \subseteq E$ is called *bounded* if $\lambda(B)$ is bounded in $\mathbb{R}$ for all $\lambda \in E'$. The set of bounded subset of E will be called the *bornology* of E (generated by E').

The dual pair (E, E') is called *complete* if the bornology on E is complete, i.e. for every bounded set B there exists a bounded absolutely convex set $A \supseteq B$ such that the normed space E_A generated by A, see [10], 8.3 or [5], 2.1.15, is complete.

Let τ be a topology on E, which is compatible with the bornology generated by E', i.e. has as von Neumann bornology exactly this bornology. Then a curve $c : \mathbb{R} \to (E,\tau)$ will be called *topologically real analytic* if it is locally given by a power series converging with respect to τ.

A curve $c : \mathbb{R} \to E$ will be called *bornologically real analytic* if it factors locally over a topologically real analytic curve into E_B for some bounded absolutely convex set $B \subseteq E$.

1.3 Review of the smooth and holomorphic setting. We will make use of the cartesian closedness of smooth maps between convenient vector spaces [14] and that of holomorphic maps between such spaces [15]. Let us recall some facts from those theories.

First the smooth theory, where we refer to [5]. *Separated preconvenient vector spaces* can be defined as those dual pairs (E, E') for which E' consists exactly of the linear functionals which are bounded with respect to the bornology on E generated by E'. To each dual pair (E, E') one can naturally associate a preconvenient vector space (E, E^b), where E^b denotes the space of linear functionals which are bounded for the bornology generated by E'. The space (E, E^b) is the dual pair with the finest structure, which has as underlying space E and which has the same bornology. On every dual pair there is a *natural locally convex topology*, namely the Mackey topology associated with E'. The associated bornological topology given by the absolutely convex bornivorous subsets of E is the natural topology of (E, E^b). A curve $c : \mathbb{R} \to E$ is called *smooth* if $\lambda \circ c : \mathbb{R} \to \mathbb{R}$ is smooth. If (E, E') is complete and τ is any topology on E that is compatible with the bornology, then c is smooth if and only if c has derivatives of arbitrary order with respect to τ or, equivalently, for every k the curve c factors locally as a $\mathcal{L}ip^k$-mapping over E_B for some bounded absolutely convex set $B \subseteq E$.

A *convenient vector space* or *convenient dual pair* is a separated preconvenient vector space (E, E'), which is complete, so that $E' = E^b$ and the natural topology is bornological. Since the completeness condition depends only on the bornology, (E, E') is complete if and only if (E, E^b) is convenient.

A set $U \subseteq E$ is called *c^∞-open* if the inverse image $c^{-1}(U) \subseteq \mathbb{R}$ is open for every smooth curve c or, equivalently, the intersection $U_B := U \cap E_B$ is open in the normed space E_B for every bounded absolutely convex set $B \subseteq E$. If E is a metrizable or a Silva locally convex space and E' its topological dual then its topology coincides with the c^∞-topology.

A mapping $f : U \to F$ into another dual pair (F, F') is called *smooth (or C^∞)* if $f \circ c$ is a smooth curve for every smooth curve c having values in U. For Banach or even Fréchet spaces this notion coincides with the classically considered notions. The *space of smooth mappings* from U to F will be denoted by $C^\infty(U, F)$. On $C^\infty(\mathbb{R}, \mathbb{R})$ we consider the Fréchet topology of uniform convergence on compact subsets of all derivatives separately. On $C^\infty(U, F)$ one considers the dual induced by the family of mappings $C^\infty(c,\lambda) : C^\infty(U,F) \to C^\infty(\mathbb{R},\mathbb{R})$ for $c \in C^\infty(\mathbb{R},U)$ and $\lambda \in F'$. This makes $C^\infty(U,F)$ into a complete dual pair provided F is complete, and so one can pass to the associated convenient vector space. If E and F are finite dimensional the bornological topology of $C^\infty(U,F)$ is the usual topology of uniform convergence on compact subsets of U of all derivatives separately. For this space the following *exponential law* is valid: For every c^∞-open set V of a convenient vector space a mapping $f : V \times U \to F$ is smooth if and only if the associated mapping $\check{f} : V \to C^\infty(U,F)$ is a well defined smooth map.

Now the holomorphic theory developed in [15]. Let $\mathbb{D}$ denote the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ in $\mathbb{C}$. For a complex dual pair (E,E') a map $c : \mathbb{D} \to E$ is called a *holomorphic curve* if $\lambda \circ c : \mathbb{D} \to \mathbb{C}$ is a holomorphic function for every $\lambda \in E'$. If (E,E') is complete and τ is any topology on E that is compatible with the bornology, then c is holomorphic if and only if c is complex differentiable with respect to τ or, equivalently, the mapping c factors locally as a holomorphic curve over E_B for some bounded absolutely convex set $B \subseteq E$. A mapping $f : U \to F$ between complete complex dual pairs is called *holomorphic* if $f \circ c : \mathbb{D} \to F$ is a holomorphic curve for every holomorphic curve c having values in U. This is true if and only if it is a smooth mapping for the associated real vector spaces and the derivative at every point in U is $\mathbb{C}$-linear. For Banach or even Fréchet spaces this notion coincides with classically considered notions. Let $\mathcal{H}(U,F)$ denote the *vector space of holomorphic maps* from U to F. Then $\mathcal{H}(U,F)$ is a closed subspace of $C^\infty(U,F)$, since $f \mapsto f'(x)(v)$ is continuous on the latter space. So one equips $\mathcal{H}(U,F)$ with the convenient vector space structure induced from $C^\infty(U,F)$. If E is finite dimensional, then the bornological topology on $\mathcal{H}(U,F)$ is the topology of uniform convergence on compact subsets of U, see 3.2. For this space one has again an *exponential law*: For every c^∞-open subset V of a complex convenient vector space a mapping $f : V \times U \to F$ is holomorphic if and only if the associated mapping $\check{f} : V \to \mathcal{H}(U,F)$ is a well defined holomorphic map. This is a slight generalization of [15], 2.14, with the same proof as given there.

1.4. Lemma. *For a formal power series $\sum_{k\ge 0} a_k t^k$ with real coefficients the following conditions are equivalent.*

(1) *The series has positive radius of convergence.*
(2) *$\sum a_k r_k$ converges absolutely for all sequences (r_k) with $r_k\, t^k \to 0$ for all $t > 0$.*
(3) *The sequence $(a_k r_k)$ is bounded for all (r_k) with $r_k\, t^k \to 0$ for all $t > 0$.*
(4) *For each sequence (r_k) satisfying $r_k > 0$, $r_k r_\ell \ge r_{k+\ell}$, and $r_k\, t^k \to 0$ for all $t > 0$ there exists an $\varepsilon > 0$ such that $(a_k\, r_k\, \varepsilon^k)$ is bounded.*

This bornological description of real analytic curves will be rather important for the theory presented here, since condition (3) and (4) are linear conditions on the coefficients of a formal power series enforcing local convergence.

Proof. (1) $\Rightarrow$ (2). $\sum a_k r_k = \sum (a_k t^k)(r_k t^{-k})$ converges absolutely for some small t.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) is clear.

(4) $\Rightarrow$ (1). If the series has radius of convergence 0, then we have $\sum_k |a_k|\,(\frac{1}{n^2})^k = \infty$ for all n. There are $k_n \nearrow \infty$ with

$$\sum_{k=k_{n-1}}^{k_n-1} |a_k|\,(\tfrac{1}{n^2})^k \geq 1.$$

We put $r_k := (\frac{1}{n})^k$ for $k_{n-1} \leq k < k_n$, then $\sum_k |a_k|\, r_k (\frac{1}{n})^k = \infty$ for all n, so $(a_k r_k (\frac{1}{2n})^k)_k$ is not bounded for any n, but $r_k\, t^k$, which equals $(\frac{t}{n})^k$ for $k_{n-1} \leq k < k_n$, converges to 0 for all $t > 0$, and the sequence (r_k) is subadditive as required. □

1.5. Theorem (Description of real analytic curves). *Let (E, E') be a complete dual pair. A curve $c : \mathbb{R} \to E$ is real analytic if and only if c is smooth, and for each sequence (r_k) with $r_k\, t^k \to 0$ for all $t > 0$, and each compact set K in $\mathbb{R}$, the set*

$$\left\{ \frac{1}{k!}\, c^{(k)}(a)\, r_k : a \in K, k \in \mathbb{N} \right\}$$

is bounded, or equivalently the set corresponding to 1.4.(4) is bounded.

Proof. Since both conditions can be tested by applying $\lambda \in E'$ and we have $(\lambda \circ c)^{(k)}(a) = \lambda(c^{(k)}(a))$ we may assume that $E = \mathbb{R}$.

($\Rightarrow$). Clearly c is smooth.

Claim. There exist $M, \rho > 0$ with $|\frac{1}{k!} c^{(k)}(a)| < M\rho^k$ for all $k \in \mathbb{N}$ and $a \in K$.

This will give that $|\frac{1}{k!}\, c^{(k)}(a)\, r_k \varepsilon^k| \leq M r_k (\varepsilon\rho)^k$ which is bounded since $r_k(\varepsilon\rho)^k \to 0$, as required.

To show the claim we argue as follows. Since the Taylor series of c converges at a there are constants M_a, ρ_a satisfying the claimed inequality for fixed a. An elementary computation shows that for all a' with $|a - a'| \leq \frac{1}{2\rho_a}$ we have

$$\left| \frac{c^{(k)}(a')}{k!} \right| \leq M_a \rho_a^k\, \frac{1}{k!}\, \frac{\partial^k}{\partial t^k}\bigg|_{t=\frac{1}{2}}\, \frac{1}{1-t}\ ,$$

hence the condition is satisfied for all those a' with some new constants M'_a, ρ'_a. Since K is compact the claim follows.

($\Leftarrow$). Let

$$a_k := \sup_{a\in K}\left|\frac{1}{k!}\,c^{(k)}(a)\right|.$$

Using 1.4 (4$\Leftarrow$1) these are the coefficients of a power series with positive radius ρ of convergence. Hence the remainder of the Taylor series goes locally to zero. □

Although topological real analyticity is a strictly stronger than real analyticity, cf. 1.2, sometimes the converse is true as the following slight generalization of [2], Lemma 7.1 shows.

1.6. Theorem. *Let (E,E') be a complete dual pair and assume that a Baire vector space topology on E' exists for which the point evaluations ev_x for $x\in E$ are continuous. Then any real analytic curve $c:\mathbb{R}\to E$ is locally given by its Mackey convergent Taylor series, and hence is bornologically real analytic and topologically real analytic for every locally convex topology compatible with the bornology.*

Proof. Since c is real analytic, it is smooth and all derivatives exist in E, since (E,E') is complete, by 1.3.

Let us fix $t_0\in\mathbb{R}$, let $a_n := \frac{1}{n!}c^{(n)}(t_0)$. It suffices to find some $r>0$ for which $\{r^n a_n : n\in\mathbb{N}_0\}$ is bounded; because then $\sum t^n a_n$ is Mackey-convergent for $|t|<r$, and its limit is $c(t_0+t)$ since we can test this with functionals.

Consider the sets $A_r := \{\lambda\in E' : |\lambda(a_n)|\le r^n \text{ for all } n\in\mathbb{N}\}$. These A_r are closed in the Baire topology, since the point evaluations at a_n are continuous. Since c is real analytic, $\bigcup_{r>0}A_r = E'$, and by the Baire property there is an $r>0$ such that the interior U of A_r is not empty. Let $\lambda_0\in U$, then for all λ in the open neighborhood $U-\lambda_0$ of 0 we have $|\lambda(a_n)|\le|(\lambda+\lambda_0)(a_n)|+|\lambda_0(a_n)|\le 2r^n$. The set $U-\lambda_0$ is absorbing, thus for every $\lambda\in E'$ some multiple $\varepsilon\lambda$ is in $U-\lambda_0$ and so $\lambda(a_n)\le\frac{2}{\varepsilon}r^n$ as required. □

1.7. Lemma. *Let (E,E') be a complete dual pair, τ a topology on E compatible with the bornology induced by E', and let $c:\mathbb{R}\to E$ be a curve. Then the following conditions are equivalent.*

(1) *The curve c is topologically real analytic.*
(2) *The curve c is bornologically real analytic.*
(3) *The curve c extends to a holomorphic curve from some open neighborhood U of $\mathbb{R}$ in $\mathbb{C}$ into the complexification $(E_{\mathbb{C}},E'_{\mathbb{C}})$.*

Proof. (1) $\Rightarrow$ (3). For every $t\in\mathbb{R}$ one has for some $\delta>0$ and all $|s|<\delta$ a converging power series representation $c(t+s)=\sum_{k=1}^{\infty}x_k s^k$. For any complex number z with

$|z| < \delta$ the series converges in $(E_{\mathbb{C}}, E'_{\mathbb{C}})$, hence c can be locally extended to a holomorphic curve into $E_{\mathbb{C}}$. By the 1-dimensional uniqueness theorem for holomorphic maps, these local extensions fit together to give a holomorphic extension as required.

(3) $\Rightarrow$ (2). A holomorphic curve factors locally over $(E_{\mathbb{C}})_B$, where B can be chosen of the form $B \times \sqrt{-1}B$. Hence the restriction of this factorization to $\mathbb{R}$ is real analytic into E_B.

(2) $\Rightarrow$ (1). Let c be bornologically real analytic, i.e. c is locally real analytic into some E_B, which we may assume to be complete. Hence c is locally even topologically real analytic in E_B by 1.6 and so also in E. $\square$

1.8. Lemma. *Let E be a regular (i.e. every bounded set is contained and bounded in some step E_α) inductive limit of complex locally convex spaces $E_\alpha \subseteq E$, let $c : \mathbb{C} \supseteq U \to E$ be a holomorphic mapping, and let $W \subseteq \mathbb{C}$ be open and such that the closure $\overline{W}$ is compact and contained in U. Then there exists some α, such that $c|W : W \to E_\alpha$ is well defined and holomorphic.*

Proof. Since W is relatively compact, $c(W)$ is bounded in E. It suffices to show that for the absolutely convex closed hull B of $c(W)$ the Taylor series of c at each $z \in W$ converges in E_B, i.e. that $c|W : W \to E_B$ is holomorphic. This follows from the

Vector valued Cauchy inequalities. If $r > 0$ is smaller than the radius of convergence at z of c then

$$\frac{r^k}{k!} c^{(k)}(z) \in B$$

where B is the closed absolutely convex hull of $\{ c(w) : |w - z| = r\}$. (By the Hahn-Banach theorem this follows directly from the scalar valued case.)

Thus we get

$$\sum_{k=n}^{m} \left(\frac{w-z}{r}\right)^k \cdot \frac{r^k}{k!} c^{(k)}(z) \in \sum_{k=n}^{m} \left(\frac{w-z}{r}\right)^k \cdot B$$

and so

$$\sum_k \frac{c^{(k)}(z)}{k!} (w-z)^k$$

is convergent in E_B for $|w - z| < r$. Since B is contained and bounded in some E_α one has $c|W : W \to E_B = (E_\alpha)_B \to E_\alpha$ is holomorphic. $\square$

This proof also shows that holomorphic curves with values in complex convenient vector spaces are topologically and bornologically holomorphic (compare with 1.3).

1.9. Theorem (Linear real analytic mappings). *Let (E, E') be a complete dual pair. For any linear functional $\lambda : E \to \mathbb{R}$ the following assertions are equivalent.*

(1) *λ is bounded.*
(2) *$\lambda \circ c \in C^\omega(\mathbb{R}, \mathbb{R})$ for each real analytic $c : \mathbb{R} \to E$.*

This will be generalized in 2.7 to non-linear mappings.

Proof. ($\Uparrow$). Let λ satisfy (2) and suppose that there is a bounded sequence (x_k) such that $\lambda(x_k)$ is unbounded. By passing to a subsequence we may suppose that $|\lambda(x_k)| > k^{2k}$. Let $a_k := k^{-k}\, x_k$, then $(r^k\, a_k)$ is bounded and $(r^k\, \lambda(a_k))$ is unbounded for all $r > 0$. Hence the curve $c(t) := \sum_{k=0}^\infty t^k\, a_k$ is given by a Mackey convergent power series. So $\lambda \circ c$ is real analytic and near 0 we have $\lambda(c(t)) = \sum_{k=0}^\infty b_k\, t^k$ for some $b_k \in \mathbb{R}$. But

$$\lambda(c(t)) = \sum_{k=0}^{N} \lambda(a_k) t^k + t^N \lambda\left(\sum_{k>N} a_k t^{k-N}\right)$$

and $t \mapsto \sum_{k>N} a_k t^{k-N}$ is still a Mackey converging power series in E. Comparing coefficients we see that $b_k = \lambda(a_k)$ and consequently $\lambda(a_k) r^k$ is bounded for some $r > 0$, a contradiction.

($\Downarrow$). Let $c : \mathbb{R} \to E$ be real analytic. By theorem 1.5 the set

$$\{\frac{1}{k!}\, c^{(k)}(a)\, r_k : a \in K, k \in \mathbb{N}\}$$

is bounded for all compact sets $K \subset \mathbb{R}$ and for all sequences (r_k) with $r_k\, t^k \to 0$ for all $t > 0$. Since c is smooth and bounded linear mappings are smooth ([5], 2.4.4), the function $\lambda \circ c$ is smooth and $(\lambda \circ c)^{(k)}(a) = \lambda(c^{(k)}(a))$. By applying 1.5 we obtain that $\lambda \circ c$ is real analytic. $\square$

1.10. Lemma. *Let (E, E^1) and (E, E^2) be two complete dual pairs with the same underlying vector space E. Then following statements are equivalent:*

(1) *They have the same bounded sets.*
(2) *They have the same smooth curves.*
(3) *They have the same real analytic curves.*

Proof. (1) $\Leftrightarrow$ (2). This was shown in [14].

(1) $\Rightarrow$ (3). This follows from 1.5, which shows that real analyticity is a bornological concept.

(1) $\Leftarrow$ (3). This follows from 1.9. $\square$

1.11. Lemma. *If a cone of linear maps $T_\alpha : (E, E') \to (E_\alpha, E'_\alpha)$ between complete dual pairs generates the bornology on E, then a curve $c : \mathbb{R} \to E$ is C^ω resp. C^∞ provided all the composites $T_\alpha \circ c : \mathbb{R} \to E_\alpha$ are.*

Proof. The statement on the smooth curves is shown in [5]. That on the real analytic curves follows again from the bornological condition of 1.5. □

2. Real analytic mappings

Parts of 2.1 to 2.5 can be found in [2]. For x in any vector space E let x^k denote the element $(x, \dots, x) \in E^k$.

2.1. Lemma (Polarization formulas). *Let $f : E \times \dots \times E \to F$ be an k-linear symmetric mapping between vector spaces. Then we have:*

$$f(x_1, \dots, x_k) = \tfrac{1}{k!} \sum_{\varepsilon_1, \dots, \varepsilon_k = 0}^{1} (-1)^{k - \Sigma\varepsilon_j} f\left((x_0 + \textstyle\sum \varepsilon_j x_j)^k\right). \tag{1}$$

$$f(x^k) = \tfrac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \tbinom{k}{j} f((a + jx)^k). \tag{2}$$

$$f(x^k) = \tfrac{k^k}{k!} \sum_{j=0}^{k} (-1)^{k-j} \tbinom{k}{j} f((a + \tfrac{j}{k} x)^k). \tag{3}$$

$$f(x_1^0 + \lambda x_1^1, \dots, x_k^0 + \lambda x_k^1) = \sum_{\varepsilon_1, \dots, \varepsilon_k = 0}^{1} \lambda^{\Sigma\varepsilon_j} f(x_1^{\varepsilon_1}, \dots, x_k^{\varepsilon_k}). \tag{4}$$

Formula (4) will mainly be used for $\lambda = \sqrt{-1}$ in the passage to the complexification.

Proof. (1). (see [17]). By multilinearity and symmetry the right hand side expands to

$$\sum_{j_0 + \dots + j_k = k} \frac{A_{j_0, \dots, j_k}}{j_0! \cdots j_k!} f(\underbrace{x_0, \dots, x_0}_{j_0}, \dots, \underbrace{x_k, \dots, x_k}_{j_k}),$$

where the coefficients are given by

$$A_{j_0, \dots, j_k} = \sum_{\varepsilon_1, \dots, \varepsilon_k = 0}^{1} (-1)^{k - \Sigma\varepsilon_j} \varepsilon_1^{j_1} \cdots \varepsilon_k^{j_k}.$$

The only nonzero coefficient is $A_{0,1,\dots,1} = 1$.

(2). In formula (1) we put $x_0 = a$ and all $x_j = x$.

(3). In formula (2) we replace a by ka and pull k out of the k-linear expression $f((ka + jx)^k)$.

(4) is obvious. □

2.2. Lemma (Power series). *Let E be a real or complex Fréchet space and let f_k be a k-linear symmetric scalar valued bounded functional on E, for each $k \in \mathbb{N}$. Then the following statements are equivalent:*

(1) $\sum_k f_k(x^k)$ *converges pointwise on an absorbing subset of E.*
(2) $\sum_k f_k(x^k)$ *converges uniformly and absolutely on some neighborhood of 0.*
(3) $\{f_k(x^k) : k \in \mathbb{N}, x \in U\}$ *is bounded for some neighborhood U of zero.*
(4) $\{f_k(x_1, \dots, x_k) : k \in \mathbb{N}, x_j \in U\}$ *is bounded for some neighborhood U of 0.*

If any of these statements are satisfied over the reals, then also for the complexification of the functionals f_k.

Proof. (1) ⇒ (3) The set $A_{K,r} := \{x \in E : |f_k(x^k)| \le Kr^k \text{ for all } k\}$ is closed in E since every bounded multi linear mapping is continuous. The union $\bigcup_{K,r} A_{K,r}$ is E, since the series converges pointwise on an absorbing subset. Since E is Baire there are $K > 0$ and $r > 0$ such that the interior U of $A_{K,r}$ is non void. Let $x_0 \in U$ and let V be an absolutely convex neighborhood of 0 contained in $U - x_0$

From 2.1 (3) we get for all $x \in V$ the following estimate:

$$\begin{aligned} |f(x^k)| &\le \tfrac{k^k}{k!} \sum_{j=0}^{k} \tbinom{k}{j} |f((x_0 + \tfrac{j}{k}x)^k)| \\ &\le \tfrac{k^k}{k!} 2^k K r^k \le K(2re)^k. \end{aligned}$$

Now we replace V by $\frac{1}{2re}V$ and get the result.

(3) ⇒ (4). From 2.1 (1) we get for all $x_j \in U$ the estimate:

$$\begin{aligned} |f(x_1,\dots,x_k)| &\le \tfrac{1}{k!} \sum_{\varepsilon_1,\dots,\varepsilon_k=0}^{1} |f\left((\textstyle\sum \varepsilon_j x_j)^k\right)| \\ &\le \tfrac{1}{k!} \sum_{\varepsilon_1,\dots,\varepsilon_k=0}^{1} (\textstyle\sum \varepsilon_j)^k \, |f\left(\left(\frac{\sum \varepsilon_j x_j}{\sum \varepsilon_j}\right)^k\right)| \\ &\le \tfrac{1}{k!} \sum_{\varepsilon_1,\dots,\varepsilon_k=0}^{1} (\textstyle\sum \varepsilon_j)^k \, C \\ &\le \tfrac{1}{k!} \sum_{j=0}^{k} \tbinom{k}{j} j^k \, C \le C(2e)^k. \end{aligned}$$

Now we replace U by $\frac{1}{2e}U$ and get (4).

(4) ⇒ (2). The series converges on rU uniformly and absolutely for any $0 < r < 1$.

(2) ⇒ (1) is clear.

(4), real case, ⇒ (4), complex case, by 2.1.(4) for $\lambda = \sqrt{-1}$. □

2.3. Theorem (Holomorphic functions on Fréchet spaces). *Let $U \subseteq E$ be open in a complex Fréchet space E. The following statements on $f : U \to \mathbb{C}$ are equivalent:*

1. *f is holomorphic along holomorphic curves.*
2. *f is smooth and the derivative $df(z) : E \to \mathbb{C}$ is $\mathbb{C}$-linear for all $z \in U$.*
3. *f is smooth and is locally given by its pointwise converging Taylor series.*
4. *f is smooth and is locally given by its uniformly and absolutely converging Taylor series.*
5. *f is locally given by a uniformly and absolutely converging power series.*

Proof. (1) ⇔ (2) [15], 2.12.

(1) ⇒ (3). Let $z \in U$ be arbitrary, without loss of generality $z = 0$, and let $b_n := \frac{f^{(n)}(z)}{n!}$ be the n-th Taylor coefficient of f at z. Then $b_n : E^n \to \mathbb{C}$ is symmetric, n-linear and bounded and the series $\sum_{n=0}^{\infty} b_n(v,..,v)t^n$ converges to $f(z + tv)$ for small t. Hence the set of those v for which the series $\sum_{n=0}^{\infty} b_n(v,..,v)$ converges is absorbing. By 2.2, (1) ⇒ (2) it converges on a neighborhood of 0 to $f(z + v)$.

(3) ⇒ (4) follows from 2.2,(2) ⇒ (3).

(4) ⇒ (5) is obvious.

(5) ⇒ (1) is the chain rule for converging power series, which easily can be shown using 2.2, (2) ⇒ (4). □

2.4. Theorem (Real analytic functions on Fréchet spaces). *Let $U \subseteq E$ be open in a real Fréchet space E. The following statements on $f : U \to \mathbb{R}$ are equivalent:*

1. *f is smooth and is real analytic along topologically real analytic curves.*
2. *f is smooth and is real analytic along affine lines.*
3. *f is smooth and is locally given by its pointwise converging Taylor series.*
4. *f is smooth and is locally given by its uniformly and absolutely converging Taylor series.*
5. *f is locally given by a uniformly and absolutely converging power series.*
6. *f extends to a holomorphic mapping $\tilde{f} : \tilde{U} \to \mathbb{C}$ for an open subset $\tilde{U}$ in the complexification $E_{\mathbb{C}}$ with $\tilde{U} \cap E = U$.*

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3). Repeat the proof of 2.3, (1) $\Rightarrow$ (3).

(3) $\Rightarrow$ (4) follows from 2.2,(2) $\Rightarrow$ (3).

(4) $\Rightarrow$ (5) is obvious.

(5) $\Rightarrow$ (6). Locally we can extend converging power series into the complexification by 2.2. Then we take the union $\tilde{U}$ of their domains of definition and use uniqueness to glue $\tilde{f}$ which is holomorphic by 2.3.

(6) $\Rightarrow$ (1). Obviously f is smooth. Any topologically real analytic curve c in E can locally be extended to a holomorphic curve in $E_{\mathbb{C}}$ by 1.3. So $f \circ c$ is real analytic. $\square$

2.5. The assumptions "f is smooth" cannot be dropped in 2.4.1 even in finite dimensions, as shown by the following example, due to [3].

Example. *The mapping $f : \mathbb{R}^2 \to \mathbb{R}$, defined by*

$$f(x, y) := \frac{xy^{n+2}}{x^2 + y^2}$$

is real analytic along real analytic curves, is n-times continuous differentiable but is not smooth and hence not real analytic.

Proof. Take a real analytic curve $t \mapsto (x(t), y(t))$ into $\mathbb{R}^2$. The components can be factored as $x(t) = t^n u(t)$, $y(t) = t^n v(t)$ for some n and real analytic curves u, v with $u(0)^2 + v(0)^2 \neq 0$. The composite $f \circ (x, y)$ is then the function

$$t \mapsto t^n \frac{uv^{n+2}}{u^2 + v^2}(t),$$

which is obviously real analytic near 0. The mapping f is n-times continuous differentiable, since it is real analytic on $\mathbb{R}^2 \backslash \{0\}$ and the directional derivatives of order i

are $(n+1-i)$-homogeneous, hence continuously extendable to $\mathbb{R}^2$. But f cannot be $(n+1)$-times continuous differentiable, otherwise the derivative of order $n+1$ would be constant, and hence f would be a polynomial. □

2.6. Definition (Real analytic mappings). Let (E, E') be a dual pair. Let us denote by $C^\omega(\mathbb{R}, E)$ the *space of all real analytic curves.*

Let $U \subseteq E$ be c^∞-open, and let (F, F') be a second dual pair. A mapping $f : U \to F$ will be called *real analytic* or C^ω for short, if f is real analytic along real analytic curves and is smooth (i.e. is smooth along smooth curves); so $f \circ c \in C^\omega(\mathbb{R}, F)$ for all $c \in C^\omega(\mathbb{R}, E)$ with $c(\mathbb{R}) \subseteq U$ and $f \circ c \in C^\infty(\mathbb{R}, F)$ for all $c \in C^\infty(\mathbb{R}, E)$ with $c(\mathbb{R}) \subseteq U$. Let us denote by $C^\omega(U, F)$ the *space of all real analytic mappings* from U to F.

2.7. Hartogs' Theorem for real analytic mappings. *Let (E, E') and (F, F') be complete dual pairs, let $U \subseteq E$ be c^∞-open, and let $f : U \to F$. Then f is real analytic if and only if f is smooth and $\lambda \circ f$ is real analytic along each affine line in E, for all $\lambda \in F'$.*

Proof. One direction is clear, and by definition 2.6 we may assume that $F = \mathbb{R}$.

Let $c : \mathbb{R} \to U$ be real analytic. We show that $f \circ c$ is real analytic by using lemma 1.5. So let (r_k) be a sequence such that $r_k r_\ell \geq r_{k+\ell}$ and $r_k\, t^k \to 0$ for all $t > 0$ and let $K \subset \mathbb{R}$ be compact. We have to show, that there is an $\varepsilon > 0$ such that the set

$$\Big\{\tfrac{1}{\ell!}(f \circ c)^{(\ell)}(a)\, r_l\, (\tfrac{\varepsilon}{2})^\ell : a \in K, \ell \in \mathbb{N}\Big\}$$

is bounded.

By theorem 1.5 the set

$$\Big\{\tfrac{1}{n!}c^{(n)}(a)\, r_n : n \geq 1, a \in K\Big\}$$

is contained in some bounded absolutely convex subset $B \subseteq E$, such that E_B is a Banach space. Clearly for the inclusion $i_B : E_B \to E$ the function $f \circ i_B$ is smooth and real analytic along affine lines. Since E_B is a Banach space, by 2.4, (2) ⇒ (4) $f \circ i_B$ is locally given by its uniformly and absolutely converging Taylor series. Then by 2.2, (2) ⇒ (4) there is an $\varepsilon > 0$ such that the set

$$\big\{\tfrac{1}{k!} d^k f(c(a))(x_1, \dots, x_k) : k \in \mathbb{N}, x_j \in \varepsilon B, a \in K\big\}$$

is bounded, so is contained in $[-C, C]$ for some $C > 0$.

The Taylor series of $f \circ c$ at a is given by

$$(f \circ c)(a+t) = \sum_{\ell \geq 0} \sum_{k \geq 0} \tfrac{1}{k!} \sum_{\substack{(m_n) \in \mathbb{N}_0^{\mathbb{N}} \\ \sum_n m_n = k \\ \sum_n m_n\, n = \ell}} \frac{k!}{\prod_n m_n!} d^k f(c(a)) \Big(\prod_n (\tfrac{1}{n!} c^{(n)}(a))^{m_n} \Big) t^\ell,$$

where

$$\prod_n x_n^{m_n} := (\underbrace{x_1, \dots, x_1}_{m_1}, \dots, \underbrace{x_n, \dots, x_n}_{m_n}, \dots).$$

This follows easily from composing the Taylor series of f and c and ordering by powers of t. Furthermore we have

$$\sum_{\substack{(m_n) \in \mathbb{N}_0^{\mathbb{N}} \\ \sum_n m_n = k \\ \sum_n m_n\, n = \ell}} \frac{k!}{\prod_n m_n!} = \tbinom{\ell-1}{k-1}$$

by the following argument: It is the ℓ-th Taylor coefficient at 0 of the function

$$\left(\sum_{n \geq 0} t^n - 1 \right)^k = \left(\frac{t}{1-t} \right)^k = t^k \sum_{j=0}^{\infty} \binom{-k}{j} (-t)^j,$$

which turns out to be the binomial coefficient in question.

By the foregoing considerations we may estimate as follows.

$$\begin{aligned}
&\tfrac{1}{\ell!} |(f \circ c)^{(\ell)}(a)|\, r_l\, (\tfrac{\varepsilon}{2})^\ell \leq \\
&\leq \sum_{k \geq 0} \Big| \tfrac{1}{k!} \sum_{\substack{(m_n) \in \mathbb{N}_0^{\mathbb{N}} \\ \sum_n m_n = k \\ \sum_n m_n\, n = \ell}} \frac{k!}{\prod_n m_n!} d^k f(c(a)) \Big(\prod_n (\tfrac{1}{n!} c^{(n)}(a))^{m_n} \Big) \Big|\, r_\ell\, (\tfrac{\varepsilon}{2})^\ell \\
&\leq \sum_{k \geq 0} \Big| \tfrac{1}{k!} \sum_{\substack{(m_n) \in \mathbb{N}_0^{\mathbb{N}} \\ \sum_n m_n = k \\ \sum_n m_n\, n = \ell}} \frac{k!}{\prod_n m_n!} d^k f(c(a)) \Big(\prod_n (\tfrac{1}{n!} c^{(n)}(a)\, r_n\, \varepsilon^n)^{m_n} \Big) \Big| \tfrac{1}{2^\ell} \\
&\leq \sum_{k \geq 0} \tbinom{\ell-1}{k-1}\, C\, \tfrac{1}{2^\ell} = \tfrac{1}{2} C,
\end{aligned}$$

because

$$\sum_{\substack{(m_n)\in\mathbb{N}_0^{\mathbb{N}}\\ \sum_n m_n=k\\ \sum_n m_n\,n=\ell}} \frac{k!}{\prod_n m_n!}\prod_n\big(\tfrac{1}{n!}c^{(n)}(a)\,\varepsilon^n\,r_n\big)^{m_n} \in \tbinom{\ell-1}{k-1}\,(\varepsilon B)^k \subseteq (E_B)^k. \quad \square$$

2.8. Corollary. *Let (E,E') and (F,F') be complete dual pairs, let $U\subseteq E$ be c^∞-open, and let $f:U\to F$. Then f is real analytic if and only if f is smooth and $\lambda\circ f\circ c$ is real analytic for every periodic (topologically) real analytic curve $c:\mathbb{R}\to U\subseteq E$ and all $\lambda\in F'$.*

Proof. By 2.7 f is real analytic if and only if f is smooth and $\lambda\circ f$ is real analytic along topologically real analytic curves $c:\mathbb{R}\to E$. Let $h:\mathbb{R}\to\mathbb{R}$ be defined by $h(t)=t_0+\varepsilon\cdot\sin t$. Then $c\circ h:\mathbb{R}\to\mathbb{R}\to U$ is a (topologically) real analytic, periodic function with period 2π, provided c is (topologically) real analytic. If $c(t_0)\in U$ we can choose $\varepsilon>0$ such that $h(\mathbb{R})\subseteq c^{-1}(U)$. Since *sin* is locally around 0 invertible, real analyticity of $\lambda\circ f\circ c\circ h$ implies that $\lambda\circ f\circ c$ is real analytic near t_0. Hence the proof is completed. $\square$

2.9. Corollary (Reduction to Banach spaces). *Let (E,E') be a complete dual pair, let $U\subset E$ be c^∞-open, and let $f:U\to\mathbb{R}$ be a mapping. Then f is real analytic if and only if the restriction $f:E_B\supset U\cap E_B\to\mathbb{R}$ is real analytic for all bounded absolutely convex subsets B of E.*

So any result valid on Banach spaces can be translated into a result valid on complete dual pairs.

Proof. By theorem 2.7 it suffices to check f along bornologically real analytic curves. These factor by definition locally to real analytic curves into some E_B. $\square$

2.10. Corollary. *Let U be a c^∞-open subset in a complete dual pair (E,E') and let $f:U\to\mathbb{R}$ be real analytic. Then for every bounded B there is some $r_B>0$ such that the Taylor series*

$$y\mapsto\sum \tfrac{1}{k!}d^kf(x)(y^k)$$

converges to $f(x+y)$ uniformly and absolutely on r_BB.

Proof. Use 2.9 and 2.4.(4). $\square$

2.11. Scalar analytic functions on convenient vector spaces E are in general not germs of holomorphic functions from $E_{\mathbb{C}}$ to $\mathbb{C}$:

Example. *Let* $f_k : \mathbb{R} \to \mathbb{R}$ *be real analytic functions with radius of convergence at zero converging to 0 for* $k \to \infty$. *Let* $f : \mathbb{R}^{(\mathbb{N})} \to \mathbb{R}$ *be the mapping defined on the countable sum* $\mathbb{R}^{(\mathbb{N})}$ *of the reals by* $f(x_0, x_1, ...) := \sum_{k=1}^{\infty} x_k f_k(x_0)$. *Then* f *is real analytic, but there is no complex valued holomorphic mapping* $\tilde{f}$ *on some neighborhood of 0 in* $\mathbb{C}^{(\mathbb{N})}$ *which extends* f, *and the Taylor series of* f *is not pointwise convergent on any* c^∞*-open neighborhood of 0.*

Proof. **Claim.** f is real analytic.
Since the limit $\mathbb{R}^{(\mathbb{N})} = \varinjlim_n \mathbb{R}^n$ is regular, every smooth curve (and hence every real analytic curve) in $\mathbb{R}^{(\mathbb{N})}$ is locally smooth (resp. real analytic) into $\mathbb{R}^n$ for some n. Hence $f \circ c$ is locally just a finite sum of smooth (resp. real analytic) functions and is therefore smooth (resp. real analytic).

Claim. f has no holomorphic extension.
Suppose there exists some holomorphic extension $\tilde{f} : U \to \mathbb{C}$, where $U \subseteq \mathbb{C}^{(\mathbb{N})}$ is c^∞-open neighborhood of 0, and is therefore open in the locally convex Silva topology by [5], 6.1.4.ii. Then U is even open in the box-topology [10], 4.1.4, i.e. there exist $\varepsilon_k > 0$ for all k, such that $\{(z_k) \in \mathbb{C}^{(\mathbb{N})} : |z_k| \leq \varepsilon_k \text{ for all } k\} \subseteq U$. Let U_0 be the open disk in $\mathbb{C}$ with radius ε_0 and let $\tilde{f}_k : U_0 \to \mathbb{C}$ be defined by $\tilde{f}_k(z) := \tilde{f}(z, 0, ..., 0, \varepsilon_k, 0, ...)\frac{1}{\varepsilon_k}$, where ε_k is inserted instead of the variable x_k. Obviously $\tilde{f}_k$ is an extension of f_k, which is impossible, since the radius of convergence of f_k is less than ε_0 for k sufficiently large.

Claim. The Taylor series does not converge.
If the Taylor series would be pointwise convergent on some U, then the previous arguments would show that the radii of convergence of the f_k were bounded from below. □

3. Function spaces in finite dimensions

3.1. Spaces of holomorphic functions. For a complex manifold N (always assumed to be separable) let $\mathcal{H}(N, \mathbb{C})$ be the *space of all holomorphic* functions on N with the topology of uniform convergence on compact subsets of N.

Let $\mathcal{H}_b(N, \mathbb{C})$ denote the Banach space of bounded holomorphic functions on N equipped with the supremum norm.

For any open subset W of N let $\mathcal{H}_{bc}(W \subseteq N, \mathbb{C})$ be the closed subspace of $\mathcal{H}_b(W, \mathbb{C})$ of all holomorphic functions on W which extend to continuous functions on the closure $\overline{W}$.

For a poly-radius $r = (r_1, \dots, r_n)$ with $r_i > 0$ and for $1 \leq p \leq \infty$ let ℓ^p_r denote the real Banach space $\{\, x \in \mathbb{R}^{\mathbb{N}^n} : \|(x_\alpha r^\alpha)_{\alpha \in \mathbb{N}^n}\|_p < \infty \,\}$.

3.2. Theorem (Structure of $\mathcal{H}(N,\mathbb{C})$ for complex manifolds N). *The space $\mathcal{H}(N,\mathbb{C})$ of all holomorphic functions on N with the topology of uniform convergence on compact subsets of N is a (strongly) nuclear Fréchet space and embeds as a closed subspace into $C^\infty(N,\mathbb{R})^2$.*

Proof. By taking a countable covering of N with compact sets, one obtains a countable neighborhood basis of 0 in $\mathcal{H}(N,\mathbb{C})$. Hence $\mathcal{H}(N,\mathbb{C})$ is metrizable.

That $\mathcal{H}(N,\mathbb{C})$ is complete, and hence a Fréchet space, follows since the limit of a sequence of holomorphic functions with respect to the topology of uniform convergence on compact sets is again holomorphic.

The vector space $\mathcal{H}(N,\mathbb{C})$ is a subspace of $C^\infty(N,\mathbb{R}^2) = C^\infty(N,\mathbb{R})^2$ since a function $N \to \mathbb{C}$ is holomorphic if and only if it is smooth and the derivative at every point is $\mathbb{C}$-linear. It is a closed subspace, since it is described by the continuous linear equations $df(x)(\sqrt{-1}\cdot v) = \sqrt{-1}\cdot df(x)(v)$. Obviously the identity from $\mathcal{H}(N,\mathbb{C})$ with the subspace topology to $\mathcal{H}(N,\mathbb{C})$ is continuous, hence by the open mapping theorem [10], 5.5.2, for Fréchet spaces it is an isomorphism.

That $\mathcal{H}(N,\mathbb{C})$ is nuclear and unlike $C^\infty(N,\mathbb{R})$ even strongly nuclear can be shown as follows. For N equal to the open unit disk $\mathbb{D} \subseteq \mathbb{C}$ this result can be found in [10], 21.8.3.b. More generally for $N = \mathbb{D}^n$ one has that $\mathcal{H}(N,\mathbb{C}) = \bigcap_{0<r<1} \ell^1_{(r,..,r)} \otimes \mathbb{C}$ as vector spaces. The identity from the right to the left is obviously continuous, if the intersection is supplied with the projective limit topology induced from the Banach spaces $\ell^1_{(r,..,r)} \otimes \mathbb{C}$, a Fréchet topology. Hence again by the open mapping theorem it is an isomorphism. Using now the Grothendieck-Pietsch criterion, cf. [10], 21.8.2, one concludes that $\mathcal{H}(\mathbb{D}^n,\mathbb{C})$ is strongly nuclear, see also [30], p. 530. For an arbitrary N the space $\mathcal{H}(N,\mathbb{C})$ carries the initial topology induced by the linear mappings $u_* : \mathcal{H}(N,\mathbb{C}) \to \mathcal{H}(u(U),\mathbb{C})$ for all charts (u,U) of N, for which we may assume $u(U) = \mathbb{D}^n$, and hence by the stability properties of strongly nuclear spaces, cf. [10], 21.1.7, $\mathcal{H}(N,\mathbb{C})$ is strongly nuclear. □

3.3 Spaces of germs of holomorphic functions. For a subset $A \subseteq N$ let $\mathcal{H}(A \subseteq N,\mathbb{C})$ be the space of *germs along A of holomorphic functions* $W \to \mathbb{C}$ for open sets W in N containing A. We equip $\mathcal{H}(A \subseteq N,\mathbb{C})$ with the locally convex topology induced by the inductive cone $\mathcal{H}(W,\mathbb{C}) \to \mathcal{H}(A \subseteq N,\mathbb{C})$ for all W. This is Hausdorff, since iterated derivatives at points in A are continuous functionals and separate points. In particular $\mathcal{H}(W \subseteq N,\mathbb{C}) = \mathcal{H}(W,\mathbb{C})$ for W open in N. For $A_1 \subset A_2 \subset N$ the "restriction" mappings $\mathcal{H}(A_2 \subset N,\mathbb{C}) \to \mathcal{H}(A_1 \subset N,\mathbb{C})$ are continuous.

The structure of $\mathcal{H}(A \subseteq S^2,\mathbb{C})$, where $A \subseteq S^2$ is a subset of the Riemannian sphere, has been studied by [29], [26], [31], [12], and [8].

3.4. Theorem (Structure of $\mathcal{H}(K \subseteq N, \mathbb{C})$ for compact subsets K of complex manifolds N). *The following inductive cones are cofinal to each other.*

$$\mathcal{H}(K \subseteq N, \mathbb{C}) \leftarrow \{\mathcal{H}(W, \mathbb{C}), N \supseteq W \supseteq K\}$$
$$\mathcal{H}(K \subseteq N, \mathbb{C}) \leftarrow \{\mathcal{H}_b(W, \mathbb{C}), N \supseteq W \supseteq K\}$$
$$\mathcal{H}(K \subseteq N, \mathbb{C}) \leftarrow \{\mathcal{H}_{bc}(W \subseteq N, \mathbb{C}), N \supseteq W \supseteq K\}$$

If $K = \{z\}$ these inductive cones and the following ones for $1 \leq p \leq \infty$ are cofinal to each other.

$$\mathcal{H}(\{z\} \subseteq N, \mathbb{C}) \leftarrow \{\ell^p_r \otimes \mathbb{C},\ r \in \mathbb{R}^n_+\}$$

So all inductive limit topologies coincide. Furthermore, the space $\mathcal{H}(K \subseteq N, \mathbb{C})$ is a Silva space, i.e. a countable inductive limit of Banach spaces, where the connecting mappings between the steps are compact, i.e. mapping bounded sets to relatively compact ones. The connecting mappings are even strongly nuclear. In particular, the limit is regular, i.e. every bounded subset is contained and bounded in some step, and $\mathcal{H}(K \subseteq N, \mathbb{C})$ is complete and (ultra-)bornological (hence a convenient vector space), webbed, strongly nuclear, reflexive and its dual is a strongly nuclear Fréchet space. It is however not a Baire space.

Proof. Let $K \subseteq V \subseteq \overline{V} \subseteq W \subseteq N$, where W and V are open and $\overline{V}$ is compact. Then the obvious mappings

$$\mathcal{H}_{bc}(W \subseteq N, \mathbb{C}) \to \mathcal{H}_b(W, \mathbb{C}) \to \mathcal{H}(W, \mathbb{C}) \to \mathcal{H}_{bc}(V \subseteq N, \mathbb{C})$$

are continuous. This implies the first cofinality assertion. For $q \leq p$ and $s < r$ the obvious maps $\ell^q_r \to \ell^p_r$, $\ell^\infty_r \to \ell^1_s$, and $\ell^1_r \otimes \mathbb{C} \to \mathcal{H}_b(\{w \in \mathbb{C}^n : |w_i - z_i| < r_i\}, \mathbb{C}) \to \ell^\infty_s \otimes \mathbb{C}$ are continuous, by the Cauchy inequalities. So the remaining cofinality assertion follows.

Let us show next that the connecting mapping $\mathcal{H}_b(W, \mathbb{C}) \to \mathcal{H}_b(V, \mathbb{C})$ is strongly nuclear (hence nuclear and compact). Since the restriction mapping from $E := \mathcal{H}(W, \mathbb{C})$ to $\mathcal{H}_b(V, \mathbb{C})$ is continuous, it factors over $E \to \widetilde{E_{(U)}}$ for some 0-neighborhood U in E, where $\widetilde{E_{(U)}}$ is the completed quotient of E with the Minkowski functional of U as norm, see [10], 6.8. Since E is strongly nuclear by 3.2, there exists by definition some larger 0-neighborhood U' in E such that the natural mapping $\widetilde{E_{(U')}} \to \widetilde{E_{(U)}}$ is strongly nuclear. So the claimed connecting mapping is strongly nuclear, since it can be factorized as

$$\mathcal{H}_b(W, \mathbb{C}) \to \mathcal{H}(W, \mathbb{C}) = E \to \widetilde{E_{(U')}} \to \widetilde{E_{(U)}} \to \mathcal{H}_b(V, \mathbb{C})$$

That a Silva space is regular and complete, can be found in [4], 7.4 and 7.5.

That $\mathcal{H}(K \subseteq N, \mathbb{C})$ is ultra-bornological, webbed and strongly nuclear follows from the permanence properties of ultra-bornological spaces, [10], 13.2.5, of webbed spaces [10], 5.3.3 and of strongly nuclear spaces [10], 21.1.7.

Furthermore, $\mathcal{H}(K \subseteq N, \mathbb{C})$ is reflexive and its strong dual is a Fréchet space, since it is a Silva-space, cf. [10], 12.5.9 and p.270. The dual is even strongly nuclear, since $\mathcal{H}(K \subseteq N, \mathbb{C})$ is a nuclear Silva-space, cf. [10], 21.8.6.

The space $\mathcal{H}(K \subseteq N, \mathbb{C})$ has however not the Baire property, since it is webbed but not metrizable, cf. [10], 5.4.4. If it were metrizable then it would be of finite dimension, by [4], 7.7. This is not the case. □

Completeness of $\mathcal{H}(K \subseteq \mathbb{C}^n, \mathbb{C})$ was shown in [31], théorème II, and regularity of the inductive limit $\mathcal{H}(K \subseteq \mathbb{C}, \mathbb{C})$ can be found in [12], Satz 12.

3.5. Lemma. *For a closed subset $A \subseteq \mathbb{C}$ the spaces $\mathcal{H}(A \subseteq S^2, \mathbb{C})$ and the space $\mathcal{H}_\infty(S^2 \setminus A \subseteq S^2, \mathbb{C})$ of all germs vanishing at ∞ are strongly dual to each other.*

Proof. This is due to [12], Satz 12 and has been generalized by [8], théorème 2 bis, to arbitrary subsets $A \subseteq S^2$. □

Compare also the modern theory of hyperfunctions, cf. [11].

3.6. Theorem (Structure of $\mathcal{H}(A \subseteq N, \mathbb{C})$ for closed subsets A of complex manifolds N). *The inductive cone*

$$\mathcal{H}(A \subseteq N, \mathbb{C}) \leftarrow \{\, \mathcal{H}(W, \mathbb{C}) : A \subseteq W \underset{\text{open}}{\subseteq} N\}$$

is regular, i.e. every bounded set is contained and bounded in some step.

The projective cone

$$\mathcal{H}(A \subseteq N, \mathbb{C}) \to \{\, \mathcal{H}(K \subseteq N, \mathbb{C}) : K \text{ compact in } A\}$$

generates the bornology of $\mathcal{H}(A \subseteq N, \mathbb{C})$.

The space $\mathcal{H}(A \subseteq N, \mathbb{C})$ is Montel (hence quasi-complete and reflexive), and ultra-bornological (hence a convenient vector space). Furthermore it is webbed and conuclear.

Proof. Compare also with the proof of the more general theorem 7.3.

We choose a continuous function $f : N \to \mathbb{R}$ which is positive and proper. Then $(f^{-1}([n, n+1]))_{n \in \mathbb{N}_0}$ is an exhaustion of N by compact subsets and

$$(K_n := A \cap f^{-1}([n, n+1]))$$

is a compact exhaustion of A.

Let $\mathcal{B} \subseteq \mathcal{H}(A \subseteq N, \mathbb{C})$ be bounded. Then $\mathcal{B}|K$ is also bounded in $\mathcal{H}(K \subseteq N, \mathbb{C})$ for each compact subset K of A. Since the cone

$$\{\mathcal{H}(W, \mathbb{C}) : K \subseteq W \underset{\text{open}}{\subseteq} N\} \to \mathcal{H}(K \subseteq N, \mathbb{C})$$

is regular by 3.4, there exist open subsets W_K of N containing K such that $\mathcal{B}|K$ is contained (so that the extension of each germ is unique) and bounded in $\mathcal{H}(W_K, \mathbb{C})$. In particular we choose $W_{K_n \cap K_{n+1}} \subseteq W_{K_n} \cap W_{K_{n+1}} \cap f^{-1}((n, n+2))$. Then we put

$$W := \bigcup_n (W_{K_n} \cap f^{-1}((n, n+1))) \cup \bigcup_n W_{K_n \cap K_{n+1}}.$$

It is easily checked that W is open in N, contains A, and that each germ in $\mathcal{B}$ has a unique extension to W. $\mathcal{B}$ is bounded in $\mathcal{H}(W, \mathbb{C})$ if it is uniformly bounded on each compact subset K of W. Each K is covered by finitely many W_{K_n} and $\mathcal{B}|K_n$ is bounded in $\mathcal{H}(W_{K_n}, \mathbb{C})$, so $\mathcal{B}$ is bounded as required.

The space $\mathcal{H}(A \subseteq N, \mathbb{C})$ is ultra-bornological, Montel and in particular quasi-complete, and conuclear, as regular inductive limit of the nuclear Fréchet spaces $\mathcal{H}(W, \mathbb{C})$.

And it is webbed because it is the (ultra-)bornologification of the countable projective limit of webbed spaces $\mathcal{H}(K \subseteq N, \mathbb{C})$, cf. [10], 13.3.3 + 5.3.3. □

3.7. Lemma. *Let A be closed in $\mathbb{C}$. Then the dual generated by the projective cone*

$$\mathcal{H}(A \subseteq \mathbb{C}, \mathbb{C}) \to \{ \mathcal{H}(K \subset \mathbb{C}, \mathbb{C}),\ K \text{ compact in } A \}$$

is just the topological dual of $\mathcal{H}(A \subseteq \mathbb{C}, \mathbb{C})$.

Proof. The induced topology is obviously coarser than the given one. So let λ be a continuous linear functional on $\mathcal{H}(A \subseteq \mathbb{C}, \mathbb{C})$. Then we have $\lambda \in \mathcal{H}_\infty(S^2 \setminus A \subseteq S^2, \mathbb{C})$ by 3.5. Hence $\lambda \in \mathcal{H}(U, \mathbb{C})$ for some open neighborhood U of $S^2 \setminus A$, so again by 3.5 λ is a continuous functional on $\mathcal{H}(K \subset S^2, \mathbb{C})$, where $K = S^2 \setminus U$ is compact in A. So λ is continuous for the induced topology. □

Problem. *Does this cone generate even the topology of $\mathcal{H}(A \subseteq \mathbb{C}, \mathbb{C})$? This would imply that the bornological topology on $\mathcal{H}(A \subseteq \mathbb{C}, \mathbb{C})$ is complete and nuclear.*

3.8. Lemma (Structure of $\mathcal{H}(A \subseteq N, \mathbb{C})$ for smooth closed submanifolds A of complex manifolds N). *The projective cone*

$$\mathcal{H}(A \subseteq N, \mathbb{C}) \to \{\ \mathcal{H}(\{z\} \subseteq N, \mathbb{C}) : z \in A\}$$

generates the bornology.

Proof. Let $\mathcal{B} \subseteq \mathcal{H}(A \subseteq N, \mathbb{C})$ be such that the set $\mathcal{B}$ is bounded in $\mathcal{H}(\{z\} \subseteq N, \mathbb{C})$ for all $z \in A$. By the regularity of the inductive cone $\mathcal{H}(\{0\} \subseteq \mathbb{C}^n, \mathbb{C}) \leftarrow \mathcal{H}(W, \mathbb{C})$ we find arbitrary small open neighborhoods W_z such that the set $\mathcal{B}_z$ of the germs at z of all germs in $\mathcal{B}$ is contained and bounded in $\mathcal{H}(W_z, \mathbb{C})$.

Now choose a tubular neighborhood $p : U \to A$ of A in N. We may assume that W_z is contained in U, has fibers which are star shaped with respect to the zero-section and the intersection with A is connected. The union W of all the W_z, is therefore an open subset of U containing A. And it remains to show that the germs in $\mathcal{B}$ extend to W. For this it is enough to show that the extensions of the germs at z_1 and z_2 agree on the intersection of W_{z_1} with W_{z_2}. So let w be a point in the intersection. It can be radially connected with the base point $p(w)$, which itself can be connected by curves in A with z_1 and z_2. Hence the extension of both germs to $p(w)$ coincide with the original germ, and hence their extensions to w are equal.

That $\mathcal{B}$ is bounded in $\mathcal{H}(W, \mathbb{C})$, follows immediately since every compact subset $K \subseteq W$ can be covered by finitely many W_z. □

3.9. The following example shows that 3.8 fails to be true for general closed subsets $A \subseteq N$.

Example. *Let $A := \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. Then A is compact in $\mathbb{C}$ but the projective cone $\mathcal{H}(A \subseteq \mathbb{C}, \mathbb{C}) \to \mathcal{H}(\{z\} \subseteq \mathbb{C}, \mathbb{C})$ $(z \in A)$ does not generate the bornology.*

Proof. Let $\mathcal{B} \subseteq \mathcal{H}(A \subseteq \mathbb{C}, \mathbb{C})$ be the set of germs of the following locally constant functions $f_n : \{x + iy \in \mathbb{C} : x \neq r_n\} \to \mathbb{C}$, with $f_n(x + iy)$ being equal to 0 for $x < r_n$ and being equal to 1 for $x > r_n$, where $r_n := \frac{2}{2n+1}$, for $n \in \mathbb{N}$. Then $\mathcal{B} \subseteq \mathcal{H}(A \subseteq \mathbb{C}, \mathbb{C})$ is not bounded, otherwise there would exist a neighborhood W of A such that the germ of f_n extends to a holomorphic mapping on W for all n. Since every f_n is 0 on some neighborhood of 0, these extensions have to be zero on the component of W containing 0, which is not possible, since $f_n(\frac{1}{n}) = 1$.

But on the other hand the set $\mathcal{B}_z \subseteq \mathcal{H}(\{z\} \subseteq \mathbb{C}, \mathbb{C})$ of germs at z of all germs in $\mathcal{B}$ is bounded, since it contains only the germs of the constant functions 0 and 1. □

3.10. Spaces of germs of real-analytic functions.

Let M be a real analytic finite dimensional manifold. If $f : M \to M'$ is a mapping between two such manifolds, then f is real analytic if and only if f maps smooth curves into smooth ones and real analytic curves into real analytic ones, by 2.4.

For each real analytic manifold M of real dimension m there is a *complex manifold* $M_{\mathbb{C}}$ of complex dimension m containing M as a real analytic closed submanifold, whose germ along M is unique ([32], Proposition 1), and which can be chosen even to be a Stein manifold, see [7], section 3. The complex charts are just extensions of the real analytic charts of an atlas of M.

Real analytic mappings $f : M \to M'$ are the germs along M of holomorphic mappings $W \to M'_{\mathbb{C}}$ for open neighborhoods W of M in $M_{\mathbb{C}}$.

Let $C^\omega(M, F)$ be the *space of real analytic functions* $f : M \to F$, for any convenient vector space F, and let $\mathcal{H}(M, \mathbb{C}) := \mathcal{H}(M \subseteq M_{\mathbb{C}}, \mathbb{C})$. Furthermore, for a subset $A \subseteq M$ let $C^\omega(A \subseteq M, \mathbb{R})$ denotes the *space of germs of real analytic functions* defined near A.

3.11. Lemma. *For any subset A of M the complexification of the real vector space $C^\omega(A \subseteq M, \mathbb{R})$ is the complex vector space $\mathcal{H}(A \subseteq M_{\mathbb{C}}, \mathbb{C})$.*

Definition. For any $A \subseteq M$ of a real analytic manifold M we will topologize $C^\omega(A \subseteq M, \mathbb{R})$ as subspace of $\mathcal{H}(A \subseteq M_{\mathbb{C}}, \mathbb{C})$, in fact as the real part of it.

Proof. Let $f, g \in C^\omega(A \subseteq M, \mathbb{R})$. They are germs of real analytic mappings defined on some open neighborhood of A in M. Inserting complex numbers into the locally convergent Taylor series in local coordinates shows, that f and g can be considered as holomorphic mappings from some neighborhood W of A in $M_{\mathbb{C}}$, which have real values if restricted to $W \cap M$. The mapping $h := f + ig : W \to \mathbb{C}$ gives then an element of $\mathcal{H}(A \subseteq M_{\mathbb{C}}, \mathbb{C})$.

Conversely let $h \in \mathcal{H}(A \subseteq M_{\mathbb{C}}, \mathbb{C})$. Then h is the germ of a holomorphic mapping $\widetilde{h} : W \to \mathbb{C}$ for some open neighborhood W of A in $M_{\mathbb{C}}$. The decomposition of h into real and imaginary part $f = \frac{1}{2}(h + \bar{h})$ and $g = \frac{1}{2}(h - \bar{h})$, which are real analytic maps if restricted to $W \cap M$, gives elements of $C^\omega(A \subseteq M, \mathbb{R})$.

That these correspondences are inverse to each other follows from the fact that a holomorphic germ is determined by its restriction to a germ of mappings $M \supseteq A \to \mathbb{C}$. $\square$

3.12. Lemma. *The inclusion $C^\omega(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ is continuous.*

Proof. Consider the following diagram, where W is an open neighborhood of M in

$M_{\mathbb{C}}$.

$$\begin{array}{ccc} C^\omega(M,\mathbb{R}) & \xrightarrow{\text{inclusion}} & C^\infty(M,\mathbb{R}) \\ {\scriptstyle\text{direct summand}}\downarrow & & \downarrow{\scriptstyle\text{direct summand}} \\ \mathcal{H}(M\subseteq M_{\mathbb{C}},\mathbb{C}) & \xrightarrow{\text{inclusion}} & C^\infty(M,\mathbb{R}^2) \\ {\scriptstyle\text{restriction}}\uparrow & & \uparrow{\scriptstyle\text{restriction}} \\ \mathcal{H}(W,\mathbb{C}) & \xrightarrow{\text{inclusion}} & C^\infty(W,\mathbb{R}^2) \end{array} \quad \square$$

3.13. Theorem (Structure of $C^\omega(A\subseteq M,\mathbb{R})$ for closed subsets A of real analytic manifolds M). *The inductive cone*

$$C^\omega(A\subseteq M,\mathbb{R})\leftarrow\{\,C^\omega(W,\mathbb{R}):A\subseteq W\underset{\text{open}}{\subseteq}M\}$$

is regular, i.e. every bounded set is contained and bounded in some step.

The projective cone

$$C^\omega(A\subseteq M,\mathbb{R})\to\{\,C^\omega(K\subseteq M,\mathbb{R}):K\ \text{compact in}\ A\}$$

generates the bornology of $C^\omega(A\subseteq M,\mathbb{R})$.

If A is even a smooth submanifold, then the following projective cone also generates the bornology.

$$C^\omega(A\subseteq M,\mathbb{R})\to\{\,C^\omega(\{x\}\subseteq M,\mathbb{R}):x\in A\}$$

The space $C^\omega(\{0\}\subseteq\mathbb{R}^m,\mathbb{R})$ is also the regular inductive limit of the spaces $\ell^p_r(r\in\mathbb{R}^m_+)$ for all $1\le p\le\infty$.

For general closed $A\subseteq N$ the space $C^\omega(A\subseteq M,\mathbb{R})$ is Montel (hence quasi-complete and reflexive), and ultra-bornological (hence a convenient vector space). It is also webbed and conuclear. If A is compact then it is even a strongly nuclear Silva space and its dual is a strongly nuclear Fréchet space. It is however not a Baire space.

Proof. This follows using 3.11 from 3.4, 3.6, and 3.8 by passing to the real parts and from the fact that all properties are inherited by complemented subspaces as $C^\omega(A\subseteq M,\mathbb{R})$ of $\mathcal{H}(A\subseteq M_{\mathbb{C}},\mathbb{C})$. $\square$

3.14 Corollary. *A subset $\mathcal{B} \subseteq C^\omega(\{0\} \subseteq \mathbb{R}^m, \mathbb{R})$ is bounded if and only if $\mathcal{B}_0^{(\alpha)} := \{f^{(\alpha)}(0) : f \in \mathcal{B}\}$ is bounded in $\mathbb{R}$ for all $\alpha \in \mathbb{N}_0^m$ and the poly-radius of convergence for $f \in \mathbb{B}$ is bounded from below by some $r_0 \in \mathbb{N}_0^m$ (or equivalently there exists an $r > 0$ such that $\{\frac{f^{(\alpha)}}{\alpha!} r^{|\alpha|} : f \in \mathbb{B}, \alpha \in \mathbb{N}_0^m\}$ is bounded in $\mathbb{R}$).*

Proof. The space $C^\omega(\{0\} \subseteq \mathbb{R}^m, \mathbb{R})$ is the regular inductive limit of the spaces $\ell^p(r \in \mathbb{R}_+^m)$ for p equal to 1 or ∞ by 3.13. Hence $\mathcal{B}$ is bounded if and only if it is contained and bounded in ℓ_r^p for some $r \in \mathbb{R}_+^m$. This shows the equivalence with the first condition using $p = 1$ and the equivalence with the second condition using $p = \infty$. □

4. A uniform boundedness principle

4.1. Lemma. *Let (E, E') be a dual pairing and let $\mathcal{S}$ be a point separating set of bounded linear mappings with common domain (E, E'). Then the following conditions are equivalent.*

1. *If F is a Banach space (or even a complete dual pairing (F, F')) and $f : F \to E$ is linear and $\lambda \circ f$ is bounded for all $\lambda \in \mathcal{S}$, then f is bounded.*
2. *If $B \subseteq E$ is absolutely convex such that $\lambda(B)$ is bounded for all $\lambda \in \mathcal{S}$ and the normed space E_B generated by B is complete, then B is bounded in E.*
3. *Let (b_n) be an unbounded sequence in E with $\lambda(b_n)$ bounded for all $\lambda \in \mathcal{S}$, then there is some $(t_n) \in \ell^1$ such that $\sum t_n b_n$ does not converge in E for the weak topology induced by $\mathcal{S}$.*

Definition. We say that (E, E') satisfies the *uniform $\mathcal{S}$-boundedness principle* if these equivalent conditions are satisfied.

Proof. (1) ⇒ (3) : Suppose that (3) is not satisfied. So let (b_n) be an unbounded sequence in E such that $\lambda(b_n)$ is bounded for all $\lambda \in \mathcal{S}$, and such that for all $(t_n) \in \ell^1$ the series $\sum t_n b_n$ converges in E for the weak topology induced by $\mathcal{S}$. We define a linear mapping $f : \ell^1 \to E$ by $f(t_n) = \sum t_n b_n$. It is easy to check that $\lambda \circ f$ is bounded, hence by (1) the image of the closed unit ball, which contains all b_n, is bounded. Contradiction.

(3) ⇒ (2): Let $B \subseteq E$ be absolutely convex such that $\lambda(B)$ is bounded for all $\lambda \in \mathcal{S}$ and that the normed space E_B generated by B is complete, and suppose that B is unbounded. Then B contains an unbounded sequence (b_n), so by (3) there is some $(t_n) \in \ell^1$ such that $\sum t_n b_n$ does not converge in E for the weak topology induced by $\mathcal{S}$. But $\sum t_n b_n$ is easily seen to be a Cauchy sequence in E_B and thus converges even bornologically, a contradiction.

(2) ⇒ (1): Let the bornology of F be complete, and let $f : F \to E$ be linear such that $\lambda \circ f$ is bounded for all $\lambda \in \mathcal{S}$. It suffices to show that $f(B)$, the image of an

absolutely convex bounded set B in F with F_B complete, is bounded. Then $\lambda(f(B))$ is bounded for all $\lambda \in \mathcal{S}$, the normed space $E_{f(B)}$ is a quotient of F_B, hence complete. By (2) the set $f(B)$ is bounded. □

4.2. Lemma. *A complete dual pair (E, E') satisfies the uniform $\mathcal{S}$-boundedness principle for each point separating set $\mathcal{S}$ of bounded linear mappings on E if and only if there exists no strictly weaker ultrabornological topology than the natural bornological topology of (E, E').*

Proof. ($\Rightarrow$) Let τ be an ultrabornological topology on E which is strictly weaker than the natural bornological topology. Since every ultra-bornological space is an inductive limit of Banach spaces, cf. [10], 13.1.2, there exists a Banach space F and a continuous linear mapping $f : F \to (E, \tau)$ which is not continuous into E. Let $\mathcal{S} = \{Id : E \to (E, \tau)\}$. Now f does not satisfy 4.1.(1).

($\Leftarrow$) If $\mathcal{S}$ is a point separating set of bounded linear mappings, the ultrabornological topology given by the inductive limit of the spaces E_B with B satisfying 4.1.(2) equals the natural bornological topology of (E, E'). Hence 4.1.(2) is satisfied. □

4.3. Lemma. *Let $\mathcal{F}$ be a set of bounded linear mappings $f : E \to E_f$ between dual pairings, let $\mathcal{S}_f$ be a point separating set of bounded linear mappings on E_f for every $f \in \mathcal{F}$, and let $\mathcal{S} := \bigcup_{f\in\mathcal{F}} f^*(\mathcal{S}_f) = \{g \circ f : f \in \mathcal{F}, g \in \mathcal{S}_f\}$. If $\mathcal{F}$ generates the bornology and E_f satisfies the uniform $\mathcal{S}_f$-boundedness principle for all $f \in \mathcal{F}$, then E satisfies the uniform $\mathcal{S}$-boundedness principle.*

Proof. We check the condition (1) of 4.1. So assume $h : F \to E$ is a linear mapping for which $g \circ f \circ h$ is bounded for all $f \in \mathcal{F}$ and $g \in \mathcal{S}_f$. Then $f \circ h$ is bounded by the uniform $\mathcal{S}_f$- boundedness principle for E_f. Consequently h is bounded since $\mathcal{F}$ generates the bornology of E. □

4.4. Theorem. *A locally convex space which is webbed satisfies the uniform $\mathcal{S}$-boundedness principle for any point separating set of bounded linear functionals.*

Proof. Since the bornologification of a webbed space is webbed, cf. [10], 13.3.3 and 13.3.1, we may assume that E is bornological, and hence that every bounded linear functional is continuous, cf. [10], 13.3.1. Now the closed graph principle, cf. [10], 56.4.1, applies to any mapping satisfying the assumptions of 4.1.1. □

4.5. Theorem (Holomorphic uniform boundedness principle).
For any closed subset $A \subseteq N$ of a complex manifold N the locally convex space $\mathcal{H}(A \subseteq N, \mathbb{C})$ satisfies the uniform $\mathcal{S}$-boundedness principle for every point separating set $\mathcal{S}$ of bounded linear functionals.

Proof. This is a immediate consequence of 4.4 and 3.6. □

Direct proof of a particular case. We prove the theorem for a closed smooth submanifold $A \subseteq \mathbb{C}$ and the set $\mathcal{S}$ of all iterated derivatives at points in A.

Let us suppose first that A is the point 0. We will show that condition 4.1.3 is satisfied. Let (b_n) be an unbounded sequence in $\mathcal{H}(\{0\}, \mathbb{C})$ such that each Taylor coefficient $b_{n,k} = \frac{1}{k!}\, b_n^{(k)}(0)$ is bounded with respect to n:

$$\sup\{\, |b_{n,k}| : n \in \mathbb{N}\,\} < \infty. \tag{1}$$

We have to find $(t_n) \in \ell^1$ such that $\sum_n t_n\, b_n$ is no longer the germ of a holomorphic function at 0.

Each b_n has positive radius of convergence, in particular there is an $r_n > 0$ such that

$$\sup\{\, |b_{n,k}\, r_n^k| : k \in \mathbb{N}\,\} < \infty. \tag{2}$$

By theorem 3.4 the space $\mathcal{H}(\{0\}, \mathbb{C})$ is a regular inductive limit of spaces ℓ^∞_r. Hence a subset $\mathcal{B}$ is bounded in $\mathcal{H}(\{0\}, \mathbb{C})$ if and only if there exists an $r > 0$ such that

$$\left\{ \frac{1}{k!} b^{(k)}(0)\, r^k : b \in \mathcal{B}, k \in \mathbb{N} \right\}$$

is bounded. That the sequence (b_n) is unbounded thus means that for all $r > 0$ there are n and k such that $|b_{n,k}| > (\frac{1}{r})^k$. We can even choose $k > 0$ for otherwise the set $\{\, b_{n,k} r^k : n, k \in \mathbb{N}, k > 0\,\}$ is bounded, so only $\{\, b_{n,0} : n \in \mathbb{N}\,\}$ can be unbounded. This contradicts (1).

Hence for each m there are $k_m > 0$ such that $\mathcal{N}_m := \{\, n \in \mathbb{N} : |b_{n,k_m}| > m^{k_m}\,\}$ is not empty. We can choose (k_m) strictly increasing, for if they were bounded, $|b_{n,k_m}| < C$ for some C and all n by (1), but $|b_{n_m,k_m}| > m^{k_m} \to \infty$ for some n_m.

Since by (1) the set $\{\, b_{n,k_m} : n \in \mathbb{N}\}$ is bounded, we can choose $n_m \in \mathcal{N}_m$ such that

$$\begin{aligned} &|b_{n_m,k_m}| \geq \tfrac{1}{2} |b_{j,k_m}| \quad \text{for } j > n_m \\ &|b_{n_m,k_m}| > m^{k_m} \end{aligned} \tag{3}$$

We can choose also (n_m) strictly increasing, for if they were bounded we would get $|b_{n_m,k_m} r^{k_m}| < C$ for some $r > 0$ and C by (2). But $(\frac{1}{m})^{k_m} \to 0$.

We pass now to the subsequence (b_{n_m}) which we denote again by (b_m). We put

$$t_m := \operatorname{sign}\left(\frac{1}{b_{m,k_m}}\sum_{j<m} t_j\, b_{j,k_m}\right)\cdot\frac{1}{4^m}. \tag{4}$$

Assume now that $b_\infty = \sum_m t_m\, b_m$ converges weakly with respect to $\mathcal{S}$ to a holomorphic germ. Then its Taylor series is $b_\infty(z) = \sum_{k\geq 0} b_{\infty,k}\, z^k$, where the coefficients are given by $b_{\infty,k} = \sum_{m\geq 0} t_m\, b_{m,k}$. But we may compute as follows, using (3) and (4) :

$$\begin{aligned}
|b_{\infty,k_m}| &\geq \left|\sum_{j\leq m} t_j\, b_{j,k_m}\right| - \sum_{j>m} |t_j\, b_{j,k_m}| = \\
&= \left|\sum_{j<m} t_j\, b_{j,k_m}\right| + |t_m\, b_{m,k_m}| \qquad \text{(same sign)} \\
&\qquad\qquad - \sum_{j>m} |t_j\, b_{j,k_m}| \geq \\
&\geq 0 + |b_{m,k_m}|\cdot\left(|t_m| - 2\sum_{j>m}|t_j|\right) = \\
&= |b_{m,k_m}|\cdot\frac{1}{3\cdot 4^m} \geq \frac{m^{k_m}}{3\cdot 4^m}.
\end{aligned}$$

So $|b_{\infty,k_m}|^{1/k_m}$ goes to ∞, hence b_∞ cannot have a positive radius of convergence, a contradiction. So the theorem follows for the space $\mathcal{H}(\{t\},\mathbb{C})$.

Let us consider now an arbitrary closed smooth submanifold $A\subseteq\mathbb{C}$. By 3.8 the projective cone $\mathcal{H}(A\subseteq N,\mathbb{C})\to\{\mathcal{H}(\{z\}\subseteq N,\mathbb{C}), z\in A\}$ generates the bornology. Hence the result follows from the case where $A=\{0\}$ by 4.3. □

4.6. Theorem (Special real analytic uniform boundedness principle). *For any closed subset $A\subseteq M$ of a real analytic manifold M, the space $C^\omega(A\subseteq M,\mathbb{R})$ satisfies the uniform $\mathcal{S}$-boundedness principle for any point separating set $\mathcal{S}$ of bounded linear functionals.*

If A has no isolated points and M is 1-dimensional this applies to the set of all point evaluations ev_t, $t\in A$.

Proof. Again this follows from 4.4 using now 3.13. If A has no isolated points and M is 1-dimensional the point evaluations are separating, by the uniqueness theorem for holomorphic functions. □

Direct proof of a particular case. We show that $C^\omega(\mathbb{R},\mathbb{R})$ satisfies the uniform $\mathcal{S}$-boundedness principle for the set $\mathcal{S}$ of all point evaluations.

We check property 4.1.2. Let $\mathcal{B} \subseteq C^\omega(\mathbb{R},\mathbb{R})$ be absolutely convex such that $\mathrm{ev}_t(\mathcal{B})$ is bounded for all t and such that $C^\omega(\mathbb{R},\mathbb{R})_B$ is complete. We have to show that $\mathcal{B}$ is complete.

By lemma 3.12 the set $\mathcal{B}$ satisfies the conditions of 4.1.2 in the space $C^\infty(\mathbb{R},\mathbb{R})$. Since $C^\infty(\mathbb{R},\mathbb{R})$ satisfies the uniform $\mathcal{S}$-boundedness principle, cf. [5], the set $\mathcal{B}$ is bounded in $C^\infty(\mathbb{R},\mathbb{R})$. Hence all iterated derivatives at points are bounded on $\mathcal{B}$, and a fortiori the conditions of 4.1.2 are satisfied for $\mathcal{B}$ in $\mathcal{H}(\mathbb{R},\mathbb{C})$. By the particular case of theorem 4.5 the set $\mathcal{B}$ is bounded in $\mathcal{H}(\mathbb{R},\mathbb{C})$ and hence also in the direct summand $C^\omega(\mathbb{R},\mathbb{R})$. □

5. Cartesian closedness

5.1. Theorem. *The real analytic curves in $C^\omega(\mathbb{R},\mathbb{R})$ correspond exactly to the real analytic functions $\mathbb{R}^2 \to \mathbb{R}$.*

Proof. ($\Rightarrow$) Let $f : \mathbb{R} \to C^\omega(\mathbb{R},\mathbb{R})$ be a real analytic curve. Then $f : \mathbb{R} \to C^\omega(\{t\},\mathbb{R})$ is also real analytic. We use theorems 3.13 and 1.6 to conclude that f is even a topologically real analytic curve in $C^\omega(\{t\},\mathbb{R})$. By lemma 1.7 for every $s \in \mathbb{R}$ the curve f can be extended to a holomorphic mapping from an open neighborhood of s in $\mathbb{C}$ to the complexification (3.11) $\mathcal{H}(\{t\},\mathbb{C})$ of $C^\omega(\{t\},\mathbb{R})$.

From 3.4 it follows that $\mathcal{H}(\{t\},\mathbb{C})$ is the regular inductive limit of all spaces $\mathcal{H}(U,\mathbb{C})$, where U runs through some neighborhood basis of t in $\mathbb{C}$. Lemma 1.8 shows that f is a holomorphic mapping $V \to \mathcal{H}(U,\mathbb{C})$ for some open neighborhoods U of t and V of s in $\mathbb{C}$.

By the exponential law for holomorphic mappings (see 1.3) the canonically associated mapping $f^\wedge : V \times U \to \mathbb{C}$ is holomorphic. So its restriction is a real analytic function $\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ near (s,t).

($\Leftarrow$) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a real analytic mapping. Then $f(t,\quad)$ is real analytic, so the associated mapping $f^\vee : \mathbb{R} \to C^\omega(\mathbb{R},\mathbb{R})$ makes sense. It remains to show that it is real analytic. Since the mappings $C^\omega(\mathbb{R},\mathbb{R}) \to C^\omega(K,\mathbb{R})$ generate the bornology, by 3.13, it is by 1.11 enough to show that $f^\vee : \mathbb{R} \to C^\omega(K,\mathbb{R})$ is real analytic for each compact $K \subseteq \mathbb{R}$, which may be checked locally near each $s \in \mathbb{R}$.

$f : \mathbb{R}^2 \to \mathbb{R}$ extends to a holomorphic function on an open neighborhood $V \times U$ of $\{s\} \times K$ in $\mathbb{C}^2$. By cartesian closedness for the holomorphic setting the associated mapping $f^\vee : V \to \mathcal{H}(U,\mathbb{C})$ is holomorphic, so its restriction $V \cap \mathbb{R} \to C^\omega(U\cap\mathbb{R},\mathbb{R}) \to C^\omega(K,\mathbb{R})$ is real analytic as required. □

5.2. Remark. From 5.1 it follows that the curve $c : \mathbb{R} \to C^\omega(\mathbb{R},\mathbb{R})$ defined in 1.1 is real analytic, but it is not topologically real analytic. In particular, it does not factor locally to a real analytic curve into some Banach space $C^\omega(\mathbb{R},\mathbb{R})_B$ for a bounded subset B and it has no holomorphic extension to a mapping defined on a neighborhood of $\mathbb{R}$ in $\mathbb{C}$ with values in the complexification $\mathcal{H}(\mathbb{R},\mathbb{C})$ of $C^\omega(\mathbb{R},\mathbb{R})$, cf. 1.7.

5.3. Lemma. *For a real analytic manifold M, the bornology on the space $C^\omega(M,\mathbb{R})$ is induced by the following cone.*

$$C^\omega(M,\mathbb{R}) \xrightarrow{c^*} C^\alpha(\mathbb{R},\mathbb{R})$$

for all C^α-curves $c : \mathbb{R} \to M$, where α equals ∞ and ω.

Proof. The maps c^* are bornological since $C^\omega(M,\mathbb{R})$ is convenient by 3.13, and by the uniform $\mathcal{S}$-boundedness principle 4.6 for $C^\omega(\mathbb{R},\mathbb{R})$ and by [5], 4.4.7 for $C^\infty(\mathbb{R},\mathbb{R})$ it suffices to check that $\mathrm{ev}_t \circ c^* = \mathrm{ev}_{c(t)}$ is bornological, which is obvious.

Conversely we consider the identity mapping i from the space E into $C^\omega(M,\mathbb{R})$, where E is the vector space $C^\omega(M,\mathbb{R})$, but with the locally convex structure induced by the cone.

Claim. The bornology of E is complete.

The spaces $C^\omega(\mathbb{R},\mathbb{R})$ and $C^\infty(\mathbb{R},\mathbb{R})$ are convenient by 3.13 and 1.3, respectively. So their product

$$\prod_c C^\omega(\mathbb{R},\mathbb{R}) \times \prod_c C^\infty(\mathbb{R},\mathbb{R})$$

is also convenient. By theorem 2.4,(1) $\Leftrightarrow$ (5), the embedding of E into this product has closed image, hence the bornology of E is complete.

Now we may apply the uniform S-boundedness principle for $C^\omega(M,\mathbb{R})$ (4.6), since obviously $\mathrm{ev}_p \circ i = \mathrm{ev}_0 \circ c_p^*$ is bounded, where c_p is the constant curve with value p, for all $p \in M$. $\square$

5.4. Structure on $C^\omega(U,F)$. Let (E,E') be a dual pair of real vector spaces and let U be c^∞-open in E. We equip the space $C^\omega(U,\mathbb{R})$ of all real analytic functions (cf. 2.6) with the dual space consisting of all linear functionals induced from the families of mappings

$$C^\omega(U,\mathbb{R}) \xrightarrow{c^*} C^\omega(\mathbb{R},\mathbb{R}), \text{ for all } c \in C^\omega(\mathbb{R},U)$$
$$C^\omega(U,\mathbb{R}) \xrightarrow{c^*} C^\infty(\mathbb{R},\mathbb{R}), \text{ for all } c \in C^\infty(\mathbb{R},U).$$

For a finite dimensional vector spaces E this definition gives the same bornology as the one defined in 3.10, by lemma 5.3.

If (F, F') is another dual pair, we equip the space $C^\omega(U, F)$ of all real analytic mappings (cf. 2.6) with the dual induced by the family of mappings

$$C^\omega(U,F) \xrightarrow{\lambda_*} C^\omega(U,\mathbb{R}), \text{ for all } \lambda \in F'.$$

5.5. Lemma. *Let (E, E') and (F, F') be complete dual pairs and let $U \subseteq E$ be c^∞-open. Then $C^\omega(U, F)$ is complete.*

Proof. This follows immediately from the fact that $C^\omega(U, F)$ can be considered as closed subspace of the product of factors $C^\omega(U, \mathbb{R})$ indexed by all $\lambda \in F'$. And $C^\omega(U, \mathbb{R})$ can be considered as closed subspace of the product of the factors $C^\omega(\mathbb{R}, \mathbb{R})$ indexed by all $c \in C^\omega(\mathbb{R}, U)$ and the factors $C^\infty(\mathbb{R}, \mathbb{R})$ indexed by all $c \in C^\infty(\mathbb{R}, U)$. Since all factors are complete so are the closed subspaces. □

5.6. Lemma (General real analytic uniform boundedness principle). *Let E and F be convenient vector spaces and $U \subseteq E$ be c^∞-open. Then $C^\omega(U, F)$ satisfies the uniform $\mathcal{S}$-boundedness principle, where $\mathcal{S} := \{ev_x : x \in U\}$.*

Proof. The complete bornology of $C^\omega(U, F)$ is by definition induced by the maps $c^* : C^\omega(U, F) \to C^\omega(\mathbb{R}, F)$ ($c \in C^\omega(\mathbb{R}, U)$) together with the maps $c^* : C^\omega(U, F) \to C^\infty(\mathbb{R}, F)$ ($c \in C^\infty(\mathbb{R}, U)$). Both spaces $C^\omega(\mathbb{R}, F)$ and $C^\infty(\mathbb{R}, F)$ satisfy the uniform $\mathcal{T}$-boundedness principle, where $\mathcal{T} := \{ev_t : t \in \mathbb{R}\}$, by 4.6 and [5], 4.4.7, respectively. Hence $C^\omega(U, F)$ satisfies the uniform $\mathcal{S}$-boundedness principle by lemma 4.3, since $ev_t \circ c^* = ev_{c(t)}$. □

5.7. Definition. Let (E, E') and (F, F') be complete dual pairs. We denote by $L(E, F)$ the *space of linear real analytic mappings* from E to F, which are by 1.9 exactly the bounded linear mappings. Furthermore, if E and F are convenient vector spaces, these are exactly the morphisms in the sense of dual pairs, since f is bounded if and only if $\lambda \circ f \in E^b$ for all $\lambda \in F^b$.

5.8. Lemma (Structure on $L(E, F)$). *The following structures on $L(E, F)$ are the same:*

1. *The bornology of pointwise boundedness, i.e. the bornology induced by the cone $(ev_x : L(E, F) \to F, x \in E)$.*
2. *The bornology of uniform boundedness on bounded sets in E, i.e. a set $\mathcal{B} \subseteq L(E, F)$ is bounded if and only if $\mathcal{B}(B) \subseteq F$ is bounded for every bounded $B \subseteq E$.*
3. *The bornology induced by the inclusion $L(E, F) \to C^\infty(E, F)$.*
4. *The bornology induced by the inclusion $L(E, F) \to C^\omega(E, F)$.*

The space $L(E,F)$ will from now on be the convenient vector space having as structure that described in the previous lemma. Thus $L(E,F)$ is a convenient vector space, by [5], 3.6.3. In particular this is true for $E' = L(E,\mathbb{R})$.

So a mapping f into $L(E,F)$ is real analytic if and only if the composites $ev_x \circ f$ are real analytic for all $x \in E$, by 1.11.

Proof. That the bornology in (1), (2) and (3) are the same was shown in [5], 3.6.4 and 4.4.24. Since $C^\omega(E,F) \to C^\infty(E,F)$ is continuous by definition of the structure on $C^\omega(E,F)$ the bornology in (4) is finer than that in (1). The bornology given in (4) is complete, since the point-evaluations $ev_x : C^\omega(E,F) \to F$ are continuous, and linearity of a mapping $E \to F$ can be checked by applying them. Furthermore $L(E,F)$ with the bornology given in (4) satisfies the uniform $\mathcal{S}$-boundedness theorem, since $C^\omega(E,F)$ does, by 5.6. So the identity on $L(E,F)$ with the bornology given in (1) to that given in (4) is bounded. □

The following two results will be generalized in 6.3. At the moment we will make use of the following lemma only in case where $E = C^\infty(\mathbb{R},\mathbb{R})$.

5.9. Lemma. $L(E, C^\omega(\mathbb{R},\mathbb{R})) \cong C^\omega(\mathbb{R},E')$ *as vector spaces, for any convenient vector space* E.

Proof. For $c \in C^\omega(\mathbb{R},E')$ consider $\tilde{c}(x) := \mathrm{ev}_x \circ c \in C^\omega(\mathbb{R},\mathbb{R})$ for $x \in E$. By the uniform $\mathcal{S}$-boundedness principle 4.6 for $\mathcal{S} = \{\mathrm{ev}_t : t \in \mathbb{R}\}$ the linear mapping $\tilde{c}$ is bounded, since $\mathrm{ev}_t \circ \tilde{c} = c(t) \in E'$.

If conversely $\ell \in L(E, C^\omega(\mathbb{R},\mathbb{R}))$, we consider $\tilde{\ell}(t) = \mathrm{ev}_t \circ \ell \in E' := L(E,\mathbb{R})$ for $t \in \mathbb{R}$. Since the bornology of E' is generated by $\mathcal{S} := \{ev_x : x \in E\}$, $\tilde{\ell} : \mathbb{R} \to E'$ is real analytic, for $\mathrm{ev}_x \circ \tilde{\ell} = \ell(x) \in C^\omega(\mathbb{R},\mathbb{R})$. □

5.10. Corollary. *We have* $C^\infty(\mathbb{R}, C^\omega(\mathbb{R},\mathbb{R})) \cong C^\omega(\mathbb{R}, C^\infty(\mathbb{R},\mathbb{R}))$ *as vector spaces.*

Proof. $C^\infty(\mathbb{R},\mathbb{R})'$ is the free convenient vector space over $\mathbb{R}$ by [5], 5.1.8, and $C^\omega(\mathbb{R},\mathbb{R})$ is convenient, we have

$$\begin{aligned} C^\infty(\mathbb{R}, C^\omega(\mathbb{R},\mathbb{R})) &\cong L(C^\infty(\mathbb{R},\mathbb{R})', C^\omega(\mathbb{R},\mathbb{R})) \\ &\cong C^\omega(\mathbb{R}, C^\infty(\mathbb{R},\mathbb{R})'') \qquad \text{by lemma 5.9} \\ &\cong C^\omega(\mathbb{R}, C^\infty(\mathbb{R},\mathbb{R})), \end{aligned}$$

by reflexivity of $C^\infty(\mathbb{R},\mathbb{R})$, see [5], 5.4.16. □

5.11. Theorem. *Let (E, E') be a complete dual pair, let U be c^∞-open in E, let $f : \mathbb{R} \times U \to \mathbb{R}$ be a real analytic mapping and let $c \in C^\infty(\mathbb{R}, U)$. Then $c^* \circ \check{f} : \mathbb{R} \to C^\omega(U, \mathbb{R}) \to C^\infty(\mathbb{R}, \mathbb{R})$ is real analytic.*

This result on the mixing of C^∞ and C^ω will become quite essential in the proof of cartesian closedness. It will be generalized in 6.4, see also 8.9 and 8.14.

Proof. Let $I \subseteq \mathbb{R}$ be open and relatively compact, let $t \in \mathbb{R}$ and $k \in \mathbb{N}$. Now choose an open and relatively compact $J \subseteq \mathbb{R}$ containing the closure $\bar{I}$ of I. There is a bounded subset $B \subseteq E$ such that $c \mid J : J \to E_B$ is a $\mathcal{L}ip^k$-curve in the Banach space E_B generated by B. This is [13], Folgerung on p.114. Let U_B denote the open subset $U \cap E_B$ of the Banach space E_B. Since the inclusion $E_B \to E$ is continuous, f is real analytic as a function $\mathbb{R} \times U_B \to \mathbb{R} \times U \to \mathbb{R}$. Thus by 2.4 there is a holomorphic extension $f : V \times W \to \mathbb{C}$ of f to an open set $V \times W \subseteq \mathbb{C} \times (E_B)_\mathbb{C}$ containing the compact set $\{t\} \times c(\bar{I})$. By cartesian closedness of the category of holomorphic mappings $\check{f} : V \to \mathcal{H}(W, \mathbb{C})$ is holomorphic. Now recall that the bornological structure of $\mathcal{H}(W, \mathbb{C})$ is induced by that of $C^\infty(W, \mathbb{C}) := C^\infty(W, \mathbb{R}^2)$. And $c^* : C^\infty(W, \mathbb{C}) \to \mathcal{L}ip^k(I, \mathbb{C})$ is a bounded $\mathbb{C}$-linear map, by [5]. Thus $c^* \circ \check{f} : V \to \mathcal{L}ip^k(I, \mathbb{C})$ is holomorphic, and hence its restriction to $\mathbb{R} \cap V$, which has values in $\mathcal{L}ip^k(I, \mathbb{R})$, is (even topologically) real analytic by 1.7. Since $t \in \mathbb{R}$ was arbitrary we conclude that $c^* \circ \check{f} : \mathbb{R} \to \mathcal{L}ip^k(I, \mathbb{R})$ is real analytic. But the bornology of $C^\infty(\mathbb{R}, \mathbb{R})$ is generated by the inclusions into $\mathcal{L}ip^k(I, \mathbb{R})$, [5], 4.2.7, and hence $c^* \circ \check{f} : \mathbb{R} \to C^\infty(\mathbb{R}, \mathbb{R})$ is real analytic. □

5.12. Theorem (Cartesian closedness). *The category of real analytic mappings between complete dual pairs of real vector spaces is cartesian closed. More precisely, for complete dual pairs (E, E'), (F, F') and (G, G') and c^∞-open sets $U \subseteq E$ and $W \subseteq G$ a mapping $f : W \times U \to F$ is real analytic if and only if $\check{f} : W \to C^\omega(U, F)$ is real analytic.*

Proof. **Step 1.** The theorem is true for $G = F = \mathbb{R}$.

($\Leftarrow$) Let $\check{f} : \mathbb{R} \to C^\omega(U, \mathbb{R})$ be C^ω. We have to show that $f : \mathbb{R} \times U \to \mathbb{R}$ is C^ω. We consider a curve $c_1 : \mathbb{R} \to \mathbb{R}$ and a curve $c_2 : \mathbb{R} \to U$.

If the c_i are C^∞, then $c_2^* \circ \check{f} : \mathbb{R} \to C^\omega(U, \mathbb{R}) \to C^\infty(\mathbb{R}, \mathbb{R})$ is C^ω by assumption, hence is C^∞, so $c_2^* \circ \check{f} \circ c_1 : \mathbb{R} \to C^\infty(\mathbb{R}, \mathbb{R})$ is C^∞. By cartesian closedness of smooth mappings, $(c_2^* \circ \check{f} \circ c_1)^\wedge = f \circ (c_1 \times c_2) : \mathbb{R}^2 \to \mathbb{R}$ is C^∞. By composing with the diagonal mapping $\Delta : \mathbb{R} \to \mathbb{R}^2$ we obtain that $f \circ (c_1, c_2) : \mathbb{R} \to \mathbb{R}$ is C^∞.

If the c_i are C^ω, then $c_2^* \circ \check{f} : \mathbb{R} \to C^\omega(U, \mathbb{R}) \to C^\omega(\mathbb{R}, \mathbb{R})$ is C^ω by assumption, so $c_2^* \circ \check{f} \circ c_1 : \mathbb{R} \to C^\omega(\mathbb{R}, \mathbb{R})$ is C^ω. By theorem 5.1 the associated map $(c_2^* \circ \check{f} \circ c_1)^\wedge = f \circ (c_1 \times c_2) : \mathbb{R}^2 \to \mathbb{R}$ is C^ω. So $f \circ (c_1, c_2) : \mathbb{R} \to \mathbb{R}$ is C^ω.

($\Rightarrow$) Let $f : \mathbb{R} \times U \to \mathbb{R}$ be C^ω. We have to show that $\check{f} : \mathbb{R} \to C^\omega(U, \mathbb{R})$ is real analytic. Obviously $\check{f}$ has values in this space. We consider a curve $c : \mathbb{R} \to U$.

If c is C^∞, then by theorem 5.11 the associated mapping $(f \circ (Id \times c))^\vee = c^* \circ \check{f} : \mathbb{R} \to C^\infty(\mathbb{R}, \mathbb{R})$ is C^ω.

If c is C^ω, then $f \circ (Id \times c) : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times U \to \mathbb{R}$ is C^ω. By theorem 5.1 the associated mapping $(f \circ (Id \times c))^\vee = c^* \circ \check{f} : \mathbb{R} \to C^\omega(\mathbb{R}, \mathbb{R})$ is C^ω.

Step 2. The theorem is true for $F = \mathbb{R}$.

($\Leftarrow$) Let $\check{f} : W \to C^\omega(U, \mathbb{R})$ be C^ω. We have to show that $f : W \times U \to \mathbb{R}$ is C^ω. We consider a curve $c_1 : \mathbb{R} \to W$ and a curve $c_2 : \mathbb{R} \to U$.

If the c_i are C^∞, then $c_2^* \circ \check{f} : W \to C^\omega(U, \mathbb{R}) \to C^\infty(\mathbb{R}, \mathbb{R})$ is C^ω by assumption, hence is C^∞, so $c_2^* \circ \check{f} \circ c_1 : \mathbb{R} \to C^\infty(\mathbb{R}, \mathbb{R})$ is C^∞. By cartesian closedness of smooth mappings, the associated mapping $(c_2^* \circ \check{f} \circ c_1)^\wedge = f \circ (c_1 \times c_2) : \mathbb{R}^2 \to \mathbb{R}$ is C^∞. So $f \circ (c_1, c_2) : \mathbb{R} \to \mathbb{R}$ is C^∞.

If the c_i are C^ω, then $\check{f} \circ c_1 : \mathbb{R} \to W \to C^\omega(U, \mathbb{R})$ is C^ω by assumption, so by step 1 the mapping $(\check{f} \circ c_1)^\wedge = f \circ (c_1 \times Id_U) : \mathbb{R} \times U \to \mathbb{R}$ is C^ω. Hence

$$f \circ (c_1, c_2) = f \circ (c_1 \times Id_U) \circ (Id, c_2) : \mathbb{R} \to \mathbb{R}$$

is C^ω.

($\Rightarrow$) Let $f : W \times U \to \mathbb{R}$ be C^ω. We have to show that $\check{f} : W \to C^\omega(U, \mathbb{R})$ is real analytic. Obviously $\check{f}$ has values in this space. We consider a curve $c_1 : \mathbb{R} \to W$.

If c_1 is C^∞, we consider a second curve $c_2 : \mathbb{R} \to U$. If c_2 is C^∞, then $f \circ (c_1 \times c_2) : \mathbb{R} \times \mathbb{R} \to W \times U \to \mathbb{R}$ is C^∞. By cartesian closedness the associated mapping

$$(f \circ (c_1 \times c_2))^\vee = c_2^* \circ \check{f} \circ c_1 : \mathbb{R} \to C^\infty(\mathbb{R}, \mathbb{R})$$

is C^∞. If c_2 is C^ω, the mapping $f \circ (Id_W \times c_2) : W \times \mathbb{R} \to \mathbb{R}$ and also the flipped one $(f \circ (Id_W \times c_2))^\sim : \mathbb{R} \times W \to \mathbb{R}$ are C^ω, hence by theorem 5.11 $c_1^* \circ ((f \circ (Id_W \times c_2))^\sim)^\vee : \mathbb{R} \to C^\infty(\mathbb{R}, \mathbb{R})$ is C^ω. By corollary 5.10 the associated mapping $(c_1^* \circ ((f \circ (Id_W \times c_2))^\sim)^\vee)^\sim = c_2^* \circ \check{f} \circ c_1 : \mathbb{R} \to C^\omega(\mathbb{R}, \mathbb{R})$ is C^∞. So for both families describing the dual of $C^\omega(U, \mathbb{R})$ we have shown that the composite with $\check{f} \circ c_1$ is C^∞, so $\check{f} \circ c_1$ is C^∞.

If c_1 is C^ω, then $f \circ (c_1 \times Id_U) : \mathbb{R} \times U \to W \times U \to \mathbb{R}$ is C^ω. By step 1 the associated mapping $(f \circ (c_1 \times Id_U))^\vee = \check{f} \circ c_1 : \mathbb{R} \to C^\omega(U, \mathbb{R})$ is C^ω.

Step 3. The general case.

$$\begin{aligned}
& f : W \times U \to F \text{ is } C^\omega \\
\Leftrightarrow \quad & \lambda \circ f : W \times U \to \mathbb{R} \text{ is } C^\omega \text{ for all } \lambda \in F' \\
\Leftrightarrow \quad & (\lambda \circ f)^\vee = \lambda_* \circ \check{f} : W \to C^\omega(U, \mathbb{R}) \text{ is } C^\omega, \text{ by step 2,} \\
\Leftrightarrow \quad & \check{f} : W \to C^\omega(U, F) \text{ is } C^\omega. \quad \square
\end{aligned}$$

6. Consequences of cartesian closedness

Among all those dual pairings on a fixed vector space E that generate the same real analytic structure there is a finest one, namely that having as dual exactly the linear real analytic functionals, which are exactly the bounded ones, by 1.9. Recall that a dual pair (E, E') is called convenient if and only if it is complete and E' consists exactly of the bounded linear functionals.

6.1. Theorem. *The category of real analytic mappings between complete dual pairs is equivalent to that of real analytic mappings between convenient dual pairs. Hence the later category is also cartesian closed.*

Proof. The second category is a full subcategory of the first. A functor in the other direction is given by associating to every dual pair (E, E') the dual pair (E, E^b), where

$$\begin{aligned} E^b :&= \{\lambda : E \to \mathbb{R} : \lambda \text{ is linear and bounded}\} \\ &= \{\lambda \in C^\omega(E, \mathbb{R}) : \lambda \text{ is linear}\} \\ &= \{\lambda : E \to \mathbb{R} : \lambda \text{ is linear, } \lambda \circ c \in C^\omega(\mathbb{R}, \mathbb{R}) \text{ for all } c \in C^\omega(\mathbb{R}, E)\}. \end{aligned}$$

Functoriality follows since the real analytic mappings form a category. One composite of this functor with the inclusion functor is the identity, and the other is naturally isomorphic to the identity, since E^b and E' generate the same bornology and hence the same C^ω- and C^∞-curves, by 1.10. □

Convention. All spaces are from now on assumed to be convenient and all function spaces will be considered with their natural bornological topology.

6.2. Corollary (Canonical mappings are real analytic). *The following mappings are C^ω:*

(1) $\text{ev} : C^\omega(U, F) \times U \to F$, $(f, x) \mapsto f(x)$,
(2) $\text{ins} : E \to C^\omega(F, E \times F)$, $x \mapsto (y \mapsto (x, y))$,
(3) $(\quad)^\wedge : C^\omega(U, C^\omega(V, G)) \to C^\omega(U \times V, G)$,
(4) $(\quad)^\vee : C^\omega(U \times V, G) \to C^\omega(U, C^\omega(V, G))$,
(5) $\text{comp} : C^\omega(F, G) \times C^\omega(U, F) \to C^\omega(U, G)$, $(f, g) \mapsto f \circ g$,
(6) $C^\omega(\quad,\quad) : C^\omega(E_2, E_1) \times C^\omega(F_1, F_2) \to$
$\to C^\omega(C^\omega(E_1, F_1), C^\omega(E_2, F_2))$, $(f, g) \mapsto (h \mapsto g \circ h \circ f)$.

Proof. (1). The mapping associated to ev via cartesian closedness is the identity on $C^\omega(U, F)$, which is C^ω, thus ev is also C^ω.

(2). The mapping associated to *ins* via cartesian closedness is the identity on $E\times F$, hence *ins* is C^ω.

(3). The mapping associated via cartesian closedness is $(f;x,y)\mapsto f(x)(y)$, which is the C^ω-mapping $\operatorname{ev}\circ(\operatorname{ev}\times id)$.

(4). The mapping associated by applying cartesian closedness twice is $(f;x;y)\mapsto f(x,y)$, which is just a C^ω evaluation mapping.

(5). The mapping associated to *comp* via cartesian closedness is just $(f,g;x)\mapsto f(g(x))$, which is the C^ω-mapping $\operatorname{ev}\circ(id\times\operatorname{ev})$.

(6). The mapping associated by applying cartesian closed twice is $(f,g;h,x)\mapsto g(h(f(x)))$, which is the C^ω-mapping $\operatorname{ev}\circ(id\times\operatorname{ev})\circ(id\times id\times\operatorname{ev})$. □

6.3. Lemma (Canonical isomorphisms). *One has the following natural isomorphisms:*

1. $C^\omega(W_1,C^\omega(W_2,F))\cong C^\omega(W_2,C^\omega(W_1,F))$,
2. $C^\omega(W_1,C^\infty(W_2,F))\cong C^\infty(W_2,C^\omega(W_1,F))$.
3. $C^\omega(W_1,L(E,F))\cong L(E,C^\omega(W_1,F))$.
4. $C^\omega(W_1,\ell^\infty(X,F))\cong \ell^\infty(X,C^\omega(W_1,F))$.
5. $C^\omega(W_1,\mathcal{L}ip^k(X,F))\cong \mathcal{L}ip^k(X,C^\omega(W_1,F))$.

In (4) *X is a ℓ^∞-space, i.e. a set together with a bornology induced by a family of real valued functions on X, cf.* [5]*, 1.2.4. In* (5) *X is a $\mathcal{L}ip^k$-space, cf.* [5]*, 1.4.1. The spaces $\ell^\infty(X,F)$ and $\mathcal{L}ip^k(W,F)$ are defined in* [5]*, 3.6.1 and 4.4.1.*

Proof. All isomorphisms, as well as their inverse mappings, are given by the flip of coordinates: $f\mapsto\tilde f$, where $\tilde f(x)(y):=f(y)(x)$. Furthermore all occurring function spaces are convenient and satisfy the uniform $\mathcal{S}$-boundedness theorem, where $\mathcal{S}$ is the set of point evaluations, by 5.5, 5.8, 4.6, and by [5], 3.6.1, 4.4.2, 3.6.6, and 4.4.7.

That $\tilde f$ has values in the corresponding spaces follows from the equation $\tilde f(x)=ev_x\circ f$. One only has to check that $\tilde f$ itself is of the corresponding class, since it follows that $f\mapsto\tilde f$ is bounded. This is a consequence of the uniform boundedness principle, since

$$(\operatorname{ev}_x\circ(\,\tilde{}\,))(f)=\operatorname{ev}_x(\tilde f)=\tilde f(x)=\operatorname{ev}_x\circ f=(\operatorname{ev}_x)_*(f).$$

That $\tilde f$ is of the appropriate class in (1) and (2) follows by composing with $c_1\in C^{\beta_1}(\mathbb{R},W_1)$ and $C^{\beta_2}(\lambda,c_2):C^{\alpha_2}(W_2,F)\to C^{\beta_2}(\mathbb{R},\mathbb{R})$ for all $\lambda\in F'$ and $c_2\in C^{\beta_2}(\mathbb{R},W_2)$, where β_k and α_k are in $\{\infty,\omega\}$ and $\beta_k\le\alpha_k$ for $k\in\{1,2\}$. Then $C^{\beta_2}(\lambda,c_2)\circ\tilde f\circ c_1=(C^{\beta_1}(\lambda,c_1)\circ f\circ c_2)^\sim:\mathbb{R}\to C^{\beta_2}(\mathbb{R},\mathbb{R})$ is C^{β_1} by 5.1 and 5.10, since $C^{\beta_1}(\lambda,c_1)\circ f\circ c_2:\mathbb{R}\to W_2\to C^{\alpha_1}(W_1,F)\to C^{\beta_1}(\mathbb{R},\mathbb{R})$ is C^{β_2}.

That $\tilde{f}$ is of the appropriate class in (3) follows, since $L(E,F)$ is the c^∞-closed subspace of $C^\omega(E,F)$ formed by the linear C^ω-mappings.

That $\tilde{f}$ is of the appropriate class in (4) follows from (3), using the free convenient vector space $\ell^1(X)$ over the ℓ^∞-space X, see [5], 5.1.24, satisfying $\ell^\infty(X,F) \cong L(\ell^1(X),F)$.

That $\tilde{f}$ is of the appropriate class in (5) follows from (3), using the free convenient vector space $\lambda^k(X)$ over the $\mathcal{L}ip^k$-space X, see [5], 5.1.3, satisfying $\mathcal{L}ip^k(X,F) \cong L(\lambda^k(X),F)$. $\square$

Definition. A $C^{\infty,\omega}$-mapping $f : U \times V \to F$ is a mapping for which

$$\check{f} \in C^\infty(U, C^\omega(V,F)).$$

6.4. Theorem (Composition of $C^{\infty,\omega}$-mappings). *Let $f : U \times V \to F$ and $g : U_1 \times V_1 \to V$ be $C^{\infty,\omega}$, and $h : U_1 \to U$ be C^∞. Then*

$$f \circ (h \circ pr_1, g) : U_1 \times V_1 \to F, \qquad (x,y) \mapsto f(h(x), g(x,y))$$

is $C^{\infty,\omega}$.

Proof. We have to show that the mapping $x \mapsto (y \mapsto f(h(x), g(x,y)), U_1 \to C^\omega(V_1,F)$ is C^ω. It is well-defined, since f and g are C^ω in the second variable. In order to show that it is C^ω we compose with $\lambda_* : C^\omega(V_1,F) \to C^\omega(V_1,\mathbb{R})$, where $\lambda \in F'$ is arbitrary. Thus it is enough to consider the case $F = \mathbb{R}$. Furthermore, we compose with $c^* : C^\omega(V_1,\mathbb{R}) \to C^\alpha(\mathbb{R},\mathbb{R})$, where $c \in C^\alpha(\mathbb{R},V_1)$ is arbitrary for α equal to ω and ∞.

In case $\alpha = \infty$ the composite with c^* is C^∞, since the associated mapping $U_1 \times \mathbb{R} \to \mathbb{R}$ is $f \circ (h \circ pr_1, g \circ (id \times c))$ which is C^∞.

Now the case $\alpha = \omega$. Let $I \subseteq \mathbb{R}$ be an arbitrary open bounded interval. Then $c^* \circ \check{g} : U_1 \to C^\omega(\mathbb{R},G)$ is C^∞, where G is the convenient vector space containing V as an c^∞-open subset, and has values in the open set $\{\gamma : \gamma(\bar{I}) \subseteq V\} \subseteq C^\omega(\mathbb{R},G)$. Thus the composite with c^*, $\operatorname{comp} \circ (\check{f} \circ h, c^* \circ \check{g})$ is C^∞, since $\check{f} \circ h : U_1 \to U \to C^\omega(V,F)$ is C^∞, $c^* \circ \check{g} : U_1 \to C^\omega(\mathbb{R},G)$ is C^∞ and $\operatorname{comp} : C^\omega(V,F) \times \{\gamma \in C^\omega(\mathbb{R},G) : \gamma(\bar{I}) \subseteq V\} \to C^\omega(I,\mathbb{R})$ is C^ω, because it is associated to $\operatorname{ev} \circ (id \times \operatorname{ev}) : C^\omega(V,F) \times \{\gamma \in C^\omega(\mathbb{R},G) : \gamma(\bar{I}) \subseteq V\} \times I \to \mathbb{R}$. That $\operatorname{ev} : \{\gamma \in C^\omega(\mathbb{R},G) : \gamma(\bar{I}) \subseteq V\} \times I \to \mathbb{R}$ is C^ω follows, since the associated mapping is the restriction mapping $C^\omega(\mathbb{R},G) \to C^\omega(I,G)$. $\square$

6.5 Corollary. *Let $f : U \to F$ be C^ω and $g : U_1 \times V_1 \to U$ be $C^{\infty,\omega}$, then*

$$f \circ g : U_1 \times V_1 \to F$$

is $C^{\infty,\omega}$.

Let $f : U \times V \to F$ be $C^{\infty,\omega}$ and $h : U_1 \to U$ be C^∞, then $f \circ (h \times id) : U_1 \times V \to F$ is $C^{\infty,\omega}$. □

The second part is a generalization of theorem 5.11.

6.6. Corollary. *Let $f : E \supseteq U \to F$ be C^ω, let $I \subseteq \mathbb{R}$ be open and bounded, and α be ω or ∞. Then $f_* : C^\alpha(\mathbb{R}, E) \supseteq \{c : c(\bar{I}) \subseteq U\} \to C^\alpha(I, F)$ is C^ω.*

Proof. Obviously $f_*(c) := f \circ c \in C^\alpha(I, F)$ is well-defined for all $c \in C^\alpha(\mathbb{R}, E)$ satisfying $c(\bar{I}) \subseteq U$.

Furthermore the composite of f_* with any C^β-curve $\gamma : \mathbb{R} \to \{c : c(\bar{I}) \subseteq U\} \subseteq C^\alpha(\mathbb{R}, E)$ is a C^β-curve in $C^\alpha(I, F)$ for β equal to ω or ∞. For $\beta = \alpha$ this follows from cartesian closedness of the C^α-maps. For $\alpha \neq \beta$ this follows from 6.5.

Finally $\{c : c(\bar{I}) \subseteq U\} \subseteq C^\alpha(\mathbb{R}, E)$ is c^∞-open, since it is open for the topology of uniform convergence on compact sets which is coarser than the bornological and hence than the c^∞-topology on $C^\alpha(\mathbb{R}, E)$. Here is the only place where we make use of the boundedness of I. □

6.7. Lemma (Free convenient vector space). *Let $U \subseteq E$ be c^∞-open in a convenient vector space E. There exists a free convenient vector spaces $Free(U)$ over U, i.e. for every convenient vector space F, one has a natural isomorphism $C^\omega(U, F) \cong L(Free(U), F)$*

Proof. Consider the Mackey closure $Free(U)$ of the linear subspace of $C^\omega(U, \mathbb{R})'$ generated by the set $\{ev_x : x \in U\}$. Let $\iota : U \to Free(U) \subseteq C^\omega(U, \mathbb{R})'$ be the mapping given by $x \mapsto \mathrm{ev}_x$. This mapping is C^ω, since $ev_f \circ \iota = f$ for every $f \in C^\omega(U, \mathbb{R})$.

Obviously every real analytic mapping $f : U \to F$ extends to the linear bounded mapping $\tilde{f} : C^\omega(U, \mathbb{R})' \to F''$, $\lambda \mapsto (l \mapsto \lambda(l \circ f))$. Since $\tilde{f}$ coincides on the generators ev_x with f, it maps the Mackey closure $Free(U)$ into the Mackey closure of $F \supseteq f(U)$. Since F is complete this is again F. Uniqueness of $\tilde{f}$ follows, since every linear real analytic mapping is bounded, hence it is determined by its values on the subset $\{\mathrm{ev}_x : x \in U\}$ that spans the linear subspace having as Mackey closure $Free(U)$. □

6.8. Lemma (Derivatives). *The derivative d, where*

$$df(x)(v) := \frac{d}{dt}\,|_{t=0}\, f(x + tv),$$

is bounded and linear $d : C^\omega(U,F) \to C^\omega(U, L(E,F))$.

Proof. The differential $df(x)(v)$ makes sense and is linear in v, because every real analytic mapping f is smooth. So it remains to show that $(f,x,v) \mapsto df(x)(v)$ is real analytic. So let f, x, and v depend real analytically (resp. smoothly) on a real parameter s. Since $(t,s) \mapsto x(s) + tv(s)$ is real analytic (resp. smooth) into $U \subseteq E$, the mapping $r \mapsto ((t,s) \mapsto f(r)(x(s)+tv(s))$ is real analytic into $C^\omega(\mathbb{R}^2,F)$ (resp. $C^\infty(\mathbb{R}^2,F)$. Composing with

$$\frac{d}{dt}\,|_{t=0}\colon C^\omega(\mathbb{R}^2,F) \to C^\omega(\mathbb{R},F) \qquad (\text{resp. } : C^\infty(\mathbb{R}^2,F) \to C^\infty(\mathbb{R},F))$$

shows that $r \mapsto (s \mapsto d(f(r))(x(s))(v(s)))$, $\mathbb{R} \to C^\omega(\mathbb{R},F)$ is real analytic. Considering the associated mapping on $\mathbb{R}^2$ composed with the diagonal map shows that $(f,x,v) \mapsto df(x)(v)$ is real analytic. □

The following examples as well as several others can be found in [5], 5.3.6.

6.9 Example. *Let* $T : C^\infty(\mathbb{R},\mathbb{R}) \to C^\infty(\mathbb{R},\mathbb{R})$ *be given by* $T(f) = f'$. *Then the continuous linear differential equation* $x'(t) = T(x(t))$ *with initial value* $x(0) = x_0$ *has a unique smooth solution* $x(t)(s) = x_0(t+s)$ *which is however not real analytic.*

Proof. A smooth curve $x : \mathbb{R} \to C^\infty(\mathbb{R},\mathbb{R})$ is a solution of the differential equation $x'(t) = T(x(t))$ if and only if

$$\frac{\partial}{\partial t}\hat{x}(t,s) = \frac{\partial}{\partial s}\hat{x}(t,s).$$

Hence we have

$$\frac{d}{dt}\hat{x}(t, r-t) = 0,$$

i.e. $\hat{x}(t, r-t)$ is constant and hence equal to $\hat{x}(0,r) = x_0(r)$. Thus $\hat{x}(t,s) = x_0(t+s)$.

Suppose $x : \mathbb{R} \to C^\infty(\mathbb{R},\mathbb{R})$ were real analytic. Then the composite with

$$ev_0 : C^\infty(\mathbb{R},\mathbb{R}) \to \mathbb{R}$$

were a real analytic function. But this composite is just $x_0 = ev_0 \circ x$, which is not in general real analytic. □

6.10 Example. *Let E be either $C^\infty(\mathbb{R},\mathbb{R})$ or $C^\omega(\mathbb{R},\mathbb{R})$. Then the mapping exp_* : $E \to E$ is C^ω, has invertible derivative at every point, but the image does not contain an open neighborhood of $exp_*(0)$.*

Proof. That exp_* is C^ω was shown in 6.6. Its derivative is given by

$$(exp_*)'(f)(g) : t \mapsto g(t)e^{f(t)}$$

and hence is invertible with $g \mapsto (t \mapsto g(t)e^{-f(t)})$ as inverse mapping. Now consider the real analytic curve $c : \mathbb{R} \to E$ given by $c(t)(s) = 1 - (ts)^2$. One has $c(0) = 1 = exp_*(0)$, but $c(t)$ is not in the image of exp_* for any $t \neq 0$, since $c(t)(\frac{1}{t}) = 0$ but $exp_*(g)(t) = e^{g(t)} > 0$ for all g and t. □

7. Spaces of sections of vector bundles

7.1. Vector bundles. Let (E, p, M) be a real analytic finite dimensional vector bundle over a real analytic manifold M, where E is their total space and $p : E \to M$ is the projection. So there is an open cover $(U_\alpha)_\alpha$ of M and vector bundle charts ψ_α satisfying

$$\begin{array}{ccc} E|U_\alpha := p^{-1}(U_\alpha) & \xrightarrow{\psi_\alpha} & U_\alpha \times V \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle pr_1} \\ U_\alpha & = & U_\alpha. \end{array}$$

Here V is a fixed finite dimensional real vector space, called the standard fiber. We have $(\psi_\alpha \circ \psi_\beta^{-1})(x, v) = (x, \psi_{\alpha\beta}(x)v)$ for transition functions $\psi_{\alpha\beta} : U_{\alpha\beta} = U_\alpha \cap U_\beta \to GL(V)$, which are real analytic.

If we extend the transition functions $\psi_{\alpha\beta}$ to $\widetilde{\psi}_{\alpha\beta} : \widetilde{U}_{\alpha\beta} \to GL(V_\mathbb{C}) = GL(V)_\mathbb{C}$, we see that there is a holomorphic vector bundle $(E_\mathbb{C}, p_\mathbb{C}, M_\mathbb{C})$ over a complex (even Stein) manifold $M_\mathbb{C}$ such that E is isomorphic to a real part of $E_\mathbb{C}|M$, compare 3.10. The germ of it along M is unique.

Real analytic sections $s : M \to E$ coincide with certain germs along M of holomorphic sections $W \to E_\mathbb{C}$ for open neighborhoods W of M in $M_\mathbb{C}$.

7.2. Spaces of sections. For a holomorphic vector bundle (F, q, N) over a complex manifold N we denote by $\mathcal{H}(F)$ the vector space of all holomorphic sections $s : N \to F$, equipped with the compact open topology, a nuclear Fréchet topology, since it is initial with respect to the cone

$$\mathcal{H}(F) \to \mathcal{H}(F|U_\alpha) \xrightarrow{(pr_2 \circ \psi_\alpha)_*} \mathcal{H}(U_\alpha, \mathbb{C}^k) = \mathcal{H}(U_\alpha, \mathbb{C})^k,$$

of mappings into nuclear spaces, see 3.2.

For a subset $A \subseteq N$ let $\mathcal{H}(F|A)$ be the space of germs along A of holomorphic sections $W \to F|W$ for open sets W in N containing A. We equip $\mathcal{H}(F|A)$ with the locally convex topology induced by the inductive cone $\mathcal{H}(F|W) \to \mathcal{H}(F|A)$ for all W. This is Hausdorff since jet prolongations at points in A separate germs.

For a real analytic vector bundle (E, p, M) let $C^\omega(E)$ be the space of real analytic sections $s : M \to E$. Furthermore let $C^\omega(E|A)$ denote the space of germs at a subset $A \subseteq M$ of real analytic sections defined near A. The complexification of this real vector space is the complex vector space $\mathcal{H}(E_\mathbb{C}|A)$, because germs of real analytic sections $s : A \to E$ extend uniquely to germs along A of holomorphic sections $W \to E_\mathbb{C}$ for open sets W in $M_\mathbb{C}$ containing A, compare 3.11.

We topologize $C^\omega(E|A)$ as subspace of $\mathcal{H}(E_\mathbb{C}|A)$.

For a smooth vector bundle (E, p, M) let $C^\infty(E)$ denote the nuclear Fréchet space of all smooth sections with the topology of uniform convergence on compact subsets, in all derivatives separately, see [18] and [5], 4.6.

7.3. Theorem (Structure on spaces of germs of sections). *If (E, p, M) is a real analytic vector bundle and A a closed subset of M, then the space $C^\omega(E|A)$ is convenient. Its bornology is generated by the cone*

$$C^\omega(E|A) \xrightarrow{(\psi_\alpha)_*} C^\omega(U_\alpha \cap A \subseteq U_\alpha, \mathbb{R})^k,$$

where $(U_\alpha, \psi_\alpha)_\alpha$ is an arbitrary real analytic vector bundle atlas of E. If A is compact, the space $C^\omega(E|A)$ is nuclear.

The corresponding statement for smooth sections is also true, see [5], 4.6.23.

Proof. We show the corresponding result for holomorphic germs. By taking real parts the theorem then follows.

So let (F, q, N) be a holomorphic vector bundle and let A be a closed subset of N. Then $\mathcal{H}(F|A)$ is a bornological locally convex space, since it is an inductive limit of the Fréchet spaces $\mathcal{H}(F|W)$ for open sets W containing A. If A is compact, $\mathcal{H}(F|A)$ is nuclear as countable inductive limit.

Let $(U_\alpha, \psi_\alpha)_\alpha$ be a holomorphic vector bundle atlas for F.

Then we consider the cone

$$\mathcal{H}(F|A) \xrightarrow{(\psi_\alpha)_*} \mathcal{H}(U_\alpha \cap A \subseteq U_\alpha, \mathbb{C}^k) = \mathcal{H}(U_\alpha \cap A \subseteq U_\alpha, \mathbb{C})^k.$$

Obviously each mapping is continuous, so the cone induces a bornology which is coarser than the given one, and which is complete by 3.13.

It remains to show that every subset $\mathcal{B} \subseteq \mathcal{H}(F|A)$, such that $(u_\alpha)_*(\mathcal{B})$ is bounded in every $\mathcal{H}(U_\alpha \cap A \subseteq U_\alpha, \mathbb{C})^k$, is bounded in $\mathcal{H}(F|W)$ for some open neighborhood W of A in N.

Since all restriction mappings to smaller subsets are continuous, it suffices to show the assertions of the theorem for some refinement of the atlas (U_α). Let us pass first to a relatively compact refinement. By topological dimension theory there is a further refinement such that any $\dim_\mathbb{R} N + 2$ different sets have empty intersection. We call the resulting atlas again (U_α). Let (K_α) be a cover of N consisting of compact subsets $K_\alpha \subseteq U_\alpha$ for all α.

For any finite set $\mathcal{A}$ of indices let us consider now all non empty intersections $U_\mathcal{A} := \bigcap_{\alpha\in\mathcal{A}} U_\alpha$ and $K_\mathcal{A} := \bigcap_{\alpha\in\mathcal{A}} K_\alpha$. Since by 3.4 (or 3.6) the space $\mathcal{H}(A \cap K_\mathcal{A} \subseteq U_\mathcal{A}, \mathbb{C})$ is a regular inductive limit there are open sets $W_\mathcal{A} \subseteq U_\mathcal{A}$ containing $A \cap K_\mathcal{A}$, such that $\mathcal{B}|(A \cap K_\mathcal{A})$ (more precisely $(\psi_\mathcal{A})_*(\mathcal{B}|(A \cap K_\mathcal{A}))$ for some suitable vector bundle chart mappings $\psi_\mathcal{A}$) is contained and bounded in $\mathcal{H}(W_\mathcal{A}, \mathbb{C})^k$. By passing to smaller open sets we may assume that $W_{\mathcal{A}_1} \subseteq W_{\mathcal{A}_2}$ for $\mathcal{A}_1 \supseteq \mathcal{A}_2$. Now we define the subset

$$W := \bigcup_{\mathcal{A}} \widehat{W}_\mathcal{A}, \text{ where } \widehat{W}_\mathcal{A} := W_\mathcal{A} \setminus \bigcup_{\alpha\notin\mathcal{A}} K_\alpha$$

W is open since (K_α) is a locally finite cover. For $x \in A$ let $\mathcal{A} := \{\alpha : x \in K_\alpha\}$, then $x \in \widehat{W}_\mathcal{A}$.

Now we show that every germ $s \in \mathcal{B}$ has a unique extension to W. For every $\mathcal{A}$ the germ of s along $A \cap K_\mathcal{A}$ has a unique extension $s_\mathcal{A}$ to a section over $W_\mathcal{A}$ and for $\mathcal{A}_1 \subseteq \mathcal{A}_2$ we have $s_{\mathcal{A}_1}|W_{\mathcal{A}_2} = s_{\mathcal{A}_2}$. We define the extension s_W by $s_W|\widehat{W}_\mathcal{A} = s_\mathcal{A}|\widehat{W}_\mathcal{A}$. This is well defined since one may check that $\widehat{W}_{\mathcal{A}_1} \cap \widehat{W}_{\mathcal{A}_2} \subseteq \widehat{W}_{\mathcal{A}_1\cap\mathcal{A}_2}$.

$\mathcal{B}$ is bounded in $\mathcal{H}(F|W)$ if it is uniformly bounded on each compact subset K of W. This is true since each K is covered by finitely many W_α and $\mathcal{B}|A \cap K_\alpha$ is bounded in $\mathcal{H}(W_\alpha, \mathbb{C})$. $\square$

7.4. Let $f : E \to E'$ be a real analytic vector bundle homomorphism, i.e. we have a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ M & \xrightarrow[\underline{f}]{} & M' \end{array}$$

of real analytic mappings such that f is fiberwise linear.

Lemma. *If f is fiberwise invertible, then $f^* : C^\omega(E') \to C^\omega(E)$, given by*

$$(f^*s)(x) := (f_x)^{-1}(s(\underline{f}(x))),$$

is continuous and linear.

If $\underline{f} = Id_M$ then the mapping $f_* : C^\omega(E) \to C^\omega(E')$, given by $s \mapsto f \circ s$, is continuous and linear.

Proof. Extend f to the complexification. Here for the compact open topology on the corresponding spaces of holomorphic sections the assertion is trivial. □

7.5 Real analytic mappings are dense. Let (E, p, M) be a real analytic vector bundle. Then there is another real analytic vector bundle (E', p', M) such that the Whitney sum $E \oplus E' \to M$ is real analytically isomorphic to a trivial bundle $M \times \mathbb{R}^k \to M$. This is seen as follows: By [7], Theorem 3, there is a closed real analytic embedding $i : E \to \mathbb{R}^k$ for some k. Now the fiber derivative along the zero section gives a fiberwise linear and injective real analytic mapping $E \to \mathbb{R}^k$ which induces a real analytic embedding j of the vector bundle (E, p, M) into the trivial bundle $M \times \mathbb{R}^k \to M$. The standard inner product on $\mathbb{R}^k$ gives rise to the real analytic orthogonal complementary vector bundle $E' := E^\perp$ and a real analytic Riemannian metric on the vector bundle E.

Hence an embedding of the real analytic vector bundle into another one induces a linear embedding of the spaces of real analytic sections onto a direct summand.

We remark that in this situation the orthogonal projection onto the vertical bundle VE within $T(M \times \mathbb{R}^k)$ gives rise to a real analytic linear connection (covariant derivative) $\nabla : C^\omega(TM) \times C^\omega(E) \to C^\omega(E)$. If $c : \mathbb{R} \to M$ is a smooth or real analytic curve in M, we have the parallel transport $Pt(c, t)v \in E_{c(t)}$ for all $v \in E_{c(0)}$ and $t \in \mathbb{R}$ which is smooth or real analytic, respectively, on $\mathbb{R} \times E_{c(0)}$. It is given by the differential equation $\nabla_{\partial_t} Pt(c, t)v = 0$.

7.6 Corollary. *If ∇ is a real analytic linear connection on a vector bundle (E, p, M), then the following cone generates the bornology on $C^\omega(E)$.*

$$C^\omega(E) \xrightarrow{Pt(c,\quad)^*} C^\alpha(\mathbb{R}, E_{c(0)}),$$

for all $c \in C^\alpha(\mathbb{R}, M)$ and $\alpha = \omega, \infty$.

Proof. The bornology induced by the cone is coarser that the given one by 7.4. A still coarser bornology is induced by all curves subordinated to some vector bundle atlas. Hence by theorem 7.3 it suffices to check for a trivial bundle, that this bornology coincides with the given one. So we assume that E is trivial. For the constant parallel transport the result follows from lemma 5.3. The change to an arbitrary real analytic parallel transport can be absorbed into a C^α-isomorphism of each vector bundle c^*E separately. $\square$

7.7. Lemma (Curves in spaces of sections).
1. For a real analytic vector bundle (E,p,M) a curve $c:\mathbb{R}\to C^\omega(E)$ is real analytic if and only if the associated mapping $\hat{c}:\mathbb{R}\times M\to E$ is real analytic.

The curve $c:\mathbb{R}\to C^\omega(E)$ is smooth if and only if $\hat{c}:\mathbb{R}\times M\to E$ satisfies the following condition:

> *For each n there is an open neighborhood U_n of $\mathbb{R}\times M$ in $\mathbb{R}\times M_{\mathbb{C}}$ and a (unique) C^n-extension $\tilde{c}:U_n\to E_{\mathbb{C}}$ such that $\tilde{c}(t,\quad)$ is holomorphic for all $t\in\mathbb{R}$.*

2. For a smooth vector bundle (E,p,M) a curve $c:\mathbb{R}\to C^\infty(E)$ is smooth if and only if $\hat{c}:\mathbb{R}\times M\to E$ is smooth.

The curve $c:\mathbb{R}\to C^\infty(E)$ is real analytic if and only if $\hat{c}$ satisfies the following condition:

> *For each n there is an open neighborhood U_n of $\mathbb{R}\times M$ in $\mathbb{C}\times M$ and a (unique) C^n-extension $\tilde{c}:U_n\to E_{\mathbb{C}}$ such that $\tilde{c}(\quad,x)$ is holomorphic for all $x\in M$.*

Proof. 1. By theorem 7.3 we may assume that M is open in $\mathbb{R}^m$, and we may consider $C^\omega(M,\mathbb{R})$ instead of $C^\omega(E)$. The statement on real analyticity follows from cartesian closedness, 5.12.

To prove the statement on smoothness we note that $C^\omega(M,\mathbb{R})$ is the real part of $\mathcal{H}(M\subseteq\mathbb{C}^m,\mathbb{C})$ by 3.11, which is a regular inductive limit of spaces $\mathcal{H}(W,\mathbb{C})$ for open neighborhoods W of M in $\mathbb{C}^m$ by 3.6. By [13], Folgerung on p. 114, c is smooth if and only if for each n, locally in $\mathbb{R}$ it factors to a C^n-curve into some $\mathcal{H}(W,\mathbb{C})$, which sits continuously embedded in $C^\infty(W,\mathbb{R}^2)$. So the associated mapping $\mathbb{R}\times M_{\mathbb{C}}\supseteq J\times W\to\mathbb{C}$ is C^n and holomorphic in the second variables, and conversely.

2. By 7.3 we may assume that M is open in $\mathbb{R}^m$, and we may consider $C^\infty(M,\mathbb{R})$ instead of $C^\infty(E)$. The statement on smoothness follows from cartesian closedness of smooth mappings, similarly as the C^ω-statement above.

To prove the statement on real analyticity we note that $C^\infty(M,\mathbb{R})$ is the projective limit of the Banach spaces $C^n(M_i,\mathbb{R})$, where M_i is a covering of M by compact cubes. By lemma 1.11 the curve c is real analytic if and only if it is real analytic into each $C^n(M_i,\mathbb{R})$, by 1.6 and 1.7 it extends locally to a holomorphic curve $\mathbb{C}\to C^n(M_i,\mathbb{C})$. Its associated mappings fit together to the required C^n-extension $\tilde{c}$. $\square$

7.8. Corollary. *Let (E,p,M) and (E',p',M) be real analytic vector bundles over a compact manifold M. Let $W \subseteq E$ be an open subset such that $p(W) = M$, and let $f : W \to E'$ be a fiber respecting real analytic (nonlinear) mapping.*

Then $C^\infty(W) := \{s \in C^\infty(E) : s(M) \subseteq W\}$ is open and not empty in the convenient vector space $C^\infty(E)$. The mapping $f_* : C^\infty(W) \to C^\infty(E')$ is real analytic with derivative $(d_v f)_* : C^\infty(W) \times C^\infty(E) \to C^\infty(E')$, where the vertical derivative $d_v f : W \times_M E \to E'$ is given by

$$d_v f(u,w) := \frac{d}{dt}\Big|_0 f(u+tw).$$

Then $C^\omega(W) := \{s \in C^\omega(E) : s(M) \subseteq W\}$ is open and not empty in the convenient vector space $C^\omega(E)$ and the mapping $f_* : C^\omega(W) \to C^\omega(E')$ is real analytic with derivative $(d_v f)_* : C^\omega(W) \times C^\omega(E) \to C^\omega(E')$.

Proof. The set $C^\infty(W)$ is open in $C^\infty(E)$ since it is open in the compact-open topology. Then $C^\omega(W)$ is open in $C^\omega(E)$ since $C^\omega(E) \to C^\infty(E)$ is continuous by 3.12 and 7.3.

Now we prove the statement for $C^\omega(W)$, the proof for $C^\infty(W)$ is then similar.

We check that f_* maps C^ω-curves to C^ω-curves and maps smooth curves to smooth curves.

If $c : \mathbb{R} \to C^\omega(W) \subseteq C^\omega(E)$ is C^ω, then $\hat{c} : \mathbb{R} \times M \to E$ is C^ω by lemma 7.7. So $(f_* \circ c)\hat{}= f \circ \hat{c} : \mathbb{R} \times M \to E'$ is also C^ω, hence $f_* \circ c : \mathbb{R} \to C^\omega(E')$ is C^ω.

If $c : \mathbb{R} \to C^\omega(W) \subseteq C^\omega(E)$ is smooth, for each n there is an open neighborhood U_n of $\mathbb{R} \times M$ in $\mathbb{R} \times M_{\mathbb{C}}$ and a C^n-extension $\tilde{c} : U_n \to E_{\mathbb{C}}$ of $\hat{c}$ such that $\tilde{c}(t, \quad)$ is holomorphic. The mapping $f : W \to E'$ has also a holomorphic extension $\tilde{f} : E_{\mathbb{C}} \supseteq W_{\mathbb{C}} \to E'_{\mathbb{C}}$. Then $\tilde{f} \circ \tilde{c}$ is an extension of $(f_* \circ c)\hat{}$ satisfying the condition in lemma 7.7, so $f_* \circ c : \mathbb{R} \to C^\omega(E')$ is smooth. □

8. Manifolds of analytic mappings

8.1. Infinite dimensional real analytic manifolds. A chart (U,u) on a set $\mathcal{M}$ is a bijection $u : U \to u(U) \subseteq E_U$ from a subset $U \subseteq \mathcal{M}$ onto a c^∞-open subset of a convenient vector space E_U. Two such charts are called C^ω-compatible, if the chart change mapping $u \circ v^{-1} : v(U \cap V) \to u(U \cap V)$ is a C^ω-diffeomorphism between c^∞-open subsets of convenient vector spaces. A C^ω-atlas on $\mathcal{M}$ is a set of pairwise C^ω-compatible charts on $\mathcal{M}$ which cover $\mathcal{M}$. Two such atlases are equivalent if their union is again a C^ω-atlas. A C^ω-structure on $\mathcal{M}$ is an equivalence class of C^ω-atlases. A C^ω-manifold $\mathcal{M}$ is a set together with a C^ω-structure on it.

The natural topology on $\mathcal{M}$ is the identification topology, where a subset $W \subseteq \mathcal{M}$ is open if and only if $u(U \cap W)$ is c^∞-open in E_U for all charts in a C^ω-atlas belonging to the structure. In the finite dimensional treatment of manifolds one requires that this topology has some properties: Hausdorff, separable or metrizable or paracompact.

In infinite dimensions it is not yet clear what the most sensible requirements are. Hausdorff does not imply regular. If $\mathcal{M}$ is Hausdorff and regular, and if all modeling vector spaces admit smooth bump functions, any locally finite open cover of $\mathcal{M}$ admits a subordinated smooth partition of unity.

Mappings between C^ω-manifolds are called C^∞ or C^ω if they are continuous and their chart representations are smooth or real analytic, respectively.

The final topology with respect to all smooth curves coincides with the identification topology on a C^ω-manifold. So the following two statements hold:

A mapping $f : \mathcal{M} \to \mathcal{N}$ between C^ω-manifolds is C^∞ if $f \circ c$ is C^∞ for each C^∞-curve in $\mathcal{M}$.

f is C^ω if it is C^∞ and $f \circ c$ is C^ω for each C^ω-curve in $\mathcal{M}$.

The tangent bundle $T\mathcal{M} \to \mathcal{M}$ of a C^ω-manifold $\mathcal{M}$ is the vector bundle glued from the sets $u(U) \times E_U$ via the transition functions $(x, y) \mapsto ((u \circ v^{-1})(x), d(u \circ v^{-1})(x)y)$ for all charts (U, u) and (V, v) in a C^ω-atlas of $\mathcal{M}$.

8.2. Theorem (Manifold structure of $C^\omega(M, N)$). *Let M and N be real analytic manifolds, let M be compact. Then the space $C^\omega(M, N)$ of all real analytic mappings from M to N is a real analytic manifold, modeled on spaces $C^\omega(f^*TN)$ of real analytic sections of pullback bundles along $f : M \to N$ over M.*

Proof. Choose a real analytic Riemannian metric on N. See 7.5 for a sketch how to find one. Let $\exp : TN \supseteq U \to N$ be the real analytic exponential mapping of this Riemannian metric, defined on a suitable open neighborhood of the zero section. We may assume that U is chosen in such a way that $(\pi_N, \exp) : U \to N \times N$ is a real analytic diffeomorphism onto an open neighborhood V of the diagonal.

For $f \in C^\omega(M, N)$ we consider the pullback vector bundle

$$\begin{array}{ccccc} M \times_N TN & = & f^*TN & \xrightarrow{\pi_N^* f} & TN \\ & & {\scriptstyle f^*\pi_N}\downarrow & & \downarrow{\scriptstyle \pi_N} \\ & & M & \xrightarrow{f} & N. \end{array}$$

Then $C^\omega(f^*TN)$ is canonically isomorphic to the space

$$C^\omega_f(M, TN) := \{h \in C^\omega(M, TN) : \pi_N \circ h = f\}$$

via $s \mapsto (\pi_N^* f) \circ s$ and $(Id_M, h) \leftarrow h$.

Now let

$$U_f := \{g \in C^\omega(M,N) : (f(x), g(x)) \in V \text{ for all } x \in M\}$$

and let $u_f : U_f \to C^\omega(f^*TN)$ be given by

$$u_f(g)(x) = (x, \exp_{f(x)}^{-1}(g(x))) = (x, ((\pi_N, \exp)^{-1} \circ (f, g))(x)).$$

Then u_f is a bijective mapping from U_f onto $\{s \in C^\omega(f^*TN) : s(M) \subseteq f^*U\}$, whose inverse is given by $u_f^{-1}(s) = \exp \circ (\pi_N^* f) \circ s$, where we view $U \to N$ as fiber bundle. Since M is compact, $u_f(U_f)$ is open in $C^\omega(f^*TN)$ for the compact open topology, thus also for the finer topology described in 7.2.

Now we consider the atlas $(U_f, u_f)_{f \in C^\omega(M,N)}$ for $C^\omega(M,N)$. Its chart change mappings are given for $s \in u_g(U_f \cap U_g) \subseteq C^\omega(g^*TN)$ by

$$\begin{aligned}(u_f \circ u_g^{-1})(s) &= (Id_M, (\pi_N, \exp)^{-1} \circ (f, \exp \circ (\pi_N^* g) \circ s)) \\ &= (\tau_f^{-1} \circ \tau_g)_*(s),\end{aligned}$$

where $\tau_g(x, Y_{g(x)}) := (x, \exp_{g(x)}(Y_{g(x)}))$ is a real analytic diffeomorphism

$$\tau_g : g^*TN \supseteq g^*U \to (g \times Id_N)^{-1}(V) \subseteq M \times N$$

which is fiber respecting over M. Thus by 7.8 the chart change $u_f \circ u_g^{-1} = (\tau_f^{-1} \circ \tau_g)_*$ is defined on an open subset and real analytic.

Finally we put the identification topology from this atlas onto the space $C^\omega(M,N)$, which is obviously finer than the compact open topology and thus Hausdorff.

The equation $u_f \circ u_g^{-1} = (\tau_f^{-1} \circ \tau_g)_*$ shows that the real analytic structure does not depend on the choice of the real analytic Riemannian metric on N. □

Remark. If N is a finite dimensional vector space, then the structure of a real analytic manifold on $C^\omega(M,N)$ described here coincides with that of the space $C^\omega(M \times N \to M)$ of sections discussed in 7.2, because the exponential mapping of any euclidean structure is the affine structure of N.

8.3. Theorem (C^ω-manifold structure on $C^\infty(M,N)$). *Let M and N be real analytic manifolds, with M compact. Then the smooth manifold $C^\infty(M,N)$ with the structure from* [18]*, 10.4 is even a real analytic manifold.*

Proof. For a fixed real analytic exponential mapping on N the charts (U_f, u_f) (from 8.2 with C^ω replaced by C^∞, which agrees with those from [18], 10.4, see also [5], 4.7) for $f \in C^\omega(M,N)$ form a smooth atlas for $C^\infty(M,N)$, since $C^\omega(M,N)$ is dense in $C^\infty(M,N)$ by [7], Proposition 8. The chart changings $u_f \circ u_g^{-1} = (\tau_f^{-1} \circ \tau_g)_*$ are real analytic by 7.8. □

8.4 Remark. If M is not compact, $C^\omega(M,N)$ is dense in $C^\infty(M,N)$ for the Whitney-C^∞-topology by [7], Proposition 8. This is not the case for the $\mathcal{D}$-topology from [18], in which $C^\infty(M,N)$ is a smooth manifold. The charts U_f for $f \in C^\omega(M,N)$ do not cover $C^\infty(M,N)$.

8.5. Theorem. *Let M and N be real analytic manifolds, where M is compact, the two infinite dimensional real analytic vector bundles $TC^\omega(M,N)$ and $C^\omega(M,TN)$ over $C^\omega(M,N)$ are canonically isomorphic. The same assertion is true for $C^\infty(M,N)$.*

Proof. Let us fix an exponential mapping exp on N. It gives rise to the canonical atlas (U_f, u_f) for $C^\omega(M,N)$ from 8.2. $TC^\omega(M,N)$ is defined as the vector bundle glued from the transition functions $(r,s) \mapsto (u_f(u_g^{-1}(r)), d(u_f \circ u_g^{-1})(r)s)$. Then $T(\exp)$ composed with the canonical flip on T^2N is an exponential mapping for TN, which gives rise to the canonical atlas $(U_{0\circ f}, u_{0\circ f})$ for $C^\omega(M,TN)$, where 0 is the zero section of TN. Via some canonical identifications the two sets of transition functions are the same, as is shown in great detail in [18], 10.11–10.13 for the analogous situation for smooth mappings. □

8.6. Lemma (Curves in spaces of mappings). *Let M and N be finite dimensional real analytic manifolds with M compact.*
1. A curve $c : \mathbb{R} \to C^\omega(M,N)$ is real analytic if and only if the associated mapping $\hat{c} : \mathbb{R} \times M \to N$ is real analytic.
The curve $c : \mathbb{R} \to C^\omega(M,N)$ is smooth if and only if $\hat{c} : \mathbb{R} \times M \to N$ satisfies the following condition:

> *For each n there is an open neighborhood U_n of $\mathbb{R} \times M$ in $\mathbb{R} \times M_{\mathbb{C}}$ and a (unique) C^n-extension $\tilde{c} : U_n \to N_{\mathbb{C}}$ such that $\tilde{c}(t, \quad)$ is holomorphic for all $t \in \mathbb{R}$.*

2. A curve $c : \mathbb{R} \to C^\infty(M,N)$ is smooth if and only if $\hat{c} : \mathbb{R} \times M \to N$ is smooth.
The curve $c : \mathbb{R} \to C^\infty(M,N)$ is real analytic if and only if $\hat{c}$ satisfies the following condition:

> *For each n there is an open neighborhood U_n of $\mathbb{R}\times M$ in $\mathbb{C}\times M$ and a (unique) C^n-extension $\tilde{c} : U_n \to N_{\mathbb{C}}$ such that $\tilde{c}(\quad, x)$ is holomorphic for all $x \in M$.*

Proof. This follows from 7.7 and the chart structure on $C^\omega(M,N)$. □

8.7. Corollary. *Let M and N be real analytic finite dimensional manifolds with M compact. Let (U_α, u_α) be a real analytic atlas for M and let $i : N \to \mathbb{R}^n$ be a closed real analytic embedding into some $\mathbb{R}^n$. Let $\mathcal{M}$ be a possibly infinite dimensional real analytic manifold.*

Then $f : \mathcal{M} \to C^\omega(M,N)$ is real analytic or smooth if and only if

$$C^\omega(u_\alpha^{-1}, i) \circ f : \mathcal{M} \to C^\omega(u_\alpha(U_\alpha), \mathbb{R}^n)$$

is real analytic or smooth, respectively.

Furthermore $f : \mathcal{M} \to C^\infty(M,N)$ is real analytic or smooth if and only if

$$C^\infty(u_\alpha^{-1}, i) \circ f : \mathcal{M} \to C^\infty(u_\alpha(U_\alpha), \mathbb{R}^n)$$

is real analytic or smooth, respectively.

Proof. By 8.1 we may assume that $\mathcal{M} = \mathbb{R}$. Then we can use lemma 8.6 on all appearing function spaces. □

8.8. Theorem (Exponential law). *Let $\mathcal{M}$ be a (possibly infinite dimensional) real analytic manifold, and let M and N be finite dimensional real analytic manifolds where M is compact.*

Then real analytic mappings $f : \mathcal{M} \to C^\omega(M,N)$ and real analytic mappings $\widehat{f} : \mathcal{M} \times M \to N$ correspond to each other bijectively.

Proof. Clearly we may assume that $\mathcal{M}$ is a c^∞-open subset in a convenient vector space. Lemma 8.7 then reduces the assertion to cartesian closedness, which holds by 5.12. □

8.9. Corollary. *If M is compact and M, N are finite dimensional real analytic manifolds, then the evaluation mapping* $\operatorname{ev} : C^\omega(M,N) \times M \to N$ *is real analytic.*

If P is another compact real analytic manifold, then the composition mapping $\operatorname{comp} : C^\omega(M,N) \times C^\omega(P,M) \to C^\omega(P,N)$ *is real analytic.*

In particular $f_* : C^\omega(M,N) \to C^\omega(M,N')$ and $g^* : C^\omega(M,N) \to C^\omega(P,N)$ are real analytic for $f \in C^\omega(N,N')$ and $g \in C^\omega(P,M)$.

Proof. The mapping $\operatorname{ev}^\vee = Id_{C^\omega(M,N)}$ is real analytic, so ev is it by 8.8. The mapping $\operatorname{comp}^\wedge = \operatorname{ev} \circ (Id_{C^\omega(M,N)} \times \operatorname{ev}) : C^\omega(M,N) \times C^\omega(P,M) \times P \to C^\omega(M,N) \times M \to N$ is real analytic, so also comp. □

8.10. Lemma. *Let M_i and N_i are finite dimensional real analytic manifolds with M_i compact. Then for $f \in C^\infty(N_1,N_2)$ the push forward $f_* : C^\infty(M,N_1) \to C^\infty(M,N_2)$ is real analytic if and only if f is real analytic. For $f \in C^\infty(M_2,M_1)$ the pullback $f^* : C^\infty(M_1,N) \to C^\infty(M_2,N)$ is, however, always real analytic.*

Proof. If f is real analytic and if $g \in C^\omega(M,N_1)$, then the mapping $u_{f\circ g} \circ f_* \circ u_g^{-1}$ is a push forward by a real analytic mapping, which is real analytic by 7.8.

Obviously the canonical maps $\text{const} : N_1 \to C^\infty(M, N_1)$ and $\text{ev}_x : C^\infty(M, N_2) \to N_2$ are real analytic. If f_* is real analytic, also $f = \text{ev}_x \circ f_* \circ \text{const}$ is it.

For the second statement choose real analytic atlases (U^i_α, u^i_α) of M_i such that $f(U^2_\alpha) \subseteq U^1_\alpha$ and a closed real analytic embedding $j : N \to \mathbb{R}^n$. Then the diagram

$$\begin{array}{ccc} C^\infty(M_1, N) & \xrightarrow{\quad f^* \quad} & C^\infty(M_2, N) \\ {\scriptstyle C^\infty((u^1_\alpha)^{-1}, j)} \downarrow & & \downarrow {\scriptstyle C^\infty((u^2_\alpha)^{-1}, j)} \\ C^\infty(u^1_\alpha(U^1_\alpha), \mathbb{R}^n) & \xrightarrow[(u^2_\alpha \circ f \circ (u^1_\alpha)^{-1})^*]{} & C^\infty(u^2_\alpha(U^2_\alpha), \mathbb{R}^n) \end{array}$$

commutes, the bottom arrow is a continuous and linear mapping, so it is real analytic. Thus by 8.7 the mapping f_* is real analytic. □

8.11. Theorem (Real analytic diffeomorphism group). *For a compact real analytic manifold M the group $\text{Diff}^\omega(M)$ of all real analytic diffeomorphisms of M is an open submanifold of $C^\omega(M, M)$, composition and inversion are real analytic.*

Proof. $\text{Diff}^\omega(M)$ is open in $C^\omega(M, M)$ in the compact open topology, thus also in the finer manifold topology. The composition is real analytic by 8.9, so it remains to show that the inversion *inv* is real analytic.

Let $c : \mathbb{R} \to \text{Diff}^\omega(M)$ be a C^ω-curve. Then the associated mapping $\hat{c} : \mathbb{R} \times M \to M$ is C^ω by 8.8 and $(inv \circ c)^\wedge$ is the solution of the implicit equation $\hat{c}(t, (inv \circ c)^\wedge(t, x)) = x$ and therefore real analytic by the finite dimensional implicit function theorem. Hence $inv \circ c : \mathbb{R} \to \text{Diff}^\omega(M)$ is real analytic by 8.8 again.

Let $c : \mathbb{R} \to \text{Diff}^\omega(M)$ be a C^∞-curve. Then by lemma 8.6 the associated mapping $\hat{c} : \mathbb{R} \times M \to M$ has a unique extension to a C^n-mapping $\mathbb{R} \times M_\mathbb{C} \supseteq J \times W \to M_\mathbb{C}$ which is holomorphic in the second variables (has $\mathbb{C}$-linear derivatives), for each $n \geq 1$. The same assertion holds for the curve $inv \circ c$ by the finite dimensional implicit function theorem for C^n-mappings. □

8.12. Theorem (Lie algebra of the diffeomorphism group). *For a compact real analytic manifold M the Lie algebra of the real analytic infinite dimensional Lie group $\text{Diff}^\omega(M)$ is the convenient vector space $C^\omega(TM)$ of all real analytic vector fields on M, equipped with the negative of the usual Lie bracket. The exponential mapping $\text{Exp} : C^\omega(TM) \to \text{Diff}^\omega(M)$ is the flow mapping to time 1, and it is real analytic.*

Proof. The tangent space at Id_M of $\text{Diff}^\omega(M)$ is the space $C^\omega(TM)$ of real analytic vector fields on M, by 8.5. The one parameter subgroup of a tangent vector is the flow $t \mapsto Fl^X_t$ of the corresponding vector field $X \in C^\omega(TM)$, so $\operatorname{Exp}(X) = Fl^X_1$ which exists since M is compact.

In order to show that $\operatorname{Exp} : C^\omega(TM) \to \operatorname{Diff}^\omega(M) \subseteq C^\omega(M, M)$ is real analytic, by the exponential law 8.8 it suffices to show that the associated mapping

$$\operatorname{Exp}^\wedge = Fl_1 : C^\omega(TM) \times M \to M$$

is real analytic. This follows from the finite dimensional theory of ordinary real analytic and smooth differential equations.

For $X \in C^\omega(TM)$ let L_X denote the left invariant vector field on $\operatorname{Diff}^\omega(M)$. Its flow is given by $Fl^{L_X}_t(f) = f \circ \operatorname{Exp}(tX)$. The usual proof of differential geometry shows that $[L_X, L_Y] = \frac{d}{dt}|_0 (Fl^{L_X}_t)^* L_Y$, thus for $e = Id_M$ we have

$$\begin{aligned}
[L_X, L_Y](e) &= (\tfrac{d}{dt}|_0 (Fl^{L_X}_t)^* L_Y)(e) \\
&= \tfrac{d}{dt}|_0 (T(Fl^{L_X}_{-t}) \circ L_Y \circ Fl^{L_X}_t)(e) \\
&= \tfrac{d}{dt}|_0 T(Fl^{L_X}_{-t})(L_Y(e \circ Fl^X_t) \\
&= \tfrac{d}{dt}|_0 T(Fl^{L_X}_{-t})(T(Fl^X_t) \circ Y) \\
&= \tfrac{d}{dt}|_0 (T(Fl^X_t) \circ Y \circ Fl^X_{-t}) \\
&= \tfrac{d}{dt}|_0 (Fl^X_{-t})^* Y = -[X, Y]. \qquad \square
\end{aligned}$$

8.13. Example. *The exponential map* $\operatorname{Exp} : C^\omega(TS^1) \to \operatorname{Diff}^\omega(S^1)$ *is neither locally injective nor surjective on any neighborhood of the identity.*

Proof. The proof of [24], 3.3.1 for the group of smooth diffeomorphisms of S^1 can be adapted to the real analytic case:

$$\varphi_n(\theta) = \theta + \tfrac{2\pi}{n} + \tfrac{1}{2^n} \sin n\theta$$

is Mackey convergent (in U_{Id}) to Id_{S^1} in $\operatorname{Diff}^\omega(S^1)$ and is not in the image of the exponential mapping. $\square$

8.14. Remarks. For a real analytic manifold M the group $\operatorname{Diff}(M)$ of all smooth diffeomorphisms of M is a real analytic open submanifold of $C^\infty(M, M)$ and is a smooth Lie group by [18], 11.11. The composition mapping is not real analytic by 8.10. Moreover it does not carry any real analytic Lie group structure by [22], 9.2, and it has no complexification in general, see [24], 3.3. The mapping

$$Ad \circ \operatorname{Exp} : C^\infty(TM) \to \operatorname{Diff}(M) \to L(C^\infty(TM), C^\infty(TM))$$

is not real analytic, see [19], 4.11.

For $x \in M$ the mapping $ev_x \circ \mathrm{Exp} : C^\infty(TM) \to \mathrm{Diff}(M) \to M$ is not real analytic, since $(ev_x \circ \mathrm{Exp})(tX) = Fl^X_t(x)$ which is not real analytic in t for general smooth X.

The exponential mapping $\mathrm{Exp} : C^\infty(TM) \to \mathrm{Diff}(M)$ is in a very strong sense not surjective: In [6] it is shown, that $\mathrm{Diff}(M)$ contains an arcwise connected free subgroup on $2^{\aleph_0}$ generators which meets the image of Exp only at the identity.

The real analytic Lie group $\mathrm{Diff}^\omega(M)$ is *regular* in the sense of [22], 7.6, where the original concept of [23] is weakened. This condition means that the mapping associating the evolution operator to each time dependent vector field on M is smooth. It is even real analytic, compare the proof of theorem 8.12.

8.15. Theorem (Principal bundle of embeddings). *Let M and N be real analytic manifolds with M compact. Then the set $Emb^\omega(M,N)$ of all real analytic embeddings $M \to N$ is an open submanifold of $C^\omega(M,N)$. It is the total space of a real analytic principal fiber bundle with structure group $\mathrm{Diff}^\omega(M)$, whose real analytic base manifold is the space of all submanifolds of N of type M.*

Proof. The proof given in [18], section 13 or [5], 4.7.8 is valid with the obvious changes. One starts with a real analytic Riemannian metric and uses its exponential mapping. The space of embeddings is open, since embeddings are open in $C^\infty(M,N)$, which induces a coarser topology. □

8.16. Theorem (Classifying space for $\mathrm{Diff}^\omega(M)$). *Let M be a compact real analytic manifold. Then the space $Emb^\omega(M,\ell^2)$ of real analytic embeddings of M into the Hilbert space ℓ^2 is the total space of a real analytic principal fibre bundle with structure group $\mathrm{Diff}^\omega(M)$ and real analytic base manifold $B(M,\ell^2)$, which is a classifying space for the Lie group $\mathrm{Diff}^\omega(M)$.*

Proof. The construction in 8.15 carries over to the Hilbert space $N = \ell^2$ with the appropriate changes to obtain a real analytic principal fibre bundle. Its total space is continuously contractible and so the bundle is classifying, see the argument in [20], section 6. □

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