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Reflection Groups on Riemannian Manifolds

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Preface

The aim of this diploma thesis is to study reflection groups on Riemannian manifolds M.

A reflection on a M is an isometry, whose tangent map in some point is an Euclidean reflection. A reflection group G is a discrete subgroup of the isometry group of M, that is generated by reflections.

We prove that algebraically G is a quotient of a Coxeter group and conversely every quotient of a countably generated Coxeter group may be realized as a reflection group on an appropriate Riemannian manifold.

A special case of reflections are disecting ones, i.e. the fixed point set "disects" the manifold into two connected components. If G is generated by disecting reflections we can tell more about its algebraic structure, then G is a Coxeter group. Examples come from simply connected manifolds. On a simply connected manifold every reflection is disecting, thus G a Coxeter group.

An important concept is the Weyl chamber, a connected component of the complement of the fixed point sets of all reflections. G acts on Weyl chambers. Of particular interest are those G, which act simply transitively on chambers. Then every chamber is a fundamental domain for the action of G and it has the structure of a manifold with corners. We can reconstruct M using only a chamber and the structure of G.

If G is generated by disecting reflections, then G acts simply transitively on chambers. We also give a partial characterization of reflection groups, that are Coxeter groups: if G acts simply transitively on chambers and is a Coxeter group, then G is generated by disecting reflections.

I give a short description of the structure of the text.

The first chapter contains background material on different topics, that will be needed in the text. We review group actions on manifolds, some useful theorems from Riemannian geometry, density and transversality theorems from differential topology and some facts about Coxeter groups. References are [10], [8], [3] for the various topics.

The second chapter introduces manifolds with corners. We define the tangent bundle, sprays and the exponential mapping. Manifolds with corners will be used in the last chapter. A more detailed exposition of manifolds with corners can be found in [11] or [9].

In the third chapter we discuss discrete groups of isometries, a slightly more general object than reflection groups. We are interested in fundamental domains and how to geometrically locate generating sets. For this and the remaining chapters [1] is the main source.

The fourth chapter introduces reflections, reflection groups and chambers. After proving some basic properties, we study groups that act simply transitively on chambers.

The fifth chamber is devoted to disecting reflections and groups generated by those. We investigate the algebraic structure of these groups and provide examples by showing that reflections on a simply connected manifold are disecting.

The sixth and last chapter is devoted to the problem of reconstructing a manifold from a chamber and the reflection group. We also show how given a suitable manifold with corners and a group, we can construct a Riemannian manifold such that this group acts on it as a reflection group.

Danke. Ich möchte mich bei meinem Betreuter Peter Michor bedanken, zu dem ich jederzeit gehen konnte und der stets Zeit hatte sich meiner Fragen anzunehmen.

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Chapter 1

Preliminaries

1.1 Transformation Groups

A *G-action* of a Lie group G on a manifold M is a smooth map $G \times M \to M$ such that $g.(h.x) = (g.h).x$ and $e.x = x$ for $g, h \in G$, $x \in M$ and e denoting the neutral element of G.

Let G act on the manifolds M and N and $f : M \to N$ be a map. f is called equivariant, if $f(g.x) = g.f(x)$ for $g \in G$, $x \in M$.

For $x \in M$ we denote by $G.x := \{g.x : g \in G\}$ the *G-orbit* of x and by $G_x := \{g : g.x = x\}$ the *isotropy group* of x. For $g \in G$ the set $M^g := \{x : g.x = x\}$ is the *fixed point set* of g. A point $x \in M$ is called a *regular point* of the G-action, if there exists an open neighborhood U of x such that for all $y \in U$, G_x is conjugate to a subgroup of G_y, i.e. $G_x \subseteq g.G_y.g^{-1}$ for some $g \in G$. Otherwise x is called *singular*. We denote by M_{reg} and M_{sing} the sets of all regular and singular points.

For a point $x \in M$ a subset $S \subseteq M$ is called a *slice* at x, if there exists an open G-invariant neighbornood U of $G.x$ and a smooth G-equivariant retraction $r : U \to G.x$ such that $S = r^{-1}(x)$.

An action is called *proper*, if the map $G \times M \to M \times M$, given by $(g, x) \mapsto (g.x, x)$ is proper, i.e. if the preimage of compacts sets is compact.

Theorem 1.1. *If G is a proper action on M, then each point admits slices.*

Proof. See [10, 6.26.]. □

Theorem 1.2. *M_{reg} is open and dense in M.*

Proof. See [10, 29.14.]. □

Theorem 1.3. *If $x \in M$ is a regular point, G_x is compact and S a slice at x, then $G_s = G_x$ for all $s \in S$, if the slice is chosen to be small enough.*

Proof. See [10, 6.16.]. □

A subset $F \subset M$ is called a *fundamental domain* for the G-action, if each orbit $G.x$ meets F exactly once.

The following lemma shows the existence of fundamental domains.

Lemma 1.4. *Let $A \subseteq B \subseteq M$ be two subsets of M. If A meets every orbit at most once and B at least once, then there exists a fundamental domain F lying between A and B, $A \subseteq F \subseteq B$.*

Proof. For each orbit $G.x$ that doesn't meet A pick a point $y_{G.x} \in B \cap G.x$. Set $F := A \cup \bigcup \{y_{G.x}\}$. □

1.2 Riemannian Geometry

We will assume the reader is familiar with the basic definitions and results on Riemannian manifolds. In particular we will use geodesics, isometries and facts about completeness throughout the text.

Unless stated otherwise, all geodesics on Riemannian manifolds are assumed to be parametrized by arc length.

Now we will state some theorems that belong to Riemannian geometry but are not always stated in textbooks. Let (M, γ) be a connected Riemannian manifold.

Theorem 1.5. *Let $N \subseteq M$ a submanifold. Take some $p \in M$ and let c be a geodesic from a point $c(0) \in N$ to p, such that c is the shortest curve from N to p. Then $c'(0) \in T_{c(0)}N^{\perp}$.*

Note that the theorem doesn't state anything about the existence of a shortest curve from N to p. The next theorem describes fixed point sets of isometries.

Theorem 1.6. *Let $s \in \mathrm{Isom}(M)$. Then every connected component N of the fixed point set M^s is a closed, totally geodesic submanifold and for any $x_0 \in N$ the tangent space T_xN is described by $T_xN = \mathrm{Eig}(1, T_xs)$, where $\mathrm{Eig}(1, T_xs)$ denotes the eigenspace of T_xs corresponding to the eigenvalue 1.*

The last theorem states that isometries are uniquely determined by the tangent mapping in a fixed point.

Theorem 1.7. *Let (M, γ) be a connected Riemannian manifold and $s, t \in \mathrm{Isom}(M)$ two isometries. If $x \in M^s \cap M^t$ is a common fixed point and the tangent mappings at x coincide, $T_xs = T_xt$, then s and t must be equal, $s = t$.*

1.3 Differential Topology

Let M, N be manifolds with corners.

For a continuous map $f : M \to N$ denote the *graph of f* by

$$\Gamma_f := \{(x, f(x)) : x \in M\}$$

Definition 1.8. A basis for the *C^0-topology* on $C(M, N)$ shall consist of sets of the form

$$\{g \in C(M, N) : \Gamma_g \subset U\},$$

where $U \subseteq M \times N$ is open.

This topology is also called the *Whitney*, *strong* or *fine* topology on $C(M,N)$. For compact M this topology coincides with the compact-open topology.

There are two other ways to describe this topology. (1) Choose a metric d on N. For a function $f \in C(M,N)$ the sets

$$\{g \in C(M,N) : d(f(x), g(x)) < \delta(x)\},$$

where $\delta : M \to \mathbb{R}_{>0}$ is continuous, form a neighborhood basis for f. (2) If $(K_i)_{i\in I}$ is a locally finite family of compact subsets of M and $(V_i)_{i\in I}$ is a family of open subsets of N, then a basis of the C^0-topology is given by the sets

$$\{g \in C(M,N) : g(K_i) \subset V_i, i \in I\}$$

Theorem 1.9. *$C^\infty(M,N)$ is dense in $C(M,N)$ in the C^0-topology.*

Proof. This is proven in [8, Ch. 2.2] for manifolds without boundary. The same proof however works also for manifolds with corners. □

Remark 1.10. About smooth homotopies. Let $f_0, f_1, f_2 : M \to N$ be smooth maps. Assume that f_0 and f_1 are homotopic via a smooth homotopy $H_1 : M \times [0,1] \to N$ and f_1, f_2 are homotopic via a smooth homotopy H_2. Let $\alpha : [0,1] \to [0,1]$ be a smooth function, s.t. $\alpha|_{[0,\epsilon]} = 0$ and $\alpha|_{[1-\epsilon,1]} = 1$ for some small $\epsilon > 0$. Then

$$H : (s,t) \mapsto \begin{cases} H_1(s, \alpha(2t)) & , t \in [0, \frac{1}{2}] \\ H_2(s, \alpha(2t-1)) & , t \in [\frac{1}{2}, 1] \end{cases}$$

defines a smooth homotopy from f_0 to f_2. Thus the relation *being homotopic via a smooth homotopy* is transitive.

Theorem 1.11. *If two points in M can be connected via a continuous path, they can also be connected via a smooth path.*

Proof. Let $c : [0,1] \to M$ be a continuous path in M. We show, that there exists a smooth path with the same endpoints. For each $t \in [0,1]$ choose coordinate charts (U_t, ϕ_t) around $c(t)$ such that $\phi_t(U_t)$ is a convex set in $\mathbb{R}^n$. Because $c([0,1])$ is compact there are finitely many $t_1, \ldots, t_N$, such that $c([0,1])$ is covered by $U_{t_1}, \ldots, U_{t_N}$. They shall be ordered such that $c(0) \in U_{t_1}$, $U_{t_i} \cap U_{t_{i+1}} \neq \emptyset$ and $c(1) \in U_{t_N}$. Some intersection is allways nonempty, since $c([0,1])$ is connected. Define $x_0 = c(0)$, $x_N = c(1)$ and choose $x_i \in U_{t_i} \cap U_{t_{i+1}}$ for $i = 1, \ldots, N-1$. Then x_i and x_{i+1} may be connected via a smooth path. Since a path is a homotopy between two maps of a one-element set, the remark about homotopies tells that there exists a smooth path between x_0 and x_N.

□

Theorem 1.12. *If $f, g : M \to N$ are smooth maps that are near enough in the C^0-topology, they are homotopic via a smooth homotopy. If furthermore $f = g$ on some set A, then the homotopy can be chosen to fix A.*

Proof. See [4, theorem 12.9]. The theorem is stated slightly differently, but the same proof works, if we adjust the assumptions and consequences accordingly. An important property of the theorem is that "near enough" doesn't vary for different f, g. Consider the first alternative description of the C^0-topology and choose a metric d on N. Then there exists a function $\delta : N \to \mathbb{R}_{>0}$ such that the theorem applies for pairs f, g for which $d(f(x), g(x)) < \delta(x)$ for all $x \in M$. □

Theorem 1.13. *If c_0, c_1 are two smooth paths in M with the same endpoints, that are homotopic via an end-point preserving homotopy, then there is also a smooth end-point preserving homotopy between c_0 and c_1.*

Proof. Let H be the endpoint preserving homotopy.

$$H(u,0) = c_0(u), \qquad H(u,1) = c_1(u), \qquad H(0,v) = x, \qquad H(1,v) = y.$$

A homotopy is a continuous path in the space $C([0,1],M)$ of paths. Let H_t denote the path $H(.,t)$. Since the image of H in $C([0,1],M)$ is compact, there are finitely many points $0 = t_0 < \ldots < t_n = 1$ sucht that the sets

$$U_{\frac{\delta}{3}}(t_i) := \{g : d(H_{t_i}(x), g(x)) < \frac{1}{3}\delta(x)\}$$

cover the image of H and $t_{i+1} \in U_{\frac{\delta}{3}}(t_i)$. δ is the function from the proof of theorem 1.12. Choose a smooth path $d_i \in U_{\frac{\delta}{3}}(t_i)$ for $i = 1,\ldots,n-1$ with the same endpoints as $d_0 := c_0$ and $d_n := c_1$. Then d_i and d_{i+1} are homotopic via a smooth endpoint preserving homotopy by theorem 1.12. □

Definition 1.14. Let $L \subseteq N$ be a submanifold with corners, $f : M \to N$ and $A \subseteq M$. We say that f is *transverse to L along A*, if for all $x \in A$

$$T_x f.T_x M + T_{f(x)}L = T_{f(x)}N.$$

Theorem 1.15 (Transversality theorem for maps)**.** *Let M, N be manifolds without boundary, $f : M \to N$ smooth and $L \subseteq N$ a submanifold. Then there exists a map $g : M \to N$ arbitrarily near to f in the C^0-topology, that is transverse to L. If f is already transverse to L along a closed set A, then we can choose g such that $g|_A = f|_A$.*

Proof. This is proved in [4, Theorem 14.7]. □

Theorem 1.16 (Whitney's extension theorem)**.** *Let M be a manifold with corners, $\widetilde{M}$ a manifold without boundary of the same dimension containing M as a closed submanifold with corners. Let N be a manifold without boundary and $f : M \to N$ smooth. Then there exists an extension $\tilde{f} : \widetilde{U} \to N$ where $\widetilde{U}$ is open in $\widetilde{M}$ and $M \subset \widetilde{U}$.*

Proof. Embed $\widetilde{M}$ via Whitney's embedding theorem in some $\mathbb{R}^K$ and embed N in some $\mathbb{R}^L$. Choose a tubular neighborhood $\widetilde{N}$ around N in $\mathbb{R}^L$ and a retraction $r : \widetilde{N} \to N$. $\widetilde{N}$ is open in $\mathbb{R}^L$. Now apply Whitney's extension theorem to obtain a smooth map $g : V \subseteq \mathbb{R}^K \to \widetilde{N} \subset \mathbb{R}^L$, that extends f. Then $r \circ g|_{V \cap \widetilde{M}}$ is the required extension. □

Corollary 1.17. *In theorem 1.15 M may be a manifold with corners.*

Theorem 1.18. *Let M be a connected manifold and N a submanifold of M, such that each connected component of N has codimension ≥ 2. Then $M \setminus N$ is connected as well.*

Proof. Let $x, y \in M \setminus N$ and let $\tilde{c}$ be a path in M connecting x and y. By corollary 1.17 there exists a path c transverse to N. c still connects x and y, because $\{x,y\} \cap N = \emptyset$. Since N has codimension at least 2, c has to avoid N alltogether by dimension. Thus $M \setminus N$ is connected. □

1.4 Coxeter Groups

For more material about Coxeter groups see [3] or [5].

Definition 1.19. A pair (G, S) consisting of a group G and a set of idempotent generators S is called a *Coxeter system*, if it satisfies the following condition:

> For s, s' in S, let $n_{s,s'}$ be the order of ss' and let I be the set of pairs (s, s'), such that $n_{s,s'}$ is finite. The generating set S and the relations $(ss')^{n_{s,s'}} = e$ for (s, s') in I form a presentation of the group G.

In this case we also say, that G is a *Coxeter group*.

Equivalently we could define (G, S) to be a Coxeter system, if G is a quotient of the free group with the set S of generators by the normal subgroup generated by the elements s^2 and $(ss')^{n_{s,s'}}$.

In the following, unless stated otherwise, let (G, S) be a Coxeter system.

Definition 1.20. The *lenght* of an element $g \in G$ is the smallest integer q, such that g is a product of q elements of S. We denote the length of g by $l(g)$.

We can characterize a Coxeter system (G, S) by certain partitions of G. First we state some properties of a Coxeter system.

Theorem 1.21. *Set $P_s^+ = \{g \in G : l(sg) > l(g)\}$. Then*
(1) $\bigcap_{s \in S} P_s^+ = \{e\}$
(2) $G = P_s^+ \cup sP_s^+$ and $P_s^+ \cap sP_s^+ = \emptyset$ for $s \in S$.
(3) Let $s, s' \in S$ and $g \in G$. If $g \in P_s^+$ and $gs' \notin P_s^+$, then $s = gs'g^{-1}$.

Proof. See [3, Ch. IV, §1, 7.]. □

Conversely we can reconstruct the Coxeter system, if we are given a family of subsets $(P_s)_{s \in S}$ with similar properties.

Theorem 1.22. *Let G be a group with a generating set S of idempotents. Let $(P_s)_{s \in S}$ be a family of subsets of G, which satisfy*
(1) $e \in P_s$ for all $s \in S$.
(2) $P_s \cap sP_s = \emptyset$ for $s \in S$.
(3) Let $s, s' \in S$ and $g \in G$. If $g \in P_s$ and $gs' \notin P_s$, then $s = gs'g^{-1}$.
Then (G, S) is a Coxeter system and $P_s^+ = P_s$.

Proof. See [3, Ch. IV, §1, 7.]. □

Given a Coxeter System (G, S) (or more generally a finitely presented group) the *Word Problem* is said to be solvable, if given two words in the generating set S, there is an algorithm to determine whether or not they represent the same element in G. The word problem is solvable for Coxeter groups and we will now give an algorithm. This algorithm is due to Tits [14] and is described in [2, p. 45] and [5, p. 40].

Given a word w in the generating set S, we define a set $D(w)$ of all words which can be derived from w by applying any sequence of operations of the following two types

(D1) For any $s \neq t$ in S such that $n_{s,t}$ is finite, replace any occurence in w of the word $stst\cdots$ of length $n_{s,t}$ with the word $tsts\cdots$ of length $n_{s,t}$.

(D2) Cancel any adjacent occurences of the same letter $s \in S$.

It is clear that $D(w)$ is finite for any given w.

Theorem 1.23. *Two words w,w' represent the same element in G if and only if $D(w) \cap D(w') \neq \emptyset$.*

Proof. A proof is given in [5, p. 40]. □

Chapter 2

Manifolds with Corners

In this chapter we introduce manifolds with corners. Since they are used in chapter 6, but they are not the main object of interest in this work, we will not always provide proofs with all details. For more information on manifolds with corners and more detailed proofs one may consult [11] or [9].

2.1 Quadrants

Definition 2.1. A *quadrant* $Q \subseteq \mathbb{R}^n$ is a subset of the form

$$Q = \{x \in \mathbb{R}^n : l^1(x) \geq 0, \dots, l^k(x) \geq 0\},$$

where $\{l^1, \dots, l^k\}$ is a linearly independent subset of $(\mathbb{R}^n)^*$.

In the following we will assume that a quadrant Q is given by the functionals $l^1, \dots, l^k$ without explicitly mentioning them. Up to multiplication by a positive scalar, the functionals $l^1, \dots, l^k$ are determined uniquely by Q. This can be seen most easily by looking at the walls of Q. The walls determine the kernels of the functionals and thus the functionals itself up to a constant.

Definition 2.2. Let Q be a quadrant and $x \in Q$. Define the *index* of x, $\mathrm{ind}_Q(x)$, to be the number of linear functionals l^i, such that $l^i(x) = 0$.

The index of a point is well defined, because of the above remarks. Informally, the index describes in which part of a quadrant a point lies.

Definition 2.3. Let Q be a quadrant. Define the *border* of Q to be the set

$$\partial Q = \{x \in Q : l^1(x) = 0 \text{ or } \dots \text{ or } l^k(x) = 0\}$$

The border of Q as defined above coincides with the topological border $\overline{Q} \backslash Q^\circ$ of Q, if we view Q as a subset of $\mathbb{R}^n$. ∂Q is a disjoint union of finitely many plane submanifolds. If we order them by dimension, then

$$\partial Q = \bigcup_{i=1}^{k} \{x \in Q : \mathrm{ind}_Q(x) = i\}$$

The next step is to consider (smooth) functions on quadrants.

Definition 2.4. Let $U \subseteq Q$ be an open subset of a quadrant, $f : U \to \mathbb{R}^p$ a continuous function and $0 \leq r \leq \infty$. We say that $f \in C^r(U, \mathbb{R}^p)$, if f has continous partial derivatives up to order r.

Theorem 2.5 (Application of Whitney's extension theorem)**.** *Let $U \subseteq Q$ be open in the quadrant $Q \subseteq \mathbb{R}^n$ and $f : U \to \mathbb{R}^p$. Then $f \in C^r(U, \mathbb{R}^p)$ iff f can be extended to a function $\tilde{f} \in C^r(\tilde{U}, \mathbb{R}^p)$, where $\tilde{U} \supseteq U$ is open in $\mathbb{R}^n$.*

Definition 2.6. Let $U \subseteq Q$ and $U' \subseteq Q'$ be open subsets of the quadrants Q, Q' respectively. A C^1 function $f : U \to U'$ is called a *diffeomorphism*, if f is bijective and Df has everywhere maximal rank.

A bijective function $f \in C^1(U, U')$ is a diffeomorphism if and only if $f^{-1} \in C^1(U', U)$. This follows from the inverse function theorem and the fact that even for points $x \in \partial Q$ on the border of the quadrant, knowing the function on the quadrant is enough to determine $D_x f$.

The following lemma shows, that the index of a point is invariant under diffeomorphisms.

Lemma 2.7. *Let $U \subseteq Q$ and $U' \subseteq Q'$ be open subsets of the quadrants Q, Q' respectively, $f : U \to U'$ a diffeomorphism and $x \in U$. Then*

$$\operatorname{ind}_Q(x) = \operatorname{ind}_{Q'}(f(x)).$$

Proof. Assume there is some $x \in U$ with $\operatorname{ind}_Q(x) \neq \operatorname{ind}_{Q'}(f(x))$. Assume w.l.o.g that $\operatorname{ind}_Q(x) < \operatorname{ind}_{Q'}(f(x))$. Otherwise apply the argument to f^{-1} and $f(x)$. Extend $\{l^1, \ldots, l^k\}$ to a basis $\{l^1, \ldots, l^n\}$ of $(\mathbb{R}^n)^*$ and let $\{x_1, \ldots, x_n\}$ be the dual basis. Do the same for $\{l'^1, \ldots, l'^{k'}\}$ to get a basis $\{l'^1, \ldots, l'^n\}$ and the dual basis $\{x'_1, \ldots, x'_n\}$. Set $s := \operatorname{ind}_Q(x)$ and $s' := \operatorname{ind}_{Q'}(f(x))$ and rearrange the vectors, such that $l^1(x) = \ldots = l^s(x) = 0$ and $l'^1(f(x)) = \ldots = l'^{s'}(f(x)) = 0$.

Because f is a diffeomorphism, the vectors $\{D_x f.x_{s+1}, \ldots, D_x f.x_n\}$ are linearly independent. Because $s > s'$, they cannot all be contained in $\operatorname{span}\{x'_{s+1}, \ldots, x'_n\}$. Without loss let $D_x f.x_{s+1} = \sum_{i=1}^n \lambda^i x'_i$ and $\lambda^i \neq 0$ for some $i \leq s'$. Without loss let $\lambda^1 \neq 0$, so that $D_x f.x_{s+1} = \lambda^1 x'_1 + \ldots$. Now take $h \in \mathbb{R}$ and linearize f

$$\begin{aligned} f(x + hx_{s+1}) &= f(x) + h \cdot D_x f.x_{s+1} + |h| \cdot \|x_{s+1}\| \cdot r_x(hx_{s+1}) \\ &= f(x) + h \cdot (\lambda^1 x'_1 + \sum_{i>1} \lambda^i x'_i) + |h| \cdot \|x_{s+1}\| \cdot r_x(hx_{s+1}) \end{aligned}$$

and $\lim_{h\to 0} r_x(hx_{s+1}) = 0$. Apply l'^1:

$$\begin{aligned} l'^1(f(x + hx_{s+1})) &= l'^1(f(x)) + hl'^1(\lambda^1 x'_1) + |h| \cdot \|x_{s+1}\| \cdot l'^1(r_x(hx_{s+1})) \\ &= h\lambda^1 + |h| \cdot \|x_{s+1}\| \cdot l'^1(r_x(hx_{s+1})) \\ &= h \left(\lambda^1 + \underbrace{(-1)^{\operatorname{sgn}(h)} \|x_{s+1}\| \cdot l'^1(r_x(hx_{s+1}))}_{\to 0 \text{ for } h \to 0} \right) \end{aligned} \tag{2.1}$$

Since $l^{s+1}(x) > 0$ and $x \in U$, the vector $x + hx_{s+1}$ will remain in U, as long as h is small enough. On the other hand, the right hand side of (2.1) can be made both, positive and negative, no matter how small h is. But this means that $f(x + hx_{s+1})$ is not allways in Q'. This contradicts the definition of f. Thus $s \neq s'$ is impossible. ∎

2.2 Manifolds

In this section we introduce the notion of a manifold with corners. It is very similar to that of a manifold without boundary, but now we require each point to have a neighborhood, that is locally homeomorphic to an open subset of a quadrant.

Definition 2.8. A *manifold with corners in the weak sense* of dimension n is a topological space M, that is separable and Hausdorff, equipped with an *atlas* $(U_i, u_i, Q_i)_{i \in I}$, i.e. an open covering $(U_i)_{i \in I}$ of M and maps $u_i : U_i \to Q_i$ satisfying

1. $u_i(U_i)$ is an open subset of the quadrant $Q_i \subseteq \mathbb{R}^n$.
2. $u_i : U_i \to u_i(U_i)$ is a homeomorphism.
3. If $U_i \cap U_j \neq \emptyset$, then $u_j^{-1} \circ u_i : u_i(U_i \cap U_j) \to u_j(U_i \cap U_j)$ is a C^∞ map.

The 3-tuple (U, u, Q) consisting of an open set $U \subseteq M$, a quadrant $Q \subseteq \mathbb{R}^n$ and a map $u : U \to Q$ is called a *chart*. The chart is called *centered* at $x \in M$, if $u(x) = 0$.

The index of a point was defined in 2.2. Using charts we can assign an index to each point of a manifold. This is well defined because of lemma 2.7.

Definition 2.9. Let M be a manifold with corners, $x \in M$ and (U, u, Q) a chart around x. The *index* of x is defined as

$$\mathrm{ind}_M(x) := \mathrm{ind}_Q(u(x))$$

$x \in M$ is called *inner point*, if $\mathrm{ind}_M(x) = 0$. The *border* of M is the set $\partial M := \{x \in M : \mathrm{ind}_M(x) > 0\}$.

Submanifolds are defined essentially in the same way as for manifolds without boundary.

Definition 2.10. Let M be a manifold with corners in the weak sense and $N \subseteq M$. N is called a *submanifold* of M of dimension k, if for all $y \in N$ there exists a chart (U, u, Q) of M centered at y and a quadrant $Q' \subseteq Q \cap \mathbb{R}^k$ such that $u(N \cap U) = u(U) \cap Q'$.

If M is a manifold with boundary, then ∂M is a submanifold without boundary. A similar result holds in the case that M is a manifold with corners, but now we have to partition ∂M into appropriate parts, each of which will be a submanifold without boundary. The right partition is given by the index function.

Theorem 2.11. *Let M be a manifold with corners in the weak sense of dimension n and $0 \leq j \leq n$. Then*

$$\partial^j M := \{x \in M : \mathrm{ind}_M(x) = j\}$$

is a $(n - j)$-dimensional submanifold of M without boundary.

Proof. Pick an $x \in \partial^j M$ and a chart (U, u, Q) of M, centered a t x. Let $Q = \{x \in \mathbb{R}^n : l^1(x) \geq 0, \dots, l^j(x) \geq 0\}$. Then the points of M with index j, that lie in the neighborhood of x are exactly those, whose image under u lies in the kernel of all l^i. So $u(\partial^j M \cap U) = u(U) \cap \bigcap_{i=1}^{j} \ker l^i$. □

Definition 2.12. A *face of index j* is the closure of a connected component of $\partial^j M$. A *boundary hypersurface* is a face of index 1.

At last we are ready to define the object of our interest.

Definition 2.13. A manifold with corners is a manifold with corners in the weak sense, such that each boundary hypersurface is a submanifold.

Remark 2.14. Obviously any point in a manifold with corners in the weak sense has an open neighborhood, which is a manifold with corners. The only thing, that can stop a manifold with corners in the weak sense from being a manifold with corners is that two of the local boundary hypersurfaces near some point are in the closure of the same component of $\partial_1 M$.

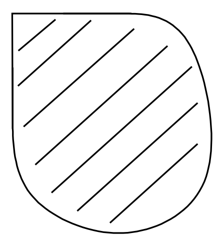


Figure 2.1: Manifold with corners in the weak sense.

2.3 Tangent Bundle

We want to define the tangent bundle TM. Let $(U_i, u_i)_{i \in I}$ be a maximal atlas for M. On the set

$$\{(i, x, v) : i \in I, x \in U_i, v \in \mathbb{R}^n\}$$

we define the relation

$$(i, x, v) \sim (j, y, w) \Leftrightarrow x = y \text{ and } D(u_j \circ u_i^{-1})_{u_i(x)}.v = w$$

It can be verified, that this is an equivalence relation. The tangent bundle shall consist of all equivalence classes with respect to this relation.

$$TM := \{[i, x, v] : i \in I, x \in U_i, v \in \mathbb{R}^n\}$$

In other words, we assign to each point and each chart a copy of $\mathbb{R}^n$ and make sure that for two different charts the vectors transform accordingly. We need a maximal atlas to ensure TM to be well defined. Otherwise it wouldn't be clear that the definition is independent of the chosen atlas.

Define the maps

$$\pi_M : \begin{cases} TM & \to & M \\ [i, x, v] & \mapsto & x \end{cases} \qquad \psi_i : \begin{cases} U_i \times \mathbb{R}^n & \to & \pi_M^{-1}(U_i) \\ (x, v) & \mapsto & [i, x, v] \end{cases}$$

Theorem 2.15. *$(TM, \pi_M, M, \mathbb{R}^n)$ is a vector bundle with the atlas given by (U_i, ψ_i).*

This means the following: TM is a smooth manifold with corners, π_M is smooth and ψ_i are diffeomorphisms such that the following diagram is commutative.

$$\begin{array}{ccc} E|_{U_i} := \pi_M^{-1}(U_i) & \xleftarrow{\;\psi_i\;} & U_i \times \mathbb{R}^n \\ {\scriptstyle \pi_M}\downarrow & \swarrow{\scriptstyle \mathrm{pr}_1} & \\ U_i & & \end{array}$$

Set $U_{ij} := U_i \cap U_j$ and define $\psi_{ij}(x) : \mathbb{R}^n \to \mathbb{R}^n$ implicitly by

$$\psi_i^{-1} \circ \psi_j : \begin{cases} U_{ij} \times \mathbb{R}^n & \to & U_{ij} \times \mathbb{R}^n \\ (x, y) & \mapsto & (x, \psi_{ij}(x)(y)) \end{cases}$$

Then ψ_{ij} has to be a smooth map $\psi_{ij} : U_{ij} \to \mathrm{GL}(\mathbb{R}, n)$, in particular $\psi_i^{-1} \circ \psi_j$ must be fibrewise linear.

Definition 2.16. A tangent vector $\xi \in M$ is called *inner*, if there is a smooth curve $c : [0, \epsilon) \to M$ with $\dot{c}(0) = \xi$. The set of all inner tangent vectors is denoted by ${}^iTM$.

The condition to be inner can be formulated using coordinates. Let (U, u, Q) be a chart around $x \in M$, where x is the basepoint of ξ. Then $(u(x), (u \circ c)'(0))$ is a coordinate expression for ξ. If $l_i(u(x)) = 0$, then

$$\frac{l_i(u \circ c(t)) - l_i(u(x))}{t} \geq 0$$

implies $l_i((u \circ c)'(0)) \geq 0$.

Conversly given any coordinate representation $(u(x), v)$ of $\xi \in T_xM$ with $l_i(u(x)) = 0$ implying $l_i(v) \geq 0$, it is not difficult to construct a curve having this vector as it's tangent vector (take e.g. the straight line in Q). So we conclude

Lemma 2.17. *A tangent vector $\xi \in T_xM$ is inner if and only if in a coordinate expression $(u(x), v)$ we have $l_i(v) \geq 0$ whenever $l_i(u(x)) = 0$.*

Definition 2.18. We call $\xi \in T_xM$ *strictly inner*, if $l_i(u(x)) = 0$ implies $l_i(v) > 0$ with strict inequality.

Strictly inner tangent vectors are those, which are inner, but are not tangent vectors of curves lying completely in ∂M.

We define a similar notion for the second tangent bundle.

Definition 2.19. A vector $\xi \in T^2M$ is said to be an *inner tangent vector to* ${}^iTM$, if there is a curve $c : [0, \epsilon) \to^i TM$ with $\dot{c}(0) = \xi$. The set of all vectors inner to ${}^iTM$ is denoted by ${}^iT^2M$.

We can rewrite this definition in coordinates as well. Let $(x, v; h, k)$ represent an element of T^2M. It is inner to ${}^iTM$ if and only if

$$\begin{array}{l} l_i(x) > 0 \text{ then } l_i(v), l_i(h), l_i(k) \text{ are arbitrary.} \\ l_i(x) = 0, l_i(v) > 0 \text{ then } l_i(h) \geq 0, l_i(k) \text{ arbitrary.} \\ l_i(x) = 0, l_i(v) = 0 \text{ then } l_i(h) \geq 0, l_i(k) \geq 0 \end{array} \tag{2.2}$$

A vector field $\xi : M \to TM$ is called *inner*, if $\xi(M) \subseteq {}^iTM$. An inner vector field admits a local flow in the following sense: there exists a set $W \subset M \times \mathbb{R}$ and a map $\phi : W \to M$ with the following properties

$$\phi(x, 0) = x$$
$$\phi(\phi(x, s), t) = \phi(x, s + t)$$
$$\frac{d}{dt}\phi(x, t) = \xi(\phi(x, t))$$

and for each $x \in M$ there exists $\epsilon_x > 0$ sucht that $\{x\}\times[0, \epsilon_x) \subset W$. Existence is guarantueed only for positive times. Since solutions of ODEs depend smoothly on initial data W is an open neighborhood of the set $M \times \{0\}$.

Using a partition of unity we can construct a strictly inner vector field ξ on M. By multiplying ξ with a small function we can adapt ξ such that the local flow is defined at least on the set $M \times [0, \epsilon)$, where ϵ doesn't depend on x any more. Take $t \in [0, \epsilon)$. Since ξ is strictly inner we have $\alpha(M, t) \subseteq M^\circ$. Also $\alpha(., t)$ is a diffeomorphism onto it's image, since the inverse map can be written as $\alpha(., -t)$, which is well defined on the image. So $M \cong \alpha(M, t) \subseteq M^\circ$ and we see that every manifold with corners is a submanifold of a manifold without boundary of the same dimension.

Let us forget about the vector field ξ and write $M \subseteq \widetilde{M}$. Then $N := M \cup \overline{M}^c$, where the closure and complemet are taken in $\widetilde{M}$, is another manifold without corners and M is a closed submanifold of N. Thus we have proven the following

Theorem 2.20. *Every manifold with corners is a closed submanifold of a manifold without boundary of the same dimension.*

Beeing closed is important in order to apply Whitney's extension theorem.

2.4 Sprays

Definition 2.21. A *Spray* is a vector field $\xi : TM \to T^2M$ on the tangent bundle that satisfies the following conditions

(S1) $T\pi_M \circ \xi = \mathrm{Id}_{TM}$

(S2) $\pi_{TM} \circ \xi = \mathrm{Id}_{TM}$

(S3) $T\mu_t \circ \xi = \frac{1}{t}\xi \circ \mu_t$, where $\mu_t : TM \to TM$ is the scalar multiplication by t.

A spray ξ is called *inner*, if $\xi({}^iTM) \subseteq {}^iT^2M$ and it is called *tangential* if ξ is tangent to each boundary face, i.e. $\xi(T\partial^jM) \subseteq T^2\partial^jM$.

Let ξ be an inner spray. In general ξ is not an inner vector field on TM, so we cannot construct integral curves for all points of TM. But we can do so for inner tangent vectors $v \in {}^iTM$. The conditions (2.2) just say that starting in v the integral curve $\phi(v, t)$ of ξ will stay in ${}^iTM$ for sufficiently small t. Since the solutions of ODEs depend smoothly on the initial data we get a smooth map $\phi : W \subseteq {}^iTM \times \mathbb{R} \to {}^iTM$ with the following properties

$$\phi(v, 0) = v$$
$$\phi(\phi(v, t), s) = \phi(v, s + t)$$
$$\frac{d}{dt}\phi(v, t) = \xi(\phi(v, t))$$

This corresponds to the local flow of the vector field. For each $v \in^i TM$ we have $\{v\} \times [0, \epsilon_v) \subset W$ for sufficiently small ϵ_v.

Let $\chi := \pi_M \circ \phi$. Then ξ describes a second order ODE for χ in the following way: we have

$$\chi(v,0) = \pi_M(v)$$
$$\frac{d}{dt}\chi(v,0) = v$$
$$\frac{d^2}{d^2t}\chi(v,t) = \xi(\chi(v,t))$$

and another property holds

$$\chi(sv,t) = \chi(v,st) \tag{2.3}$$

This can be seen by taking second order derivatives with respect to t.

$$\frac{d^2}{d^2t}\chi(sv,t) = \frac{d}{dt}\phi(sv,t) = \xi(\phi(sv,t))$$
$$\frac{d^2}{d^2t}\chi(v,st) = \frac{d}{dt}s\phi(v,st) = T\mu_s.\xi(\phi(v,st))s = \xi(s\phi(v,st))$$

So both sides satisfy the second order ODE $\ddot{x} = \xi(\dot{x})$ and have the same value and derivative for $t = 0$.

We define the exponential map

$$\exp : \begin{cases} U & \rightarrow & M \\ v & \mapsto & \chi(v,1) \end{cases}$$

U is the set of all $v \in {}^iTM$ such that $\chi(v,1)$ is defined. (2.3) tells us that U is a neighborhood of the zero section in ${}^iTM$. The expoential map is locally a diffeomorphism as shown by the next theorem

Theorem 2.22. *Let $\xi : TM \to T^2M$ be an inner spray. Then for each $x \in M$ there exists an open neighborhood V_x of 0_x in ${}^iT_xM$ such that $\exp_x : V_x \to M$ is a diffeomorphism onto it's image.*

If ξ is moreover tangential then $\exp_x$ restriced to a tangent space of a boundary face equals the exponential map of the spray resptricted to the boundary face.

Proof. See [11, 2.10.] □

We can use sprays to introduce Riemannian metrics on manifolds with corners. A Riemannian metric γ on M is as usual a smooth, symmetric $\binom{0}{2}$-tensor field on M assigning each $x \in M$ a positive definite bilinear form $\gamma_x : T_xM \times T_xM \to \mathbb{R}$.

We may extend γ to a Riemannian metric $\widetilde{\gamma}$ on a manifold without boundary $\widetilde{M}$ of the same dimension. There we can define the geodesic spray $\widetilde{\xi}$ by

$$\widetilde{\xi}(v) = \ddot{c}(v,0)$$

where $c(v,t)$ is the geodesic with initial velocity vector $v \in T\widetilde{M}$. The exponential map of this spray is exactly the usual exponential map of Riemannian geometry. Now we restrict $\widetilde{\xi}$ to TM and get a spray ξ on M. This is an inner tangential spray, if all faces are totally geodesic submanifolds. We can reformulate theorem 2.22 for Riemannian manifolds.

Theorem 2.23. *Let (M, γ) be a Riemannian manifold with corners such that each face of M is a totally geodesic submanifold.*

Then the geodesic spray is inner and tangential. For each $x \in M$ there exists an open ball B_x in (T_xM, γ_x) such that the exponential map $\exp_x : B_x \cap {}^iT_xM \to M$ is a diffeomorphism onto it's image.

${}^iT_xM$ is a quadrant in $\mathbb{R}^n$. It boundary hypersurfaces are given by the inverse images under $\exp_x$ of the faces of M.

Chapter 3

Discrete Groups of Isometries

In the following let (M, γ) be a complete, connected Riemannian manifold.

We denote by $\mathrm{Isom}(M)$ the group of all isometries on M. The following theorem summarizes the facts, that we will need about it.

Theorem 3.1. *(1) The compact-open topology and the pointwise-open topology coincide on* $\mathrm{Isom}(M)$.

(2) With this topology $\mathrm{Isom}(M)$ *can be given the structure of a Lie group such that the action* $(g, x) \mapsto g.x$ *on* M *is smooth.*

(3) If $(g_n)_{n\in\mathbb{N}}$ *is a sequence in* $\mathrm{Isom}(M)$*, such that* $(g_n.x)_{n\in\mathbb{N}}$ *converges for some* $x \in M$*, then there exists a convergent subsequence* $(g_{n_k})_{k\in\mathbb{N}}$.

(4) The isotropy group of each point of M *is compact.*

Proof. (1) See [7, Ch. IV, §2].

(2) See [13] or [10, 28.1.].

(3) See [7, Ch. IV, Theorem 2.2.].

(4) We can view the isotropy group at $x \in M$ as a subgroup of the orthogonal group $O(T_xM, \gamma_x)$. Since $\mathrm{Isom}(M)_x$ is closed and $O(T_xM, \gamma_x)$ is compact, $\mathrm{Isom}(M)_x$ is also compact.

□

Theorem 3.2. *Let* G *be a closed subgroup of* $\mathrm{Isom}(M)$*. Then the* G*-action on* M *is proper. In particular the action admits slices.*

Proof. See [10, 6.25.] □

3.1 Discreteness

We will be interested in discrete subgroups of $\mathrm{Isom}(M)$. So for the rest of the chapter let G be a discrete subgroup of $\mathrm{Isom}(M)$.

G is *discrete*, if the induced topology is the discrete one, i.e. if every set is open and closed. Another characterization is that G does not contain any accumulation points. $g \in G$ is an *accumulation point*, if we can find a sequence $g_n \to g$ with $g_n \in G \setminus \{g\}$. The same holds for every metric space.

Some consequences of the above theorem are:

Lemma 3.3. *G is closed in* $\mathrm{Isom}(M)$.

Proof. If $g \in \mathrm{Isom}(M)$ is an accumulation point of G, then we can find a sequence $(g_n)_{n\in\mathbb{N}}$ with $g_n \to g$ and $g_n \neq g_m$ for $n \neq m$. But then $g_n.g_{n+1}^{-1} \to e$ and $g_n.g_{n+1}^{-1} \neq e$, so e is an accumulation point of G. This contradicts G being discrete. □

Corollary 3.4. *The isotropy group G_x of each point is finite.*

Proof. G_x is discrete and compact, so it must be finite. □

Corollary 3.5. *Let $(g_n)_{n\in\mathbb{N}}$ be a sequence in G, such that $(g_n.x)_{n\in\mathbb{N}}$ converges in M for some $x \in M$. Then there exists a constant subsequence $(g_{n_k})_{k\in\mathbb{N}}$ of $(g_n)_{n\in\mathbb{N}}$.*

Proof. A convergent subsequence in a discrete closed set must be constant. □

The following theorem gives a different characterization of discrete subgroups of $\mathrm{Isom}(M)$.

Theorem 3.6. *G is discrete if and only if each orbit is discrete in M. In this case the isotropy group of each regular point is trivial.*

Proof. Assume G is discrete and take $x \in M$. Assume the orbit $G.x$ is not discrete. Then it has an accumulation point $g.x$ that is the limit of a sequence $(g_n.x)_{n\in\mathbb{N}}$ with $g_n.x \neq g.x$ for all $n \in \mathbb{N}$. But $g_n.x \to g.x$ and corollary 3.5 imply the existence of a constant subsequence $(g_{n_k})_{k\in\mathbb{N}}$, i.e. $g_{n_k}.x = g.x$, a contradiction. Thus each orbit is discrete.

Now assume all orbits are discrete. Let $\overline{G}$ be the closure of G in $\mathrm{Isom}(M)$ and $x \in M$ a regular point for the $\overline{G}$-action. By theorem 3.1 (3) G-orbits are closed, so $G.x = \overline{G}.x$. By theorem 3.2 the $\overline{G}$-action admits slices. Let $S \subseteq M$ be a slice at x and $r : U \to \overline{G}.x$ the corresponding retraction. Since $\overline{G}.x$ is discrete, $S = r^{-1}(x)$ is open in M and $G_s = G_x$ for all $s \in S$ by theorem 1.3. Thus any $g \in G_x$ is the identity on S. Since S is open and g an isometry this means $g = \mathrm{Id}_M$ or $g = e$ in G, so the isotropy group of x is trivial. We have a continuous, bijective map $\overline{G} = G/G_x \to \overline{G}.x = G.x$ into a discrete set. Thus $G = \overline{G}$ and G is discrete in $\mathrm{Isom}(M)$. □

We shall say that G *acts discretely on* M. It should be noted that in this case a point is regular if and only if it's isotropy group is trivial.

$$M_{reg} = \{x \in M : G_x = \{e\}\}$$

The following lemma contains some consequences of discreteness that will be used later.

Lemma 3.7. *Let $G \subset \mathrm{Isom}(M)$ act discretely on M and let $x \in M$. Then the set $\{d(x, g.x) : g \in G\}$ is a discrete and closed subset of $\mathbb{R}$. In particular*
(1) There exists an element $g \in G$ such that

$$d(y, g.x) = \inf_{h\in G} d(y, h.x) = d(y, G.x)$$

(2) The set $\{d(x, g.x) : g \in G\}$ is a locally finite subset of $\mathbb{R}$.
This also holds if we replace G by any subset $H \subseteq G$.

Proof. Assume for contradiction that $a \in \mathbb{R}$ is an accumulation point of

$$\{d(x, g.x) : g \in G\}.$$

Then there exists a sequence $(g_n)_{n\in\mathbb{N}}$ such that $d(x, g_n.x) \to a$. It follows that the set $\{g_n.x : n \in \mathbb{N}\} \subset M$ is bounded and by the completeness of M it's closure is compact. If the set $\{g_n.x : n \in \mathbb{N}\}$ were finite, the sequence $(d(x, g_n.x))_{n\in\mathbb{N}}$ would, in order to converge, equal a for large n. But this is excluded by the definition of an accumulation point. So the set $\{g_n.x : n \in \mathbb{N}\}$ is infinite. Since it's closure is compact it must have an accumulation point in M. This accumulation point would also be an accumulation point of $G.x$. But this is impossible since $G.x$ is discrete and closed. Therefore $\{d(x, g.x) : g \in G\}$ must also be discrete and closed.

If H is a subset of G, then $\{d(x, h.x) : h \in H\} \subseteq \{d(x, g.x) : g \in G\}$. Therefore $\{d(x, h.x) : h \in H\}$ is discrete and closed as well, since it is a subset of a discrete, closed set. So the lemma holds for H as well. □

3.2 Dirichlet Domains

Definition 3.8. Let $G \subset \mathrm{Isom}(M)$ be a subgroup of the group of isometries, that acts discretely on M. Let x_0 be a regular point for the action. The *closed Dirichlet domain* for x_0 ist the set

$$D(x_0) := \{y \in M : d(y, x_0) \leq d(y, g.x_0) \text{ for all } g \in G\}$$

The open interior $D(x_0)^\circ$ is called the *open Dirichlet domain* for x_0.

Definition 3.9. Let $x_0, x_1 \in M$ be two different points. The *central hypersurface* is the set

$$H_{x_0,x_1} := \{y \in M : d(y, x_0) = d(y, x_1)\}$$

Lemma 3.10. *For $y \in H_{x_0,x_1}$ let c_0 be a minimal geodesic from y to x_0. Then c_0 meets H_{x_0,x_1} only at y.*

Proof. Let $c_0(t_0) = x_0$. Then $t_0 = d(y, x_0) = d(y, x_1)$. Suppose for contradiction that $c_0(t) \in H_{x_0,x_1}$ for some $t > 0$. Then

$$d(y, x_1) \leq d(y, c_0(t)) + d(c_0(t), x_1)$$

Assume equality holds. Take a minimal geodesic b from $c_0(t)$ to x_1. Then the concatenation $c_1 := c_0|_{[0,t]} \cdot b$ is, because of equality above, a minimal geodesic from y to x_1. But, because a geodesic is uniquely determined by it's initial velocity vector, we get $c_1 = c_0$ on $[0, t_0]$. Here $t > 0$ was used. In particular $x_1 = c_1(t_0) = c_0(t_0) = x_0$, a contradiction.

So the inequality $d(y, x_1) < d(y, c_0(t)) + d(c_0(t), x_1)$ is strict. Then we get

$$\begin{aligned} t_0 = d(y, x_1) &< d(y, c_0(t)) + d(c_0(t), x_1) \\ &= d(y, c_0(t)) + d(c_0(t), x_0) = d(y, x_0) = t_0 \end{aligned}$$

But this is impossible. Therefore c_0 meets H_{x_0,x_1} only at y. □

Lemma 3.11. *Let $G \subset \mathrm{Isom}(M)$ act discretely on M and let $x_0 \in M$ be a regular point. Then*

$$D(x_0)^\circ = \{y \in M : d(y, x_0) < d(y, g.x_0) \text{ for all } g \in G \setminus \{e\}\}$$

and $\overline{D(x_0)^\circ} = D(x_0)$, i.e. $D(x_0)$ is a regular closed set.

Proof. First we show that

$$A := \{y \in M : d(y, x_0) < d(y, g.x_0) \text{ for all } g \in G \setminus \{e\}\}$$

is open. Take $y \in A$. Lemma 3.7 shows the existence of a $g \in G \setminus \{e\}$, such that $d(y, g.x_0) = \inf_{h \in G \setminus \{e\}} d(y, h.x_0)$. Set $\epsilon := \frac{1}{2}(d(y, g.x_0) - d(y, x_0))$. We claim that $B_\epsilon(y) \subseteq A$. Take $z \in B_\epsilon(y)$. Then for all $h \in G \setminus \{e\}$

$$\begin{aligned} d(z, x_0) &\leq d(z, y) + d(y, x_0) \\ &< \epsilon + d(y, x_0) = d(y, g.x_0) - \epsilon \\ &\leq d(y, h.x_0) - d(y, z) \leq d(z, h.x_0). \end{aligned} \tag{3.1}$$

So $z \in A$. Hence A is open. Because $A \subset D(x_0)$, this shows that $A \subseteq D(x_0)^\circ$.

Now we show that $D(x_0)^\circ \subseteq A$. Let $y \in D(x_0)^\circ$ and assume that $d(y, x_0) = d(y, g.x_0)$ for some $g \in G \setminus \{e\}$. This means $y \in H_{x_0, g.x_0}$. Pick a minimal geodesic c_0 from y to $g.x_0$. Because y is an inner point of $D(x_0)$, there exists a $t_0 > 0$ such that $d(c_0(t_0), x_0) \leq d(c_0(t_0), g.x_0)$. Let $t_1 > t_0$ be such that $c_0(t_1) = g.x_0$. Then $d(c_0(t_1), x_0) > 0 = d(c_0(t_1), g.x_0)$ because $g.x_0 \neq x_0$. By the intermediate value theorem there exists a $t \in [t_0, t_1]$ such that $d(c_0(t), x_0) = d(c_0(t), g.x_0)$. But this would imply that c_0 meets $H_{x_0, g.x_0}$ at least twice in the points $y, c_0(t)$. This is impossible by lemma 3.10. So we must have $d(y, x_0) < d(y, g.x_0)$ with a strict inequality. Therefore $y \in A$.

Since $D(x_0)$ is closed, we have $\overline{D(x_0)^\circ} \subseteq D(x_0)$. Now take $x \in D(x_0)$ and let c be a minimal geodesic from x to x_0. If $t > 0$ and $g \neq e$, then

$$d(c(t), x_0) = d(x, x_0) - d(x, c(t)) \leq d(x, g.x_0) - d(x, c(t)) \leq d(c(t), g.x_0)$$

Assume equality holds in both cases. Then $d(x, g.x_0) = d(x, c(t)) + d(c(t), g.x_0)$. Thus, if c_1 is a minimal geodesic from $c(t)$ to $g.x_0$, then $c|_{[0,t]} \cdot c_1$ is a minimal geodesic from x to $g.x_0$. Because c and $c|_{[0,t]} \cdot c_1$ coincide on $[0, t]$, they coincide everywhere and so $d(x, x_0) = d(x, g.x_0)$ implies $x_0 = g.x_0$ or $g = e$, which is a contradiction. So the inequality is strict and we have $c(t) \in D(x_0)^\circ$. Thus

$$x = \lim_{t \to 0+} c(t) \in \overline{D(x_0)^\circ}.$$

□

Lemma 3.12. *Let $G \subset \mathrm{Isom}(M)$ act discretely on M and let $x_0 \in M$ be a regular point. Then*

(1) $g.D(x_0) = D(g.x_0)$ and $g.H_{x_0, h.x_0} = H_{g.x_0, g.h.x_0}$ for $g, h \in G$.

(2) If $g.D(x_0) = D(x_0)$ for some $g \in G$, then $g = e$.

(3) $M = \bigcup_{g \in G} g.D(x_0)$

(4) There exists a fundamental domain F for the action of G satisfying

$$D(x_0)^\circ \subseteq F \subseteq D(x_0).$$

Proof. (1)

$$\begin{aligned} y \in D(g.x_0) &\Leftrightarrow d(y, g.x_0) \le d(y, h.g.x_0) \quad \forall h \in G \\ &\Leftrightarrow d(g^{-1}.y, x_0) \le d(y, h.x_0) \quad \forall h \in G \\ &\Leftrightarrow g^{-1}.y \in D(x_0) \\ &\Leftrightarrow y \in g.D(x_0) \end{aligned}$$

The other statement can be proved in the same way.

(2) If $g.D(x_0) = D(x_0)$ then $g.x_0 \in D(x_0)$. By definition this means $d(g.x_0, x_0) \le d(g.x_0, h.x_0)$ for all $h \in G$. By setting $h = g$ we get $g.x_0 = x_0$ or $g \in G_{x_0}$. So $g = e$ by lemma 3.6.

(3) Take $y \in M$. By lemma 3.7 there exists a $g \in G$ such that $d(y, g.x_0) = d(y, G.x_0)$. Then $y \in D(g.x_0) = g.D(x_0)$.

(4) We want to apply lemma 1.4. Take $y \in D(x_0)^\circ$ and $g \in G \setminus \{e\}$. Then

$$d(g.y, x_0) = d(y, g^{-1}.x_0) > d(y, x_0) = d(g.y, g.x_0)$$

This implies $g.y \notin D(x_0)^\circ$ and therefore $D(x_0)^\circ$ meets every orbit of G at most once. It follows from (3) that $D(x_0)$ meets every orbit at least once. So lemma 1.4 guarantees us the existence of a fundamental domain F with $D(x_0)^\circ \subseteq F \subseteq D(x_0)$.

□

Lemma 3.13. *Let $G \subset \mathrm{Isom}(M)$ act discretely on M and let $x_0 \in M$ be a regular point. Then*

(1) $D(x_0)^\circ$ is the connected component containing x_0 of

$$M \setminus \bigcup_{g \in G \setminus \{e\}} H_{x_0, g.x_0} \subseteq M_{reg}$$

(2) G acts simply transitively on the set $\{D(g.x_0) : g \in G\}$ of all closed Dirichlet domains.

Proof. (1) If $y \notin \bigcup_{g \in G \setminus \{e\}} H_{x_0, g.x_0}$ then $d(y, x_0) \neq d(y, g.x_0)$ for all $g \in G \setminus \{e\}$. So if $g.y = y$ for some $g \neq e$, then

$$d(y, x_0) = d(g.y, x_0) = d(y, g^{-1}.x_0) \neq d(y, x_0),$$

a contradiction. Therefore $G_y = \{e\}$ and y is regular.

We note first that $D(x_0)^\circ$ is path connected, because if $y \in D(x_0)^\circ$ then the minimal geodesic between y and x_0 lies in $D(x_0)^\circ$. Next, $M \setminus \bigcup_{g \in G \setminus \{e\}} H_{x_0, g.x_0}$ is a disjoint union of sets of the form

$$\bigcap_{h_1 \in H_1} \{y : d(y, x_0) < d(y, h_1.x_0)\} \cap \bigcap_{h_2 \in H_2} \{y : d(y, x_0) > d(h_2, g.x_0)\}$$

where H_1, H_2 is a partition of G. We get $D(x_0)^\circ$ by setting $H_1 = G$, $H_2 = \emptyset$. By a proof similar that of lemma 3.11 it follows that these sets are all open. So $D(x_0)^\circ$ must be the whole connected component.

(2) Transitivity follows from the identity $D(g.x_0) = g.D(x_0)$ of lemma 3.12 (1). Part (2) of the same lemma shows that the action is free. So the action is simply transitive.

□

3.3 Lemma of Poincaré

Definition 3.14. Let $G \subset \mathrm{Isom}(M)$ act discretely on M. Let $x_0 \in M$ be a regular point and $g \in G \setminus \{e\}$. The set $H_{x_0,g.x_0} \cap D(x_0)$ is called *wall* of the closed Dirichlet domain $D(x_0)$, if it contains a non-empty open subset of $H_{x_0,g.x_0}$.

Two closed Dirichlet domains are called *neighbors* if they contain a common wall.

The action of G on the set of closed Dirichlet domains maps neighbors to neighbors. This is a direct consequence of lemma 3.12 (1).

Lemma 3.15 (Lemma of Poincaré)**.** *Let $G \subset \mathrm{Isom}(M)$ act discretely on M and let $D := D(x_0)$ be the closed Dirichlet domain of a regular point $x_0 \in M$. Let $g_1.D, g_2.D, \ldots$ be all the neighbors of D. Then the elements $g_1, g_2, \ldots$ generate the group G.*

Proof. Claim. For each $g \in G$, there exists a sequence $e = h_0, h_1, \ldots, h_n = g$ such that $D(h_i.x_0)$ and $D(h_{i+1}.x_0)$ are neighbors for each i. We call this a *Dirichlet neighbors chain* from x_0 to $g.x_0$.

The claim proves the lemma as follows. For a $g \in G$, pick a Dirichlet neighbors chain $h_0, \ldots, h_n$. Since $D(h_1.x_0)$ is a neighbor of $D = D(x_0)$, we have $D(h_1.x_0) = g_{i_1}.D(x_0) = g_{i_1}.D$ for some i_1. Since $g_{i_1}.D$ and $D(h_2.x_0)$ are neighbors, so are D and $g_{i_1}^{-1}.D(h_2.x_0)$. So there exists some i_2 such that $g_{i_1}^{-1}.D(h_2.x_0) = g_{i_2}.D$. Therefore $D(h_2.x_0) = g_{i_1}.g_{i_2}.D$. Continuing in the same fashion we finally obtain $D(g.x_0) = D(h_n.x_0) = g_{i_1} \ldots g_{i_n}.D$. By lemma 3.13 (2) we get $g = g_{i_1} \ldots g_{i_n}$.

We prove the claim by induction on $\{d_g := d(x_0, g.x_0) : g \in G\}$. Induction is admissible here because this is a locally fninie set by lemma 3.7.

Let $g \in G$ and assume that there exists a Dirichlet neighbors chain from x_0 to $h.x_0$ whenever $d_h < d_g$. Assume $d(g_1.x_0, g_2.x_0) < d_g$ for some $g_1, g_2 \in G$. By induction hypothesis take a Dirichlet neighbors chain $h_0, \ldots, h_n$ from x_0 to $g_1^{-1}.g_2$. Then $g_1.h_0, \ldots, g_0.h_n$ is a Dirichlet neighbors chain from $g_1.x_0$ to $g_2.x_0$. We conclude that a Dirchlet neighbors chain from $g_1.x_0$ to $g_2.x_0$ exists, whenever $d(g_1.x_0, g_2.x_0) < d_g$. Consider a minimal geodesic c from x_0 to $g.x_0$ of length d_g. We distuinguish three cases.

Case 1. Suppose that c meets $\bigcup_{e \neq k \in G} H_{x_0,k.x_0}$ in $x = c(t_1) \in H_{x_0,k.x_0}$ at distance $t_1 < \frac{1}{2} d_g$. Then $t_1 = d(x_0, x) = d(x, k.x_0)$. By triangle inequality

$$d_k = d(x_0, k.x_0) \leq d(x_0, x) + d(x, k.x_0) = 2t_1 < d_g$$

Thus by induction there exists a Dirichlet neighbors chain from x_0 to $k.x_0$. Next

$$d(k.x_0, g.x_0) \leq d(k.x_0, x) + d(x, g.x_0) = t_1 + d(x, g.x_0) = d_g$$

Assume equality holds and let c_1 be a minimal geodesic from x to $k.x_0$. Then the curve

$$c_2 := (-c|_{[t_1, d_g]}) \cdot c_1 : t \mapsto \begin{cases} c(d_g - t) & t \in [0, d_g - t_1] \\ c_1(t - (d_g - t_1)) & t \in [d_g - t_1, d_g] \end{cases}$$

is a minimal geodesic from $g.c_0$ to $k.x_0$ and coincides with $-c$ on the interval $[0, d_g - t_1]$. Thus they must coincide everywhere, in particular $x_0 = (-c)(d_g) =$

$c_2(d_g) = k.x_0$. This is impossible. So $d(k.x_0, g.x_0) < d_g$ and by induction we get a Dirichlet neighbors chain from $k.x_0$ to $g.x_0$. Together we get a Dirichlet neighbors chain from x_0 via $k.x_0$ to $g.x_0$, as required.

Case 2. Suppose that c meets $\bigcup_{e \neq k \in G} H_{x_0,k.x_0}$ for the first time at $x = c(\frac{1}{2}d_g) \in H_{x_0,g.x_0}$ and that x lies in no other central hypersurface. Because the curve c lies in no other central hypersurface on it's way between x_0 and x, we have $d(x, x_0) < d(x, k.x_0)$ for all $k \in G \setminus \{e, g\}$. By lemma 3.7 we get a $k \in G$ such that $d(x, h.x_0) = \inf_{k \in G \setminus \{e,g\}} d(x, k.x_0)$ and we set $\epsilon = \frac{1}{2}(d(x, h.x_0) - d(x, x_0))$. Then it can be shown as in (3.1) that each $y \in U := B_\epsilon(x)$ satisfies $d(y, x_0) < d(y, k.x_0)$ for all $k \in G \setminus \{e, g\}$. In particular, if $y \in U \cap H_{x_0,g.x_0}$ then $y \in D(x_0)$, so $U \cap H_{x_0,g.x_0} \subset D(x_0) \cap H_{x_0,g.x_0}$. Thus $D(x_0) \cap H_{x_0,g.x_0}$ is a wall and $D(x_0)$ and $D(g.x_0)$ are neighbors. The Dirichlet neighbors chain from x_0 to $g.x_0$ is given by e, g.

Case 3. Suppose that c meets $\bigcup_{e \neq k \in G} H_{x_0,k.x_0}$ for the first time at $x = c(\frac{1}{2}d_g) \in H_{x_0,g.x_0} \cap H_{x_0,k.x_0}$ for $k \neq g$. Then

$$d_k = d(x_0, k.x_0) \leq d(x_0, x) + d(x, k.x_0) = d(x_0, x) + d(x, g.x_0) = d_g$$

Assume equality holds and let c_1 be a minimal geodesic from x to $k.x_0$. Then the concatenated curve $c|_{[0,\frac{1}{2}d_g]} \cdot c_1$ would be a minimal geodesic from x_0 to $k.x_0$, that coincides with c on the interval $[0, \frac{1}{2}d_g]$. This is impossible, since $g.x_0 \neq k.x_0$. Thus $d_k < d_g$ and by induction there is a Dirichlet neighbors chain from x_0 to $k.x_0$. Next,

$$d(k.x_0, g.x_0) \leq d(k.x_0, x) + d(x, g.x_0) = d(x_0, x) + d(x, g.x_0) = d_g$$

Again, assume equality holds. Then the curve $((-c)|_{[0,\frac{1}{2}d_g]}) \cdot c_1$ would be a minimal geodesic from $g.x_0$ to $k.x_0$ and coincide with $-c$ on the interval $[0, \frac{1}{2}d_g]$. This would imply that $x_0 = k.x_0$, which is not possible. Thus $d(k.x_0, g.x_0) < d_g$ and we get a Dirichlet neighbors chain from $k.x_0$ to $g.x_0$. Together we get a Dirichlet neighbors chain from x_0 via $k.x_0$ to $g.x_0$, as required. This finishes the prove of the claim.

□

Chapter 4

Discrete Groups of Reflections

Unless stated otherwise (M, γ) shall be a complete, connected Riemannian manifold.

4.1 Reflections

First we recall the definition of a reflection in the Euclidean space $\mathbb{R}^n$.

Definition 4.1. A linear map s on $\mathbb{R}^n$ is a *Euclidean reflection*, if for some vector $0 \neq X \in \mathbb{R}^n$

$$s.X = -X \quad \text{and} \quad s|X^\perp = \mathrm{Id}$$

Now we can define, what a reflection on a Riemannian manifold shall mean.

Definition 4.2. A *reflection* s on M is an isometry, $s \in \mathrm{Isom}(M)$, such that for some fixed point x of s the tangent mapping $T_x s$ is a reflection in the Euclidean space $(T_x M, \gamma_x)$ in the sense of definition 4.1.

We note that if $M = \mathbb{R}^n$, then every Euclidean reflection is a reflection in the sense of definition 4.2, since for linear maps s we have $T_X s = s$ at every point $X \in \mathbb{R}^n$.

Next we prove some basic properties of reflections.

Lemma 4.3. *Let s be a reflection on M. Then*

(1) s is an involution, $s \circ s = \mathrm{Id}_M$.

(2) Every connected component N of the fixed point set M^s is a closed totally geodesic submanifold and for each $x \in N$ the tangent mapping $T_x s$ equals the identity on $T_x N$ and $-\mathrm{Id}$ on $T_x N^\perp$.

(3) Every connected component N of M^s determines s completely as follows: For $y \in M$ there exists $x \in N$ such that $d(y, x) = d(y, N)$. Let $t \mapsto \exp(t.Y_x)$ be a minimal geodesic from x to y, which reaches y at $t = 1$. Then $s(y) = \exp(-Y_x)$.

(4) At least one connected component of M^s has codimension 1.

(5) For any $y \in M \setminus M^s$ we have $M^s \subseteq H_{y,s.y}$.

Proof. (1) Let $x \in M^s$ be a point, such that $T_x s$ is a Euclidean reflection. Then $T_x s \circ T_x s = \mathrm{Id}_{T_x M}$. Thus by theorem 1.7 we have $s \circ s = \mathrm{Id}_M$.

(2) That every connected component N of M^s is a totally geodesic submanifold is shown in theorem 1.6. Furthermore for each $x \in N$, we have $T_x N = \mathrm{Eig}(1, T_x s)$. So $T_x s|_{T_x N} = \mathrm{Id}$. Because of $(T_x s)^2 - \mathrm{Id} = 0$ the only possible eigenvalues are -1 and 1. Since $T_x s$ is an orthogonal transformation, we conclude that $\mathrm{Eig}(-1, T_x s) = T_x N^\perp$. So $T_x s|_{T_x N^\perp} = -\mathrm{Id}$.

(3) Take $y \in M$. From theorem 1.6 be know that N is closed, so there exists $x \in N$ with $d(y,x) = d(y,N)$. Let $c(t) = \exp(t.Y_x)$ be a minimal geodesic from x to $y = c(1)$ and $Y_x \in T_x M$. Then by theorem 1.5, $Y_x = c'(0) \in T_x N^\perp$. So

$$s(y) = s(\exp(Y_x)) = \exp(T_x s.Y_x) = \exp(-Y_x).$$

(4) Let $x \in M^s$ be a point sucht that $T_x s$ is a Euclidean reflection. We show that the connected component N of M^s containing x has codimension 1:

$$\dim N = \dim T_x N = \dim \mathrm{Eig}(1, T_x s) = n - 1,$$

because $T_x s$ is a Euclidean reflection.

(5) Take $x \in M^s$. Then $d(x,y) = d(s.x, s.y) = d(x, s.y)$. Thus $x \in H_{y,s.y}$. □

Of all components of the fixed point set M^s, those of codimension 1 will play the most important part, since they are able to cut the manifold M into smaller pieces. Thus we give them a name.

Definition 4.4. Let s be a reflection on M. A connected component of M^s of codimension 1 is called a *reflection hypersurface of s*.

Another basic property of reflexions is that conjugation by an isometry again creates a reflexion.

Lemma 4.5. *Let s be a reflexion on M and g an isometry. Then $g.s.g^{-1}$ is also a reflection and $M^{g.s.g^{-1}} = g.M^s$.*

Proof. Let $x \in M$ be a fixed point of s, such that $T_x s$ is a Euclidean reflection. Then $g.x$ is a fixed point of $g.s.g^{-1}$ and

$$T_{g.x}(g.s.g^{-1}) = (T_x g).(T_x s).(T_x g)^{-1}$$

is a Euclidean reflection. The other statement follows from

$$x \in g.M^s \Leftrightarrow s.g^{-1}.x = g^{-1}.x \Leftrightarrow g.s.g^{-1}.x = x \Leftrightarrow x \in M^{g.s.g^{-1}}$$

□

Example 4.6. Some simple examples of reflections are given in figure 4.1. In the left picture we have $M = S^1$ and the reflection corresponds to complex conjugation $z \mapsto \overline{z}$. The fixed point set $M^s = \{-1, 1\}$ is the union of two reflection hypersurfaces. Another example of a reflection is shown in the right picture.

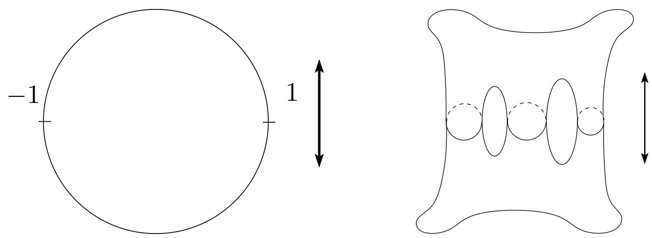


Figure 4.1: Reflection on S^1 and on a surface of genus 2.

4.2 Reflection Groups

Now we are ready to define the main object of our interest.

Definition 4.7. Let $G \subset \mathrm{Isom}(M)$ be a discrete group of isometries acting on M. We call G a *reflection group*, if G is generated by all reflections contained in G.

In chapter 3 we used the group G to partition M into Dirichlet domains. Now G contains distinguished elements, namely the reflections, so we will use them to partition M. Let H_1 be the union of all reflection hypersurfaces of all reflections in G.

Definition 4.8. A *(Weyl) chamber* is the closure $\overline{B}$ of a connected component B of the complement $M \setminus H_1$. The open interior C° of a Weyl chamber C is called an *open (Weyl) chamber*.

The next lemma showes that H_1 is closed, so that the connected component B of $M \setminus H_1$ is open. If $C = \overline{B}$ is a chamber and if B' is another connected component of $M \setminus H_1$, then $C \cap B' = \emptyset$, since $B \cap B' = \emptyset$ and B' is open. We see that $C = B \cup (C \cap H_1)$ consists of points of B and points of H_1 and that two chambers intersect only along H_1.

Lemma 4.9. *Let G be a reflection group. The family of all reflection hypersurfaces is locally finite. In particular the union of all reflection hypersurfaces is closed.*

Proof. Assume the family is not locally finite near $x \in M$. Then we can choose a sequence $(x_n)_{n\in\mathbb{N}}$, such that $x_n \to x$, and x_n lie in distinct reflection hypersurfaces, the hypersurface of x_n belonging to the reflection s_n. For a fixed reflection s, the set of reflection hypersurfaces is a locally finite set, since M^s is closed. So we can choose our sequence in such a way, that the s_n are distinct as well.

Now

$$d(s_n.x, x) \leq d(s_n.x, s_n.x_n) + d(s_n.x_n, x) = 2d(x_n, x)$$

shows that $s_n.x \to x$. From Corollary 3.5 we infer that there exists a constant subsequence $(s_{n_k})_{k\in\mathbb{N}}$. But this is a contradiction, since we assumed all s_n to be distinct. Therefore the family of all reflection hypersurfaces must be locally finite. □

Lemma 4.10. *Let G be a reflection group. Then the union of all reflexion hypersurfaces is invariant under G. Furthermore G acts on the set of chambers.*

Proof. Take some $g \in G$. If a point x lies in a reflection hypersurface of the reflection s, then $g.x$ lies in a reflection hypersurface of $g.s.g^{-1}$.

Let H_1 be the union of all reflection hypersurfaces, B a connected component of $M \setminus H_1$ and $g \in G$. Since g preserves H_1, g must also preserve the complement $M \setminus H_1$. Thus $g.B$ is again a connected complement of $M \setminus H_1$. So $g.C = \overline{g.B}$ is the closure of a connected component of $M \setminus H_1$, thus a chamber. □

Now we can examine the relationship between Dirichlet domains considered in chapter 3 and Weyl chambers. It turns out that Weyl chambers are exactly unions of Dirichlet domains. Define the *normalizer* $N_G(C)$ of a chamber C to be the subgroup of all isometries that fix C.

$$N_G(C) := \{g \in C : g.C = C\}$$

Theorem 4.11. *Let C be a chamber and $x \in C$ a regular point. Then*

$$C = \bigcup_{g \in N_G(C)} D(g.x),$$

in particular $D(x) \subseteq C$. Furthermore G acts transitively on the set of all chambers.

Proof. First let us introduce some notation. Let H_1 be the union of all reflexion hypersurfaces and $H(x) := \bigcup_{g \neq e} H_{x,g.x}$. Then $H_1 \subseteq H(x)$ by lemma 4.3 (5). Denote by B the (open) connected component of $M \setminus H_1$, whose closure is C. x is regular and so its isotropy group is trivial. In particular x cannot lie in any reflection hypersurface. Therefore $x \in B$. (See remarks before lemma 4.9).

Step 1. $D(x) \subseteq C$.
We have the following inclusions $D(x)^\circ \subseteq M \setminus H(x) \subseteq M \setminus H_1$ (lemma 3.13 (1)). Since $D(x)^\circ$ is connected, it lies in the same connected component of $M \setminus H_1$ as x. So $D(x)^\circ \subseteq B$. Now $D(x) = \overline{D(x)^\circ} \subseteq \overline{B} = C$.

Step 2. If $D(g.x) \subseteq C$ then $g.C = C$.
Let $g \in G$ and $D(g.x) \subseteq C$. Assume $g.C$ is a chamber different from C. Then $C \cap g.C \subseteq H_1$, which is impossible since $g.x \in C \cap g.C$ and $g.x_0$ is regular point and thus $g.x \notin H_1$.

Step 3. If $g \in N_G(C)$ then $D(g.x) \subseteq C$.
This follows from $D(g.x) = g.D(x) \subseteq g.C = C$ by step 1. As a consequence we have

$$\bigcup_{g \in N_G(C)} D(g.x) \subseteq C$$

Step 4. $B \subseteq \bigcup_{g \in N_G(C)} D(g.x)$.
Let $y \in B$. Since the Dirichlet domains cover M, $y \in D(g.x)$ for some $g \in G$. We will show that $g \in N_G(C)$. $g.x$ is regular, so it lies in some connected component B' of $M \setminus H_1$. Step 1 shows that $y \in D(g.x) \subseteq \overline{B'}$. If B and B' were different connected components, then $\overline{B'} \cap B = \emptyset$, a contradiction, since we have $y \in B \cap \overline{B'}$. Thus $B = B'$ and $D(g.x) \subseteq C$. Using step 2 we conclude that $g.C = C$ and $g \in N_G(C)$.

Step 5. $\bigcup_{g\in N_G(C)} D(g.x)$ is closed.
Let y be an element of the closure of this set. Then there exists a sequence $(g_n.x_n)_{n\in\mathbb{N}}$ with $g_n \in N_G(C)$ and $x_n \in D(x)$ such that $g_n.x_n \to y$. From $d(x, x_n) \le d(g_n^{-1}.x, x_n) = d(x, g_n.x_n)$ we see that the set $\{x_n : n \in \mathbb{N}\}$ is bounded. M is assumed to be a complete Riemannian manifold, hence $\{x_n : n \in \mathbb{N}\}$ is precompact. Thus we can choose a convergent subsequence, which we again denote by $(x_n)_{n\in\mathbb{N}}$. Let $z \in D(x)$ be the limit, $x_n \to z$. Then

$$d(g_n.z, y) \le d(g_n.z, g_n.x_n) + d(g_n.x_n, y) = d(z, x_n) + d(g_n.x_n, y) \to 0$$

and we see that $g_n.z \to y$. Corollary 3.5 tells us, that there exists a constant subsequence of $(g_{n_k})_{k\in\mathbb{N}}$. So $y = g_{n_k}.z$ or $y \in D(g_{n_k}.x)$.

Taking the closure in step 4 concludes the proof of the first statement.

Step 6. Transitivity
Let C' be another chamber and B' the corresponding connected component. If $g.x_0 \notin B'$ for some $g \in G$, then $D(g.x_0) \cap B' = \emptyset$ by step 1. Since the Dirichlet domains cover M, there exists a $g \in G$, such that $g.x_0 \in B'$. Then $g.C = C'$, since g maps chambers to chambers. □

Next we want to understand better, how reflections generate the group G, more precisely, how many reflections are needed to generate the group. For this we need some definitions first.

Definition 4.12. Let C be a Weyl chamber and s a reflection on M. A connected component of $C \cap M^s$ is called a *wall* of C, if it contains a non-empty open subset of M^s of codimension 1.

Two walls are called *neighbors*, if their intersection contains a connected component of codimension 2.

Some remarks are in order:

1. To each wall corresponds a unique reflection. By definition, to each wall F, there exists a reflection s and a chamber C, such that $F \subseteq C \cap M^s$. Let $x \in U \subseteq F$, where U is an open subset of M^s of codimension 1. Then, since s is a reflection, T_xs is completely determined by F. More precisely, we must have $T_xs|_{T_xU} = \mathrm{Id}$ and $T_xs|_{T_xU^\perp} = -\mathrm{Id}$. Because U is open in M^s, it is also open in F, so $T_xU = T_xF$. So F completely determines T_xs and thus s as well. So we can say that F *is a wall corresponding to the reflection* s.

2. Let F be a common wall of the chambers C, C'. Let s be the reflection corresponding to F. Then $C' = s.C$. This is most easily seen in normal coordinates. Take a point in the interior of the wall, $x \in F^\circ$, where the interior is understood with respect to M^s. Then in normal coordinates s is simply a Euclidean reflection with respect to the hyperplane T_xF. x lies in both chambers and each chamber is the closure of a connected component of $M \setminus H_1$. Thus in normal coordinates, each side of the hyperplane T_xF must belong to one of the chambers and s maps one to the other.

The folowing theorem is an analogue to the lemma of Poincaré (lemma 3.15).

Theorem 4.13. *Let G be a reflexion group on M, C a chamber, $F_1, F_2, \ldots$ the walls of C and let s_i be the reflection with respect to the wall F_i.*

Then the reflections $s_1, s_2, \ldots$ generate G and they satisfy the following relations
(1) $s_i^2 = e$
(2) $(s_i.s_j)^{n_{i,j}} = e$ *for some natural numbers* $n_{i,j}$*, if the walls* F_i, F_j *are neighbors.*

Proof. **Step 1.** Let C be a chamber, $g \in G$ and F a wall of C corresponding to the reflection s. Then $g.F$ is a wall of $g.C$ corresponding to the reflection $g.s.g^{-1}$.
F is a connected component of $C \cap M^s$. Then $g.F$ is a connected component of $g.(C \cap M^s) = g.C \cap M^{g.s.g^{-1}}$. Let $U \subseteq F$ be an open subset of M^s of codimension 1. Then $g.U$ is open in $M^{g.s.g^{-1}}$ and has codimension 1. Thus $g.F$ is a wall of $g.C$.

Step 2.
Let s' be some reflection. Let C' be a chamber, that has a wall F' that corresponds to s'. Since G acts transitively on the set of all chambers, there exists a $g \in G$ with $C = g.C'$. Then $g.F'$ is a wall of C corresponding to the reflection $g.s'.g^{-1}$. So $g.s'.g^{-1} = s_i$ or $s' = g^{-1}.s_i.g$ for some i. The first part of the theorem will be proven, after the following claim has been shown.

Claim. *We can choose* g*, that lies in the subgroup generated by* $s_1, s_2, \ldots$*.*

Step 3.
Let $\mathscr{H}$ be the set of all reflection hypersurfaces. Since they are totally geodesic submanifolds, any intersection is again a totally geodesic submanifold. If H_1 and H_2 are two reflection hypersurfaces of the distinct reflections s_1, s_2, then their intersection $H_1 \cap H_2$ has codimension 2. To see this, look at H_1, H_2 in normal coordinates. There they look like hyperplanes in the Euclidean space. If their intersection has codimension 1, then H_1 and H_2 must be locally equal, which would imply $T_x s_1 = T_x s_2$ for some $x \in H_1 \cap H_2$ and so $s_1 = s_2$ by theorem 1.7. Let

$$N := \bigsqcup_{H \in \mathscr{H}} H \sqcup \bigsqcup_{H_1, H_2 \in \mathscr{H}} H_1 \cap H_2$$

be the disjoint sum of all hypersurfaces and all intersections of hypersurfaces. Because G is discrete, G is countable and each reflection has at most countably many reflection hypersurfaces, thus $\mathscr{H}$ is countable and N is separable. Thus N is again a manifold. There exists a natural map $\iota : N \to M$, which embeds each connected component of N into M.

Step 4. Proof of the claim.
Now choose regular points $x \in C$ and $x' \in C'$ and a smooth path c from x to x'. We can assume that the curve c is transverse to ι. By dimension, c has to avoid all intersections $H_1 \cap H_2$. Thus c changes chambers only transversally through open interiors of walls. By compactness, c passes through only a finite number of chambers $C = C_1, \ldots, C_n = C'$. We prove the claim by induction. If $i = 1$, then $C = e.C$. Now assume that $C = g.C_i$ and g lies in the subgroup of G, generated by $s_1, s_2, \ldots$. The curve passes from C_i to C_{i+1} through a wall F belonging to the reflection s. Because of $C = g.C_i$, $g.F$ is a wall of C belonging to a reflection s_j for some j. So $s = g^{-1}.s_j.g$. Thus s also belongs to the subgroup generated by $s_1, s_2, \ldots$. Because F is a common wall of C_i and C_{i+1}, we have $C_i = s.C_{i+1}$. So $C = g.C_i = g.s.C_{i+1}$. Thus the claim is established.

Step 5. Second part of the theorem.
Let F_i, F_j be neighbors. Choose $x \in F_i \cap F_j$. Then $s_i, s_j \in G_x$. G_x is finite because of corollary 3.4, thus each element, in particular $s_i.s_j$ has finite order. □

Implicitly in the proof was another proof for the fact, that G acts transitively on the set of all chambers. Namely given two chambers C, C', we picked a curve from C to C', transverse to all walls, and successively constructed an element g with $C = g.C'$ in the proof.

Remark 4.14. Last few theorems may be generalized in the following way: Let G be a reflection group and choose a subset S of reflections, that generates G and is invariant under G, i.e. $g.s.g^{-1} \in S$ for $s \in S$ and $g \in G$. let H_1' be the union of reflection hypersurfaces of all reflections in S. Then we can define a *generalized (Weyl) chamber* as the closure of a connected component of $M \setminus H_1'$.

Then lemma 4.9 holds, since we are dealing with possibly fewer reflection hypersurfaces. The statement of lemma 4.10 still holds, since we built it into our assumptions. The proof of theorem 4.11 remains the same, since it uses only lemma 4.10 and topological arguments.

If we reread the proof of theorem 4.13, then we have to pick in step 2 a reflection s' in S and following the proof we show that we may write s' using the reflections $s_1, s_2, \dots$ corresponding to the walls of our generized chamber. Since the reflections in S generate G, so do the reflections $s_1, s_2, \dots$.

Thus all the theorems remain valid if we replace chambers by generalized chambers. We will need these generalized versions later, when we discuss reflection groups generated by disecting reflections.

4.3 Simple Transitive Actions on Chambers

In the following section we will consider reflection groups G, that act simply transitively on the set of all chambers.

In this case the Weyl chambers of the action have some interesting properties.

Theorem 4.15. *Let G act simply transitively on the set of all chambers and let C be a chamber. Then*
(1) $C = D(x)$ for any $x \in C^\circ$ and $C^\circ \subseteq M_{reg}$
(2) $M_{reg} = \bigcup g.C^\circ$

Proof. (1) First let $x \in C^\circ$ be a regular point. Because G acts simply transitively on the set of all chambers, we have $N_G(C) = \{\mathrm{Id}\}$. Thus $C = D(x)$. This implies $C^\circ = D(x)^\circ$, which shows that C° consists of regular points. Thus $C = D(x)$ for any $x \in C^\circ$.

(2) By (1) we have the inclusion $\bigcup g.C^\circ \subseteq M_{\mathrm{reg}}$. Let H_1 be the union of all reflection hypersurfaces. Let B be the (open) connected component of $M \setminus H_1$, such that $C = \overline{B}$. Then we have the inclusion

$$M_{\mathrm{reg}} \subseteq M \setminus H_1 \subseteq \bigcup g.B \subseteq \bigcup g.C^\circ$$

which concludes the proof. □

Remark 4.16. The previous and the following theorems can be found in [1, Corollary 3.8.]. However part (3) of [1, Corollary 3.8.] is wrong. It states that every central hypersurface $H_{x,g.x}$ of a regular point x and $e \neq g \in G$ is a reflection hypersurface. This cannot be true, since for any reflection s the set $H_{x,s.x}$ is disecting but the reflection hypersurface doesn't need not to be.

The next theorem shows that chambers are in fact fundamental domains for the action of G. This result is a big improvement to the situation of general discrete isometry groups, where fundamental domains were located somewhere between open and closed Dirichlet domains.

Theorem 4.17. *Let G act simply transitively on the set of all chambers and let C be a chamber. Then the natural projection $\pi : M \to M/G$ induces a homeomorphism $\pi|_C : C \to M/G$.*

Proof. By theorem 4.15 (1) C is a closed Dirichlet domain. Lemma 3.12 tells us that a Dirichlet domain meets every orbit. Thus $\pi|_C$ is onto.

Next we show that $\pi|_C$ is one-to-one. Assume $\pi(x) = \pi(y)$ for $x, y \in C$. This means $y = g.x$ for some $g \in G$. Thus the chambers C and $g.C$ have the point y in common. Choose a curve c from a point $c(0) \in C^\circ$ to a point $c(1) \in g.C^\circ$, that is traverse to all reflection hypersurfaces, avoids intersections of reflection hypersurfaces and stays near y (near enough, such that c intersects only reflection surfaces, that pass through y). Let $s_1, \dots, s_n$ be the reflexions corresponding to the reflexion hypersurfaces, that are crossed by c, in the order of crossing. Then $g' := s_n \dots s_1$ also maps C to C' and keeps y unchanged, $g'.y = y$. Since G acts simply transitively on the set of all chambers, we have $g = g'$ and $x = g^{-1}.y = y$. Thus $\pi|_C$ is one-to-one.

Let $O \subset C$ be open in C. We need to show that $\pi(O)$ is open in M/G. Since $\pi(O)$ is open if and only if $\pi^{-1}(\pi(O)) = G.O$ is open in M, we will show that $G.O$ is open. First let $x \in O$ and choose an open ball $B_\epsilon(x)$ around x, such that $B_\epsilon(x) \cap C \subseteq O$ and let $\epsilon > 0$ be small enough, that $B_\epsilon(x)$ meets only reflection hypersurfaces, that pass through x. Let $\mathscr{C}$ be the set of all chambers, that contain x. Then

$$B_\epsilon(x) = \bigcup_{C' \in \mathscr{C}} B_\epsilon(x) \cap C' = \bigcup_{g.C \in \mathscr{C}} B_\epsilon(x) \cap g.C$$

We saw in the last paragraph, that g can be chosen such that $g.x = x$. So

$$B_\epsilon(x) = \bigcup_{g.C \in \mathscr{C}} B_\epsilon(x) \cap g.C = \bigcup_{g.C \in \mathscr{C}} g.(B_\epsilon(g^{-1}.x) \cap C) \subseteq \bigcup_{g.C \in \mathscr{C}} g.O \subseteq G.O$$

Now given any $g.x \in G.O$ with $x \in O$ we choose $\epsilon > 0$ as above. Then $B_\epsilon(g.x) = g.B_\epsilon(x) \subseteq g.G.O \subseteq G.O$. Thus $G.O$ is open. This proves that $\pi|_C$ is a homeomorphism. □

Corollary 4.18. *C is a continuous retract of M. For fundamental groups this implies $\pi_1(C) \leq \pi_1(M)$.*

Proof. The retraction $r : M \to C$ is given by $r := (\pi|_C)^{-1} \circ \pi$. The statement about fundamental groups follows from right-invertibility of r, $r \circ \iota = \mathrm{Id}_C$, where $\iota : C \to M$ is the inclusion map. □

This corollary allows a simper proof of [1, Proposition 2.14.] than given there, see theorem 5.15.

We can give a more explicit description of the retraction $r : M \to C$. By definition $r : (\pi|_C)^{-1} \circ \pi$, so $r(x)$ is the unique point $r(x) \in G.x \cap C$ that lies in C and the orbit $G.x$. So on any chamber C' with $C = g.C'$ we have $r|_{C'}(x) = g.x$, the retraction is an isometry on every chamber.

Theorem 4.19. *Let G act simply transitively on the set of all chambers and let C be a chamber. Then C is geodesically convex in the sense that for two points $x, y \in C$ there is a minimal geodesic joining x and y that lies in C.*

Proof. Given $x, y \in C$ we have to show that there exist a minimal geodesic from x to y lying in C. Let $\widetilde{c}$ be a minimal geodesic from x to y and set $c := r.\widetilde{c}$. Since r acts piecewise isometrically on $\widetilde{c}$ we have length$(\widetilde{c})$ = length(c). So c is also a minimal geodesic from x to y and lies in C. □

We also have a way to describe the isotropy group of a point. The following theorem is a generalization of [1, Theorem 3.10 (2)], where it was proven only for groups generated by disecting reflections.

Theorem 4.20. *Let G act simply transitively on the set of all chambers and let $x \in M$. The isotropy group G_x of x is the group generated by reflections with respect to those walls W with $x \in W$.*

Proof. Let G'_x be the group generatd by reflections with respect to those walls W with $x \in W$. The inclusion $G'_x \subseteq G_x$ is obvious. Take $g \in G_x$. Let B_ϵ be an open ball around x, small enough, that it doesn't intersect any other walls, than those, which contain x. Let $y \in B_\epsilon$ be a regular point. Because $g.x = x$ and g is an isometry, we have $g.B_\epsilon = B_\epsilon$, in particular $g.y \in B_\epsilon$. Choose a curve c from y to $g.y$ that remains in B_ϵ, is transverse to all walls and avoids intersections of walls. Let $s_k \cdots s_1$ be the word assigned to c, by adding a reflection from the left, whenever c crosses the wall corresponding to that reflection. Since B_ϵ intersects only walls, that contain x, we have $s := s_k \cdots s_1 \in G'_x$. s maps the chamber that contains y to the chamber that contains $g.y$ by construction. Since G acts simply transitively on the set of all chambers, we have $g = s \in G'_x$. Thus $G_x = G'_x$. □

Remark 4.21. Let C be a chamber, $x \in C$ a regular point and S the set of all reflections with respect to the walls of C. Denote by $\langle S \rangle$ the free group with alphabet S. Using the canonical projection $\langle S \rangle \to G$ we can treat the elements of $\langle S \rangle$ as elements of G. We will now present a construction that will enable us to interpret curves in M as words in $\langle S \rangle$ and visualize words in $\langle S \rangle$ as curves.

First we start with a word $w \in \langle S \rangle$ and we will construct a curve γ_w that starts at x, ends at $w.x$, is transverse to all walls and avoids intersections of walls. To do so we first choose smooth curves c_s from x to $s.x$ for each $s \in S$ with the following properties: c_s crosses the wall between C and $s.C$ exactly once, is transverse to this wall and avoids all other walls. An example of such a curve is a minimal geodesic between x and $s.x$. We'll define γ_w inductively. We have $w = s_1 \cdots s_n$ with $s_i \in S$. Set $w_i = s_1 \cdots s_i$. Start with $\gamma_1 := c_{s_1}$ and define inductively $\gamma_{i+1} := \gamma_i \cdot w_i.c_{s_{i+1}}$. At last take $\gamma_w := \gamma_n$. We will have to do some smoothing at the connection points, but this does not affect the other properties of γ_w.

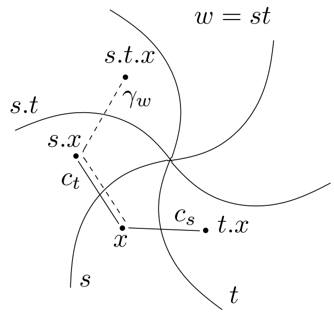


Figure 4.2: Construction of γ_w

Now we start with a curve γ that starts in a regular point $\gamma(0) \in C$ and ends in a regular point in some chamber $g.C$. Furthermore we require γ to be transverse to all walls and avoid intersections of walls. To such a γ we will assign a word $w_\gamma \in \langle S \rangle$ such that $w = g$ in G. Since γ crosses walls only transversally, it crosses only finitely many walls. Starting from the chamber C, we may walk along γ and build a list of chambers $C = C_0, \ldots, C_n$, where γ passes through. Note that the chambers C_i and C_{i+1} are neighbors. We build the word w_γ inductively. Let w_1 be the letter corresponding to the wall between C_0 and C_1. Inductively let w_i be the word that maps C_0 to C_i and w'_{i+1} be the wall between $w_i^{-1}.C_i$ and $w_i^{-1}.C_{i+1}$. Define $w_{i+1} := w_i.w'_{i+1}$. Then w_{i+1} maps C_0 to C_{i+1}. At last set $w_\gamma := w_n$.

The constructions fit together in the sense that for a word $w \in \langle S \rangle$, we have $w_{\gamma_w} = w$. In other words, starting from a word $w \in \langle S \rangle$ we can assign to it a curve γ_w and the word w_{γ_w} assigned to this curve equals the one we started with.

There is another way to visualize this construction. On the manifold we label each wall of C with the element of S it belongs to. Now we spread the labels using the elements of G to all walls of all chambers. Then each chamber has a complete set of labels on it's walls. If we are given a curve we simply write down, which labels the curve passes. For the other direction, starting with a word we construct a curve that crosses the labels determined by the word. One has to be careful to distinguish the label of a wall and the reflection the wall belongs to. In general they coincide only for the chamber C, where we started.

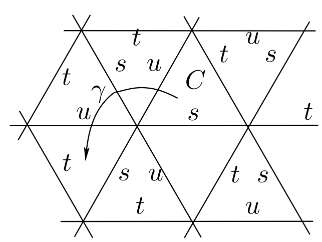


Figure 4.3: Curve γ with word $w_\gamma = usu$

Chapter 5

Disecting Reflections

Unless stated otherwise, (M, γ) shall be a complete, connected Riemannian manifold.

5.1 Disecting Reflections

A special class of isometries are those, whose fixed point set disects the manifold.

Definition 5.1. An isometry $s \in \text{Isom}(M)$ is called *disecting*, if $M \setminus M^s$ is not connected.

It turns out that all disecting isometries are in fact reflections.

Lemma 5.2. *Every disecting isometry is a reflection.*

Proof. By theorem 1.6 M^s is a disjoint union of closed, totally geodesic submanifolds. If no connected component had codimension 1, $M \setminus M^s$ would be connected by theorem 1.18. This cannot be, since s is assumed to be disecting. So let N be a connected component of M^s of codimension 1. Take any $x \in N$. Then $T_x s|_{T_x N} = \text{Id}$ and $T_x s$ is a nontrivial isometry on the 1-dimensional subspace $T_x N^\perp$. Thus $T_x s|_{T_x N^\perp} = -\text{Id}$ and s is a reflecion. $\square$

Although it may be suggested by intuition, not all reflections are disecting.

Example 5.3. We consider the real projective plane $\mathbb{RP}^2$ with the metric induced from S^2 and a reflection s with respect to the line H (see figure 5.1). We

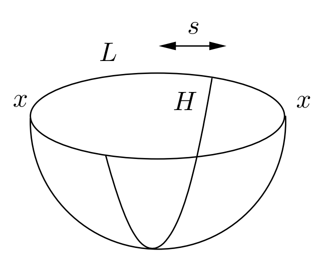


Figure 5.1: Reflection on $\mathbb{RP}^2$

choose H to be orthogonal to the line at infinity L. On L we have to identify antipodally. The fixed point set of s consists of H and the single point x. Since we identify antipodally on L, the line H doesn't disect $\mathbb{RP}^2$.

For disecting reflections we may sharpen the results of lemma 4.3.

Lemma 5.4. *Let s be a disecting reflection on M. Then*
(1) The fixed point set M^s disects M into exactly two pieces. s permutes these pieces.
(2) The fixed point set M^s is a disjoint union of codimension 1 submanifolds.
(3) For any $x \in M \setminus M^s$ we have $M^s = H_{x,s.x}$.

Proof. Take some $x \in M \setminus M^s$. Then $M^s \subset H_{x,s.x}$ by lemma 4.3 (5). Now we apply the results on discrete groups of isometries to the discrete group $\{e, s\}$. Note that, because $G_y = \{e\}$ for each $y \in M \setminus M^s$, the point x is a regular point for the action. By lemma 3.11

$$\begin{aligned} D(x)^\circ &= \{y \in M : d(y,x) < d(y,s.x)\} \\ D(s.x)^\circ &= \{y \in M : d(y,s.x) < d(y,x)\} \end{aligned}$$

and we get a decomposition of $M \setminus H_{x,s.x} = D(x)^\circ \cup D(s.x)^\circ$ into disjoint sets. Lemma 3.13 (1) tells us, that $D(x)^\circ$ and $D(s.x)^\circ$ are connected. So $H_{x,s.x}$ decomposes M into the 2 connected components $D(x)^\circ$ and $D(s.x)^\circ$.

Let M^s_1 be the union of all codimension 1 connected components of M^s. Assume that $M^s_1 \neq H_{x,s.x}$. Take a $y \in H_{x,s.x} \setminus M^s_1$. Pick a minimal geodesic c_0 from y to x and a minimal geodesic c_1 from y to $s.x$. By lemma 3.10 c_0 meets $H_{x,s.x}$ only in y, in particular c_0 does not meet M^s_1. The same holds for c_1. So we can connect the points x, $s.x$ and any $y \in H_{x,s.x} \setminus M^s_1$ by continuous paths lying in $M \setminus M^s_1$. Thus the set

$$\begin{aligned} M \setminus M^s_1 &= (M \setminus H_{x,s.x}) \cup (H_{x,s.x} \setminus M^s_1) \\ &= D(x)^\circ \cup D(s.x)^\circ \cup (H_{x,s.x} \setminus M^s_1) \end{aligned}$$

is connected. In the next paragraph we show, that this cannot be.

Denote by $M^s_{\geq 2}$ the union of all connected components of M^s of codimension at least 2. Then

$$M \setminus M^s = (M \setminus M^s_1) \setminus M^s_{\geq 2}$$

Because s is disecting $M \setminus M^s$ is disconnected. But then $M \setminus M^s_1$ cannot be connected by theorem 1.18, because we obtain $M \setminus M^s$ from $M \setminus M^s_1$ by removing submanifolds of codimension ≥ 2 and this does not change connectivity. This contradicts the previous paragraph. Therefore $M^s_1 = H_{x,s.x}$.

Because $M^s \subseteq H_{x,s.x} = M^s_1 \subseteq M^s$, we see that $M^s = M^s_1 = H_{x,s.x}$ and thus M^s is a disjoint union of codimension 1 submanifolds. This proves (2) and (3). Also because $M^s = H_{x,s.x}$, the first paragraph shows that M^s disects M into exactly two components, namely $D(x)^\circ$ and $D(s.x)^\circ$ and for every $x \in M \setminus M^s$, the points x and $s.x$ lie in diferent components. So s permutes them. Thus (1) is also proved. ∎

We may note that for an isometry s, the condition $M^s = H_{x,s.x}$ from Lemma 5.4 (3) is even equivalent for beeing a disecting reflection.

Corollary 5.5. *Let s be an isometry and $x \in M \setminus M^s$. Then s is a disecting reflection if and only if $M^s = H_{x,s.x}$.*

Proof. We saw in the proof of Lemma 5.4, that $M \setminus H_{x,s.x} = D(x)^\circ \cup D(s.x)^\circ$ is disconnected. The other direction was shown in Lemma 5.4 (3). □

Lemma 5.6. *Let s be a disecting reflection and g an isometry. Then $g.s.g^{-1}$ is again a disecting reflection.*

Proof. $g.s.g^{-1}$ is a reflection because of lemma 4.5. Let M_+ and M_- be the connected components of $M \setminus M^s$. Then $g.M_+$, $g.M_-$ are the connected components of $M \setminus M^{g.s.g^{-1}} = M \setminus g.M^s$. □

5.2 Riemann Coxeter Manifolds

Definition 5.7. Let G be a reflection group on M, that is generated by disecting reflections. Then we call the pair (M, G) a Riemann Coxeter manifold.

The reason for the name is that if G is generated by disecting reflections, G is a Coxeter group. This is the statement of the next theorem.

Theorem 5.8. *Let (M, G) be a Riemann Coxeter manifold, C a chamber and let S be the set of reflections with respect to the walls of C. Then (G, S) is a Coxeter system. Furthermore G acts simply transitively on the set of all chambers.*

Proof. We first look at generalized Weyl chambers and then show that these coincide with the usual ones. Let T' be the set of disecting reflections in G. In the end we will see, that all reflections in G are disecting. Since G is generated by disecting reflections and because of lemma 5.6, T' is invariant under G, so T' satisfies the conditions in remark 4.14 and we may look at generalized chambers with respect to T'.

Let C' be a generalized chamber. Let S' be the set of reflections with respect to the walls of C'. S' generates G by theorem 4.13. For $s \in S'$ let M_+^s be the connected component of $M \setminus M^s$ with $C' \subseteq \overline{M_+^s}$ and M_-^s the other connected component. Define the sets

$$P_s := \{g \in G : g.C' \subseteq \overline{M_+^s}\}$$

We want to apply theorem 1.22 to show that (G, S') is a Coxeter group. Obviously $e \in P_s$. If $g \in P_s$, then $s.g.C' \subseteq \overline{M_-^s}$, thus $s.g \notin P_s$. Thus the first two properties hold.

Now let $s, s' \in S'$ and $g \in G$ with $g \in P_s$ and $g.s' \notin P_s$. We have to conclude that $s = g.s'.g^{-1}$. Let F be a wall of C' corresponding to the reflection s'. Then $g.F$ is a wall of $g.C'$ corresponding to the reflection $g.s'.g^{-1}$. The chambers C' and $s'.C'$ are neighbors and have the common wall F. Thus $g.C'$ and $g.s'.C'$ share the wall $g.F$. Because $g.C \subseteq \overline{M_+^s}$, $g.s'.C' \subseteq \overline{M_-^s}$ and because $g.F$ lies in both sets, it follows that $g.F \subseteq M^s$. This implies that $g.F$ is a wall with respect to the reflection s. Since each wall corresponds to exactly one reflection, we have $s = g.s'.g^{-1}$. Thus we have verified that (G, S') is a Coxeter group.

We know that G acts transitively on the set of all generalized chambers from theorem 4.11. Assume that $g.C' = C'$ for some $g \in G$. Then $g \in P_s$ for all

$s \in S'$. Thus $g = e$ by theorem 1.21. Therefore G acts simply transitively on the set of generalized chambers.

Now we show that the generalized chambers coincide with the "ordinary" chambers as in definition 4.8. Assume G contains a non-disecting reflection s. Then some part of a reflection hypersurface of s would be in the interior of a generalized chamber C'. Since s maps generalized chambers to generalized chambers, this would imply that $s.C' = C'$. But this contradicts the fact that the action of G is simply transitive. So all reflections are disecting and thus the generalized chambers coincide with the "ordinary" ones. This completes the proof. □

Remark 5.9. The proof given in [1, Theorem 3.5.] for this theorem is incomplete. In the proof it is required, that the reflections with respect to the walls of one chamber are disecting. The assumption is however, that only some generating set of reflections is disecting. These generating reflections need not to bound a chamber as is shown in the following example.

Example 5.10. We consider $\mathbb{R}^2$ and the group generated by the reflections along the dotted lines. The group G generated by these reflections is given by the presentation $\langle t_1, t_2, t_3 | t_i^2 = e, (t_i.t_j)^3 = e\rangle$. The chambers are equilateral triangles and we see that the reflections s_1, s_2, s_3 don't bound a chamber.

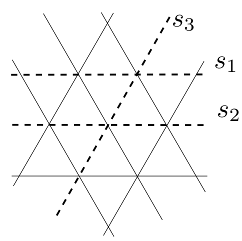


Figure 5.2: Reflection group on $\mathbb{R}^2$.

Theorem 5.11. *Let G be a reflection group on M that acts simply transitively on the set of all chambers. Let S be the set of reflections with respect to the walls of a chamber C. If (G, S) is a Coxeter system, then the reflections in S are disecting, i.e. (M, G) is a Riemann Coxeter Manifold.*

Proof. Take $s \in S$ and assume that s is not disecting. Let $x \in C$ be a regular point. If $M \setminus M^s$ is not disconnected, then we can find a curve c from x to $s.x$ that does not cross M^s. We can assume c to be transverse to all walls and avoid intersections of walls. Let $s_1 \cdots s_n$ be the word assigned to c as in remark 4.21. Because G acts simply transitively on the set of all chambers we have $s = s_1. \cdots .s_n$ in G.

We use the algorithm in theorem 1.23 to reduce the word $s_1 \cdots s_n$. The theorem states that we can obtain s from $s_1 \cdots s_n$ by applying a finite sequence of operations (D1) and (D2). We now show the following: given a word w such that the curve assigned to w does not cross M^s, we can apply to w an operation (D1) or (D2) to obtain a new word w' and the curve assigned to w' doesn't meet M^s either. If we've shown this, the contradiction is immediate: starting with

$s_1 \cdots s_n$ we obtain a sequence of words, whose curves don't cross M^s, but the curve corresponding to the last element s obviously crosses M^s.

(D1). Replacing $stst\cdots$ with $tsts\cdots$ amounts to changing the curve in the way shown in figure 5.3 We see that either both curves cross M^s or none does. This is a heuristic argument, which may be made rigorous in the following way. First we need to translate the statement "curve c assigned to $s_1 \cdots s_n$ crosses M^s" into a statement about words. Assume c crosses M^s while passing from chamber $s_1.\cdots.s_i.C$ to $s_1.\cdots.s_{i+1}.C$ for some $1 \le i < n$. It does so via the reflection $(s_1.\cdots.s_i).s_{i+1}.(s_1.\cdots.s_i)^{-1}$. Thus we have

$$(s_1.\cdots.s_i).s_{i+1}.(s_1.\cdots.s_i)^{-1} = s$$
$$s_1.\cdots.s_i.s_{i+1} = s.s_1.\cdots.s_i$$

What we've just said may be read in both directions, so we obtain the following criterium:

> The curve assigned to $s_1 \cdots s_n$ crosses M^s if and only if
>
> $$s_1.\cdots.s_i.s_{i+1} = s.s_1.\cdots.s_i$$
>
> for some $1 \le i < n$.

Using this criterium and guided by the idea of figure 5.3 the statement may be proven.

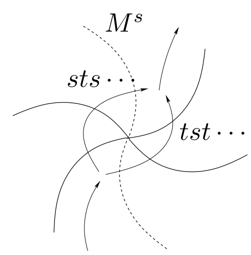


Figure 5.3: Case (D1).

(D2). The operation (D2) simply omits crossings of walls, so applying (D2) to a word, whose curve does not cross M^s certainly yields another such word. □

Remark 5.12. A proof of this theorem is found as part of the proof of [1, Theorem 4.5.]. Unfortunately this proof is wrong. To a curve c in the proof a word $s_1 \ldots s_n$ is assigned. However then the notions of c not crossing a reflection hypersurface of a reflection s and the word not containing the letter s are mixed. The word may contain s and the curve still not cross a reflection hypersurface belonging to s. See remark 4.21

Together the last two theorems give the following result

Theorem 5.13. *Let G be a reflection group on M that acts simply transitively on the set of all chambers. Let S be the set of reflections with respect to the walls of a chamber C. Then the following are equivalent*
(1) (G, S) is a Coxeter system.
(2) G is generated by disecting reflections.

5.3 Simply Connected Manifolds

An application of the theory developed so far is given by simply connected manifolds: on them every reflection is disecting.

Theorem 5.14. *Let M be a simply connected, connected, complete Riemannian manifold. Then any reflection s on M is disecting and its fixed point set M^s is a connected, orientable, totally geodesic and closed hypersurface.*

Proof. Pick $x \in M \setminus M^s$ and let H be a connected component of M^s of codimension 1. Because H is closed, there exists $y \in H$, such that $d(x,y) = d(x,H)$. Choose a minimal geodesic c^+ from x to y. It hits H orthogonally by minimality (theorem 1.5). Define the continuation $c^- := -s.c^+$, which is a minimal geodesic from y to $s.x$. Then c^- also hits H orthogonally. In y the curves c^+ and c^- have the same tangent vectors. Because in coordinates a geodesic is defined via a second order ODE, this suffices for the concatenation $c_0 := c^+ \cdot c^-$ to be smooth in y. So we have a geodesic c_0 from x to $s.x$, that hits H exactly once.

Assume that $M \setminus H$ is connected. Then there exists a curve c_1 from x to $s.x$, that doesn't meet H. Because we work in a manifold, we can assume that this curve is smooth (theorem 1.11). Since M is simply connected, the curves c_0 and c_1 are homotopic. Let $h : [0,1] \times [0,1] \to M$ be an endpoint preserving homotopy

$$h(u,0) = c_0(u), \qquad h(u,1) = c_1(u), \qquad h(0,v) = x, \qquad h(1,v) = s.x.$$

We probably have to reparametrize c_0 and c_1 first in order to be defined on $[0,1]$. Again we can assume that h is smooth (theorem 1.13). Denote by $A := \{0,1\} \times [0,1] \cup [0,1] \times \{0,1\}$ the boundary of the square $[0,1] \times [0,1]$. From the construction of c_0 we see that $h(\frac{1}{2},0) \in H$ and this is the only point of A, that gets mapped to H.

Now we alter h in order to be transverse to H. Using Whitney's extension theorem 1.16 we extend h to a smooth function $\widetilde{h} : U \to M$, where U is open in $\mathbb{R}^2$ and $[0,1] \times [0,1] \subset U$. Because H has codimension 1 and c_0 is orthogonal to H in $y = c_0(\frac{1}{2})$, the map $\widetilde{h}$ is transverse to H in $(\frac{1}{2},0)$. Since this is the only point from A that gets mapped to H, we see that $\widetilde{h}$ is transverse to H along A. By the transversality theorem 1.15 there exists a map $g : U \to M$, transverse to H with $g|_A = \widetilde{h}|_A$. It doesn't matter how near g is to $\widetilde{h}$, all we need is that they coincide along A.

Because g is transverse to H, we conclude that $g^{-1}(H)$ is a 1-dimensional submanifold of U. Let N be the connected component of $g^{-1}(H)$ that intersects A. We note two facts. First $N \cap A = \{(\frac{1}{2},0)\}$ consists of only one point. To this we will derive a contradiction. Second, because $c_0'(\frac{1}{2})$ doesn't lie in T_yH, N cannot be tangent to A. So N must have points inside and outside of $[0,1]\times[0,1]$.

Connected 1-dimensional manifolds (without boundary) are easy to classify: N is diffeomorphic either to the circle S^1 or to $\mathbb{R}$ (see e.g. [12, Appendix]). Assume that $N \cong S^1$. Then $N \setminus \{(\frac{1}{2},0)\}$ would still be connected and lie in at least two components of $U \setminus A$, namely $(0,1) \times (0,1)$ and $U \setminus [0,1] \times [0,1]$. This is a contradiction.

Now assume that $N \cong \mathbb{R}$. Remember that N has points inside and outside the square $[0,1] \times [0,1]$ and crosses the border only once. So $N \cap [0,1] \times [0,1]$ is

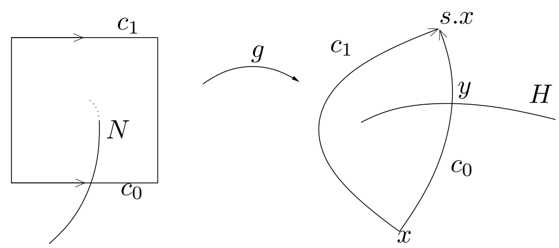


Figure 5.4: Submanifold hitting the boundary once.

homeomorphic to a semiopen interval $[0, \infty)$. But on the other hand N is closed in U, because H is closed in M. This implies that $N \cap [0,1] \times [0,1]$ is compact. This is also a contradiction.

We see that our assumption, that $M \setminus H$ is connected, is false. Since $M \setminus M^s = (M \setminus H) \setminus (M^s \setminus H)$ and $M^s \setminus H$, having codimension at least one, cannot exhaust a connected component of $M \setminus H$, we see that $M \setminus M^s$ is disconnected as well. Thus s is a disecting reflection.

Now we show that $H = M^s$ and M^s is connected. Lemma 5.4 (1) tells us that $M \setminus M^s = U_1 \cup U_2$ consists of exactly two connected components. By the same exhaustion argument $M \setminus H$ cannot have more connected components, thus it has also exactly two, $M \setminus H = V_1 \cup V_2$ and $U_1 \subseteq V_1$, $U_2 \subseteq V_2$. Lemma 5.4 (1) also tells us that s maps U_1 to U_2 and vice versa. Thus it also interchanges V_1 and V_2, since they are connected. But this shows, that V_1 and V_2 cannot contain any fixed points. Thus $H = M^s$. □

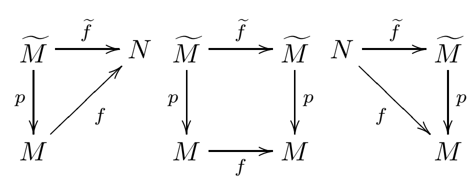
Theorem 5.15. *Let G be a reflection group on M and M simply connected. Then every chamber is simply connected.*

Proof. Since every reflection on M is disecting, (M, G) is a Riemann Coxeter manifold. Let C be a chamber. Then $\pi_1(C) \leq \pi_1(M)$ by corollary 4.18. Thus since M is simply connected, so is C. □

5.4 Lifting to the Universal Cover

In algebraic topology (see, e.g. [6, p.63 ff.]) it is proven that every pathconnected, locally pathconnected and semi-locally simply connected space has a universal cover. A connected manifold M satisfies these properties and we will denote by $p : \widetilde{M} \to M$ its universal cover.

Theorem 5.16. *Let M be a connected manifold. Then $\widetilde{M}$ can be given the structure of a differentiable manifold, such that $p : \widetilde{M} \to M$ is a surjective submersion and a local diffeomorphism. If in one of the*



following diagrams f is smooth, then $\widetilde{f}$ is smooth as well.

Proof. We give $\widetilde{M}$ the differential structure of M such that p is a local diffeomorphism. □

If (M, γ) is a connected Riemannian manifold, then using p we can pull back the Riemannian metric γ to a Riemannian metric $\widetilde{\gamma} := p^*\gamma$ on $\widetilde{M}$, thus turning $(\widetilde{M}, \widetilde{\gamma})$ into a Riemannian manifold.

Theorem 5.17. *Let (M, γ) be a connected Riemannian manifold and $(\widetilde{M}, \widetilde{\gamma})$ defined as above. Then*
(1) $p : \widetilde{M} \to M$ is a local isometry.
(2) Lifts of isometries are isometries.
(3) The group Γ of all deck transformations consists of isometries.
(4) Lifts of geodesics are geodesics.
(5) If M is complete, so is $\widetilde{M}$.

Proof. (1) p is a local diffeomorphism and preserves the Riemannian metric by definition:

$$\widetilde{\gamma}_{\widetilde{x}}(\widetilde{X}, \widetilde{Y}) = \gamma_{p.\widetilde{x}}(T_{\widetilde{x}}p.\widetilde{X}, T_{\widetilde{x}}p.\widetilde{Y}) \text{ for } \widetilde{X}, \widetilde{Y} \in T_{\widetilde{x}}\widetilde{M}$$

(2) Let $\widetilde{g} : \widetilde{M} \to \widetilde{M}$ be a lift of the isometry $g : M \to M$ and $\widetilde{x} \in \widetilde{M}$. Locally around $\widetilde{x}$, the map $\widetilde{g}$ is given by $\widetilde{g} = p^{-1}.g.p$. So $\widetilde{g}$ is a local isometry. Being a lift of a diffeomorphism, $\widetilde{g}$ is again a diffeomorphism, thus $\widetilde{g}$ is an isometry.

(3) Every deck transformation is a lift of the identity Id_M and thus an isometry.

(4) Let $\widetilde{c} : I \to \widetilde{M}$ be a lift of the geodesic $c : I \to M$. Being a geodesic is a local property and locally $\widetilde{c}$ is given by $\widetilde{c} = p^{-1}.c$.

(5) Given a point $\widetilde{x} \in \widetilde{M}$ and a tangent vector $\widetilde{X} \in T_{\widetilde{x}}\widetilde{M}$ we must show that there exists a geodesic defined on whole $\mathbb{R}$ through $\widetilde{x}$ with this initial velocity vector $\widetilde{X}$. Since M is complete, the geodesic $c : \mathbb{R} \to M$ through $c(0) = p.\widetilde{x}$ with initial velocity $\dot{c}(0) = T_{\widetilde{x}}p.\widetilde{X}$ is defined on whole $\mathbb{R}$. Lifting it back to $\widetilde{M}$ gives us the required geodesic through $\widetilde{x}$, defined on whole $\mathbb{R}$.

□

From now on M shall again be a connected, complete Riemannian manifold.

Lemma 5.18. *Let s be a reflection on M. Then*
(1) A lift $\widetilde{s}$ of s is a reflection on $\widetilde{M}$ if and only if there exists a fixed point $\widetilde{x} \in \widetilde{M}$ of $\widetilde{s}$, such that $p.\widetilde{x}$ lies in a reflection hypersurface of s.
(2) Given x in a reflection hypersurface of s and $\widetilde{x} \in p^{-1}(x)$ we may lift s to a reflection $\widetilde{s}$ with $\widetilde{s}.\widetilde{x} = \widetilde{x}$. In particular, every reflection on M may be lifted to a reflection on $\widetilde{M}$.

Proof. (1) Assume $\widetilde{s}$ is a reflection. Let $\widetilde{x}$ be a fixed point of $\widetilde{s}$ lying in a reflection hypersurface. Let $\widetilde{U}$ be a neighborhood of $\widetilde{x}$, such that $p|_{\widetilde{U}}$ is an isometry. Then

$$M^s \cap p.\widetilde{U} = M^{p.\widetilde{s}.p^{-1}} \cap p.\widetilde{U} = p.(M^{\widetilde{s}} \cap \widetilde{U})$$

has codimension 1 in M. Thus $p.\widetilde{x}$ lies in a reflection hypersurface of s.

For the other direction assume that $\widetilde{x}$ is a fixed point of $\widetilde{s}$ and $p.\widetilde{x}$ lies in a reflection hypersurface. Then

$$T_{\widetilde{x}}\widetilde{s} = (T_{\widetilde{x}}p)^{-1}.(T_{p.\widetilde{x}}s).(T_{\widetilde{x}}p)$$

is a Euclidean reflection. Thus $\widetilde{s}$ is a reflection on $\widetilde{M}$.

(2) Let $\widetilde{s}$ be any lift of s. Then $\widetilde{s}.\widetilde{x}$ also lies in the fibre over x. Let $g \in \Gamma$ be the deck transformation with $g.\widetilde{x} = \widetilde{s}.\widetilde{x}$. Then $g^{-1}.\widetilde{s}$ is also a lift of s and satisfies $g^{-1}.\widetilde{s}.\widetilde{x} = \widetilde{x}$. From (1) we see that $g^{-1}.\widetilde{s}$ is a reflection with the desired property.

□

Let $\widetilde{G}$ be the group generated by all reflections, which are lifts of reflections in G. Then $\widetilde{G}$ is a reflection group. Discreteness is not a problem, since given an orbit of $\widetilde{G}$, it locally looks like an orbit of G and thus is discrete.

Lemma 5.19. *(1)* $\widetilde{G}.\Gamma = \Gamma.\widetilde{G}$ *in* $\text{Isom}(\widetilde{M})$.
(2) $\widetilde{G}.\Gamma$ *is a group and consists of all lifts of isometries in* G.
(3) $\Gamma \trianglelefteq \widetilde{G}.\Gamma$
(4) $\widetilde{G} \cap \Gamma \trianglelefteq \widetilde{G}$
(5) $G \cong \widetilde{G}.\Gamma/\Gamma \cong \widetilde{G}/(\widetilde{G} \cap \Gamma)$

Proof. (1) If $\widetilde{s}$ is a reflection on $\widetilde{M}$, that is a lift of the reflection s, then so is $\gamma.\widetilde{s}.\gamma^{-1}$ for any deck transformation $\gamma \in \Gamma$. So if $\gamma.\widetilde{s} \in \Gamma.\widetilde{G}$ and $\widetilde{s}$ is a reflection that is a lift of a reflection in G, then $\gamma.\widetilde{s} = (\gamma.\widetilde{s}.\gamma^{-1}).\gamma \in \widetilde{G}.\Gamma$. Now given an arbitrary $\widetilde{g} \in \widetilde{G}$ we can write it as a product of reflections that are lifts of reflections and shift γ step-by-step over to the right side. Thus $\Gamma.\widetilde{G} \subseteq \widetilde{G}.\Gamma$. The other inclusion is proved the same way by shifting γ to the left.

(2) Because of (1), it is clear that $\widetilde{G}.\Gamma$ is a group. Let $\widetilde{g}.\gamma \in \widetilde{G}.\Gamma$. We can write $\widetilde{g}$ as $\widetilde{g} = \widetilde{s}_1 \dots \widetilde{s}_n$, where $\widetilde{s}_1, \dots, \widetilde{s}_n$ are lifts of $s_1, \dots, s_n \in G$. Then $\widetilde{g}$ is a lift of $g = s_1 \dots s_n$ and $\widetilde{g}.\gamma$ is another lift of g. Thus $\widetilde{G}.\Gamma$ contains only lifts of elements of G.
Now take $g \in G$. We can write g as a product of reflections, $g = s_1 \dots s_n$. Each s_i may be lifted to a reflection $\widetilde{s}_i$ on $\widetilde{M}$. Then $\widetilde{g} = \widetilde{s}_1 \dots \widetilde{s}_n$ is a lift of g and every other lift is given by $\gamma.\widetilde{g}$ with $\gamma \in \Gamma$. Thus $\widetilde{G}.\Gamma$ contains all lifts.

(3) Let $\widetilde{g} \in \widetilde{G}.\Gamma$ be a lift of $g \in G$. We claim that $\widetilde{g}.\Gamma$ and $\Gamma.\widetilde{g}$ both consist of all lifts of g. Obviously those sets contain only lifts of g. Let $\overline{g}$ be some lift of g. Pick a point $x \in M$ and $\widetilde{x} \in p^{-1}(x)$. Let $\gamma \in \Gamma$ be the deck transformation taking $\widetilde{g}.\widetilde{x}$ to $\overline{g}.\widetilde{x}$. Then $\overline{g} = \gamma.\widetilde{g}$. Let $\gamma' \in \Gamma$ be the deck transformation taking $\widetilde{x}$ to $\widetilde{g}^{-1}.\overline{g}.\widetilde{x}$. Then $\overline{g} = \widetilde{g}.\gamma'$. Thus $\overline{g}$ is contained in both $\widetilde{g}.\Gamma$ and $\Gamma.\widetilde{g}$.

(4) Let $\widetilde{s} \in \widetilde{G}$ be a reflection, that is a lift of a reflection s. Let $\widetilde{x}$ be a fixed point of $\widetilde{s}$, such that $x = p.\widetilde{x}$ lies in a reflection hypersurface of s. We show that given $\gamma \in \widetilde{G} \cap \Gamma$ we can find $\gamma' \in \widetilde{G} \cap \Gamma$ such that $\gamma.\widetilde{s} = \widetilde{s}.\gamma'$. Lemma 5.18 (2) tells us that there exists a reflection $\overline{s}$ with $\overline{s}.(\widetilde{s}.\gamma.\widetilde{x}) = \widetilde{s}.\gamma.\widetilde{x}$, that is a lift of s. Because a reflexion is involutive, $\gamma' := \overline{s}.\widetilde{s}.\gamma$ is a lift of Id_M and thus a deck transformation. So we have $\gamma.\widetilde{s} = \gamma'.\widetilde{s}$. Using an inductive argument as in (1) we have shown that $(\widetilde{G} \cap \Gamma).\widetilde{g} \subseteq \widetilde{g}.(\widetilde{G} \cap \Gamma)$. The other inclusion can be proven in a similar way.

(5) Let $\phi : G \to \widetilde{G}.\Gamma/\Gamma$ be defined by $g \mapsto [\widetilde{g}]$, where $\widetilde{g}$ is any lift of g. This is well defined, since any two lifts of g differ only by an element of Γ. Assume

$\phi.g = [\mathrm{Id}_{\widetilde{M}}]$, then $\widetilde{g} \in \Gamma$ for any lift $\widetilde{g}$ of g. So $\widetilde{g}$ is a lift of the identity, which means $g = e$ in G. Thus ϕ is one-to-one. Now take an element $[\widetilde{g}.\gamma] \in \widetilde{G}.\Gamma/\Gamma$ in the image. Then $\widetilde{g}$ and also $\widetilde{g}.\gamma$ is the lift of some $g \in G$. With this g we have $\phi.g = [\widetilde{g}.\gamma]$. Thus ϕ is an isomorphism.
To show the other isomorphism, define $\psi : G \to \widetilde{G}/(\widetilde{G} \cap \Gamma)$ again by $g \mapsto [\widetilde{g}]$, where $\widetilde{g}$ is any lift of g, that lies in $\widetilde{G}$. If we write $g = s_1 \dots s_n$ as a product of reflections and lift these reflections to reflections $\widetilde{s}_1, \dots, \widetilde{s}_n$, then $\widetilde{g} = \widetilde{s}_1 \dots \widetilde{s}_n$ is a lift of g, that lies in $\widetilde{G}$. Therefore such a lift exists. If $\overline{g} \in \widetilde{G}$ is another lift of g, then $\overline{g} = \widetilde{g}.\gamma$ with $\gamma \in \Gamma$. So $\widetilde{g}^{-1}.\overline{g} \in \widetilde{G} \cap \Gamma$. Thus ψ is well defined. That ψ is one-to-one and onto can be proven exactly as above. Thus $G \cong \widetilde{G}/(\widetilde{G} \cap \Gamma)$.
□

The following theorem is a result about the structure of an arbitrary reflection group. Note that G is not even required to act simply transitively on the set of all chambers.

Theorem 5.20. *Every reflection group is the quotient of a Coxeter group.*

Proof. Let G be a reflection group on M. Then $\widetilde{G}$ is a reflection group on a simply connected manifold and thus a Coxeter group. Now use lemma 5.19 (5). □

Next we want to state a criterium to decide, when G acts simply transitively on the set of all chambers. Let H_1 denote the union of all reflection hypersurfaces in M and let $\widetilde{H}_1$ be the same for the group $\widetilde{G}$ in $\widetilde{M}$.

Lemma 5.21. $p(\widetilde{H}_1) = H_1$ *and* $p^{-1}(H_1) = \widetilde{H}_1$.

Proof. Take $\widetilde{x} \in \widetilde{H}_1$. Let $\widetilde{s}$ be a reflection, such that $\widetilde{x}$ lies in a reflection hypersurface of $\widetilde{s}$. Then $\widetilde{s}$ is a lift of a map s in G. s is also a reflection, since $p.\widetilde{x}$ lies in the codimension 1 component, that is obtained by projecting down the reflection hypersurface of $\widetilde{s}$. So $p.\widetilde{x} \in H_1$ which implies $p(\widetilde{H}_1) \subseteq H_1$ and $\widetilde{H}_1 \subseteq p^{-1}(p.\widetilde{H}_1) \subseteq p^{-1}(H_1)$.

For any $x \in H_1$ lemma 5.18 (2) shows that $p^{-1}(x) \in \widetilde{H}_1$. This shows the inclusion, $p^{-1}(H_1) \subseteq \widetilde{H}_1$. Because p is surjective, we have $H_1 = p(p^{-1}(H_1)) \subseteq p.\widetilde{H}_1$. This concludes the proof. □

Lemma 5.22. *Assume that the chambers of M are simply connected. Let $\widetilde{C}$ be a chamber in $\widetilde{M}$ for the reflection group $\widetilde{G}$. Then $p.\widetilde{C}$ is a chamber in M and $p|_{\widetilde{C}}$ is a homeomorphism from a chamber in $\widetilde{M}$ onto a chamber in M.*

Proof. $p.\widetilde{C}$ is contained in some chamber C, because $p.\widetilde{H}_1 \subseteq H_1$. Since C is a pathconnected and simply connected set, $p^{-1}(C)$ is a disjoint union of sets, each homeomorphic to C via p. Because $p^{-1}(H_1) = \widetilde{H}_1$, each of these sets lies in some chamber of $\widetilde{M}$. From this and $p.\widetilde{C} \subseteq C$ it follows that $p.\widetilde{C} = C$ and $p|_{\widetilde{C}}$ is a homeomorphism. □

Theorem 5.23. *Assume that the chambers of M are simply connected. Then G acts simply transitively on the set of all chambers if and only if* $\Gamma \subseteq \widetilde{G}$.

Proof. Assume G acts simply transitively on the set of all chambers. Let $\gamma \in \Gamma$ and take a chamber $\widetilde{C}$ in $\widetilde{M}$. Lemma 5.22 implies that $\gamma.\widetilde{C}$ is another chamber. Since $\widetilde{G}$ acts simply transitively on the set of all chambers, we have $\gamma.\widetilde{C} = \widetilde{g}.\widetilde{C}$ for some $\widetilde{g} \in \widetilde{G}$. The element $\widetilde{g}$ is the lift of some $g \in G$ and since $\widetilde{C}$ and $\widetilde{g}.\widetilde{C}$ both cover the same chamber $p.\widetilde{C}$ it follows that g maps $p.\widetilde{C}$ to itself. By assumption $g = \mathrm{Id}_M$. Thus $\widetilde{g}$, being a lift of the identity, is a deck transformation. Since deck transformations are uniquely defined by giving the image of a point, we conclude that $\widetilde{g} = \gamma$. So $\Gamma \subseteq \widetilde{G}$.

Now assume that $\Gamma \subseteq \widetilde{G}$ and let $g \in G$ map a chamber C to itself. Let $\widetilde{C}$ be a chamber covering C. Pick a regular point $x \in C$ and points $\widetilde{x}, \widetilde{y} \in \widetilde{C}$ with $p.\widetilde{x} = x$ and $p.\widetilde{y} = g.x$. Let $\widetilde{g}$ be a lift of g with $\widetilde{g}.\widetilde{x} = \widetilde{y}$. Because $\Gamma.\widetilde{G} = \widetilde{G}$ all lifts of maps in G are contained in $\widetilde{G}$. Thus $\widetilde{g} \in \widetilde{G}$. From $\widetilde{g}.\widetilde{C} = \widetilde{C}$ it follows that $\widetilde{g} = \mathrm{Id}_{\widetilde{M}}$. Thus also $g = \mathrm{Id}_M$ and we see that G acts simply transitively on the set of all chambers. ☐

Chapter 6

Riemannian chambers

In this chapter we want to discuss the following questions: Let G be a reflection group on a manifold M that acts simply transitively on the set of all chambers. Can we reconstruct M, if we are given G and a chamber? The other question is, which groups will we encounter as reflection groups on Riemannian manifolds. The answer to these questions is given in theorem 6.9 and theorem 6.10.

First we cite a theorem, proved by Vinberg in [15, Theorem 1]. The theorem helps to motivate the definition of a Riemannian chamber.

Theorem 6.1 (Theorem 1 in [15])**.** *Let Q be a quadrant in $\mathbb{R}^n$ and G the reflection group generated by reflections with respect to the walls of Q. Then the following are equivalent:*
(1) G is a finite Coxeter group with Weyl chamber Q.
(2) The angle between two walls is of the form π/n for $n \in \mathbb{N}$.

Definition 6.2. A *Riemannian chamber* is a complete, connected Riemannian manifold (C, γ) with corners, such that the following conditions are satisfied

(C1) Each face is a totally geodesic submanifold.

(C2) The angle between neighboring walls W_i and W_j is a constant of the form $\pi/n_{i,j}$ for $n_{i,j} \in \mathbb{N}$.

Let $V \subseteq T^iC$ be a small enough open set as in theorem 2.23. Then for $x \in C$, $\exp_x : V_x := V \cap {}^iT_xC \to W_x \subseteq C$ is a diffeomorphism. By theorem 2.23, V_x is the intersection of an open ball B_x in (T_xC, γ_x) with a quadrant Q. The walls of Q contain the inverse images under $\exp_x$ of the walls of C containing x. The angles between the walls of Q are of the form π/n by (C2). By theorem 6.1 this is equivalent to the fact that the group G_x, generated by reflections with respect to the walls of Q is a finite Coxeter group with Weyl chamber Q.

(C3) We consider the pullback Riemannian metric $(\exp_x |V_x)^*\gamma$ on V_x and extend it to B_x using the elements of G_x as isometries. Then the resulting metric $\widetilde{\gamma}$ must be smooth.

The main source of examples for Riemannian chambers are Weyl chambers of reflection groups.

Theorem 6.3. *Let G be a reflection group on the complete, connected Riemannian manifold M and let C be a chamber. If G acts simply transitively on the set of all chambers, then C with the induced Riemannian metric is a Riemannian chamber.*

Proof. **Step 1.** C is a manifold with corners.
Take $x \in C$ and let W be a normal neighborhood of x. Then $V := \exp^{-1}(W)$ is an open set in $\mathbb{R}^n$ and $\exp^{-1}$ maps the walls through x to hyperplanes in V. Let W be small enough, not to contain any other walls. Then $\exp^{-1}(C \cap W)$ consists of one or more quadrants in V, the quadrants beeing created by the inverse images of walls under exp.

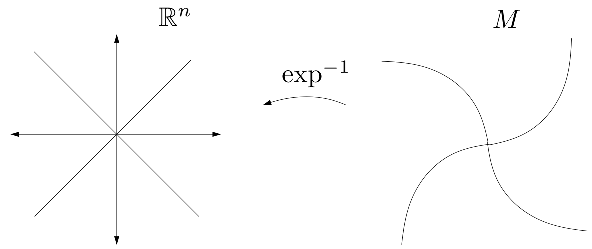


Figure 6.1: Chambers in normal coordinates

Assume $\exp^{-1}(C \cap W)$ consists of more than one quadrant. Then we could choose a curve, transverse to walls and their intersections, from one quadrant into another, both preimages of $C \cap W$. Remark 4.21 assigns this curve an element of G, that maps C to itself, but is not the identity, since in normal coordinates it maps one quadrant to different one. This is impossible, since we assumed G to act simply transitively on the set of all chambers. Thus $\exp^{-1}(C \cap W)$ consists of exactly one quadrant. We have therefore shown that C is a manifold with corners in the weak sense.

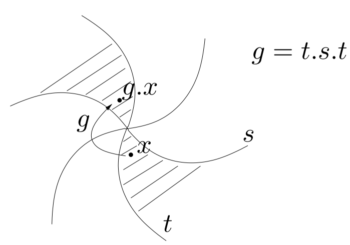


Figure 6.2: Chamber being mapped to two quadrants

Connected components of $\partial_1 C$ are open subsets of reflection hypersurfaces. So every wall lies in the reflection hypersurface of some reflection. Thus, in normal coordinates, every wall lies in a hypersurface of $\mathbb{R}^n$. Therefore, remembering remark 2.14, C is a manifold with corners.

Step 2. Angles between walls.
Let W, W' be two walls with non-empty intersection and $x \in W \cap W'$. Let s, s' be the reflection with respect to these walls. In normal coordinates around x, C

looks like a quadrant and G_x, the isotropy group of x, is a finite reflecion group with respect to the walls of the quadrant. Thus theorem 6.1 tells us that the angle between W and W' is of the form π/n for some $n \in \mathbb{N}$. The only possible n is the order of ss' in G. Therefore the angle is constant.

Step 3. Extendibility.
Take $x \in C$. The isotropy group coincides with G_x as defined in definition 6.2. This is true, since the isotropy group of x is generated by reflections in the walls containing x (theorem 4.20). Let Z be a normal neighborhood around x. Then (Z, γ) is isometrically isomorphic to the Riemannian manifold $(B_x := \exp_x^{-1}(Z), (\exp_x |B_x)^*\gamma)$. The required extension of $(\exp_x |V_x)^*\gamma$ to B_x using G_x as isometries is just $(\exp_x |B_x)^*\gamma$. ☐

Definition 6.4. Let C be a Riemannian chamber. An *equipment* of C is a group G together with a generating set S of idempotents and a surjective map $s : \mathcal{W} \to S$ from the set of walls of C that satisfies the following conditions:

(E1) If two walls W, W' are neighbors, then the order of $s(W).s(W')$ in G equals n, where π/n is the angle between W and W' in C.

(E2) For each $x \in C$, the goup G_x generated by $s(W)$ for each wall W with $x \in W$ is a finite Coxeter group, i.e. the relations described above are all relations in G_x.

The equipment is called *universal*, if s is one-to-one and the relations above generate all relations of G, i.e. if (G, S) is a Coxeter system with the relations $(s(W).s(W'))^n = e$ as above.

Remark 6.5. Note that condition (E2) describes the structure of the group G only where walls intersect. If some walls $W_1, \dots, W_k$ don't intersect G may or may not have a relation involving the elements $s(W_1), \dots, s(W_k)$. For example given a manifold with boundary but no corners, the map $s : \mathcal{W} \to S$ assigns to each connected component of the boundary a generator in S. In this case the conditions (E1) and (E2) are void, since no walls intersect. So the group G may be arbitrary as long as the generating set S consists of idempotents.

Theorem 6.6. *Let G be a reflection group on the complete, connected Riemannian manifold M and let C be a chamber. Then G is an equipment of C via the canonical map s, that assigns each wall the reflection with respect to that wall.*

Proof. Let S be the set of reflections with respect to the walls of C. S is a generating set of G by theorem 4.13. The map $s : \mathcal{W} \to S$ is surjective by definition of S. Given two neighboring walls W, W', let $x \in W \cap W'$ be a point in the intersection. Then the isotropy group G_x, when viewed as a subgroup of $O(T_x M, \gamma_x)$ is a reflection group and the order of $s(W).s(W')$ in G coincides with the order of $T_x s(W).T_x s(W')$ in G_x. The latter equals n, if we assume the angle between W and W' to be π/n. Thus G is an equipment of C. ☐

Lemma 6.7. *Every Riemannian chamber C carries a universal equipment. The universal equipment is unique.*

Proof. Let W be the set of walls of C. For each wall W we take a generator s. Let G be the group generated by these generators with relations $s^2 = e$ and $(s.s')^n = e$, if the corresponding walls W, W' are neighbors and have angle π/n.

The map $s : \mathcal{W} \to S$ is obviously one-to-one and onto. The resulting group G is a Coxeter group, thus the order of $s(W).s(W')$ is n as required. So G is a universal Coxeter equipment of C.

The uniqueness is clear, since the universal equipment is completely defined in terms of W. □

Lemma 6.8. *Let G be a finite reflection group in $\mathbb{R}^n$, let C be a chamber and B_ϵ an open ball around 0. Define the equivalence relation $(g, x) \sim (h, y) \Leftrightarrow x = y$ and $g^{-1}.h.x = x$. Then $\mathbb{R}^n \cong G \times C/\sim$ and $B_\epsilon \cong G \times (C \cap B_\epsilon)/\sim$.*

Proof. We prove only $\mathbb{R}^n \cong G \times C/\sim$. The other statement is proven in the same way. Define the map $p : G \times C \to \mathbb{R}^n$ by $(g, x) \mapsto g.x$. Then p is continuous and respects the equivalence relation. Thus we get a continous map $\overline{p} : G \times C/\sim \to \mathbb{R}^n$. The set C is the fundamental domain for the action of G, thus p and $\overline{p}$ are onto. If $g.x = h.y$, then $x = y$, since C is a fundamental domain. Therefore $[g, x] = [h, y]$ and $\overline{p}$ is one-to-one.

To show that $\overline{p}^{-1}$ is continous as well, let $g_n.x_n \to y$ converge in $\mathbb{R}^n$. We have to show that $[g_n, x_n]$ converges in $G \times C/\sim$. Since G is finite, we can split the sequence into a finite number of subsequences (g_{n_k}, x_{n_k}), such that g_{n_k} is constant in each subsequence. Then $x_{n_k} \to g_{n_k}^{-1}.y$, so (g_{n_k}, x_{n_k}) converges in $G \times C$. We only need to show, that these limits coincide in $G \times C/\sim$. That C is closed implies $g_{n_k}^{-1}.y \in C$ and since it is a fundamental domain, it meets every orbit exactly once, so $g_{n_k}^{-1}.y = g_{n_l}^{-1}.y$. Thus $[g_{n_k}, g_{n_k}^{-1}.y] = [g_{n_l}, g_{n_l}^{-1}.y]$ and all is proven. □

Now we will prove the main theorem which states, that the manifold M may be reconstructed knowing only a chamber C and the reflection group G.

Theorem 6.9. *Given a Riemannian chamber C, an equipment G with the map $s : \mathcal{W} \to S$, there exists a complete, connected Riemannian manifold $\mathcal{U}(G, C)$ with G acting on it as a reflection group with the following properties:*

(1) G acts simply transitively on the set of all chambers.

(2) C is isometrically isomorphic to a chamber of $\mathcal{U}(G, C)$.

(3) Let M be a complete, connected Riemannian manifold with a reflection group G, that acts simply transitively on the set of all chambers and C a chamber. Then M is isometrically isomorphic to $\mathcal{U}(G, C)$.

Proof. To each $x \in C$ we assign a subgroup G_x of G in the following way: G_x is generated by elements $\{s(W) : W \text{ is a wall of } C \text{ with } x \in W\}$. Later we will see that G_x is the isotropy group of x for the action of G on $\mathcal{U}(G, C)$. Define an equivalence relation on $G \times C$ by

$$(g, x) \sim (h, y) \iff x = y \text{ and } g^{-1}.h \in G_x$$

Now set $\mathcal{U}(G, C) := G \times C/\sim$. Then $\mathcal{U}(G, C)$ is the quotient of the disjoint union of $|G|$ copies of C, which are glued together along walls. G acts on $\mathcal{U}(G, C)$ by $g.[h, x] := [g.h, x]$. It is easy to check that this is well defined.

Let $x \in C$. We will define a chart around $[e, x] \in \mathcal{U}(G, C)$. Using the exponential map $\exp_x$ on C, we get a neighborhood W of x in C and an open set V in $\mathbb{R}^n$ around 0, such that $V = Q \cap B_\epsilon$ is the intersection of a quadrant and an open ball. The walls of the quadrant are the preimages under $\exp_x$ of the walls of C and $\exp_x$ restricts to a diffeomorphism $\exp_x : V \to W$. The group G'_x as defined in definition 6.2 is isomorphic to G_x, since G_x doesn't contain any other relations than those prescribed in definition 6.4. In $\mathcal{U}(G, C)$ the set $\mathcal{U}(G_x, W) := G_x \times W/\sim$ is a neighborhood of $[e, x]$ and by lemma 6.8, $G'_x \times V/\sim$ is a neighborhood of 0 in $\mathbb{R}^n$. Since $G_x \cong G'_x$ and $V \cong W$ and because the equivalence relation is the same in both cases, we see that $\exp_x$ extends to a homeomorphism $B_\epsilon \to \mathcal{U}(G_x, W)$. Given any other element $[g, x]$ we can map it using the action of G to $[e, x]$. Thus $\mathcal{U}(G, C)$ is locally Euclidean.

We have to show that chart changes are smooth. Let $x, y \in C$ and denote by $\phi_x : B_x \to \mathcal{U}(G_x, W_x)$ the chart around $[e, x]$ as constructed in the preceding paragraph. Then $g.\phi_x$ is a chart around $[g, x]$. Assume $[k, z]$ is in the domain of both, the chart around $[g, x]$ and $[h, y]$. Because of

$$(h.\phi_y)^{-1}.(g.\phi_x) = (h.\phi_y)^{-1}.(k.\phi_z)^{-1}.(k.\phi_z)(g.\phi_x)$$

it is sufficient to consider the case $[g, x] \in h.\mathcal{U}(G_y, W_y)$. Then $[h^{-1}.g, x] \in \mathcal{U}(G_y, W_y)$ and $h^{-1}.g \in G_y$. This means, the equivalence classes $[g, y] = [h, y]$ are the same.

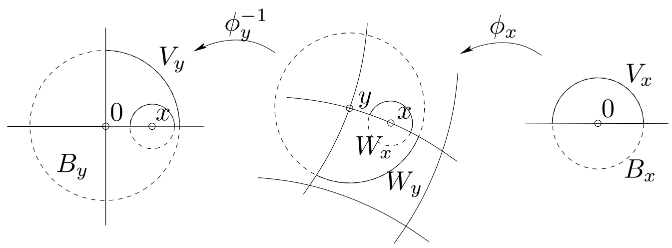


Figure 6.3: Chart changes.

First we show that the chart change $(g.\phi_y)^{-1}.(g.\phi_x) = \phi_y^{-1}.\phi_x$ is smooth. By definition $\phi_x|_{V_x} = \exp_x$ is an isometry between the Riemannian manifolds $(V_x, (\exp_x)^*\gamma)$ and (W_x, γ) and the same holds for $\phi_y|_{V_y}$. Thus $\phi_y^{-1}.\phi_x|_{V_x})$ is also an isometry, if we make V_x small enough, such that $\phi_x(V_x) \subseteq \phi_y(V_y)$. Furthermore this isometry preserves walls, so we get a natural inclusion $G_y \subseteq G_x$. Since both, the Riemannian metrics on B_x and B_y and the maps ϕ_x, ϕ_y were obtained from their restrictions V_x, V_y and $\exp_x$, $\exp_y$ respectively by extension via the elements of G_x, G_y, the map $\phi_y^{-1}.\phi_x$ is a continuous isometry on whole B_y. Finally we use the fact, that every continuous isometry is smooth.

The only case left is the chart change $(g.\phi_x)^{-1}.(h.\phi_x)$, if $[g, x] = [h, x]$. We see from

$$\begin{array}{ccc} G_x \times V_x/\sim & \xrightarrow{\phi_x} & G_x \times W_x/\sim \\ \downarrow g & & \downarrow g \\ G_x \times V_x/\sim & \xrightarrow[\phi_x]{} & G_x \times W_x/\sim \end{array} \qquad \begin{array}{ccc} [h, y] & \xrightarrow{\phi_x} & [h, \exp_x .y] \\ \downarrow g & & \downarrow g \\ [g.h, y] & \xrightarrow[\phi_x]{} & [g.h, \exp_x .y] \end{array}$$

that $g.\phi_x = \phi_x$, so $(g.\phi_x)^{-1}.(h.\phi_x) = \phi_x^{-1}.\phi_x.g^{-1}.h = g^{-1}.h$ is smooth. Thus we have constructed a differentiable structure for $\mathcal{U}(G, C)$.

We equip $\mathcal{U}(G, C)$ with the Riemannian metric of C and extend it using the elements of G as isometries. $\mathcal{U}(G, C)$ is complete, since we have glued together complete Riemannian manifolds and the gluing is locally finite.

Let $t \in S \subset G$ and x a point in a wall W of C with $s(W) = t$. Then $T_x t|_{T_x W}$ is the identity. Since $s \neq \mathrm{Id}$, we see that t is a reflection. Thus G is generated by reflections. If $x \in C^\circ$, then the orbit $G.x = \bigcup_{g \in G}[g, x]$ is clearly discrete, thus G is a reflection group.

We can look at the map $\iota : C \to \mathcal{U}(G, C)$ defined by $x \mapsto [e, x]$. Since the equivalence relation restricted on the image of ι, $\{e\} \times C/\sim$ is just the trivial one, we see that ι is an embedding. Thus we may view C as a subspace of $\mathcal{U}(G, C)$.

∂C is the set of all walls of C. By the previous argument, $G.\partial C \subseteq H_1$ is contained in the union of all reflection hypersurfaces. On the other hand, C° is connected and consists of regular points, thus C is a chamber.

Now for the converse statement: let C be the chamber for the action of G on M. Then C is a Riemannian chamber and G an equipment for it. We define the map $\psi : \mathcal{U}(G, C) \to M$ by $[g, x] \mapsto g.x$. It is continuous and bijective. One-to-one can be seen as follows: $g.x = h.y$ imlies $x = y$, since C is a fundamental domain, so $h^{-1}.g \in G_x$ and $[g, x] = [h, y]$ follows. To see that ψ is an isometry, note that on $C \subset \mathcal{U}(G, C)$ the Riemannian metric coincides with that on M by definition. Everywhere else it was defined using the elements of G as isometries and ψ commutes with G. Thus ψ is an isometry. □

Now we can answer the question, which groups may act as reflection groups on Riemannian manifolds. It turns out that this class of groups is rather large.

Corollary 6.10. *Let G be a group, generated by at most countably many involutive elements. Then there exists a complete, connected Riemannian manifold M, such that G is a reflection group on M and acts simply transitively on the set of all chambers.*

Proof. Let n be the number of involutive generators of G (possibly $n = \infty$). We take C to be a Riemannian chamber with n walls, such that no 2 walls intersect, i.e. C is a manifold with boundary, but no corners.

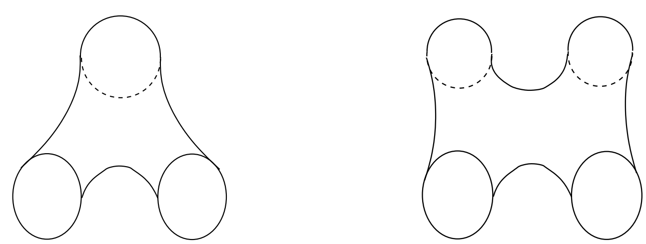


Figure 6.4: Examples of possible chambers for $n = 3$ and $n = 4$.

Assign to each wall of C a generator of G. Then G is an equipment for C, since there are no angle conditions to satisfy. By theorem 6.9 we can set $M = \mathcal{U}(G, C)$. □

Remark 6.11. This theorem answers the question, if there is a reflection group on a Riemannian manifold, that is not a Coxeter group, stated in [1, 3.6.]. The class of groups, which occur as reflection groups is quite large and the Coxeter groups are a quite special case. The other question stated there, about the classification of those groups, which are Coxeter group still remains open. A partial answer is provided by theorem 5.13.

Lemma 6.12. *Let C be a Riemannian chamber and G with the map $s : \mathcal{W} \to S$ an equipment. Let N be a normal subgroup of G and $p : G \to G/N$ the canonical projection. Then the following are equivalent*
(1) G/N with the map $p.s : \mathcal{W} \to S/N$ is an equipment of C.
(2) $G_x \cap N = \{e\}$ for all $x \in C$.

Proof. If G/N is an equipment, then the groups $(G/N)_x$ and G_x are isomorphic, since they are defined in terms of wall and angles between walls only. With the equipment map $p.s$, we have $(G/N)_x = G_x/N$. So $G_x \cong G_x/N$. This is only if $G_x \cap N = \{e\}$.

Conversely let $G_x \cap N = \{e\}$ for all $x \in C$. Then $G_x \cong G_x/N = (G/N)_x$. So (E2) is clear. (E1) follows since, if W, W' are neighbors, then $p.s(W), p.s(W') \in (G/N)_x$ for some $x \in C$. □

Theorem 6.13. *Let C be a Riemannian chamber, G an equipment and N a normal subgroup of G, such that G/N is also an equipment of N. Then $\mathcal{U}(G, C)$ is a covering space of $\mathcal{U}(G/N, C)$.*

Proof. Define the covering map to be

$$p : \begin{cases} \mathcal{U}(G, C) \to \mathcal{U}(G/N, C) \\ [g, x] \mapsto [g.N, x] \end{cases} .$$

Let $g \in G$, $x \in C$. We have to find an open neighborhood U around $[g.N, x]$, such that $p^{-1}(U)$ is a disjoint sum of open sets homeomorphic to U via p. Let $U = gN.(G/N)_x \times W_x/\sim$ be coordinate neighborhood as defined in the proof of theorem 6.9. Then the preimage is $\bigcup_{n\in N} g.n.G_x \times W_x/\sim$ is a union of open sets. Assume the union is not disjoint, i.e. $[g.n.h, y] = [g.m.k, z]$ with $y, z \in W_x$, $h, k \in G_x$ and $n, m \in N$. Then $y = z$ and $k^{-1}.m^{-1}.n.h \in G_y$. Remember that W_x intersects only walls that contain x, so $G_y \leq G_x$. $k^{-1}.m^{-1}.n.h \in G_x$ implies $m^{-1}.n \in G_x \cap N$, thus $m = n$. We see that the preimage is a disjoint union of the open sets $g.n.G_x \times W_x/\sim$. Clearly, because of $G_x \cong (G/N)_x$, $g.n.G_x \times W_x/\sim$ is homeomorphic to $gN.(G/N)_x \times W_x/\sim$. Thus p is a covering map. □

We want to apply this theorem to increase our understanding of reflection groups on simply connected manifolds.

Corollary 6.14. *Let M be a simply connected, connected, complete Riemannian manifold, G a reflection group on M and C a chamber. Then G is the universal equipment of C.*

Proof. Denote the universal equipment of C by G_{univ}. By definition the universal equipment is the equipment with the fewest possible relations, so every other equipment is a quotient of the universal equipment. So by theorem 6.13 the space $\mathcal{U}(G_{univ}, C)$ is a covering of $\mathcal{U}(G, C) \cong M$. Since M is simply connected, the covering map must be an isometry, so $\mathcal{U}(G_{univ}, C) \cong \mathcal{U}(G, C)$. This implies $G \cong G_{univ}$. □

This corollary tells us that given a reflection group G on a simply connected manifold M, all relations of G are generated by the angular relations of (E2).

Remark 6.15. The proof of this theorem given in [1, Theorem 3.10.] depends on Theorem 3.9. of the same paper. However the statement of the theorem is partly false. The map $\pi_1(C^\circ, x_0) *_e G(W) \to \pi_1(M_2, x_0)$ is not one-to-one and the map $\pi_1(C^\circ, x_0) *_e G(W)/R_a \to \pi_1(M, x_0)$ is not well defined as shown by the counterexamples below.

Take a torus and let s be the reflection as shown in figure 6.5. Then we have 2 chambers. Let w be the curve in $\pi_1(C^\circ)$ and c correspond to the element s in $G(W)$. Then the curve represented by the word wsw^-1s is nullhomotopic in $\pi_1(M_2)$. Thus the map $\pi_1(C^\circ, x_0) *_e G(W) \to \pi_1(M_2, x_0)$ is not one-to-one.

We look at the situation in figure 6.5. Then the empty word ϵ and the word $stst$ both equal e in $G(W)/R_a$. But the empty word ϵ is getting mapped to the zero curve in $\pi_1(M, x_0)$, whereas the word $stst$ is getting mapped to $c_1c_2c_3c_4$, which is clearly not null homotopic. So the choice of a representative for each element of the quotient $G(W)/R_a$ matters.

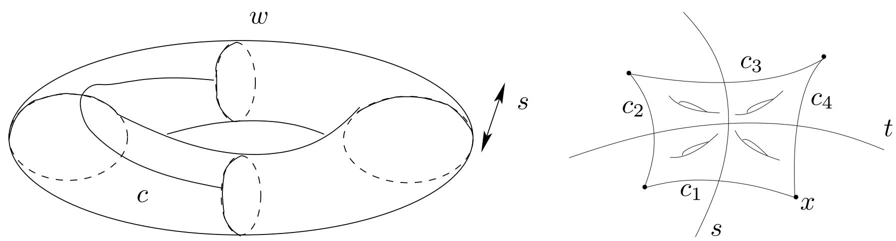


Figure 6.5: Counterexamples.

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