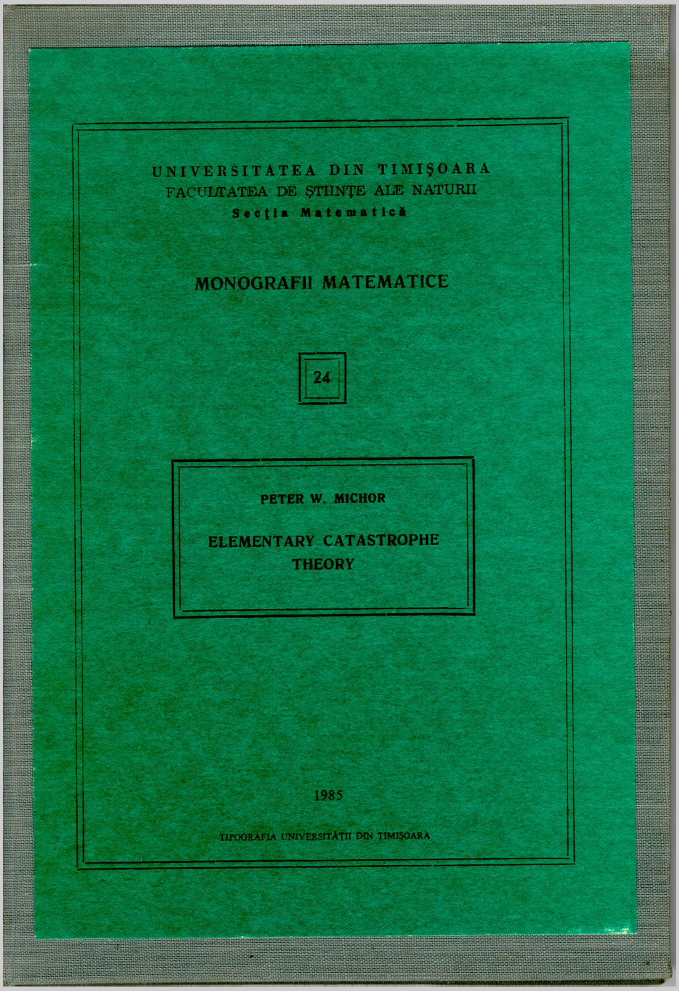
Peter W. Michor



Elementary Catastrophe Theory

Monografii Matematice, 24

Universitatea din Timisoara, Sectia Matematica

Gesamt

UNIVERSITATEA DIN TIMIŞOARA
FACULTATEA DE ŞTIINŢE ALE NATURII
Secţia Matematică

MONOGRAFII MATEMATICE

24

PETER W. MICHOR

ELEMENTARY CATASTROPHE THEORY

1985

TIPOGRAFIA UNIVERSITĂŢII DIN TIMIŞOARA

ELEMENTARY CATASTROPHE THEORY

PETER W. MICHOR

Institut für Mathematik, Universität Wien, Strudlhofgasse 4,
A-1090 Wien, Austria.

The following are notes of lectures held at the University of Vienna in 1975/76 and at the University of Linz in February 1976. There the main part of these notes has been written and revised by Prof. J. B. Cooper, who also invited me to Linz. I thank him and the other members of the enthusiastic and inspiring audience at Linz who helped to bring the presentation into a somewhat final form. The notes were multiplied with the title:

Classification of elementary catastrophes of codimension $\leqslant 6$.
Institutsbericht Nr. 51 of Johannes Kepler Universität Linz,
Institut für Mathematik. 1976.

After that I received letters of D. Siersma and Tim Poston, pointing out mistakes. The lectures were next given at the University of Klagenfurt in 1977/78, where also a multiplied version of the (corrected) notes appeared under the title:

Elementary catastrophe theory. Institut für Mathematik,
Universität für Bildungswissenschaften, Klagenfurt, 1978.

Then the lectures were held at the University of Mannheim in 1979, where §10 was added.

Prof. M. Craioveanu suggested to publish these notes at the University of Timisoara. Since there is a slightly increasing demand for them and am tired of photocopying. I happily accepted this offer and I thank Prof. Craioveanu for his offer.

This book also contains a reproduction of my paper: The division theorem on Banach spaces, with the kind permission of the Austrian Academy of Sciences.

In the exposition I have mostly followed [23] (see the list of references at the end) but also have taken many details of proofs from [2], [8], [11], [21] The exposition of §9 differs to some extent from that of [23], since in the latter there are some serious gaps. Let me indicate, that there I do not not need any kind of the rather difficult concept of stratification.

CONTENTS

§0. THOM'S THEOREM:

Suppose that one has a physical system described by external parameters which range through an open subset of $\mathbb{R}^4$ (e.g. space-time) and n internal parameters. Assume that the behaviour of the system is determined by a potential i.e. a smooth function f from $\mathbb{R}^n \times \mathbb{R}^4$ into $\mathbb{R}$ so that for a given $y \in \mathbb{R}^4$ this function assumes a local minimum (with respect to the internal parameters). Then the possible states lie in the set

$$M_f := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^4 : \frac{\partial f}{\partial x_1}(x,y) = \cdots = \frac{\partial f}{\partial x_n}(x,y) = 0\}.$$

One hopes that, under general conditions on f, M_f is a submanifold of $\mathbb{R}^n \times \mathbb{R}$. Motivated by these considerations, R. THOM formulated the following theorem which has since been proved by the combined efforts of MATHER, MALGRANGE and others.

Theorem: The following assertion is true for any r if $n=1$, for $r \leq 6$ if $n=2$, for $r \leq 5$ if $n \geq 3$:

there is an open, dense subset $\mathcal{F}$ of $C^\infty(\mathbb{R}^n \times \mathbb{R}^r, \mathbb{R})$ so that for each $f \in \mathcal{F}$,

(a) M_f is an r-dimensional manifold;

(b) if χ_f denotes the restriction of the projection $\pi : \mathbb{R}^n \times \mathbb{R}^r \longrightarrow \mathbb{R}^r$, then every singularity of χ_f is locally equivalent to one of a finite set of types - called the elementary catastrophes;

(c) χ_f is locally stable with respect to small changes in f;

(d) the number of elementary catastrophes is just r if $n=1$ and is given by the following table if $n \geq 3$ ($n=2$):

r	1	2	3	4	5	6	7	8
	1	2	5	7	11	(14)∞	∞	∞

The purpose of this course is to give a complete proof of this theorem.

To clarify the above formulation, we recall some notation. On the space $C^\infty(\mathbb{R}^m, \mathbb{R})$ of smooth functions from $\mathbb{R}^m$

into $\mathbb{R}$ we employ the <u>Whitney topology</u> which is defined later (cf. §9.1).
$\chi_f : M_f \to \mathbb{R}^r$ is <u>singular</u> at $(x,y) \in M_f$, if the rank of $d\chi_f(x,y)$ is not maximal.
Two functions f,g from $\mathbb{R}^m$ into $\mathbb{R}^r$ are <u>locally equivalent</u> at x, x' if there are open neighbourhoods U, U' of x, x' in $\mathbb{R}^m$ resp. open neighbourhoods V,V' of $f(x)$, $f(x')$ in $\mathbb{R}^r$ and diffeomorphisms $\phi : U \to U'$, $\psi : V \to V'$ with $\phi(x)=x'$ and $\psi(f(x))=g(x')$ so that the diagram

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\phi"'] & V \arrow[d, "\psi"] \\ U' \arrow[r, "g"] & V' \end{tikzcd}$$

commutes.
A function f is <u>locally stable</u> if there is a neighbourhood of f (in the Whitney topology) which contains only functions which are locally equivalent to f.

§1. <u>THE RING OF GERMS OF DIFFERENTIABLE FUNCTIONS</u>:

1.1. A <u>germ of a smooth function</u> at $0 \in \mathbb{R}^n$ is an equivalence class of smooth functions from a neighbourhhod of 0 in $\mathbb{R}^n$ with values in $\mathbb{R}$ under the relation

$$(f : U \to \mathbb{R}) \sim (g : V \to \mathbb{R}) \Leftrightarrow \bigvee_{\text{nbd. } W \text{ of } 0} f|_W = g|_W$$

The space of germs of functions is a commutative ring with unit (more precisely an $\mathbb{R}$-algebra) - we denote it by $\mathcal{E}_n(\mathbb{R}^n, 0)$ or more simply by $\mathcal{E}_n$.
Then $f \in \mathcal{E}_n$ is invertible if and only if $f(0) \neq 0$.

1.2. The partial derivatives (w.r.t. some basis $\{x_1, \cdots, x_n\}$ of $\mathbb{R}^n$) are clearly well-defined and themselves elements of $\mathcal{E}_n$. We denote them by $\partial f / \partial x_i$ $(i=1, \cdots, n)$.
<u>Def.</u> We denote by $\mathcal{M}_n$ $(= \mathcal{M}_n^1)$ the ideal of germs f so that $f(0)=0$.
<u>Def.</u> A <u>local ring</u> is a commutative ring with unit which possesses exactly one maximal ideal M (then, of course, the quotient ring A/M is a field).

Lemma: $\mathcal{E}_n$ is a local ring and $\mathcal{M}_n$ is its maximal ideal.
Proof: It is clear that $\mathcal{M}_n$ is a maximal ideal (since it is the kernel of the ring homomorphisms $f \longmapsto f(0)$ from $\mathcal{E}_n$ onto the field $\mathbb{R}$). On the other hand, any proper ideal is contained in $\mathcal{M}_n$ since it cannot contain any invertible element.

1.3. Def. We denote by $\mathcal{M}_n^k$ $(k \in \mathbb{N})$ the ideal of k-flat germs in $\mathcal{E}_n$ i.e. the germs f so that f and its partial derivatives up to order $k-1$ vanish at the origin.
For example, the coordinate functions $x_1, \cdots, x_n$ are elements of $\mathcal{M}_n^1$ and a monomial $x_1^{i_1} \; \cdots \; x_n^{i_n}$ $(i_n \in \mathbb{N})$ is in $\mathcal{M}_n^k$ where $k = i_1 + \cdots + i_n$.

Proposition: (a) $\mathcal{M}_n^1$ is generated by $\{x_1, \cdots, x_n\}$ (as an $\mathcal{E}_n$-module);
(b) $\mathcal{M}_n^k = [\mathcal{M}_n]^k$ that is, the ideal generated by the products $f_1 . \cdots . f_k$ (where each f_i is in $\mathcal{M}_n$).
Proof: Let $f : U \longrightarrow \mathbb{R}$ be a representant of a germ in $\mathcal{E}_n$ where U is a convex neighbourhood of 0. Then if $[0,x]$ denotes the segment from 0 to x we have

$$\begin{aligned} f(x) &= f(0) + \int_{[0,x]} df \\ &= f(0) + \sum_{i=1}^{n} \int_0^1 x_i \; \partial f / \partial x_i (\lambda . x) \, d\lambda \\ &= f(0) + \sum_{i=1}^{n} x_i . h_i(x) \end{aligned}$$

where $h_i \; : x \longmapsto \int_0^1 \partial f / \partial x_i (\lambda \, x) \, d\lambda$.
Then $h_i \in \mathcal{E}_n$ and so (a) follows immediately (since if $f \in \mathcal{M}_n$ then $f(0) = 0$).
If $f \in \mathcal{M}_n^k$, then $h_i \in \mathcal{M}_n^{k-1}$ and so $\mathcal{M}_n^k \subseteq \mathcal{M}_n . \mathcal{M}_n^{k-1}$. The result then follows by a simple induction proof.
The converse inclusions are trivial.

1.4. Def. We call the quotient space $\mathcal{E}_n / \mathcal{M}_n^{k+1}$ (written $J^k(\mathcal{E}_n, 0)$ or simply J_n^k) the algebra of k-jets of smooth functions at the origin. The Taylor expansion shows that J_n^k is canonically isomorphic to the space of polynomials of

degree $\leqslant k$ (the latter with multiplication obtained by cutting terms of degree $> k$ in the usual product).
Similarly, $\mathcal{M}_n^k/\mathcal{M}_n^{k+1}$ is canonically isomorphic to the vector space of homogeneous polynomials of degree k in n-variables.
Thus if $j^k : \mathcal{E}_n \longrightarrow J_n^k$ is the canonical projection, we can identify $j^k f$ with its Taylor polynomial of order k at 0.
J_n^k is a (finite-dimensonal) local ring and j^k is a local ring morphism.

1.5. <u>Nakayama's Lemma</u>: Let A be a local ring with maximal ideal I (1.2). If M is an A-module, M', M'' submodules of M with M' finitely generated, then

$$M' \subseteq M'' + I.M' \implies M' \subseteq M''.$$

<u>Proof</u>: Let $N := (M' + M'')/M''$. Then

$$N \subseteq (M'' + I.M')/M'' = I.(M' + M'')/M'' = I.N.$$

We must show that $N=0$ (i.e. we have reduced to the case $M''=0$).
Let $(n_1, \cdots, n_p)$ generate N. Since $I.N \supseteq N$ there exist $(a_{ij}) \in I$ with

$$n_i = \sum_{j=1}^{p} a_{ij} n_j \quad (1 \leqslant i,j \leqslant p).$$

i.e. $(I - A).n = 0$ where A is the matrix (a_{ij}) and n is the column vector with entries $(n_1, \cdots, n_p)$. Now the determinant $\det(I-A)$ has the form $1+a$ (where $a \in I$) and so is invertible. Therefore the inverse matrix $(I - A)^{-1}$ exists (it is calculated exactly as in elementary Linear Algebra) and so $n_1 = n_2 = \cdots = n_p = 0$ i.e. $N = \{0\}$.

1.6. <u>Proposition</u>: Let $I \subseteq \mathcal{E}_n$ be an ideal. Then the following are equivalent:

(a) $I \supseteq \mathcal{M}_n^k$;

(b) $j^k(I) \supseteq \mathcal{M}_n^k/\mathcal{M}_n^{k+1}$ i.e. $I + \mathcal{M}_n^{k+1} \supseteq \mathcal{M}_n^k$.

<u>Proof</u>: (a) $\Rightarrow$ (b) is trivial.
(b) $\Rightarrow$ (a) follows from 1.5 (take $M' = \mathcal{M}_n^k$, $M'' = I$).

Corollary: (a) $f_1, \cdots, f_p$ generate $\mathcal{M}_n^k$ if and only if $j^k f_1, \cdots, j^k f_p$ generate the vector space $\mathcal{M}_n^k / \mathcal{M}_n^{k+1}$ (of homogeneous polynomials of degree k in n variables);
(b) if I is an ideal in $\mathcal{E}_n$, then the following are equivalent:

(i) there exists a k with $I \supseteq \mathcal{M}_n^k$;

(ii) I has finite codimension in $\mathcal{E}_n$ (as an $\mathbb{R}$-vector space).

Proof: (a) Apply the proposition to the ideal $I = \langle f_1, \cdots, f_p \rangle_{\mathcal{E}_n}$ generated by $f_1, \cdots, f_p$ in $\mathcal{E}_n$.
(b) (i)$\Rightarrow$(ii) is trivial.
(ii)$\Rightarrow$(i) consider the chain

$$\mathcal{E}_n \supseteq I + \mathcal{M}_n \supseteq I + \mathcal{M}_n^2 \supseteq \cdots \supseteq I + \mathcal{M}_n^r \supseteq I + \mathcal{M}_n^{r+1} \supseteq \cdots$$

Since I has finite codimension, there is an r with

$$I + \mathcal{M}_n^r = I + \mathcal{M}_n^{r+1}$$

i.e. $\mathcal{M}_n^r \subseteq I + \mathcal{M}_n^r \subseteq I + \mathcal{M}_n^{r+1}$ and we can apply the Proposition.

1.7. We denote by $\mathcal{M}_n^\infty$ the ideal $\bigcap_{k=1}^{\infty} \mathcal{M}_n^k$ of <u>flat functions</u>. It is not finitely generated. Put $J_n^\infty := \mathcal{E}_n / \mathcal{M}_n^\infty$.

Proposition: $J_n^\infty \cong \mathbb{R}[[x_1, \cdots, x_n]]$ the ring of formal power series in n variables.

Proof: The isomorphism is induced by associating to f its Taylor series. We need only show that this is surjective. This is a special case of the following Lemma.

Lemma: For $\alpha \in \mathbb{N}_*^n$, let $f_\alpha : U \longrightarrow \mathbb{R}$ a smooth function defined on a neighbourhood of 0 in $\mathbb{R}^p$. Then there exists a smooth function $f : V \longrightarrow \mathbb{R}$ where V is an open neighbourhood of 0 in $\mathbb{R}^n \times \mathbb{R}^p$ so that $\frac{\partial^\alpha}{\partial x^\alpha} f(0,y) = f_\alpha(y)$ ($\alpha \in \mathbb{N}_*^n$, $y \in \mathbb{R}^p$).

Proof: Without loss of generality we can assume that each f_α is defined on $\mathbb{R}^p$ and has compact support. Let g be a smooth function from $\mathbb{R}^n$ into $[0,1]$ so that

$$\rho : x \longmapsto \begin{cases} 1 & \text{if } \|x\| \leq 1/2 \\ 0 & \text{if } \|x\| > 1. \end{cases}$$

We shall show that we can find a sequence (t_α) indexed by $\mathbb{N}^n_*$ so that the sum

$$\sum_\alpha \frac{\partial^\beta}{\partial x^\beta}((f_\alpha(y)/\alpha!)x^\alpha \rho(t_\alpha . x))$$

converges uniformly for each multi-index β.
Then if

$$f : (x,y) \longmapsto \sum_\alpha \frac{f_\alpha(y)}{\alpha!} x^\alpha . \rho(t_\alpha . x)$$

we can differentiate term by term to obtain

$$\frac{\partial^\beta}{\partial x^\beta} f(0,y) = \sum_\alpha f_\alpha(y) \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \frac{\partial^\gamma}{\partial x^\gamma} \frac{x^\alpha}{\alpha!}\Big|_{x=0} \frac{\partial^{\beta-\gamma}}{\partial x^{\beta-\gamma}} \rho(t_\alpha . x)\Big|_{x=0} .$$

$$= f_\beta(y).$$

To determine (t_α), we manipulate as follows:

$$f(x,y) = \sum_\alpha (1/t_\alpha)^{|\alpha|} . \frac{f_\alpha(y)}{\alpha!} (t_\alpha . x)^\alpha \rho(t_\alpha . x)$$

$$= \sum_\alpha \left(\frac{1}{t_\alpha}\right)^{|\alpha|} f_\alpha(y) \psi_\alpha(t_\alpha . x)$$

where $\psi_\alpha : y \longmapsto y^\alpha \rho(y)$ vanishes for $\|y\| > 1$.

Hence, since f_α has compact support

$$M_\alpha := \max \left\{ \frac{\partial^\beta}{\partial x^\beta} (f_\alpha(y) \psi_\alpha(x)) : |\beta| \leq |\alpha| \right\} < \infty .$$

Then we have

$$\sum \left| \frac{\partial^\beta}{\partial x^\beta} ((f_\alpha(y)/\alpha!) x^\alpha \rho(t_\alpha . x)) \right| \leq \sum_\alpha \left(\frac{1}{t_\alpha}\right)^{|\alpha|} (t_\alpha)^{|\beta|} M_\alpha$$

$$\leq \sum_\alpha M_\alpha / t_\alpha \quad (|\beta| \leq |\alpha|)$$

and so it suffices to choose (t_α) so that $\sum_\alpha \frac{M_\alpha}{t_\alpha} < \infty$.

1.8, <u>Corollary</u>: (a) $j^\infty_n : \mathcal{E}_n \longrightarrow J^\infty_n$ is a surjective $\mathbb{R}$ - algebra homomorphism ;
(b) J^∞_n is a local ring with maximal ideal $\mathcal{M}_n / \mathcal{M}^\infty_n$;
(c) J^∞_n is a Noether ring with unique prime decomposition (see Jacobson).

1.9. <u>Lemma</u>: $\dim_{\mathbb{R}} J^k_n = \dim_{\mathbb{R}} \mathcal{E}_n / \mathcal{M}^{k+1}_n = \frac{(n+k)!}{n!k!}$.

<u>Proof</u>: By induction on n and k. The cases $n=0$ and $k=0$ are trivial.

In general we have that $\mathcal{E}_n/\mathcal{M}_n^{k+1}$, the space of polynomials of degree k in $x_1, \cdots, x_n$, is the direct sum of the space of polynomials in degree $\leq k$ in $x_1, \cdots, x_{n-1}$ and x_n times the space of polynomials of degree $\leq k-1$ in $x_1, \cdots, x_n$). Hence its dimension is

$$\frac{(n+k-1)!}{(n-1)!\,k!} + \frac{(n+k-1)!}{n!\,(k-1)!} = \frac{(n+k)!}{n!\,k!} .$$

§2. THE GROUP OF LOCAL DIFFEOMORPHISMS OF $\mathbb{R}^n$:

2.1. A germ of a local diffeomorphism is an equivalence class of functions $\phi : U \longrightarrow U'$ where U, U' are open neighbourhoods of 0 and $\phi(0)=0$ so that ϕ is a diffeomorphism on some open neighbourhood of 0 (equivalently, $D\phi(0)$ is invertible). The equivalence relation is defined exactly as in 1.1. The set of such germs is a group (with the multiplication induced by composition of functions) which we denote by $L(\mathbb{R}^n, 0)$ or L_n.

2.2. The group of k-jets of local diffeomorphisms: If $\phi \in L_n$, $k \in \mathbb{N}$, the Taylor expansion of ϕ up to order k has the form

$$P_1 + P_2 + \cdots + P_k + \mathcal{E},$$

where $P_1 = D\phi(0) \in GL(n, \mathbb{R})$ and P_r is a homogeneous polynomial of degree r from $\mathbb{R}^n$ into $\mathbb{R}^n$. The coordinate functions of the remainder term $\mathcal{E}$ are elements of $\mathcal{M}_n^{k+1}$. The germ ϕ is k-flat with respect to the identity if $P_1 = Id_{\mathbb{R}^n}$, $P_2 = \cdots = P_k = 0$, i.e. if the coordinate functions of $\phi - Id_{\mathbb{R}^n}$ are in $\mathcal{M}_n^{k+1}$.

Lemma: The set of k-flat germs is a normal subgroup of L_n.

Proof: We have displayed above a natural mapping from L_n onto the space of polynomials

$$P_1 + \cdots + P_k$$

of degree k from $\mathbb{R}^n$ into $\mathbb{R}^n$ with P_1 invertible. Now this latter space is a group when multiplication is

defined as follows: if P, Q are such polynomials, let $P \circ Q$ be the usual composition of P and Q. The terms of degree $> k$ are dropped to obtain the group product $P.Q$.
Now the mapping mentioned above from L_n onto the polynomials is a group homomorphism and the space of k-flat germs is precisely its kernel.

Def. The quotient group of L_n with respect to the normal subgroup of k-flat germs is called the group of k-jets of local diffeomorphisms at 0 and is denoted by L_n^k. We write j^k for the canonical projection from L_n onto L_n^k. The proof of the Lemma shows that L_n^k is naturally isomorphic to the groups of polynomials of degree $\leq k$ with invertible linear parts.

2.3. Proposition: The group L_n^k has a natural Lie group structure.

Proof: L_n^k is an open subset of the finite dimensional vector space $\mathcal{L}_n^k$ of all polynomials $P_1 + \cdots + P_k$ of degree $\leq k$ without constant term (for L_n^k is the set of polynomials for which $\det P_1 \neq 0$). Thus L_n^k has a global coordinate system, defined by the coefficients of the polynomials P_r $(1 \leq r \leq k)$. The product $L_n^k \times L_n^k \longrightarrow L_n^k$ is defined by algebraic operations on the coefficients and so is analytic. Hence L_n^k is a Lie group (analyticity of inversion follows from an elementary result on Lie groups - see Cohn, Lie groups, p. 44).

Remarks: 1) the group L_n^1 is just $GL(n, \mathbb{R})$;
2) for $k' \geq k$ there is a natural projection from $L_n^{k'}$ into L_n^k and this is a Lie group homomorphism.

2.4. The group L_n operates in a natural way on $\mathcal{E}_n$. If $\phi \in L_n$ then the mapping

$$\phi^* : f \longmapsto f \circ \phi$$

is a ring automorphism of $\mathcal{E}_n$ and the mapping $\phi \longmapsto \phi^*$

is a group antihomomorphism from L_n into $\mathrm{Aut}_{Ring}(\mathcal{E}_n)$. In particular, we have $\phi^*(\mathcal{M}_n^k) = \mathcal{M}_n^k$ for each ϕ since a ring automorphism preserves the unique maximal ideal and its powers.

Def. Two germs $f,g \in \mathcal{E}_n$ are (right)equivalent (written $f \sim g$) if there is a $\phi \in L_n$ so that $f \circ \phi = g$ i.e. if f and g are in the same L_n-orbit in $\mathcal{E}_n$.

2.5. Since $\phi^*(\mathcal{M}_n^{k+1}) = \mathcal{M}_n^{k+1}$ for each $\phi \in L_n$, $\mathcal{M}_n^{k+1}$ is an L_n-submodule and so $J_n^k = \mathcal{E}_n / \mathcal{M}_n^{k+1}$ is an L_n-module and $j^k : \mathcal{E}_n \longrightarrow J_n^k$ is an L_n-module homomorphism. It is clear that the k-flat germs act trivially on J_n^k i.e. we have the following factorisation

$$\begin{array}{ccc} L_n & \longrightarrow & \mathrm{Aut}_{Ring}(J_n^k) \\ {\scriptstyle j^k}\downarrow & \nearrow & \\ L_n^k & & \end{array}$$

On the other hand, a $\phi \in L_n^k$ operates on $f \in J_n^k$ as follows: one forms the composition $f \circ \phi$ and drops the terms of degree $\geq k$, in other words, one has the following commutative diagram:

$$\begin{array}{ccc} L_n & \longrightarrow & \mathrm{Aut}_{Ring}(\mathcal{E}_n) \\ {\scriptstyle j^k}\downarrow & & \downarrow \\ L_n^k & \longrightarrow & \mathrm{Aut}_{Ring}(J_n^k) \end{array}$$

in symbols, $j^k(f \circ \phi) = j^k(f).\phi = j^k(f).j^k(\phi)$.

Remark: The operation of L_n on $\mathcal{M}_n^k/\mathcal{M}_n^{k+1}$ can be factorised as follows:

$$\begin{array}{ccc} L_n & \longrightarrow & \mathrm{Aut}\,(\mathcal{M}_n^k/\mathcal{M}_n^{k+1}) \\ {\scriptstyle j^1}\downarrow & \nearrow & \\ L_n^1 = GL(n,\mathbb{R}). & & \end{array}$$

2.6. Infinitesimal generation of L_n, L_n^k: We write $(t,x_1,\cdots,x_n)$ for a point in $\mathbb{R} \times \mathbb{R}^n$. Let X be a smooth vector field on an open neighbourhood of $\mathbb{R} \times \{0\}$ in $\mathbb{R} \times \mathbb{R}^n$ of the form

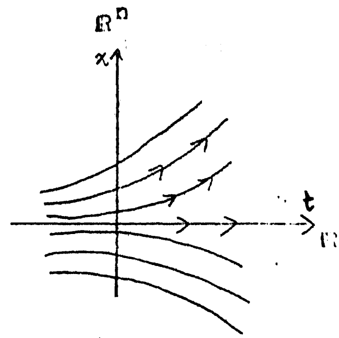
$$X(t,x) = \partial/\partial t + \sum_{i=1}^{n} X_i(t,x)\,\partial/\partial x_i$$

where $X(t,0) = 0 \quad (t \in \mathbb{R})$.

The integral curves of X are functions $u : \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{R}^n$ with $du/ds = X(u(s))$, i.e. solutions of the ordinary differential equations

$$\frac{du_0}{ds} = 1 \; ; \quad \frac{du_i}{ds}(t,x) = X_i(t,x) \quad (i=1,\cdots,n).$$

The assumption $X_i(t,0)=0$ ensures that $\mathbb{R} \times \{0\}$ (i.e. the curve $u_0 = \mathrm{id}_{\mathbb{R}}$, $u_i = 0$ $(i=1,\cdots,n)$) is a solution. We denote by $t \longmapsto (t, \phi(t,x))$ the solution of the equation with initial condition $u(0)=(0,x)$. Then by the theory of ordinary differential equations there exists an open set $U \subseteq \mathbb{R} \times \mathbb{R}^n$ containing $[0,1] \times \{0\}$ so that



a) U is the union of the integral curves of X which pass through $(0,x) \in U$;

2) every integral curve in U is defined on a neighbourhood of the interval $[0,1]$.

3) for $t \in [0,1]$ the mapping

$$\phi_t : x \longmapsto \phi(t,x)$$

is a diffeomorphisms from U_0 onto U_t where

$$U_t := \{x \in \mathbb{R}^n : (t,x) \in U\}.$$

In addition $\phi_t(0)=0$ and $\phi_0 = \mathrm{Id}_{\mathbb{R}^n}$.

We have thus defined a mapping $X \longmapsto \phi_1$ which maps certain germs of vector field along $[0,1] \times \{0\} \subseteq \mathbb{R} \times \mathbb{R}^n$ into elements of L_n, i.e. it maps $VL([0,1] \times \{0\}) \longrightarrow L_n$ (where VL denotes the set of vector fields satisfying the conditions imposed above on X). This mapping is not surjective since one can calculate that $\det D\,\phi_t(0) > 0$ (since ϕ is infinitesimally orientation-preserving!).

2.7. Lemma: Every $\phi \in L_n$ with $\det D\,\phi(0) > 0$ is in the range of the above mapping.

Proof: We can write $\phi(x) = Ax + \psi(x)$ where $A := D\phi(0)$ and the coordinates of ψ are in $\mathcal{M}_n^2$. Since $\det A > 0$, there is a curve $t \longmapsto A(t)$ from $\mathbb{R}$ into $GL(n,\mathbb{R})$ with $A(0) = Id$, $A(1) = A$. Let ϕ be the mapping

$$(t,x) \longmapsto A(t).x + t\psi(x) .$$

For each $t \in \mathbb{R}$, $x \longmapsto \phi(t,x)$ is the germ of a diffeomorphism at 0 and $\phi_0 = Id$. Consider the vector field

$$X(t,x) = \partial/\partial t + \sum_{i=1}^{n} \partial\phi_i/\partial t \; \partial/\partial x_i$$

where $\phi = (\phi_1, \cdots, \phi_n)$. Then by the construction, $t \longmapsto (t, \phi(t,x))$ is the integral curve of X.

2.8. Remark: If the components of the vector field X (of 2.6) are k-flat in each point of $\mathbb{R} \times \{0\}$ $(k \geq 1)$, then the germs of the diffeomorphisms ϕ_t are k-flat w.r.t the identity.

2.9. We can identify $\mathcal{L}_n$ ($= \mathcal{M}_n \times \cdots \times \mathcal{M}_n$ - n factors) with the germs of vector fields at zero which vanish at the origin (by the mapping $(X_1, \cdots, X_n) \longmapsto \sum X_i \partial/\partial x_i$). If $X \in \mathcal{L}_n$, the vector field $\bar{X} := \partial/\partial t + X$ defines a one-parameter subgroup ϕ_t of L_n by the process described in 2.6 (it is a subgroup since the X_i are independent of t). The mapping

$$X \longmapsto \bar{X} \longmapsto \phi_1$$

from $\mathcal{L}_n$ into L_n is called the exponential mapping and one writes $\phi_1 = \exp X$. Clearly, $\phi_t = \exp(tX)$.

2.10. $\mathcal{L}_n^k$ is the tangent space to the Lie group L_n^k at $Id_{\mathbb{R}^n}$ (since L_n^k is open in $\mathcal{L}_n^k$). $\mathcal{L}_n^k$ is thus the underlying vector space of the Lie algebra of L_n^k. We have a natural group homomorphism $j^k : L_n \longrightarrow L_n^k$ (2.2) and a natural projection $j^k : \mathcal{L}_n \longrightarrow \mathcal{L}_n^k$ (2.3).

Proposition: The following diagram is commutative:

$$\begin{array}{ccc} \mathcal{L}_n & \xrightarrow{\exp} & L_n \\ {\scriptstyle j^k}\downarrow & & \downarrow{\scriptstyle j^k} \\ \mathcal{L}_n^k & \xrightarrow{\exp} & L_n^k \end{array}$$

<u>Proof</u>: If $X \in \mathcal{L}_n$ and $\phi_t = \exp tX$, define

$$\phi_t^k := j^k(\exp tX) \ (\in L_n^k).$$

$\{\phi_t^k\}$ is a one-parameter subgroup of L_n^k and

$$\frac{\partial}{\partial t}\phi_t^k\Big|_{t=0} = \frac{\partial}{\partial t} j^k\phi_t\Big|_{t=0} = j^k \partial/\partial t\, \phi_t\Big|_{t=0} = j^k X$$

and this characterises the exponential $\exp t\xi$ where $\xi := j^k X$. Thus $\phi_t^k = \exp t\xi$.

Thus we have the following method for calculating the exponential of $\xi = j^k X \in \mathcal{L}_n^k$. We consider the vector field defined by X in $\mathbb{R}^n$, integrate it and obtain a one-parameter subgroup $\{\phi_t\}$ of local diffeomorphisms. Then $\exp t\xi = j^k\phi_t$.

<u>Remark</u>: We can give $\mathcal{L}_n$ a natural Lie algebra structure as follows: if $X,Y \in \mathcal{L}_n$ one can take the Lie bracket $[X,Y]$ of the associated vector fields. Then $[X,Y] \in \mathcal{L}_n$. One can show that the Lie bracket in the Lie algebra $\mathcal{L}_n^k$ is given by the formula $[j^k X, j^k Y] = j^k[X,Y]$.

2.11. For $f \in \mathcal{E}_n$, $X \in \mathcal{L}_n$ ($X = \sum_i X_i\, \partial/\partial x_i$ with $X_i \in \mathcal{M}_n$) we put

$$F_t(x) \ (= F(t,x)) := f \circ \exp tX.$$

Then $F_t \in \mathcal{E}_n$ for each t and $F_0 = f$.

<u>Proposition</u>: $\partial F/\partial t\big|_{t=0} = \sum X_i\, \partial f/\partial x_i \ (= Xf)$.

(this follows from the definition of an integral curve of a vector field).

We call $f \mapsto Xf$ <u>the infinitesimal action of X on</u> f.

The mapping $t \longmapsto j^k F_t$ is a smooth curve in J_n^k through $j^k f$ and it is contained in the orbit of $j^k f$ under the action of L_n^k. The vector

$$\partial/\partial t\ j^k F_t\big|_{t=0} \in J_n^k$$

is thus a tangent vector to the orbit $j^k f \circ L_n^k$.

By the general theory of Lie groups, the orbit of $j^k f$ is an immersive submanifold of J_n^k and the tangent space to the orbit of $j^k f$ at $j^k f$ is exactly the set of vectors obtained above (infinitesimal action of the Lie algebra on $j^k f$).

2.12. Proposition: $T_{j^k f}(j^k f \circ L_n^k) \subseteq J_n^k$ is the subspace

$$\{ j^k(\sum_{i=1}^n X_i \, \partial f/\partial x_i) : X_i \in \mathcal{M}_n \}$$

of J_n^k.

§3. FINITELY DETERMINED GERMS:

3.1. Def. Two germs f, g in $\mathcal{E}_n$ are k-equivalent (written $f \overset{k}{\sim} g$) if $j^k f = j^k g$ in J_n^k.
Recall that f and g are right equivalent when they are in the same L_n-orbit of $\mathcal{E}_n$ (2.4). A germ $f \in \mathcal{E}_n$ is k-determined if every germ g which is k-equivalent to f is right equivalent to f (i.e. $g \overset{k}{\sim} f \Longrightarrow g \sim f$).

3.2. Lemma: Let $f \in \mathcal{E}_n$ be k-determined. Then
1.) $g \overset{k}{\sim} f \Longrightarrow g$ is k-determined;
2) $g \sim f \Longrightarrow g$ is k-determined
(i.e. the property of being k-determined is essentially a property of $j^k f$).
Proof: We need only prove 2) (since $g \overset{k}{\sim} f \Longrightarrow g \sim f$ as f is k-determined).
2) Suppose that $f = g \circ \phi_1$ $(\phi_1 \in L_n)$. If $h \overset{k}{\sim} g$ then

$$j^k h = j^k g = j^k(f \circ \phi_1^{-1}) = j^k f . j^k(\phi_1^{-1})$$

and so $j^k(h \circ \phi_1) = (j^k h).(j^k \phi_1) = j^k(f)$.
Thus $h \circ \phi_1 \overset{k}{\sim} f$ and so there is a $\phi_2 \in L_n$ with $h \circ \phi_1 \circ \phi_2 = f = g \circ \phi_1$, i.e. $h \circ (\phi_1 \circ \phi_2 \circ \phi_1^{-1}) = g$. Hence $g \sim h$.

3.3. Definition: If $f \in \mathcal{E}_n$, we define

$$\Delta(f) := \langle \partial f/\partial x_1, \ldots, \partial f/\partial x_n \rangle_{\mathcal{E}_n}$$

the ideal generated by the partial derivatives of f with respect to a given basis $\{x_1, \cdots, x_n\}$ for $\mathbb{R}^n$. $\Delta(f)$ is independent of the choice of basis.

3.4. Lemma: If $f \in \mathcal{E}_n \setminus \mathcal{M}_n$ and $f' := f - f(0)$ then $\Delta(f) = \Delta(f')$ and f is k-determined if and only if f' is.

Proof: $\Delta(f) = \Delta(f')$ since $\partial f/\partial x_i = \dfrac{\partial(f - f(0))}{\partial x_i}$.

$f \overset{k}{\sim} g \iff f' \overset{k}{\sim} g'$ and $f(0)=g(0)$.

$$f = g \circ \phi \iff f' = g' \circ \phi \quad \text{and} \quad f(0)=g(0),$$

i.e. $f \sim g \iff f' \sim g'$ and $f(0)=g(0)$ and the result follows.

Thus in examining k-determination we can restrict our attention to germs $f \in \mathcal{M}_n$.

3.5. Theorem: Let f be a germ in $\mathcal{M}_n$. Then $\mathcal{M}_n^k \subseteq \mathcal{M}_n . \Delta(f) + \mathcal{M}_n^{k+1} \iff j^k(\mathcal{M}_n^k) \subseteq j^k(\mathcal{M}_n . \Delta(f)) \implies$

$\implies f$ ist k-determined $\implies \mathcal{M}_n^{k+1} \subseteq \mathcal{M}_n . \Delta(f) + \mathcal{M}_n^{k+2} \iff$

$\iff j^{k+1}(\mathcal{M}_n^{k+1}) \subseteq j^{k+1}(\mathcal{M}_n . \Delta(f))$.

Proof: The equivalences follow from 1.6.
We now prove the first implication. Suppose that $\mathcal{M}_n^k \subseteq \mathcal{M}_n . \Delta(f)$. Take $g \in \mathcal{E}_n$ with $j^k(f) = j^k(g)$. We must show the existence of $\phi \in L_n$ so that $f \circ \phi = g$. Define

$$F : (x,t) \longmapsto (1-t)f(x) + tg(x) \quad (t \in \mathbb{R}\ ,\ x \in \mathbb{R}^n\ )$$

and denote by F^t the function $x \longmapsto F(t,x)$ so that $F^0 = f$, $F^1 = g$. To prove the result we use a homotopy-type argument. We make the following

Claim: If $t_0 \in [0,1]$ then there is a family Γ^t in L_n, defined in a neighbourhood of t_0, so that $\Gamma^{t_0} = \mathrm{id}$, $F^t . \Gamma^t = F^{t_0}$ for each t.

The result follows from this claim by a standard compactness argument.

Proof of claim: We denote for the moment by $\mathcal{E}_{n+1}$ the ring of germs from $(\mathbb{R}^n \times \mathbb{R}\ ,(0,t_0)) \to \mathbb{R}$, and by $\mathcal{M}_{n+1}$ its maximal ideal. There is a natural injection $\pi^* : \mathcal{E}_n \longrightarrow \mathcal{E}_{n+1}$ induced by the projection $\pi : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}$. Hence we can regard $\mathcal{M}_n$ as a subspace of $\mathcal{M}_{n+1}$. Now

$$\mathcal{M}_n^k \subseteq \mathcal{M}_n \langle \partial f/\partial x_1, \dots, \partial f/\partial x_n \rangle_{\mathcal{E}_n} + \mathcal{M}_n^{k+1} .$$

Hence $\mathcal{M}_n^k \cdot \mathcal{E}_{n+1} \subseteq \mathcal{M}_n \langle \partial F/\partial x_1, \dots, \partial F/\partial x_n \rangle_{\mathcal{E}_{n+1}} + \mathcal{M}_n^{k+1} \cdot \mathcal{E}_{n+1}$
$\subseteq \mathcal{M}_n \langle \partial F/\partial x_1, \dots, \partial F/\partial x_n \rangle_{\mathcal{E}_{n+1}} + \mathcal{M}_{n+1} \cdot \mathcal{M}_n^k \cdot \mathcal{E}_{n+1}$

since $\partial F/\partial x_i - \partial f/\partial x_i - t \cdot \partial/\partial x_i (g-f) \in t \cdot \mathcal{M}_n^k$.

Thus, by Nakayama, $\mathcal{M}_n^k \cdot \mathcal{E}_{n+1} \subseteq \mathcal{M}_n \langle \partial F/\partial x_1, \dots, \partial F/\partial x_n \rangle_{\mathcal{E}_{n+1}}$.

Now $\partial F/\partial t = g-f \in \mathcal{M}_n^{k+1} \subseteq \mathcal{M}_n^k \subseteq \mathcal{M}_n \langle \partial F/\partial x_1, \dots, \partial F/\partial x_n \rangle_{\mathcal{E}_{n+1}}$,

i.e. $\partial F/\partial t(x,t) = \sum_j x_j \gamma_j(x,t)$ where each $\gamma_j \in \langle \partial F/\partial x_1, \dots, \partial F/\partial x_n \rangle_{\mathcal{E}_{n+1}}$

$= \sum_{i,j} x_j a_{ij}(x,t) \partial F/\partial x_i$ for suitable (a_{ij}) in $\mathcal{E}_{n+1}$.

Let $\Psi : (\mathbb{R}^n \times \mathbb{R}, (0,t_0)) \longrightarrow (\mathbb{R}^n, 0)$ be the germ defined by the function

$$(x,t) \longmapsto \Big(\sum_{j=1}^n x_j a_{ij}(x,t) \Big)_{i=1,\dots,n}$$

and consider the ordinary differential equation

$$\partial \Gamma/\partial t\,(x,t) = -\Psi(\Gamma(x,t),t)$$

for (x,t) near $(0,t_0)$ with initial value $\Gamma(x,t_0) = x$.

There exists a smooth solution $\Gamma : (\mathbb{R}^n \times \mathbb{R}, (0,t_0)) \longrightarrow (\mathbb{R}^n, 0)$.

By uniqueness we have $\Gamma(0,t) = 0$ since $\Psi(0,t) = 0$.
A simple calculation, involving the above equation for $\partial F/\partial t$ shows that $\partial/\partial t\,(F(\Gamma(x,t),t))=0$ and so $F(\Gamma(x,t),t))$ is constant as a function of t. Since $t \longmapsto \det d\,\Gamma(\cdot,t)(0)$ is continuous and $\Gamma(\cdot,t_0) = \mathrm{Id}_{\mathbb{R}^n}$, $\Gamma(\cdot,t)$ is an element of L_n for t near t_0. The equation

$$F(\Gamma(x,t),t) = F(\Gamma(x,t_0),t_0)$$

for t near t_0 is the required result.

We now suppose that $f \in \mathcal{M}_n$ is k-determined and show that $j^{k+1}(\mathcal{M}_n^{k+1}) \subseteq j^{k+1}(\mathcal{M}_n \cdot \Delta(f))$. Let

$P := \{ g \in \mathcal{M}_n : g \overset{k}{\sim} f \} = f + \mathcal{M}_n^{k+1}$

$Q := \{ g \in \mathcal{M}_n : g \sim f \} = f \circ L_n$, the orbit of f under L_n.

Then $j^{k+1}(P) = j^{k+1}(f) + j^{k+1}(\mathcal{M}_n^{k+1}) = j^{k+1}(f) + \mathcal{M}_n^{k+1}/\mathcal{M}_n^{k+2}$

which is an affine subspace of the real vector space J_n^k, in particular, a submanifold.

$j^{k+1}(Q) = j^{k+1}(f \circ L_n) = j^{k+1}(f) \cdot j^{k+1}(L_n) = j^{k+1}(f) \cdot L_n^k$,
the orbit of $j^{k+1}(f)$ under the finite dimensional Lie group

L_n^k (cf. 2.2 2.5, 2.12) and so an immersive submanifold. Also (2.12),

$$T_{j^{k+1}f}(j^{k+1}(f).L_n^{k+1}) = \{j^{k+1}(\sum_{i=1}^{n} X_i \partial f/\partial x_i) : X_i \in \mathcal{M}_n\}$$
$$= j^{k+1}(\mathcal{M}_n.\Delta(f)) \text{ in } J_n^{k+1}.$$

Since f is k-determined, $P \subseteq Q$, in particular, $j^{k+1}(P) \subseteq j^{k+1}(Q)$. Thus $T_{j^{k+1}f}(j^{k+1}(P)) \subseteq T_{j^{k+1}f}(j^{k+1}(Q))$ i.e. $j^{k+1}(\mathcal{M}_n^{k+1}) \subseteq j^{k+1}(\mathcal{M}_n.\Delta(f))$

3.6. Corollary: 1) $f \in \mathcal{M}_n$ is finitely determined $\Longleftrightarrow$ $\Longleftrightarrow \mathcal{M}_n^k \subseteq \Delta(f)$ for some k;
2) $f \in \mathcal{M}_n \setminus \mathcal{M}_n^2 \Longrightarrow f$ is 1-determined.

Proof: 1) f is k-determined $\Longrightarrow \mathcal{M}_n^{k+1} \subseteq \mathcal{M}_n.\Delta(f) \subseteq \Delta(f)$
On the other hand, if $\mathcal{M}_n^k \subseteq \Delta(f)$ then
$$\mathcal{M}_n^{k+1} \subseteq \mathcal{M}_n.\Delta(f)$$
and so f is $(k+1)$-determined.
2) If $f \in \mathcal{M}_n \setminus \mathcal{M}_n^2$ then $\frac{\partial f}{\partial x_i}(0) \neq 0$ for some i and so $\Delta(f) = \mathcal{E}_n$. Then $\mathcal{M}_n \subseteq \mathcal{M}_n \Delta(f)$ i.e. f is 1-determined.

3.7. Def. Let $f \in \mathcal{M}_n$; $\{x_1, \cdots, x_n\}$ be a fixed basis of $\mathbb{R}^n$. The essence of f (w.r.t to these coordinates) is the smallest k for which each x_i occurs (with non-zero coefficient) in $j^k f$. We write ess(f) for the essence of f. det (f) denotes the smallest k for which f is k-determined.
Corollary: det (f) $\geq$ ess (f) (for any basis).
Proof: If $k <$ ess (f) then $g = j^k f$ doesn't contain x_i for some i. Then no power of x_i lies in $\Delta(g)$ and so $\mathcal{M}^\ell \nsubseteq \Delta(g)$ for each $\ell \geq 0$. Then g is not finitely determined (3.5), but $g \overset{k}{\sim} f$. If f were k-determined, then so would be g (3.2.) -contradiction.

§4. CODIMENSION:

4.1. Def. If $f \in \mathcal{M}_n^2$ the codimension of f(written codim f) is defined to be $\dim_{\mathbb{R}} \mathcal{M}_n / \Delta(f)$ (note that $\Delta(f) \subseteq \mathcal{M}_n$ since $\frac{\partial f}{\partial x_i} \in \mathcal{M}_n$ for each i).

4.2. Lemma: If $f \in \mathcal{M}_n^2$, then codim f = ∞ if and only if det f = ∞ and if they are finite, then det f $\leqslant$ codim f + 2.

Proof: Consider the chain of vector spaces

$$\mathcal{M}_n = \mathcal{M}_n + \Delta(f) \supseteq \mathcal{M}_n^2 + \Delta(f) \supseteq \cdots \supseteq \mathcal{M}_n^k + \Delta(f) \supseteq \cdots$$

There are two cases:

Case 1: There exists a k so that $\mathcal{M}_n^{k-1} + \Delta(f) = \mathcal{M}_n^k + \Delta(f)$. We can assume that k is the smallest such integer. Then by Nayakama's Lemma, $\mathcal{M}_n^{k-1} \subseteq \Delta(f)$ and so $\mathcal{M}_n^k \subseteq \mathcal{M}_n \Delta(f)$ i.e. f is k-determined. Then det f $\leqslant$ k and codim f is greater than the length of the non-stationary part of the chain i.e. k-2.

Case 2: The chain is strictly decreasing at each term. In this case both det f and codim f are infinite. For if det f $< \infty$, then $\mathcal{M}_n^k \subseteq \Delta(f)$ for some k (3.6) and so $\mathcal{M}_n^k + \Delta(f) = \Delta(f) = \mathcal{M}_n^{k+1} + \Delta(f)$ i.e. we would have Case 1.

Similarly codim f = ∞ since we have an infinite, strictly decreasing chain of subspaces between $\Delta(f)$ and $\mathcal{M}_n$.

4.3. We introduce following notation:

$$\Gamma_c := \{f \in \mathcal{M}_n^2 : \text{codim } f = c\}$$

$$\Omega_c := \{f \in \mathcal{M}_n^2 : \text{codim } f \leqslant c\}$$

$$\Sigma_c := \{f \in \mathcal{M}_n^2 : \text{codim } f \geqslant c\}$$

Then we have the following partition of $\mathcal{M}_n^2$:-

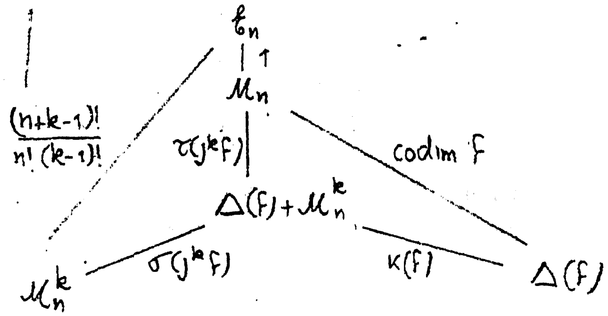
$$\mathcal{M}_n = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_c \cup \cdots \cup \Gamma_\infty .$$

4.4. Theorem: If $0 \leqslant c \leqslant k-2$, then $j^k(\mathcal{M}_n^2)$ is the disjoint union of $j^k(\Omega_c)$ and $j^k(\Sigma_{c+1})$ and $j^k(\Sigma_{c+1})$ is a closed real algebraic set in $j^k(\mathcal{M}_n^2)$.

<u>Proof</u>: If $f \in \mathcal{M}_n^2$ we define $\tau(f)$ to be $\dim_{\mathbb{R}} \mathcal{M}_n / \Delta(f) + \mathcal{M}_n^k$.
Then if $g \overset{k}{\sim} f$, $\tau(f) = \tau(g)$ (i.e. τ depends only on $j^k f$) (for $f-g \in \mathcal{M}_n^{k+1}$ and so $\partial f/\partial x_i - \partial g/\partial x_i \in \mathcal{M}_n^k$ i.e. $\Delta(f) + \mathcal{M}_n^k = \Delta(f) + \mathcal{M}_n^k$).
We claim: (i) $\tau(j^k f) \leqslant c \Rightarrow \operatorname{codim} f = \tau(f)$ i.e. $j^k f \in j^k(\Omega_c)$;

(ii) $\tau(j^k f) > c \Rightarrow \operatorname{codim} f > c$ i.e. $j^k f \in j^k(\Sigma_{c+1})$.

This implies the first statement of the Theorem.
To prove these claims, consider the following scheme (the symbols between the spaces denote codimension):



K and σ are defined by the diagram.
Note that $\tau(j^k(f))$ is always finite although codim f can be infinite.
Case (ii): we have $\operatorname{codim} f \geqslant \tau(j^k(f)) > c$ - clear.
Case (i) : we have $k-2 \geqslant c \geqslant \tau(j^k(f))$. Consider the chain

$$0 = \mathcal{M}_n/\mathcal{M}_n = \mathcal{M}_n/(\Delta(f)+\mathcal{M}_n) \leftarrow \mathcal{M}_n/(\Delta(f)+\mathcal{M}_n^2) \leftarrow \cdots \leftarrow \mathcal{M}_n/(\Delta(f)+\mathcal{M}_n^k).$$

There are $(k-1)$ steps and $\dim \mathcal{M}_n/(\Delta(f)+\mathcal{M}_n^k) \leqslant k-2$. Hence one step must be trivial, i.e. there is an $i \leqslant k$ so that

$$\Delta(f) + \mathcal{M}_n^{i-1} = \Delta(f) + \mathcal{M}_n^i \quad \text{i.e.} \quad \mathcal{M}_n^{i-1} \subseteq \Delta(f) + \mathcal{M}_n^i .$$

Hence by Nayakama's Lemma $\mathcal{M}_n^{i-1} \subseteq \Delta(f)$ and so $\mathcal{M}_n^k \subseteq \Delta(f)$ i.e. $0 = K(f)$. Therefore $\operatorname{codim} f = \tau(j^k(f))$ and so (i) holds.

We now turn to the second statement. Since $\tau(j^k(f)) > c$, we have

$$\sigma(j^k f) = \frac{(n+k-1)!}{n!\,(k-1)!} - 1 - \tau(j^k f) < K := \frac{(n+k-1)!}{n!\,(k-1)!} - 1 - c .$$

Hence $j^k(\Sigma_{c+1}) = \{ j^k f \in j^k(\mathcal{M}_n^2) : \sigma(j^k f) < K \}$

and we show that the latter set is real algebraic.
Let $\{x_1, \cdots, x_n\}$ be a basis for $\mathbb{R}^n$ and number the monomials of degree $\leqslant k$ as follows:

$$\begin{array}{cccccccccc} X_1 & X_2 & X_3 & \cdots & X_{n+1} & X_{n+2} & X_{n+3} & \cdots & X_\beta \\ 1 & x_1 & x_2 & & x_n & x_1^2 & x_1 x_2 & & x_n^k \end{array}$$

to form a basis $\{X_\gamma : 1 \leqslant \gamma \leqslant \beta = \frac{(n+k)!}{n!k!}\}$ of J_n^k.

If $z = j^k(f) \in j^k(\mathcal{M}_n^2)$ (where $f \in \mathcal{M}_n^2$), then z has a representation of the form $\sum_{j=n+2}^{\beta} a_j X_j$. Then $\partial z / \partial x_i$ has a representation of the form $\sum_{j=2}^{\bar\beta} a_{ij} X_j$ where $\bar\beta := \frac{(n+k-1)!}{n!(k-1)!}$ and each a_{ij} is a whole number $\times a_j$. Now

$$j^{k-1}(\Delta(f)) = (\Delta(f)+\mathcal{M}_n^k)/\mathcal{M}_n^k = \langle \partial z/\partial x_1, \cdots, \partial z / \partial x_n \rangle_{J_n^{k-1}}$$

is the subspace of $j^{k-1}(\mathcal{M}_n)$ generated by $\left\{ \frac{\partial z}{\partial x_i} X_j : \begin{array}{l} i=1,\cdots,n \\ j=1,\cdots,\bar\beta \end{array} \right\}$.

Now we can write $\frac{\partial z}{\partial x_i} . X_j$ as $\sum_{\ell=2}^{\bar\beta} a_{ij,\ell} X_\ell$ where each $a_{ij,\ell}$ is an a_{pq}.

Let M be the $n(\bar\beta - 1) \times (\bar\beta - 1)$ matrix $(a_{ij,\ell})$ so that its columns consist of the coordinates of a set of vectors which spans $(\Delta(f)+\mathcal{M}_n^k)/\mathcal{M}_n^k$.

Then $\sigma(z) < K \iff \dim_{\mathbb{R}} (\Delta(f)+\mathcal{M}_n^k)/\mathcal{M}_n^k < K$

i.e. if and only if the rank of M is less than K, or equivalently, if the K-minors of M vanish. Now the condition that a K-minor of M vanish is expressed by the vanishing of a polynomial in $\{a_i\}$ with whole number coefficients. Hence $j^k(\Sigma_{c+1})$ is a real algebraic variety of dimension $\frac{(n+k)!}{n!k!} - n - 1$ in the real vector space $j^k(\mathcal{M}_n^2)$.

4.5. <u>Corollary</u>: $j^k(\mathcal{M}_n^2)$ is the disjoint union

$$\Gamma_0^k \cup \Gamma_1^k \cup \cdots \cup \Gamma_{k-2}^k \cup j^k(\Sigma_{k-1})$$

where each Γ_c^k is the difference of two algebraic varieties $(j^k(\Sigma_c) \setminus j^k(\Sigma_{c+1}))$.

4.6. Theorem: If $f \in \mathcal{M}_n^2$ and $\operatorname{codim} f = c$ with $0 \leqslant c \leqslant k-2$, then $j^k(f \circ L_n) = j^k f . L_n^k$ is an immersive submanifold of $j^k(\mathcal{M}_n^2)$ with codimension c.

Proof: In 2.12 we showed that

$$T_{j^k f}((j^k f).L_n^k) = j^k(\mathcal{M}_n . \Delta(f)) \subseteq j^k(\mathcal{M}_n^2).$$

Now by 4.2 $\det f \leqslant (\operatorname{codim} f) + 2 \leqslant k$ and so f is k-determined. Hence $\mathcal{M}_n^{k+1} \subseteq \mathcal{M}_n . \Delta(f)$ (3.5).

Now the codimension of $(j^k f).L_n^k$ in $j^k(\mathcal{M}_n^2)$ is

$$\begin{aligned} &\dim j^k(\mathcal{M}_n^2) - \dim j^k(\mathcal{M}_n . \Delta(f)) \\ &= \dim (\mathcal{M}_n^2/\mathcal{M}_n^{k+1}) - \dim (\mathcal{M}_n . \Delta(f)/\mathcal{M}_n^{k+1}) \\ &= \dim (\mathcal{M}_n^2/\mathcal{M}_n . \Delta(f)) \end{aligned}$$

and since $\mathcal{M}_n / \mathcal{M}_n . \Delta(f) \cong \mathcal{M}_n/\mathcal{M}_n^2 \oplus \mathcal{M}_n^2/\mathcal{M}_n . \Delta(f)$

this is equal to

$$\begin{aligned} &\dim (\mathcal{M}_n/\mathcal{M}_n . \Delta(f)) - \dim (\mathcal{M}^n/\mathcal{M}_n^2) \\ &= \dim (\mathcal{M}_n/\Delta(f)) + \dim (\Delta(f)/\mathcal{M}_n . \Delta(f)) \\ &\quad - \dim (\mathcal{M}_n/\mathcal{M}_n^2) \\ &= c + n - n \qquad \text{(by the following Lemma).} \\ &= c. \end{aligned}$$

4.7. Lemma: If $f \in \mathcal{M}_n^2$ with $\operatorname{codim} f < \infty$, then

$$\dim_{\mathbb{R}}(\Delta(f)/\mathcal{M}_n . \Delta(f)) = n.$$

Proof: A typical element g of $\Delta(f)$ has the form $\sum_{i=1}^{n} a_i \partial f/\partial x_i$ with each $a_i \in \mathcal{E}_n$. We write $a_i = a_i' + a_i(0)$ so that each $a_i' \in \mathcal{M}_n$. Then $g = \sum_{i=1}^{n} a_i(0) \partial f/\partial x_i \pmod{\mathcal{M}_n . \Delta(f)}$ and so $\dim(\Delta(f)/\mathcal{M}_n . \Delta(f)) \leqslant n$.

We now show that $\{\partial f/\partial x_i\}$ is linearly independent mod $\mathcal{M}_n . \Delta(f)$.

If not, there would exist c_i in $\mathbb{R}$, not all zero, so that $\sum_i c_i \partial f/\partial x_i = \sum b_i \partial f/\partial x_i \quad (b_i \in \mathcal{M}_n)$.

Then $X := \sum_i (c_i - b_i) \partial/\partial x_i$ is the germ of a vector field at 0 with $X(0) \neq 0$. Then we can find new coordinates $\{y_1, \cdots, y_n\}$ in a neighbourhood of 0 so that $X = \partial/\partial y_1$. Then $Xf = 0$ i.e. $\partial f/\partial y_1 = 0$ in a neighbourhood of zero. But this implies that ess $f = \infty$, with respect to these coordinates and this gives a contradiction since det $f \geq$ ess f (3.7) and so det $f = \infty$ which would imply that codim $f = \infty$ (4.2).

§5. THE PREPARATION THEOREM:

5.1. <u>The division theorem</u>: Let $d : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a C^∞-function defined on a neighbourhood of 0 so that

$$d(t,0) = t^k \bar{d}(t)$$

for some $k \in \mathbb{N}$ and $\bar{d}$ a smooth function from $\mathbb{R}$ into $\mathbb{R}$ with $\bar{d}(0) \neq 0$. Then for any smooth $f : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ (defined on a neighbourhood of zero) there exist smooth functions $q, r : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$, defined near 0 so that

1) $f = q.d + r$ (near 0);

2) r has the form $\sum_{i=0}^{k-1} r_i(x) t^i$ for suitable smooth functions r_i.

For a proof, see [4], IV.2.1 (p. 95) or [2], §6.5 (p. 57).

5.2. <u>The preparation theorem</u>: Let $f : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$ be a smooth germ, $f^* : \mathcal{E}_p \to \mathcal{E}_n$ the induced ring homomorphism. Then if M is a finitely generated $\mathcal{E}_n$-module, the following holds:

M is finitely generated as an $\mathcal{E}_p$-module (via f^*) if and only if $\dim_{\mathbb{R}}(M/(f^*\mathcal{M}_p).M) < \infty$.

Proof: $\Rightarrow$) If M is finitely generated as $\mathcal{E}_p$-module, then there is a surjective $\mathcal{E}_p$-module homomorphism $\overset{k}{\oplus}\mathcal{E}_p = \mathcal{E}_p \oplus \mathcal{E}_p \oplus \ldots \oplus \mathcal{E}_p \longrightarrow M$. Then $\mathbb{R}^k = \overset{k}{\oplus}\mathcal{E}_p/\mathcal{M}_p \longrightarrow M/\mathcal{M}_p.M$ is also surjective, so $\dim_{\mathbb{R}} M/\mathcal{M}_p.M \leqslant k$.

$\Leftarrow$) Step 1: Let $n = p+1$, let $f: (\mathbb{R}^n = \mathbb{R}\times\mathbb{R}^p, 0) \rightarrow (\mathbb{R}^p, 0)$ be given by $f(t,x) = x$. Choose $a_1, a_2, \ldots, a_k \in M$ which generate M as $\mathcal{E}_{p+1}$-module and $M/\mathcal{M}_p.M$ as real vector space.

Then any $m \in M$ can be written in the form $m = \sum_j c_j a_j + \sum_j z_j a_j$, where $c_j \in \mathbb{R}$ and $z_j \in f^*(\mathcal{M}_p).\mathcal{E}_{p+1}$. This is seen as follows: $m = \sum_j c_j a_j + b$ for $c_j \in \mathbb{R}$ and $b \in f^*(\mathcal{M}_p).M$. Then $b = \sum_q y_q b_q$ for $y_q \in f^*(\mathcal{M}_p)$ and $b_q \in M$, in turn $b_q = \sum_j w_{jq} a_j$ for $w_{jq} \in \mathcal{E}_{p+1}$ and we may take $z_j = \sum_q y_q w_{jq} \in f^*(\mathcal{M}_p).\mathcal{E}_{p+1}$.

Now we use this for $m = t\, a_i$:

$t\, a_i = \sum_j (c_{ij} + z_{ij})\, a_j$ for $c_{ij} \in \mathbb{R}$ and $z_{ij} \in f^*(\mathcal{M}_p).\mathcal{E}_{p+1}$.

This means $\sum_j (t\,\delta_{ij} - c_{ij} - z_{ij})\, a_j = 0$.

Consider the matrix $B = (b_{ij}) = (t\,\delta_{ij} - c_{ij} - z_{ij})$. Then $B.\vec{a} = 0$, where $\vec{a}$ is the vector $(a_1, \ldots, a_k)$. Let $C(B)$ be the matrix of cofactors of B, then $C(B).B = B.C(B) = \det(B).(\delta_{ij})$. All these are germs, so let $\Delta(t,x) = \det(B) = \det(t\,\delta_{ij} - c_{ij} - z_{ij}(t,x))$.

Then $\Delta(t,x).\vec{a} = \det(B).\vec{a} = C(B).B.\vec{a} = 0$.

For $x = 0$ we get $z_{ij}(t,0) = 0$, so $\Delta(t,0) = \det(t\,\delta_{ij} - c_{ij})$ is a polynomial of degree k in t which is normed (the leading coefficient is 1). Thus there is $q \leqslant k$ such that $(t,0) = p(t).t^q$ with $p(0) = 1$.

Now we may use the division theorem 5.1: For any $f \in \mathcal{E}_{p+1}$ we have

$$f(t,x) = \Delta(t,x).q(t,x) + r_i(x).t^i .$$

So $\mathcal{E}_{p+1}/\Delta.\mathcal{E}_{p+1}$ is finitely generated over $\mathcal{E}_p$, in fact by $1, t, t^2, \ldots, t^{q-1}$. But $\Delta.\vec{a} = 0$ as we saw above, thus $\Delta.M = 0$, so M is a module over the quotient algebra $\mathcal{E}_{p+1}/\Delta.\mathcal{E}_{p+1}$ and finitely generatedly so (since M is finitely generated over $\mathcal{E}_{p+1}$), and this quotient algebra is finitely generated over $\mathcal{E}_p$, so M himself is finitely generated over $\mathcal{E}_p$.

Step 2: Let $f: (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^p, 0)$ be a germ of rank n, i.e. a germ of an embedding. The implicit function theorem gives us coordinates $(y_1, y_2, \ldots, y_p)$ of $\mathbb{R}^p$ near 0 such that $f(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0)$ in these coordinates. So f is the germ of the usual embedding $\mathbb{R}^n \rightarrow \mathbb{R}^p$. Then $f^* : \mathcal{E}_p \rightarrow \mathcal{E}_n$ is surjective and generators of M over $\mathcal{E}_n$ are also generators of M over $\mathcal{E}_p$.

<u>Step</u> 3: Now let $f\colon (\mathbb{R}^n,0) \longrightarrow (\mathbb{R}^p,0)$ be arbitrary. Then we may write f in the form $(\mathbb{R}^n,0) \xrightarrow{(id,f)} (\mathbb{R}^n \times \mathbb{R}^p,0) \xrightarrow{pr_2} (\mathbb{R}^p,0)$.
The first germ is an embedding as treated in step 2. The second germ is composition of n projections as treated in step 1. So it remains to show that the conclusion of the theorem survives composition.

Let $(\mathbb{R}^n,0) \xrightarrow{f} (\mathbb{R}^p,0) \xrightarrow{g} (\mathbb{R}^q,0)$ be germs of smooth functions such that 5.2 holds for f and for g. We have to show that 5.2 holds for $g \circ f$.

Let M be a finitely generated $\mathcal{E}_n$-module with $\dim_{\mathbb{R}} M/((g\circ f)^*\mathcal{M}_q.M) < \infty$. $g^*\mathcal{M}_q \subset \mathcal{M}_p$, so $(g\circ f)^*\mathcal{M}_q = f^*(g^*\mathcal{M}_q) \subset f^*\mathcal{M}_p$, thus $(g\circ f)^*\mathcal{M}_q.M \subset f^*\mathcal{M}_p.M$, and thus $\dim_{\mathbb{R}} M/f^*\mathcal{M}_p.M \leq \dim_{\mathbb{R}} M/(g\circ f)^*\mathcal{M}_q.M < \infty$. Since we may apply 5.2 to f this implies that M is finitely generated over $\mathcal{E}_p$ via f*.
By definition of the $\mathcal{E}_q$-action on M we have $g^*\mathcal{M}_q.M = f^*g^*\mathcal{M}_q.M$, so $\dim_{\mathbb{R}} M/g^*\mathcal{M}_p.M < \infty$ and by applying 5.2 to g we get that M is finitely generated as $\mathcal{E}_q$-module.

5.3 Suppose that $\phi : (\mathbb{R}^n,0) \longrightarrow (\mathbb{R}^p,0)$ is a smooth germ, A is a finitely generated $\mathcal{E}_p$-module, C is a finitely generated $\mathcal{E}_n$-module and B is an arbitrary $\mathcal{E}_n$-module. Consider the following scheme where β is an $\mathcal{E}_p$-module homomorphism and α is an $\mathcal{E}_p$-module homomorphism over ϕ^* (i.e. $\alpha(f.a) = \phi^*(f).\alpha(a)$ for $a \in A$ and $f \in \mathcal{E}_p$).

$$\begin{tikzcd} & B \arrow[d, "\beta"] & \\ A \arrow[r, "\alpha"] & C & \text{finitely generated} \\ \mathcal{E}_p \arrow[r, "\phi^*"] & \mathcal{E}_n & \end{tikzcd}$$

<u>Proposition</u>: $C = \alpha A + \beta B + (\phi^*\mathcal{M}_p).C \Longrightarrow C = \alpha A + \beta B$.

<u>Proof</u>: Let $C' = C/\beta B$ and denote by $\rho : C \longrightarrow C'$ the natural projection. C' is finitely generated over $\mathcal{E}_n$. Now $C' = \rho\alpha A + (\phi^*\mathcal{M}_p)C'$ and so $C'/(\phi^*\mathcal{M}_p).C'$ is finitely generated over $\mathcal{E}_p$. Let $c_1, \dots, c_r \in C'$ generate C' mod $(\phi^*.\mathcal{M}_p)C'$ over $\mathcal{E}_p$. Then if $c \in C'$, c has a representation in the form $\sum_{i=1}^{r} \phi^*(f_i).c_i$ mod $(\phi^*.\mathcal{M}_p)C'$ (with $f_i \in \mathcal{E}_p$). Writing $f = f' + f(0)$ we get

$$c = \sum_{i=1}^{r} f_i(0)c_i \quad \text{mod } (\phi^*.\mathcal{M}_p).C'$$

and so $\dim_{\mathbb{R}}(C'/(\phi^*.\mathcal{M}_p)C') < \infty$.

Then by 5.2 (with $C'=M$), C' is finitely generated over $\mathcal{E}_p$ (via ϕ^*). We can then apply Nayakama's Lemma to the equation $C' = \rho\alpha A + (\phi^*.\mathcal{M}_p).C'$ to get $C' \subseteq \rho\alpha A$ i.e. $C \subseteq \alpha A + \beta B$.

§6. <u>UNFOLDINGS</u>:

6.1. Take $f \in \mathcal{M}_n^2$. The <u>category of unfoldings of</u> f has as <u>objects</u> pairs (r, f') where $r \in \mathbb{N}_*$ and $f' : (\mathbb{R}^{n+r}, 0) \longrightarrow (\mathbb{R}, 0)$ is a germ so that $f'|_{\mathbb{R}^n\times\{0\}} = f$ i.e. the diagram

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "f"] \arrow[d, "id\times 0"] & \mathbb{R} \arrow[d, equal] \\ \mathbb{R}^n \times \mathbb{R}^r \arrow[r, "f'"] & \mathbb{R} \end{tikzcd}$$

commutes;

as <u>morphisms</u> from (s, f'') to (r, f') triples $(\phi, \bar{\phi}, \varepsilon)$ where $\phi : (\mathbb{R}^{n+s}, 0) \longrightarrow (\mathbb{R}^{n+r}, 0)$, $\bar{\phi} : (\mathbb{R}^s, 0) \longrightarrow (\mathbb{R}^r, 0)$ and $\varepsilon : (\mathbb{R}^s, 0) \longrightarrow (\mathbb{R}, 0)$ are germs so that

$$\phi|_{\mathbb{R}^n\times\{0\}} = id_{\mathbb{R}^n}, \quad \pi_r \circ \phi = \bar{\phi} \circ \pi_s, \quad f'' = f' \circ \phi + \varepsilon \circ \pi_s$$

$$\begin{tikzcd}
\mathbb{R} \arrow[d, equal] & \mathbb{R}^n \arrow[l, "f"'] \arrow[r, equal] \arrow[d, "id\times 0"] & \mathbb{R}^n \arrow[r, "f"] \arrow[d, "id\times 0"] & \mathbb{R} \arrow[d, equal] \\
\mathbb{R} & \mathbb{R}^{n+s} \arrow[l, "f''"'] \arrow[r, "\phi"] \arrow[d, "\pi_s"'] & \mathbb{R}^{n+r} \arrow[r, "f'"] \arrow[d, "\pi_r"] & \mathbb{R} \\
 & \mathbb{R}^s \arrow[r, "\bar{\phi}"] & \mathbb{R}^r &
\end{tikzcd}$$

Note that the last equation can be written in the form

$$f''(x,y) = L_{\varepsilon(y)}\{f'(\phi(x,y))\}$$

where $L_{\varepsilon(y)}$ denotes translation by $\varepsilon(y)$.

the <u>identity on</u> (r,f') is the triple $(id_{\mathbb{R}^{n+r}}, id_{\mathbb{R}^r}, 0)$;

the <u>composition</u> of two morphisms is defined by the formula

$$(\phi, \bar{\phi}, \varepsilon)\circ(\psi, \bar{\psi}, \tau) = (\phi\circ\psi, \bar{\phi}\circ\bar{\psi}, \varepsilon\circ\bar{\psi} + \tau);$$

a morphism $(\phi, \bar{\phi}, \varepsilon)$ is <u>an isomorphism</u> if and only if $r=s$ and ϕ and $\bar{\phi}$ are local diffeomorphisms.

6.2. <u>Examples</u>: 1) The <u>sum</u> of two unfoldings (r,f') and (s,f'') is defined to be the unfolding $(r+s, f'+f''-f)$, where $(f'+f''-f) : (x,u,v) \longmapsto f'(x,u) + f''(x,v) - f(x)$.

2) The <u>constant unfolding</u> (r,f) where $f : (x,u) \longmapsto f(x)$.

Then $(r,f) + (s,f') = (r+s, f')$.

3) If $f \in \mathcal{M}_n^2$ and $b_1, \cdots, b_r \in \mathcal{M}_n^2$, then (r,f') is an unfolding of f where

$$f' : (x,u) \longmapsto f(x) + b_1(x)u_1 + \cdots + b_r(x)u_r.$$

6.3. If $(\phi, \bar{\phi}, \varepsilon) : (s,f'') \longrightarrow (r,f')$ is a morphism then we can recover (s,f'') from (r,f') and $(\phi, \bar{\phi}, \varepsilon)$ (since $f''(x,u) = f'\circ\phi(x,u) + \varepsilon(u)$). We say that (s,f'') <u>is induced from</u> (r,f') by $(\phi, \bar{\phi}, \varepsilon)$. This suggests the following definition:

<u>Def</u>. An unfolding (r,f') of f is <u>versal</u> if every unfolding

of f is i uced by (r,f') (that is, for every unfolding (s,f''), there is a morphism $(\phi,\bar{\phi},\varepsilon)$ from (s,f'') into (r,f')).
A <u>universal unfolding</u> of f is a versal folding (r,f') for which r is minimal.

6.4. <u>The k-jet extension of a germ</u>: If $f \in \mathcal{M}_n$, define $j_1^k f$ to be the germ from $(\mathbb{R}^n,0)$ into $j^k(\mathcal{M}_n) \subseteq J_n^k$ of the mapping

$$x \longmapsto (\text{k-jet of } (y \mapsto f(x+y)-f(x)) \text{ at zero})$$
$$= j^k([y \longmapsto f(x+y) - f(x)]_0).$$

$j_1^k f$ is called the <u>natural k-jet extension</u> of f. $j_1^k f$ is determined by the germ f, not by its particular representation. Note that $j_1^k f(0) = j^k([y \mapsto f(0+y)-f(0)]_0) = j^k f$.

<u>Def.</u> Let X,Y be manifolds, $h: X \longrightarrow Y$ a smooth function, V an immersive submanifold of X. h is <u>transversal to</u> V at $x \in X$ (written $h \pitchfork V$ at x) if either $h(x) \notin V$ or

$$(Th)_x(T_x(X)) + T_{h(x)}(V) = T_{h(x)}(Y).$$

h is <u>transversal to</u> V (written $h \pitchfork V$) if it is transversal to V at each $x \in X$.

6.5. <u>Lemma</u>: 1) $j_1^k f$ is transversal to $j^k(\mathcal{M}_n . \Delta(f))$ in $j^k(\Delta(f))$ at zero and $\mathrm{Im}\,(Dj_1^k f(0))$ is generated by

$$\{j^k(\partial f/\partial x_1), ---, j^k(\partial f/\partial x_n)\}.$$

2) if f is k-determined, then $j_1^k f$ is the germ of an embedding from $(\mathbb{R}^n,0)$ into $(j^k(\mathcal{M}_n), j^k f)$.

<u>Proof</u>: 1) $\mathrm{Im}\,(D(j_1^k f)(0))$ is generated by the columns of the Jacobi matrix of $j_1^k f$ at 0, i.e. by $\{\frac{\partial j_1^k f}{\partial x_i}(0) : i=1,\dots,n\}$.
But $\frac{\partial}{\partial x_i} j_1^k f(0) = j_1^k(\partial f/\partial x_i)(0) = j^k(\partial f/\partial x_i)$.
Transversality: we must show that

$$\mathrm{Im}\,(D(j_1^k f)(0)) + j^k(\mathcal{M}_n . \Delta(f)) = j^k(\Delta(f)).$$

Suppose that $g \in \Delta(f)$ so that $g = \sum_i a_i \partial f/\partial x_i \,(a_i \in \mathcal{E}_n)$.
We write $a_i = a_i' + a_i(0)$ with $a_i' \in \mathcal{M}_n$. Thus

$$j^k(g) = \sum_i a_i(0) j^k(\partial f/\partial x_i) + \sum_i j^k(a_i' . \partial f/\partial x_i)$$
$$\in \operatorname{Im} D(j_1^k f)(0) + j^k(\mathcal{M}_n . \Delta(f)).$$

2) By 4.2, codim $f < \infty$ and so $\dim_{\mathbb{R}} (\Delta(f)/\mathcal{M}_n . \Delta(f)) = n$ (4.7).
Also $\mathcal{M}_n^{k+1} \subseteq \mathcal{M}_n . \Delta(f)$ (3.5). Hence

$$\dim_{\mathbb{R}} [j^k(\Delta(f))/j^k(\mathcal{M}_n . \Delta(f))]$$
$$= \dim \left[(\Delta(f) + \mathcal{M}_n^{k+1})/\mathcal{M}_n^{k+1}\right] / \left[(\mathcal{M}_n . \Delta(f) + \mathcal{M}_n^{k+1})/\mathcal{M}_n^{k+1}\right]$$
$$= \dim (\Delta(f)/\mathcal{M}_n . \Delta(f)) = n$$

and so, by 1), $\operatorname{Im} D(j_1^k f)(0)$ has dimension n, i.e. $j_1^k f$ is an immersion at 0 and so in a neighbourhood of 0.

6.6. <u>k-jet extensions of unfoldings</u>: Let (r,f') be an unfolding of $f \in \mathcal{M}_n^2$. We define $j_1^k f'$ to be the germ from $(\mathbb{R}^{n+r},0) \longrightarrow (J_n^k, j^k f)$ defined by the function

$$(x',y') \longmapsto (\text{k-jet of } (x \longmapsto f'(x'+x,y') - f'(x',y')) \text{ at } 0)$$
$$= j^k(\,[\,x \longmapsto f'(x'+x,y') - f'(x',y')]_0\,).$$

Note that the definition of 6.4 corresponds to this definition applied to the trivial unfolding $(0,f)$. As before

$$j_1^k f'(0,0) = j^k([\,x \longmapsto f'(x,0) - f'(0,0)]_0) = j^k f.$$

<u>Def</u>. An unfolding (r,f') of f is k-<u>transversal</u> $(k \geqslant 0)$ if $j_1^k f'$ is transversal to $(j^k f).L_r^k$ in $j^k(\mathcal{M}_n)$ at $0 \in \mathbb{R}^{n+r}$.
Choose a basis $\{x_1, \cdots, x_n\}$ of $\mathbb{R}^n$ and $\{y_1, \cdots, y_r\}$ of $\mathbb{R}^r$. Then $\partial f'/\partial y_j$ is a germ from $(\mathbb{R}^{n+r},0)$ into $(\mathbb{R},0)$.
We write $\partial_j f'$ for the element $\partial f'/\partial y_j \big|_{\mathbb{R}^n \times \{0\}} - \partial f'/\partial y_j (0,0)$ of $\mathcal{M}_n$ $(j=1, \cdots, r)$. Then we denote by $V_{f'}$ the $\mathbb{R}$-subspace $\langle \partial_1 f', \cdots, \partial_r f' \rangle_{\mathbb{R}}$ of $\mathcal{M}_n$.

6.7. <u>Lemma</u>: An unfolding (r,f') of $f \in \mathcal{M}_n^2$ is k-transversal if and only if $\mathcal{M}_n = \Delta(f) + V_{f'} + \mathcal{M}_n^{k+1}$:

Proof: By 6.12, $T_{j^kf}((j^kf).L_n^k) = j^k(\mathcal{M}_n . \Delta(f)) \subseteq j^k(\mathcal{M}_n)$.

$T(j_1^k f')_{(0,0)}(T_{(0,0)}(\mathbb{R}^n \times \{0\}))$ is generated by $\{\partial/\partial x_i \, j_1^k f'(0,0)\}_{i=1}^n$

and we have

$$\begin{aligned} \partial/\partial x_i \, j_1^k f'(0,0) &= j_1^k \, \partial f'/\partial x_i (0,0) \\ &= j^k([x \longrightarrow \partial f'/\partial x_i (x,0) - \partial f'/\partial x_i (0,0)]_0) \\ &= j^k(\partial f/\partial x_i). \end{aligned}$$

Hence $T(j_1^k f)_{(0,0)}(T_{(0,0)}(\mathbb{R}^n \times \{0\})) = \operatorname{Im} Dj_1^k f(0)$ (6.5).

$T(j_1^k f')_{(0,0)}(T_{(0,0)}(\{0\} \times \mathbb{R}^r))$ is generated by $\{\partial/\partial y_j \, j_1^k f'(0,0)\}_{j=1}^r$

and we have

$$\begin{aligned} \partial/\partial y_j \, j_1^k f'(0,0) &= j_1^k \, \partial f'/\partial y_j (0,0) \\ &= j^k([x \longmapsto \partial f'/\partial y_j (x,0) - \partial f'/\partial y_j (0,0)]_0) \\ &= j^k(\partial_j f'). \end{aligned}$$

Thus $T(j_1^k f')_{(0,0)}(T_{(0,0)}(\{0\} \times \mathbb{R}^r)) = j^k(V_{f'})$.

Then (r,f') is k-transversal $\Longleftrightarrow$

$$\begin{aligned} &\Longleftrightarrow \operatorname{Im} T(j_1^k f')_{(0,0)} + T_{j^kf}((j^kf).L_n^k) = j^k(\mathcal{M}_n) \\ &\Longleftrightarrow j^k(V_{f'}) + \operatorname{Im} Dj_1^k f(0) + j^k(\mathcal{M}_n . \Delta(f)) = j^k(\mathcal{M}_n) \\ &\Longleftrightarrow j^k(V_{f'}) + j^k(\Delta(f)) = j^k(\mathcal{M}_n) \quad \text{(by 6.5)} \\ &\Longleftrightarrow \mathcal{M}_n = \Delta(f) + V_{f'} + \mathcal{M}_n^{k+1}. \end{aligned}$$

6.8. Corollary: Let $b_1, \cdots, b_r$ be representatives of a basis of $\mathcal{M}_n/(\Delta(f) + \mathcal{M}_n^{k+1})$. Then if

$$f' : (x,y) \longmapsto f(x) + \sum_{j=1}^r b_j(x) y_j$$

the unfolding (r,f') is k-transversal.

Proof: $\partial_j f'(x) = \partial f'/\partial y_j (x,0) - \partial f'/\partial y_j (0,0) = b_j(x)$ and so $\{b_1, \cdots, b_r\}$ generates $V_{f'}$. Hence $\Delta(f) + \mathcal{M}_n^{k+1} + V_{f'} = \mathcal{M}_n$, i.e. (r,f') is k-transversal (6.7).

6.9. Lemma: A versal unfolding (r,f') of f is k-transversal for each $k \geq 0$. In addition $r \geq \operatorname{codim} f$.

Proof: Choose a k-transversal unfolding (s,f'') of f (this exists by 6.8). Then

$$f''(x,z) = f'(\pi_n \circ \phi(x,z), \bar{\phi}(z)) + \varepsilon(z)$$

where $\pi_n : \mathbb{R}^{n+r} \to \mathbb{R}^n$ is the canonical projection. Differentiating, we get

$$\partial_j f''(x) = \sum_{i=1}^{n} \frac{\partial f'}{\partial x_i}(x,0)\frac{\partial \phi_y}{\partial z_j}(x,0) + \sum_{\ell=1}^{r}\left(\frac{\partial f'}{\partial y_\ell}(x,0) - \frac{\partial f'}{\partial y_\ell}(0,0)\right)\frac{\partial \bar{\phi}_\ell}{\partial z_j}(0).$$

Hence $V_{f''} \leq \Delta(f) + V_{f'}$ and so

$$\mathcal{M}_n \supseteq \Delta(f) + V_{f'} + \mathcal{M}_n^{k+1} \supseteq \Delta(f) + V_{f''} + \mathcal{M}_n^{k+1} = \mathcal{M}_n$$

and so (r,f') is k-transversal.

In addition, $r \geqslant \dim_{\mathbb{R}} V_{f'} \geqslant \dim_{\mathbb{R}}(\mathcal{M}_n/(\Delta(f) + \mathcal{M}_n^{k+1})$ for $k \geqslant 0$.

Hence the chain

$$\mathcal{M}_n = \Delta(f) + \mathcal{M}_n \supseteq \Delta(f) + \mathcal{M}_n^2 \supseteq \cdots$$

must become stationary, say at the ℓ-th term. Then

$$\Delta(f) + \mathcal{M}_n^{\ell} = \Delta(f) + \mathcal{M}_n^{\ell+1}.$$

Then we have $\mathcal{M}_n^{\ell} \leq \Delta(f)$ by Nayakama's Lemma and so

$$r \geqslant \dim_{\mathbb{R}} V_{f'} \geqslant \dim_{\mathbb{R}} \mathcal{M}_n/(\Delta(f) + \mathcal{M}_n^{\ell}) = \dim_{\mathbb{R}} \mathcal{M}_n/\Delta(f)$$
$$= \operatorname{codim} f.$$

6.10. Lemma: Let $f \in \mathcal{M}_n^2$ be k-determined. Then if (r,f') and (r,f'') are k-transversal unfoldings of f, (r,f') and (r,f'') are isomorphic.

Proof: Since f is k-determined, $\mathcal{M}_n^{k+1} \subseteq \mathcal{M}_n \cdot \Delta(f) \subseteq \Delta(f)$ (3.5). On the other hand, by 6.7,

$$\mathcal{M}_n = \Delta(f) + V_{f'} + \mathcal{M}_n^{k+1} = \Delta(f) + V_{f'} = \Delta(f) + V_{f''}.$$

By 5.9, $r \geqslant \operatorname{codim} f = \dim(\mathcal{M}_n/\Delta(f)) =: c$. Let $u_1, \cdots, u_c \in \mathcal{M}_n$ be representatives of a basis for $\mathcal{M}_n/\Delta(f)$. Then $h : (\mathbb{R}^n \times \mathbb{R}^c \times \mathbb{R}^{c-r}, 0) \to (\mathbb{R}, 0)$ is an unfolding of f where

$$h : (x,v,w) \longmapsto f(x) + u_1(x)v_c + \cdots + u_c(x)v_c$$

(i.e. h is independent of w).

Now if $\overline{\partial_j f'}$ (resp. $\bar{u}_c$) denotes the image of $\partial_j f'$ (resp. u_c) in $\mathcal{M}_n/\Delta(f)$ we have

$$\overline{\partial_j f'} = \sum_{\ell=1}^{c} a_{\ell j}\, \bar{u}_\ell \quad (\text{for some } \{a_{\ell j}\}).$$

The $c \times r$ matrix $A := (a_{\ell j})_{j=1,\dots,r}^{\ell=1,\dots,c}$ has rank c (since $\{\overline{\partial_j f'}\}$ spans $\mathcal{M}_n / \Delta(f)$). Let $B := (b_{\ell j})_{j=1,\dots,r}^{\ell=c+1,\dots,r}$ be an $(r-c) \times r$-matrix so that $[A,B]$ is regular and define

$$\bar{\phi} : \mathbb{R}^r \longrightarrow \mathbb{R}^c \times \mathbb{R}^{r-c} \quad \text{by} \quad y \longmapsto ({}^tAy, {}^tBy)$$

$$\phi : \mathbb{R}^{n+r} \longrightarrow \mathbb{R}^{n+c+(r-c)} \quad \text{by} \quad \phi := \mathrm{id}_{\mathbb{R}^n} \times \bar{\phi}$$

and $h' : (\mathbb{R}^n \times \mathbb{R}^r, 0) \longrightarrow (\mathbb{R}, 0)$ by

$$h' : (x,y) \longmapsto h(x, \bar{\phi} y) = f(x) + \sum_{j=1}^{r} \Big(\sum_{\ell=1}^{c} a_{\ell j} u_\ell(x) \Big) y_j .$$

Then (r,h') is an unfolding of f and

$$\partial_j h(x) = \frac{\partial h}{\partial y_j}(x,0) - \frac{\partial h}{\partial y_j}(0,0) = \begin{cases} u_j(x) & 1 \leq j \leq c \\ 0 & c+1 \leq j \leq r \end{cases}$$

and so $\overline{\partial_j h'} = \sum_{\ell=1}^{c} a_{\ell j} \bar{u}_\ell = \overline{\partial_j f'}$.

Then $(\phi, \bar{\phi}, 0)$ is an isomorphism from (r,h') onto (r,h) and $\overline{\partial_j h'} = \sum_{\ell=1}^{c} a_{\ell j} \bar{u}_\ell = \overline{\partial f_j'}$, i.e. we have constructed an unfolding (r,h') (isomorphic to (r,h)) so that $\overline{\partial f_j'} = \overline{\partial h_j'}$ (for each j) (so that, in particular, (r,h') is k-transversal).

By symmetry, the Lemma will be proved if we can demonstrate that $(r,f') \cong (r,h')$ i.e. if we can prove the special case of the Lemma where $\overline{\partial f_j'} = \overline{\partial f_j''}$ $(j=1,\dots,r)$. We assume from now on that these equations hold.

Define

$$F^t(x,y) \; (= F(x,y,t)) := (1-t) f'(x,y) + t f''(x,y).$$

Then $\overline{\partial F_j^t} = (1-t) \overline{\partial f_j'} + t \overline{\partial_j f''} = \overline{\partial f_j'}$ for each j

and so (r, F^t) is k-transversal for each t.

<u>Claim</u> 1: for each $t_0 \in [0,1]$ there is a neighbourhood U_{t_0} of t_0 in $\mathbb{R}$ so that there is an isomorphism

$$(\phi^t, \bar{\phi}^t, \varepsilon^t) : (r, F^{t_0}) \longrightarrow (r, F^t)$$

for each $t \in U_{t_0}$.

The result follows from this claim by the usual compactness argument.

Claim 2: There exist germs

$$\left.\begin{array}{llll} \phi & : (\mathbb{R}^{n+r}\times\mathbb{R}, (0,t_0)) \longrightarrow (\mathbb{R}^{n+r},0) & \text{with} & \phi(0,t) = 0 \\ \bar{\phi} & : (\mathbb{R}^{r}\times\mathbb{R}, (0,t_0)) \longrightarrow (\mathbb{R}^{r},0) & \text{with} & \bar{\phi}(0,t) = 0 \\ \varepsilon & : (\mathbb{R}^{r}\times\mathbb{R}, (0,t_0)) \longrightarrow (\mathbb{R},0) & \text{with} & \varepsilon(0,t) = 0 \end{array}\right\} \text{for all } t.$$

so that

1) $\pi\circ\phi = \bar{\phi}\circ\pi$ where π is the projection $: \mathbb{R}^{n+r}\times\mathbb{R} \to \mathbb{R}^{r}\times\mathbb{R}$;

2) $\phi(.,t_0) = Id_{\mathbb{R}^{n+r}}$ (and so $\bar{\phi}(.,t_0) = Id_{\mathbb{R}^r}$ by 1));

3) $F(\phi(x,y,t),t) + \varepsilon(y,t) = F(x,y,t_0)$ for t near t_0.

Note that we can replace 3) by

4) $$\sum_i^n \partial F/\partial x_i(\phi(x,y,t),t).\partial\phi_i/\partial t(x,y,t) + \sum_{j=1}^r \partial F/\partial y_j(\phi(x,y,t),t).\partial\phi_j/\partial t(x,t)$$
$$+ \partial F/\partial t(\phi(x,y,t),t) + \partial\varepsilon/\partial t(y,t) = 0$$

(for the expression in 4) is the derivative of the left hand side of 3)).

Proof that Claim 2 implies Claim 1: The function

$$t \longmapsto \det D\phi(.,t)(0)$$

is continuous and so, since $\det D\phi(.,t\ )(0) = \det Id = 1$, there is an open interval U_{t_0} containing t_0 so that $\det D\bar{\phi}(.,t)(0) > 0$ on U_{t_0}. Similarly $\det D\phi(.,t)(0) > 0$ on U_{t_0}. Then $\phi(.,t)$ and $\bar{\phi}(.,t)$ are diffeomorphisms on U_{t_0} and this implies Claim 1.

Claim 3: There exist germs

$$\left.\begin{array}{llll} X & : (\mathbb{R}^{n+r}\times\mathbb{R}, (0,t_0)) \longrightarrow (\mathbb{R}^{n},0) & \text{with} & X(x,0,t)=0 \\ Y & : (\mathbb{R}^{r}\times\mathbb{R}, (0,t_0)) \longrightarrow (\mathbb{R}^{r},0) & \text{with} & Y(0,t) = 0 \\ Z & : (\mathbb{R}^{r}\times\mathbb{R}, (0,t_0)) \longrightarrow (\mathbb{R},0) & \text{with} & Z(0,t)=0 \end{array}\right\} \text{for all } t$$

so that

5) $$\sum_{i=1}^n \partial F/\partial x_i(x,y,t)X_i(x,y,t) + \sum_{j=1}^r \partial F/\partial y_j(x,y,t)Y_j(y,t) +$$
$$+ \partial F/\partial t(x,y,t) + Z(y,t) = 0 \quad \text{for } (x,y,t) \text{ near } (0,0,t_0).$$

Proof that Claim 3 implies Claim 2: Let ϕ_1, ϕ_2 be the (smooth) solutions of the differential equations

$$\partial\phi_1/\partial t(x,y\ t) = X(\ \phi_1(x,y,t), \phi_2(y,t), t)$$
$$\partial\phi_2/\partial t(y,t) = Y(\phi_2(y,t),t)$$

with initial conditions $\phi_1(x,y,t_0) = x, \quad \phi_2(y,t_0) = y$.

By uniqueness, $\phi_1(0,0,t) = 0, \quad \phi_2(0,t) = 0$ for all t.

Put $\quad \varepsilon(y,t) := \int_{t_0}^{t} Z(\ \phi_2(y,\tau)\ d\tau$,

$\quad\quad \phi(x,y,t) := (\ \phi_1(x,y,t), \phi_2(y,t))$,

$\quad\quad \bar{\phi}(y,t) := \phi_2(y,t)$.

Then a routine calculation shows that 4) (of Claim 2) is satisfied.

Also $\pi_0\phi = \pi(\phi_1, \phi_2) = \phi_2 = \phi_2 \cdot \bar{\pi}$, i.e. 1) holds.

2) holds on account of the initial conditions and so we have proved Claim 2.

<u>Proof of Claim</u> 3: We regard $\mathbb{R}^{n+r+1}$ as $\mathbb{R}^n \times \mathbb{R}^r \times \mathbb{R}$ (with typical element (x,y,t)). In this paragraph we denote by $\mathcal{E}_{r+1}$ the germs at $(0,t_0) \in \mathbb{R}^{r+1}$. Let A be a free $\mathcal{E}_{r+1}$-module with $(r+1)$ generators (a typical $a \in A$ is written as $(Y_1, \cdots, Y_r, Z)$ with $Y_i, Z \in \mathcal{E}_{r+1}$) and B be a free $\mathcal{E}_{n+r+1}$-module with n generators (a typical $b \in B$ has the form $(X_1, \cdots, X_n)$ with $X_j \in \mathcal{E}_{n+r+1}$). We construct a scheme

$$\begin{array}{ccc} & & B \\ & & \downarrow{\scriptstyle\beta} \\ A & \xrightarrow{\ \alpha\ } & C = \mathcal{E}_{n+r+1} \\ \mathcal{E}_{r+1} & \xrightarrow{\ \pi^*\ } & \mathcal{E}_{n+r+1} \end{array}$$

as in the preparation theorem - 5.3).

$\alpha : A \longrightarrow C$ is defined by

$$\alpha(Y_1, \cdots, Y_r, Z) = \sum_{j=1}^{r} \partial F/\partial y_j \cdot Y_j + Z .$$

Then α is a module homomorphism over π^* since we have, for $g \in \mathcal{E}_{r+1}$

$$\begin{aligned}\alpha(gY_1, \cdots, gY_r, gZ) &= \sum_{j=1}^{r} \partial F/\partial y_j\, g\cdot Y_j + g\cdot Z \\ &= g\cdot(\sum_{j=1}^{r} \partial F/\partial y_j \cdot Y_j + Z) = g\ \alpha(Y_1, \cdots, Y_r, Z).\end{aligned}$$

$\beta : B \longrightarrow C$ is defined by $\beta(X_1, \cdots, X_n) := \sum_{i=1}^{n} \partial F/\partial x_i \, X_i$.

Claim 4: $C = \alpha A + \beta B + (\pi^* \mathcal{M}_{r+1})C$, i.e. the hypothesis of 5.3 is satisfied.

We have $\mathcal{M}_n = \Delta(f) + V_{F^t}$ for each t and so

$$\mathcal{E}_n = \Delta(f) + V_{F^t} + \mathbb{R}.1 \,.$$

Choose $g \in C = \mathcal{E}_{n+r+1}$ and put $\tilde{g} := g|_{\mathbb{R}^n \times \{0\} \times \{t_0\}}$ so that

$$\tilde{g} = \sum_{i=1}^{n} \partial f/\partial x_i \, \tilde{X}_i + \sum_{j=1}^{r} (\partial F^{t_0}/\partial y_j|_{\mathbb{R}^n \times \{0\}} - \partial F^{t_0}/\partial y_j(0,0)).\tilde{Y}_j + s$$

for $\tilde{X}_i \in \mathcal{E}_n$, $\tilde{Y}_j \in \mathbb{R}$, $s \in \mathbb{R}$.

Let $\hat{g} := \sum_{i=1}^{n} \partial F/\partial x_i . X_i + \sum_{j=1}^{r} \partial F/\partial y_j . Y_j + Z$,

where X_i ($\in \mathcal{E}_{n+r+1}$), Y_j ($\in \mathcal{E}_{r+1}$), Z ($\in \mathcal{E}_{r+1}$) are chosen so that

$X_i(x,0,t_0) = \tilde{X}_i(x)$ (for all x),

$Y_j(0,t_0) = \tilde{Y}_j$,

$Z(0,t_0) = \sum_j -\partial F/\partial y_j(0,0,t\,).\tilde{Y}_j + s.$

Then $\hat{g}|_{\mathbb{R}^n \times \{0\} \times \{t_0\}} = g|_{\mathbb{R}^n \times \{0\} \times \{t_0\}}$ and one can prove exactly as in 1.3 that this implies that $\hat{g} - g \in \mathcal{M}_{r+1}.\mathcal{E}_{n+r+1}$. Now $\hat{g} = \beta(X_1, \cdots, X_n) + \alpha(Y_1, \cdots, Y_n, Z) \in \alpha A + \beta B$ and so $g \in \alpha A + \beta B + \mathcal{M}_{r+1}.C$, i.e. $C \subseteq \alpha A + \beta B + \mathcal{M}_{r+1}.C$.

Thus we have demonstrated Claim 4 and so can deduce that $C \subseteq \alpha A + \beta B$ (5.3). Hence $\mathcal{M}_r .C \subseteq \alpha(\mathcal{M}_r.A) + \beta(\mathcal{M}_r.B)$.

Now $\partial F/\partial t = f'' - f'$ and so vanishes on $\mathbb{R}^n \times \{0\} \times \mathbb{R}$. Hence, as above, we can deduce that $-\partial F/\partial t \in \mathcal{M}_r.\mathcal{E}_{n+r+1} \subseteq \alpha(\mathcal{M}_r A) + \beta(\mathcal{M}_r.B)$, i.e. there exist germs $X_1, \cdots, X_n \in \mathcal{M}_r.\mathcal{E}_{n+r+1}$, $Y_1, \cdots, Y_r, Z \in \mathcal{M}_r.\mathcal{E}_{r+1}$ so that

$$-\partial F/\partial t = \alpha(Y_1, \cdots, Y_r, Z) + \beta(X_1, \cdots, X_n)$$

$$= \sum_{i=1}^{n} \partial F/\partial x_i . X_i + \sum_{j=1}^{r} \partial F/\partial y_j . Y_j + Z$$

and this is exactly Claim 3.

Thus the proof of Lemma 6.10 is complete.

6.11. Theorem: Let $f \in \mathcal{M}_n^2$ be k-determined. Then an unfolding (r,f') of f is versal if and only if it is k-transversal.

Proof: By 6.9, a versal unfolding is k-transversal.
Now suppose that (r,f') is k-transversal and that (s,f'') is an arbitrary unfolding. We must construct a morphism from (s,f'') into (r,f').

$$(\mathrm{Id}_{\mathbb{R}^{n+s}} \times 0_{\mathbb{R}^r},\ \mathrm{Id}_{\mathbb{R}^s} \times 0_{\mathbb{R}^r}, 0)$$

is a morphism from (s,f'') into $(r+s,\ f''+f'-f)$ and the latter is k-transversal since one can easily calculate that

$$\partial_j (f''+f'-f) = \begin{cases} \partial_j f' & (j=1,\cdots,r) \\ \partial_j f'' & (j=r+1,\cdots,r+s) \end{cases}$$

and so $V_{f''+f'-f} = V_{f'} + V_{f''}$.
Since (r,f') is k-transversal, we have $\mathcal{M}_n = \Delta(f) + V_{f'} + \mathcal{M}_n^{k+1}$. Thus $\Delta(f) + V_{f''+f'-f} + \mathcal{M}_n^{k+1} \supseteq \mathcal{M}_n$, i.e. $(r+s, f''+f'-f)$ is k-transversal.

Similarly, $(r+s,f')$ (the unfolding $(x,y,z) \longmapsto f'(x,y)$) is k-transversal. Hence, by 6.10, $(r+s,f')$ and $(r+s,f''+f'-f)$ are isomorphic. Hence we can construct a morphism from (s,f'') into (r,f') as the composition

$$(s,f'') \to (s+r, f''+f'-f) \cong (r+s,f') \longrightarrow (r,f')$$

(the last morphism is the obvious one - forget the irrelevant coordinates!).

6.12. Theorem: $f \in \mathcal{M}_n^2$ has a versal unfolding if and only if codim $f < \infty$. Then

a) any two s-parameter versal unfoldings are isomorphic;
b) every versal unfolding is isomorphic to an unfolding of the form $(r+s,f')$ where (r,f') is universal;
c) if $b_1,\cdots,b_r \in \mathcal{M}_n$ are representatives of a basis for $\mathcal{M}_n/\Delta(f)$, then (r,f') is a universal unfolding where $f': (x,y) \mapsto f(x) + \sum_{j=1}^{r} b_j(x) y_j$.

Proof: If codim $f < \infty$ then f is k-determined for some k (4.2) and so, by 6.8, there exists a k-transversal

unfolding (r,f') and it is versal by 6.11.
On the other hand, if f has a versal unfolding, it has a universal unfolding (r,f') and then $\operatorname{codim} f \leqslant r$ (6.9).
Now suppose that $\operatorname{codim} f < \infty$.
a) By 6.9 both unfoldings are k-transversal for each k and so isomorphic by 6.10.
b) Let (s,f'') be a versal unfolding, (r,f') a universal unfolding. Then $s \geqslant r$. We can extend (r,f') in a trivial manner to an unfolding (s,f') and this unfolding is k-transversal. Then (s,f') and (s,f'') are isomorphic by 6.10.
c) By 6.8, (r,f') is k-transversal for each k and so versal. By 6.9, r is minimal and so (r,f') is universal.

6.13. Examples: 1) Take $n=1$, $f : x \longmapsto x^N$ $(N \geqslant 2)$.
Then $\Delta(f) = \langle x^{N-1} \rangle_{\mathcal{E}_1} = \mathcal{M}_1^{N-1}$ and $\mathcal{M}_1/\Delta(f) = j^{N-2}(\mathcal{M}_1)$.
This has as basis the functions $\{x, x^2, \cdots, x^{N-2}\}$ and so

$$f' : (x,y) \longmapsto x^N + x^{N-2}y_1 + \cdots \quad xy_{N-2}$$

is the universal unfolding.
2) Let f be the function

$$x \longmapsto x_1^N \pm x_2^2 \pm \cdots \pm x_n^2 .$$

Then $\Delta(f) = \langle x_1^{N-1}, x_2, x_3, \cdots, x_n \rangle$
and so $\mathcal{M}_n/\Delta(f)$ has as basis the functions $\{x_1, x_1^2, \cdots, x_1^{N-2}\}$.
Hence $f' : (x,y) \longmapsto f(x) + x_1^{N-2}y_1 + \cdots + x_1 y_{N-2}$
is the universal unfolding.
3) We generalise 2) as follows: take $f \in \mathcal{M}_k^2$ with universal unfolding

$$f(x_1, \cdots, x_k) + g(x_1, \cdots, x_k, y_1, \cdots, y_r) .$$

If $q(x_{k+1}, \cdots, x_n)$ is a non-degenrated quadratic form in further variables, then the universal unfolding of $f(x_1, \cdots, x_k) + q(x_{k+1}, \cdots, x_n)$ is

$$f(x_1, \cdots, x_k) + q(x_{k+1}, \cdots, x_n) + g(x_1, \cdots, x_k, y_1, \cdots, y_r) .$$

In particular, $\operatorname{codim} f = \operatorname{codim}(f+q)$
(choose coordinates so that q has the form $\pm x_{k+1}^2 \cdots \pm x_n^2$ and continue as in 2)).

It is thus natural to transform f so that as many variables as possible are separated into a non-degenerate quadratic form.

6.14. If $f \in \mathcal{M}_n^2$, we define the <u>corank of</u> f (written corank(f)) to be the corank of the matrix $(\partial^2 f/\partial x_i \partial x_j)_{i,j=1}^n|_0$, i.e. it corresponds to the corank of the quadratic form determined by $j^2 f$.

<u>Reduction Lemma</u>: Let $f \in \mathcal{M}_n^2$ have corank n-r. Then f is right-equivalent to a germ of the form

$$q(x_1, \cdots, x_r) + g(x_{r+1}, \cdots, x_n)$$

where q is a non-degenerate quadratic form and $j^2 g = 0$.

<u>Proof</u>: By a linear transformation we can reduce $j^2 f$ to the form

$$q(x_1, \cdots, x_r) = \pm x_1^2 \pm x_2^2 \pm \cdots \pm x_r^2 .$$

Let $h := f|_{\mathbb{R}^r \times 0}$. Then $h \in \mathcal{M}_r^2$ and $j^2 h = q$. Now

$\Delta(q) = \langle x_1, \cdots, x_r \rangle_{\mathcal{E}_r} = \mathcal{M}_r$ and so $\mathcal{M}_r^2 \subseteq \mathcal{M}_r . \Delta(q)$

and q is 2-determined by 3.5 and hence so is h by 3.2. Hence h is right equivalent to q, i.e. $q = h \circ \phi$ for some $\phi \in L_r$.

Now f is right equivalent to $f \circ (\phi \times Id_{\mathbb{R}^{n-r}})$. This means essentially that we can assume that $f|_{\mathbb{R}^r \times 0} = q$, which we now do.
Since $\Delta(q) = \mathcal{M}_r$, q is its own universal unfolding (6.12c)) and so $(n-r, f)$ is a versal unfolding of q (since there is a morphism from $(0,q)$ into $(n-r,f)$). But $(n-r,q)$ is also versal and so $(n-r,q)$ and $(n-r,f)$ are isomorphic, i.e. there is an isomorphism $(\psi, \bar{\psi}, \varepsilon)$ from $(n-r,q)$ into $(n-r,f)$. In particular

$$q(x_1, \cdots, x_r) = f \circ \psi(x_1, \cdots, x_n) + \varepsilon(x_{r+1}, \cdots, x_n)$$

and so we can choose $g = -\varepsilon$.

Now $j^1\varepsilon = 0$ for $j^1 q = 0$ and $j^1 f = 0$. $j^2\varepsilon = 0$ follows since $j^2\varepsilon \neq 0$ would imply that the corank of $q - \varepsilon$ would be smaller than that of q (since q and ε act on distinct variables) and this gives a contradiction (note that the corank is invariant under the action of L_r).

6.15 Lemma: If $f \in \mathcal{M}_n^2$ has corank r then codim f $\geqslant \frac{r^3+5r}{6}$. In particular, if $r \geqslant 3$ then codim f $\geqslant$ 7.

Proof: By 6.14, f is right equivalent to

$$g(x_1,\dots,x_r) + q(x_{r+1},\dots,x_n)$$

with $j^2g = 0$ and q a non degenerate quadratic form. By 6.13.3), codim f = codim g. Now $g \in \mathcal{M}_r^3$ and so $\Delta(g) \subseteq \mathcal{M}_r^2$. Consider $j^3\Delta(g) = (\Delta(g) + \mathcal{M}_r^4)/\mathcal{M}_r^4 = \langle j^3(\partial g/\partial x_1),\dots,j^3(\partial g/\partial x_r)\rangle_{J_r^3}$.

Over the field $\mathbb{R}$ there are no linear generators, at most r quadratic generators $j^3(\partial g/\partial x_i)$, $i = 1,\dots,r$, and at most r^2 cubic generators $x_j \cdot j^3(\partial g/\partial x_i)$. The worst case is if they are all $\mathbb{R}$ - linearly independent. Then dim $j^3\Delta(g) = r + r^2$. So in general dim $j^3\Delta(g) \leqslant r(r+1)$ and

$$\begin{aligned}\text{codim } f = \text{codim } g &= \dim_{\mathbb{R}} \mathcal{M}_r/\Delta(g) \geqslant \dim j^3(\mathcal{M}_r)/j^3\Delta(g) = \\ &= \dim j^3(\mathcal{M}_r) - \dim j^3\Delta(g) \geqslant \binom{r+3}{3} - 1 - r(r+1) \\ &= \frac{r^3+5r}{6}.\end{aligned}$$

§7. THE CLASSIFICATION:

7.1. Theorem: Every germ $f \in \mathcal{M}_n^2$ of codimension ≤ 6 is right-equivalent to one of the following (non-equivalent) germs.

Here $\sum_{i=2(3)}^{n} \varepsilon_i x_i^2 = +x_{2(3)}^2 + \cdots + x_j^2 - x_{j+1}^2 - \cdots - x_n^2 \; (j=2(3),\ldots,n)$ is the normal form of a non degenerate quadratic form in (n-1) (resp. (n-2)) variables.

Every germ is accompanied by its universal unfolding together with its common name.

Codim	f	Universal unfolding
0	$\sum_{i=1}^{n}\varepsilon_i x_i^2$	
1	$x_1^3+\sum_{i=2}^{n}\varepsilon_i x_i^2$	$x_1^3+\sum_{i=2}^{n}\varepsilon_i x_i^2+y_1x_1$ (fold)
2	$\pm x_1^4+\sum_{i=2}^{n}\varepsilon_i x_i^2$	$\pm x_1^4+\sum_{i=2}^{n}\varepsilon_i x_i^2+y_1x_1^2+y_2x_1$ (cusp)
3	$x_1^5+\sum_{i=2}^{n}\varepsilon_i x_i^2$	$x_1^5+\sum_{i=2}^{n}\varepsilon_i x_i^2+y_1x_1^3+y_2x_1^2+y_3x_1$ (swallowtail).
3	$x_1^3+x_2^3+\sum_{i=3}^{n}\varepsilon_i x_i^2$	$x_1^3+x_2^3+\sum_{i=3}^{n}\varepsilon_i x_i+y_1x_1x_2+y_2x_1+y_3x_2$ (hyperbolic umbilic)
3	$x_1^3-x_1x_2^2+\sum_{i=3}^{n}\varepsilon_i x_i^2$	$x_1^3-x_1x_2^2+\sum_{i=3}^{n}\varepsilon_i x_i^2+y_1(x_1^2+x_2^2)+y_2x_1+y_3x_2$ (elliptic umbilic)
4	$\pm x_1^6+\sum_{i=2}^{n}\varepsilon_i x_i^2$	$\pm x_1^6+\sum_{i=2}^{n}\varepsilon_i x_i^2+y_1x_1^4+y_2x_1^3+y_3x_1^2+y_4x_1$ (butterfly)
4	$\pm(x_1^2x_2+x_2^4)+\sum_{i=3}^{n}\varepsilon_i x_i^2$	$\pm(x_1^2x_2+x_2^4)+\sum_{i=3}^{n}\varepsilon_i x_i^2+y_1x_1^2+y_2x_2^2+y_3x_1+y_4x_2$ (parabolic umbilic - mushroom)
5	$x_1^7+\sum_{i=2}^{n}\varepsilon_i x_i^2$	$x_1^7+\sum_{i=2}^{n}\varepsilon_i x_i^2+y_1x_1^5+y_2x_1^4+y_3x_1^3+y_4x_1^2+y_5x_1$
5	$x_1^2x_2+x_2^5+\sum_{i=3}^{n}\varepsilon_i x_i^2$	$x_1^2x_2+x_2^5+\sum_{i=3}^{n}\varepsilon_i x_i^2+y_1x_2^3+y_2x_1^2+y_3x_2^2+y_4x_1+y_5x_2$
5	$x_1^2x_2-x_2^5+\sum_{i=3}^{5}\varepsilon_i x_i^2$	$x_1^2x_2-x_2^5+\sum_{i=3}^{5}\varepsilon_i x_i^2+y_1x_2^3+y_2x_1^2+y_3x_2^2+y_4x_1+y_5x_2$
5	$\pm(x_1^3+x_2^4)+\sum_{i=3}^{n}\varepsilon_i x_i^2$	$\pm(x_1^3+x_2^4)+\sum_{i=3}^{n}\varepsilon_i x_i^2+y_1x_1x_2^2+y_2x_1x_2+y_3x_2^2+y_4x_1+y_5x_2$

Codim	f	Universal unfolding
6	$\pm x_1^8 + \sum_{i=2}^{n} \varepsilon_i x_i^2$	$\pm x_1^8 + \sum_{i=2}^{n} \varepsilon_i x_i^2 + y_1 x_1^6 + y_2 x_1^5 + y_3 x_1^4 + y_4 x_1^3 + y_5 x_1^2 + y_6 x_1$
6	$\pm(x_1^2 x_2 + x_2^6) + \sum_{i=3}^{n} \varepsilon_i x_i^2$	$\pm(x_1^2 x_2 + x_2^6) + \sum_{i=3}^{n} \varepsilon_i x_i^2 + y_1 x_1 + y_2 x_2 + y_3 x_2^2 + y_4 x_2^3 + y_5 x_2^4 + y_6 x_2^5$
6	$x_1^3 + x_1 x_2^3 + \sum_{i=3}^{n} \varepsilon_i x_i^2$	$x_1^3 + x_1 x_2^3 + \sum_{i=3}^{n} \varepsilon_i x_i^2 + y_1 x_1 + y_2 x_1^2 + y_3 x_2 + y_4 x_2^2 + y_5 x_1 x_2 + y_6 x_1^2 x_2$

Proof: We classify according to corank.

Corank 0: In 6.14 we showed that every germ of corank 0 is equivalent to a non degerate quadratic form. Hence we need only verify that the number of minus signs in the canonical form is an invariant w.r.t. the L_n-action on $\mathcal{M}_n^2$. By 4.2, det $f \leqslant 2$ and so $f \overset{r}{\sim} j^2 f$. The L_n-action on J_n^2 factors over L_n^2 and if $P = P_1 + P_2 \in L_n^2$ (i.e. $P_1 \in GL(n,\mathbb{R})$), then the action of P_2 on $j^2 f \in j^2(\mathcal{M}_n^2)$ is cut off since the result has degree 4. Hence only $GL(n,\mathbb{R})$ acts effectively on $j^2(\mathcal{M}_n^2)$ and then the number of minus signs is clearly invariant.

Corank 1 (Cuspoids): By separating a non degenerate quadratic form we can assume that n=1. Since codim f $\leqslant$ 6 we have det $f \leqslant 8$ (4.2). Thus $f \overset{r}{\sim} j^8 f = \sum_{i=3}^{8} a_i x^i$ say. Let k be the smallest index with $a_k \neq 0$. Then $\Delta(f) = \langle x^{k-1} \rangle_{\mathcal{E}_1} = \mathcal{M}_1^{k-1}$ and so $\mathcal{M}_1^k \subseteq \mathcal{M}_1 \cdot \Delta(f)$. By 3.5, f is k-determined, i.e. $f \overset{r}{\sim} a_k x^k$. If k is even, then the substitution $x \longmapsto |a_k|^{-1/k} \cdot x$ show that $f \overset{r}{\sim} \pm x^k$, if k is odd then $x \longmapsto |a_k|^{-1/k} \operatorname{sgn} a_k \cdot x$ shows that $f \overset{r}{\sim} x^k$. Hence

$$f \overset{r}{\sim} x^3, \pm x^4, x^5, \pm x^6, x^7, \pm x^8.$$

Corank 2: Once more, after removing a non degenrate quadratic form, we can assume that n=2, i.e. f (=f(x,y)) $\in \mathcal{M}_2^2$. By 6.15, codim f $\geqslant$ 3, i.e. codim f = 3,4,5,6.

j^3f is a homogenenous polynomial of degree 3 (since $j^2f = 0$) in two variables, thus corresponding to an inhomogeneous polynomial in one variable which factors over $\mathbb{C}$. So j^3f can be decomposed (over $\mathbb{C}$) into the linear factors

$$(a_1x + b_1y)(a_2x + b_2y)(a_3x + b_3y),$$

and this decomposition is unique up to constants.

We consider four cases:

1) the vectors $\{(a_i,b_i)\}$ are pairwise linearly independent over $\mathbb{C}$;

2) two of the vectors are linearly dependent, the third is independent of the first two;

3) the vectors are pairwise linearly dependent;.

4) $j^3f = 0$.

<u>Case 1:</u>

a) the $\{a_i\}$ and $\{b_i\}$ are all real. Under the transformation

$$x \longmapsto a_1x + b_1y \ , \ y \longmapsto a_2x + b_2y$$

we see that $j^3f(x,y) \sim xy(a\,x + b\,y)$ with $a,b \neq 0$.

(for if A denotes the matrix $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, then

$ax+by = (a_3,b_3).A^{-1}\binom{x}{y}$, i.e. $(a,b).A = (a_3,b_3)$ and if $a=0$

then (a_2,b_2) and (a_3,b_3) are linearly dependent).

$$
\begin{aligned}
\text{Now}\quad xy(ax+by) &\overset{r}{\sim} (1/ab)xy(x+y) && (\text{by } (x,y) \longmapsto (ax,by)) \\
&\overset{r}{\sim} xy(x+y) && (\text{by } (x,y) \longmapsto (ab)^{-1/3}(x,y)) \\
&\overset{r}{\sim} x(x^2-y^2) && (\text{by } (x,y) \longmapsto 2^{-1/3}(x+y,x-y)) \\
&= x^3-xy^2.
\end{aligned}
$$

Now $\Delta(x^3-xy^2) = \langle 3x^2-y^2, 2xy\rangle_{\mathcal{E}_2}$ and so

$$
\begin{aligned}
\mathcal{M}_2 \cdot \Delta(x^3-xy^2) &= \langle 3x^3-xy^2, 3x^2y-y^3, 2x^2y, 2xy^2\rangle_{\mathcal{E}_2} \\
&= \langle x^3, y^3, xy^2, x^2y\rangle_{\mathcal{E}_2} = \mathcal{M}_2^3
\end{aligned}
$$

and so x^3-xy^2 is 3-determined (3.5).

Then $f \overset{r}{\sim} j^3f \overset{r}{\sim} x^3-xy^2$.

b) not all of the $\{a_i\}$, $\{b_i\}$ are real. Since j^3f is real, the factorisation must have the form

$$(a_1x + b_1y)(a_2x + b_2y)(\bar{a}_2x + \bar{b}_2y)$$

where a_1 and b_1 are real. Hence

$$\begin{aligned} j^3 f &\overset{r}{\sim} (ax+by)(x^2+y^2) \\ &\overset{r}{\sim} cx(x^2+y^2) \qquad \text{(by a rotation)} \\ &\overset{r}{\sim} x(x^2+y^2) = x^3+xy^2 \\ &\overset{r}{\sim} 2x^3+6xy = (x+y)^3+(x-y)^3 \\ &\overset{r}{\sim} x^3+y^3 \end{aligned}$$

Then $\Delta(x^3+y^3) = \langle 3x^2, 3y^2\rangle_{\mathcal{E}_2}$ and so

$$\mathcal{M}_2.\Delta(x^3+y^3) = \langle x^3, x^2y, xy^2, y^3\rangle_{\mathcal{E}_2} = \mathcal{M}_2^3$$

and so x^3+y^3 is 3-determined. Hence
$f \overset{r}{\sim} j^3 f \overset{r}{\sim} x^3+y^3$.

Case 2: Suppose that (a_1,b_1) and (a_2,b_2) are linearly independent and (a_3,b_3) is a multiple of (a_2,b_2). Then the factorisation can be arranged in the form

$$(a_1x + b_1y)(a_2x + b_2y)^2$$

where the $\{a_i\}$ and the $\{b_i\}$ are real.
Then $j^3 f \overset{r}{\sim} x^2y$ (by $(x,y) \longmapsto (a_2x+b_2y, a_1x+b_1y)$) and $\Delta(x^2y) = \langle 2xy, x^3\rangle_{\mathcal{E}_2}$ is not finitely determined (since no power of y alone can be generated). Since f is finitely determined, there exists a maximal k for which $j^k f \overset{r}{\sim} x^2y$. We can assume that $j^k f = x^2y$. Then $j^{k+1} f = x^2y+h(x,y)$, where h is a homogeneous polynomial of degree $k+1$. Applying a transformation $\Phi : (x,y) \longmapsto (x+\phi(x,y), y+\psi(x,y))$ where ϕ, ψ are homogeneous polynomials of degree $k-1 \geqslant 2$, we have

$$\begin{aligned} j^{k+1}(f \circ \Phi) = j^{k+1}f.\Phi &= (x+\phi)^2(y+\psi) + h(x,y) \\ &= (x+2x\phi)(y+\psi) + h(x,y) \\ &= x^2y + 2xy\phi + x^2\psi + h(x,y)\,. \end{aligned}$$

• We can choose ϕ and ψ so that the terms of h which are divisible by xy or x^2 vanish. Then we have

$$j^{k+1}(f\circ\Phi) = x^2y + ay^{k+1} \quad (a\neq 0).$$

Now $\Delta(x^2y + ay^{k+1}) = \langle 2xy, x^2+a(k+1)y^k\rangle_{\mathcal{E}_2}$

and so $\mathcal{M}_2.\Delta(x^2y + ay^{k+1}) = \langle x^2y, xy^2, x^3+bxy^k, x^2y+by^{k+1}\rangle_{\mathcal{E}_2}$
$\supseteq \mathcal{M}_2^{k+1}$.

By 3.5, $x^2y + ay^{k+1}$ is $(k+1)$-determined and hence so is f. Thus $f \overset{r}{\sim} x^2y + ay^{k+1} \overset{r}{\sim} x^2y \pm y^{k+1}$ (by $(x,y)\mapsto(|a|^{\frac{1}{2k+2}}.x, |a|^{\frac{-1}{k+1}}.y)$).

Now $4 \leq k+1 = \det f \leq (\operatorname{codim} f) + 2 \leq 8$. Hence we have the following possibilities:

$k = 3$: $f \overset{r}{\sim} x^2y\pm y^4 \overset{r}{\sim} \pm(x^2y + y^4)$.

$k = 4$: $f \overset{r}{\sim} x^2y\pm y^5$.

$k = 5$: $f \overset{r}{\sim} x^2y\pm y^6 \overset{r}{\sim} \pm(x^2y + y^6)$.

$k \geq 6$: $x, y, y^2, y^3, y^4, y^5, y^6 \ldots$ are linearly independent in $\mathcal{M}_2/\Delta(f)$ and so $\operatorname{codim} f \geq 7$.

Just as for the case where f has corank 0, one can show that the minus signs cannot be removed.

<u>Case</u> 3: $j^3f = (ax + by)^3$, a,b real. Then if $(\tilde{a},\tilde{b})$ and (a,b) are linearly independent, the transformation

$$\Phi : (x,y) \longmapsto (ax+by, \tilde{a}x+\tilde{b}y)$$

gives $j^3(f\circ\Phi) = j^3f.\Phi = x^3$.

x^3 has infinite codimension and so f is not 3-determined. We can assume that $j^3f = x^3$ and choose k maximal so that $j^kf \overset{r}{\sim} x^3$. Then $j^{k+1}f = x^3+h(x,y)$ where h is a homogeneous polynomial of degree $k+1$. If Ψ is the transformation $(x,y)\longmapsto(x+\psi(x,y),y)$ where ψ is homogeneous of degree $k-1\geq 2$, then

$$j^{k+1}(f\circ\Psi) = (x+\psi)^3 + h(x,y)$$
$$= x^3 + 3x^2\psi + h(x,y).$$

Now choose ψ so that the terms of h which are divisible by x^2 vanish. Then

$$j^{k+1}(f\circ\Psi) = x^3 + cxy^k + dy^{k+1} \quad ((c,d)\neq(0,0)).$$

a). $d\neq 0$: Applying $\rho : (x,y)\mapsto(x, y - \frac{c}{(k+1)d}x)$, we get

$$j^{k+1}(f\circ\psi\circ\rho) = x^3 + cx\left(y - \frac{c}{(k+1)d}x\right)^k + d\left(y - \frac{c}{(k+1)d}x\right)^{k+1}$$

$$= x^3 + cx(y^k + x.P_1(x,y)) + d\left(y^{k+1} - (k+1)y\frac{c}{(k+1)d}x + \right.$$

$$\left. + x^2 P_2(x,y)\right),$$

where P_1, P_2 are homogeneous polynomials of degree $k-1 \geq 2$,

$$= x^3 + cxy^k - cxy^k + 3x^2P(x,y) + dy^{k+1}$$

$$= x^3 + 3x^2P(x,y) + dy^{k+1},$$ where P is homogeneous of degree $\geq k-1$.

Applying $\eta : (x,y) \longmapsto (x-P(x,y),y)$ we get

$$j^{k+1}(f\circ\psi\circ\rho\circ\eta) = (x-P)^3 + 3x^2P(x,y) + dy^{k+1}$$

$$= x^3 + dy^{k+1} \overset{r}{\sim} x^3 \pm y^{k+1}.$$

Now $\Delta(x^3 \pm y^{k+1}) = \langle 3x^2, (k+1)y^k\rangle_{\mathcal{E}_2}$ and so

$\mathcal{M}_2 . \Delta(x^3 \pm y^{k+1}) = \langle x^3, xy^k, x^2y, y^{k+1}\rangle_{\mathcal{E}_2} \supseteq \mathcal{M}_2^{k+1}$

and so $x^3 \pm y^{k+1}$ is $(k+1)$-determined. Hence $f \overset{r}{\sim} x^3 \pm y^{k+1}$. Since $4 \leq k+1 = \det f \leq (\operatorname{codim} f) + 2 \leq 8$, we have the following possibilities:

$k = 3$: $f \overset{r}{\sim} x^3 \pm y^4 \overset{r}{\sim} \pm(x^3+y^4)$.

$k \geq 4$: $x, y, y^2, \cdots, y^{k-1}, xy, xy^2, \cdots, xy^{k-1}$ are linearly independent in $\mathcal{M}_n/\Delta(f)$ and so $\operatorname{codim} f = 1+2(k-1) = 2k-1 \geq 7$

b) $d=0$: Then $j^{k+1}(f\circ\Phi) = x^3 + cxy^k \quad (c \neq 0)$

$$\overset{r}{\sim} x^3 \pm xy^k.$$

Then $\Delta(x^3 \pm xy^k) = \langle 3x^2 \pm y^k, kxy^{k-1}\rangle_{\mathcal{E}_2}$.

$k = 3$: $x^3 \pm xy^3 \overset{r}{\sim} x^3 + xy^3$ is 4-determined by the following Lemma (7.2). Hence $j^4 f \overset{r}{\sim} x^3 + xy^3$ is 4-determined and so $f \overset{r}{\sim} x^3 + xy^3$.

$k \geq 4$: we consider $j^{k+2}f = x^3 \pm xy^k + P(x,y)$, where P is homogeneous of degree $k+2$. Then

$$j^{k+1}\Delta(f) = \langle j^{k+1}\partial f/\partial x, j^{k+1}\partial f/\partial y \rangle_{J_2^{k+1}}$$
$$= \langle \partial/\partial x\, j^{k+2}f, \partial/\partial y\, j^{k+2}f \rangle_{J_2^{k+1}}$$
$$= \langle 3x^2 \pm y^k + \partial P/\partial x(x,y), kxy^{k-1} + \partial P/\partial y(x,y) \rangle_{J_2^{k+1}}.$$

If $\partial P/\partial y \neq 0$ then $x, y, y^2, y^3, y^4, xy, xy^2, xy^3$ are linearly independent in $j^{k+1}\mathcal{M}_2 / j^{k+1}\Delta(f)$;
if $\partial P/\partial y = 0$ then $P = P(x) = ax^{k+2}$ and

$$x, y, y^2, y^3, x^2, x^3, x^4, xy, xy^2, xy^3$$

are linearly independent.
So $\operatorname{codim} f = \dim \mathcal{M}_2 / \Delta(f) \geq \dim j^{k+1}\mathcal{M}_2 / j^{k+1}\Delta(f) \geq 8$.

Case 4: $j^3f = 0$ and so $f \in \mathcal{M}_2^4$, i.e. $\Delta(f) \subseteq \mathcal{M}_2^3$.
$\Delta(f)$ is generated by 2 elements over $\mathcal{E}_2$, $\mathcal{M}_2^3$ by 4 (the homogeneous monomials of degree 3) which are independent over $\mathcal{E}_2$. Therefore $\dim \mathcal{M}_2^3 / \Delta(f) \geq 2$. Hence

$$\operatorname{codim} f = \dim \mathcal{M}_2 / \Delta(f) = \dim \mathcal{M}_2 / \mathcal{M}_2^3 + \dim \mathcal{M}_2^3 / \Delta(f)$$
$$\geq \binom{4}{2} - 1 + 2 = 7.$$

Corank ≥ 3: By lemma 6.15 we conclude that $\operatorname{codim} f \geq 7$.
Thus the theorem is proved.

Remark: One could continue the classification (see Siersma [15], or the papers of V.I. Arnold). In the next step one finds the germ $4y^3 - xz^2 - a_1x^2y - a_2x^3$ with $27a_2^2 - a_1^3 \neq 0$, which is 3-determined and has codimension 7. Furthermore a_1^3/a_2^2 is an invariant under smooth coordinate change, so the classification becomes infinite from codimension 7 onwards.

7.2. <u>Lemma</u>: The germ x^3+xy^3 in $\mathcal{M}_2^2$ is 4-determined.

<u>Proof</u>: Let $f \in \mathcal{M}_2$ with $j^4 f = x^3+xy^3$. Then we can write

$$f(x,y) = x^3 + xy^3 + R(x,y) \quad \text{with} \quad R \in \mathcal{M}_2^5 .$$

Put $F(x,y,t) = (1-t)f(x,y) + t(x^3+xy^3) = x^3 + xy^3 + (1-t)R(x,y)$.

Note that in the proof of 3.5 we employed the inclusion

$$\mathcal{M}_n^k \subseteq \mathcal{M}_n \langle \partial F/\partial x_1, \dots, \partial F/\partial x_n \rangle_{\mathcal{E}_{n+1}}$$

to obtain the family $\{\Gamma_t\}$ in a neighbourhood of t_0. In fact, we only used the weaker inclusion:

$$\mathcal{M}_n^{k+1} \subseteq \mathcal{M}_n \langle \partial F/\partial x_1, \dots, \partial F/\partial x_n \rangle_{\mathcal{E}_{n+1}}$$

and we shall now verify this for $n=2$, $k=4$). By Nakayama's Lemma, it is sufficient to show that

$$\mathcal{M}_2^5 \subseteq \mathcal{M}_2 \langle \partial F/\partial x, \partial F/\partial y \rangle_{\mathcal{E}_3} + \mathcal{M}_2^6 .$$

Now, $\langle \partial F/\partial x, \partial F/\partial y \rangle_{\mathcal{E}_3} = \langle 3x^2+y^3+(1-t)g(x,y), 3xy^2+(1-t)h(x,y) \rangle_{\mathcal{E}_3}$

where $g = \partial R/\partial x$, $h = \partial R/\partial y$ and so

$$\mathcal{M}_2 \cdot \langle \partial F/\partial x, \partial F/\partial y \rangle_{\mathcal{E}_3} = \langle 3x^3+xy^3+(1-t)xg(x,y), 3x^2y+y^4+(1-t)yg(x,y), 3x^2y^2+(1-t)xh(x,y), 3xy^3+(1-t)yh(x,y) \rangle_{\mathcal{E}_3} .$$

Hence we obtain the following elements (mod $\mathcal{M}_2^6$ so that we ignore terms of degree ≥ 6):

$3x^4+x^2y^3$ and $3x^2y^3$ – hence x^4, x^2y^3;

$3xy^4$ – hence xy^4;

$3x^3y+xy^4$ – together with xy^4 this gives x^3y.

Hence we have obtained a generating system of $\mathcal{M}_2^5$ with the exception of y^5. Now all terms of order 5 in $(1-t)xh(x,y)$ are divisible by x and so can be removed with the generators that we already have. Hence we obtain x^2y^2 and this, together with $3x^2y^4+y^5$, gives y^5.

7.3. Corollary: Every germ $f \in \mathcal{M}_1$ of codimension r is right equ alent to x^{r+2} and its universal unfolding is given by

$$x^{r+2} + y_1 x + y_2 x^2 + \cdots + y_r x^r .$$

Proof: Adapt the proof of 7.1 - corank 1.

7.4. Corollary: Every germ $f \in \mathcal{M}_2$ of codimension ≤ 7 is right equivalent to one of the list of 7.1 (for n=2) or one of the following germs:

codim	f	universal unfolding
7	$x_1^9+x_2^2$	$x_1^9+x_2^2+y_1 x_1+y_2 x_1^2+ \cdots +y_7 x_1^7$
7	$x_1^2 x_2+x_2^7$	$x_1^2 x_2+x_2^7+y_1 x_1+y_2 x_2+y_3 x_2^2+y_4 x_2^3+y_5 x_2^4+y_6 x_2^5+y_7 x_2^6$
7	$x_1^2 x_2-x_2^7$	$x_1^2 x_2-x_2^7+y_1 x_1+y_2 x_2+y_3 x_2^2+y_4 x_2^3+y_5 x_2^4+y_6 x_2^5+y_7 x_2^7$
7	$\pm(x_1^3+x_2^5)$	$\pm(x_1^3+x_2^5)+y_1 x_1+y_2 x_2+y_3 x_2^2+y_4 x_2^3+y_5 x_1 x_2+y_6 x_1 x_2^2+y_7 x_1 x_2^3$.

Proof: ◡onsult the proof of 7.1 with the further restriction n=2. The first germ comes from corank 1, the next two from corank 2 (case 2) and the last from case 3a).
We have shown that the next germ in corank 2, case 3b) has codimension ≥ 8. Thus it only remains to show that in case 4 of corank 2, any germ has codimension ≥ 8. This can be done by decomposing

$$j^4 f = (a_1 x+b_1 y)(a_2 x+b_2 y)(a_3 x+b_3 y)(a_4 x+b_4 y)$$

over $\mathbb{C}$ and proceeding as in the proof of 7.1, corank 2.

It turns out that the germ with the lowest codimension is right equivalent to $(x^2 \pm y^2)(x^2+\alpha y^2)$ $(\alpha \neq 0,-1,1)$ and that α is an invariant under linear transformations. This germ has codimension 8 and so for codimension >7, the classification becomes infinite. The proof of this fact is a tedious repetition of by now familiar techniques and is left to the reader.

7.5. We now examine the question of the number of L_n-orbits. We shall be mainly interested in the orbits of germs f which have codimension $\leqslant 6$ (and so are 8-determined by 4.2). f is then equivalent to its 8-jet and

$$f \circ L_n = (j^8)^{-1}(j^8 f . L_n^8) \cap \mathcal{M}_n \,.$$

We shall therefore be mainly interested in the number of L_n^8-orbits in $J = j^8(\mathcal{M}_n) \subseteq J_n^8$.

a) The open subset $j^8(\mathcal{M}_n) \setminus j^8(\mathcal{M}_n^2)$ is an orbit. For if $f \in \mathcal{M}_n \setminus \mathcal{M}_n^2$ then f is 1-determined (3.6) and so $f \overset{r}{\sim} j^1 f$. $j^1 f$ is a linear form on $\mathbb{R}^n$ and so is right equivalent to x_1 say.

b) There are $n+1$ distinct orbits consisting of equivalence classes of non degenerate quadratic forms. They are immersive submanifolds of codimension 0 in $j^8(\mathcal{M}_n^2)$ (4.6) and so have codimension n in J.

c) The orbits of corank 1 and codimension $\leqslant 6$ - according to 7.1 there are nine types: $x_1^3, \pm x_1^4, x_1^5, \pm x_1^6, x_1^7, \pm x_1^8$. The remaining $(n-1)$ variable are contained in non degenerate quadratic forms. There are n possibilities and this gives a total of $9n$ orbits whose codimensions can be read from the list in 7.1 (add n).

d) The orbits of corank 2 and codimension $\leqslant 6$. According to 7.1 there are eleven types: $x_1+x_2^3, x_1^3-x_1x_2^2, \pm(x_1^2x_2+x_2^4), x_1^2x_2 \pm x_2^5, \pm(x_1^3+x_2^4), \pm(x_1^2x_2+x_2^6), x_1^3+x_1x_2^3$. As in b) one obtains $11(n-1)$ orbits whose codimensions can be read from 7.1.

e) The remaining orbits w.r.t L_n^8 in J are contained in $\Sigma_7^8 = j^8(\Sigma_7)$ since their elements have codimension $\geqslant 7$.

7.6 We shall now decompose Σ_7^8, the set of all 8-jets of germs of codimension $\geqslant 7$ into finitely many disjoint immersive submanifolds. As we have already seen this cannot be done using orbits (since there are infinitely many of them), so we shall attempt to produce as simple a decomposition as possible. Our aim is that each of these manifolds has codimension (in J) higher than the maximal codimension appearing in 7.5.

This is possible only in the case that n = 2. In case $n \geqslant 3$ there will one type of manifolds with codimension n+6, and we will be forced to add the orbits of the germs of codimension 6 to this decomposition here to obtain a decomposition of satisfying all requirements. This distinction in codimension will be essential in §9.

a) First of all we consider the subset of Σ_7^8 consisting of the 8-jets of germs with corank 1. By 6.14, every f with corank 1 is right equivalent to

$$g(x_1) + \sum_{i=2}^{n} \varepsilon_i x_i^2 .$$

If $j^8 g \neq 0$, then f is right equivalent to a germ in the list 7.1 - hence $j^8 f \notin \Sigma_7^8$. Thus $j^8 g = 0$. The set of $z \in \Sigma_7^8$ with corank 1 can then be decomposed into the n distinct orbits of non degerate quadratic forms in n-1 variables. The tangent space of such an orbit has the form

$$T_{j^8 f}(j^8 f.L_n^8) = j^8(\mathcal{M}_n . \Delta(f))$$

by 2.12 and $\Delta(f) = \langle x_2, x_3, \cdots, x_n \rangle_{\mathcal{E}_n} = \mathcal{M}_{n-1} \cdot \mathcal{E}_n$. Hence

$$\mathcal{M}_n . \Delta(f) = \mathcal{M}_n . \mathcal{M}_{n-1} = \langle x_1 x_2, \cdots, x_1 x_n, \mathcal{M}_{n-1}^2 \rangle_{\mathcal{E}_n} .$$

The codimension of the orbit in J is

$$\dim \left[j^8(\mathcal{M}_n)/j^8(\mathcal{M}_n . \Delta(f)\right] = \dim \left[(\mathcal{M}_n/\mathcal{M}_n^9)/(\mathcal{M}_n . \Delta(f) + \mathcal{M}_n^9/\mathcal{M}_n^9) \right]$$

$$= \dim \left[\mathcal{M}_n/(\mathcal{M}_n . \Delta(f) + \mathcal{M}_n^9) \right]$$

$$= \dim \left[\mathcal{M}_n/(\langle x_1 x_2, \cdots, x_1 x_n, \mathcal{M}_{n-1}^2 \rangle_{\mathcal{E}_n} + \mathcal{M}_n^9) \right]$$

and $\{x_1, \cdots, x_n, x_1^2, x_1^3, \cdots, x_1^8\}$ is a basis for the last space over $\mathbb{R}$. Hence the codimension is precisely n+7.

b) We now consider the subset of Σ_7^8 consisting of 8-jets of germs with corank 2. Then for such an f

$$f(x_1, \cdots, x_n) \sim g(x_1, x_2) + \sum_{i=3}^{n} \varepsilon_i x_i^2$$

with $j^2 g = 0$. We consider the four cases investigated in the proof of 7.1.

<u>Case</u> 1: a) and b) produce germs in list 7.1 and so $j^8 f \notin \Sigma_7^8$.

<u>Case</u> 2: Since $f \in \Sigma_7$, $j^k g \overset{r}{\sim} x_1^2 x_2$ (k=3,4,5,6) - otherwise we would produce a germ in the list. In particular, $j^6 g \overset{r}{\sim} x_1^2 x_2$. Consider the n-1 orbits in $j^6(\mathcal{M}_n)$ generated by $x_1^2 x_2 + \sum_{i=3}^{n} \varepsilon_i x_i^2$. Then

$$\Delta(j^6 f) = \langle 2x_1x_2, x_1^2, x_3, \cdots, x_n\rangle \quad \text{and so}$$

$$\mathcal{M}_n \cdot \Delta(j^6 f) = \langle x_1^2x_2, x_1x_2^2, x_1^3, x_1^2x_2, \mathcal{M}_n \cdot \mathcal{M}_{n-2}\rangle_{\mathcal{E}_n} .$$

The elements $\{x_1, \cdots, x_n, x_1^2, x_1x_2, x_2^2, x_2^3, \cdots, x_2^6\}$ define a basis for $\mathcal{M}_n/(\mathcal{M}_n \cdot \Delta(j^6 f) + \mathcal{M}_n^7)$ and so, just as in a), each of these orbits has codimension $n+7$ in $j^6(\mathcal{M}_n)$.

Now consider the canonical projection $\pi_6^8 : j^8(\mathcal{M}_n) \longrightarrow j^6(\mathcal{M}_n)$.

For the preimages under π_6^8 of these $n-1$ orbits in $j^8(\mathcal{M}_n)$ we have

$$(\pi_6^8)^{-1}((x_1^2x_2 + \sum_{i=3}^{n} \varepsilon_i x_i^2).L_n^6) = ((x_1^2x_2 + \sum_{i=3}^{n} \varepsilon_i x_i^2).L_n^6) \times \mathcal{M}_n^7/\mathcal{M}_n^9$$

and these preimages have the same codimension in $j^8(\mathcal{M}_n)$, namely $n+7$. These are the $n-1$ immersive submanifolds in which we decompose the germs of case 2.

Case 3: $j^3 g \overset{r}{\sim} x_1^3$. Since $f \in \Sigma_7$ we have $j^4 g \overset{r}{\sim} x_1^3$ (otherwise we produce a germ in the list). Now consider the $n-1$ orbits generated by the germs $x_1^3 + \sum_{i=3}^{n} \varepsilon_i x_i^2$ in $j^4(\mathcal{M}_n)$.

$$\Delta(j^4 f) = \langle 3x_1^2, x_3, \cdots, x_n\rangle_{\mathcal{E}_n} \quad \text{and so}$$

$$\mathcal{M}_n \cdot \Delta(j^4 f) = \langle x_1^3, x_1^2x_2, \mathcal{M}_n \cdot \mathcal{M}_{n-2}\rangle_{\mathcal{E}_n}$$

and the elements $\{x_1, \cdots, x_n, x_1^2, x_2^2, x_2^3, x_2^4, x_1x_2, x_1x_2^2, x_1x_2^3\}$ define a basis for $\mathcal{M}_n/(\mathcal{M}_n \cdot \Delta(j^4 f) + \mathcal{M}_n^5)$. Hence these orbits have codimension $n+7$. Then we decompose the germs of case 3 into the immersive submanifolds formed by the preimages

$$(\pi_4^8)^{-1}((x_1^3 + \sum_{i=3}^{n} \varepsilon_i x_i^2).L_n^4) = (x_1^3 + \sum_{i=3}^{n} \varepsilon_i x_i^2).L_n^4 \times \mathcal{M}_n^5/\mathcal{M}_n^9$$

of these orbits. They have codimension $n+7$.

Case 4: $j^3 g = 0$. Hence $j^3 f \overset{r}{\sim} \sum_{i=3}^{n} \varepsilon_i x_i^2$. Consider the $n-1$ orbits generated by the germs $\sum_{i=3}^{n} \varepsilon_i x_i^2$ in $J^3(\mathcal{M}_n)$. Then

$$\Delta(j^3 f) = \langle x_3, \cdots, x_n\rangle_{\mathcal{E}_n} = \mathcal{M}_{n-2} \quad \text{and so}$$

$\mathcal{M}_n \cdot \Delta(j^3 f) = \mathcal{M}_n \cdot \mathcal{M}_{n-2}$. The elements

$$\langle x_1, \cdots, x_n, x_1^2, x_1x_2, x_2^2, x_1^3, x_1^2x_2, x_1x_2^2, x_2^3\rangle$$

define a basis for $\mathcal{M}_n/(\mathcal{M}_n \cdot \Delta(j^3 f) + \mathcal{M}_n^4)$. Hence each of these orbits has codimension $n+7$ in $j^3(\mathcal{M}_n)$. Once again, we decompose the germs of case 4 into the preimages, under the projection π_3^8, of these $n-1$ orbits - they are immersive submanifolds of codimension $n+7$.

c) We now consider the subset $\Lambda \subseteq \Sigma^8_7$ consisting of the elements corank $\geqslant 3$. This set is empty for $n \leqslant 2$.
We claim that Λ is a finite disjoint union of immersive submanifolds of codimension $n+6$, if $n \geqslant 3$.
To see this let $f \in \mathcal{M}_n^2$ with $j^8 f \in \Lambda$, i.e. with corank $\geqslant 3$.
Then $j^2 f \in J_n^2$ and $j^2 f$ is a degenerate quadratic form of rank $n - \varrho$ (which may be zero). In the L_n^2-orbit of $j^2 f$ we may find a qadratic form which looks like

$$q(x_1,\dots,x_n) = \pm x_1^2 \dots \pm x_{n-\varrho}^2 .$$

Its matrix is of the form $\begin{pmatrix} \pm 1 & & & 0 \\ & \ddots & & \\ & & \pm 1 & \\ 0 & & & 0 \ddots 0 \end{pmatrix}$

where the number of minus signs in the main diagonal is a L_n^2-invariant of the orbit.
Now let q' be near q in J_n^2. Then the matrix of q' looks like

$$\begin{pmatrix} A & B \\ B^t & C \end{pmatrix}, \text{ where A and C are symmetric and } \det A \neq 0.$$

Then

$$\operatorname{rank} q' = \operatorname{rank}\begin{pmatrix} A & B \\ B^t & C \end{pmatrix} = \operatorname{rank}\begin{pmatrix} A & 0 \\ -B^t A^{-1} & I \end{pmatrix}\begin{pmatrix} A & B \\ B^t & C \end{pmatrix}$$

$$= \operatorname{rank}\begin{pmatrix} I & A^{-1}B \\ 0 & -B^t A^{-1} B + C \end{pmatrix}.$$

So $\operatorname{rank} q' = n-\varrho = \operatorname{rank} q$ iff $C = B^t A^{-1} B$. So to stay in the same rank class one has to suppress the free choice for the entries in symmetric C, i.e. one has to suppress $\frac{1}{2}\varrho(\varrho+1)$ variables. So the codimension of the orbit $j^2 f \,.\, L_n^2$ is $\frac{1}{2}\varrho(\varrho+1)$. There are finitely many orbits in J_n^2.
If $\pi_2^8 : j^8(\mathcal{M}_n) \longrightarrow j^2(\mathcal{M}_n)$ denotes the canonical projection "truncate", then Λ is the disjoint union of the inverse images under π_2^8 of all the L_n^2-orbits of quadratic forms of corank $\geqslant 3$ in $j^2(\mathcal{M}_n)$. These have codimension

$n + \frac{1}{2}\varrho(\varrho+1) \geqslant n + \frac{1}{2}\, 3.4 = n+6$ in $j^2(\mathcal{M}_n)$, so the inverse images have the same codimension in J.

7.5 For later reference we collect again the decompositions of $J = j^8(\mathcal{M}_n)$ into immersive submanifolds, which we will use.
If $n = 1$, then for any k the space $j^k(\mathcal{M}_1)$ consists of finitely many orbits of codimension $\leqslant n+k-1 = k$, and $\Sigma^k_{k-1} = \{0\}$ has codimension $n+k = k+1$.

If $n = 2$ then the open subset $J \setminus \Sigma_7^8$ is the disjoint union of the finitely many orbits of codimension $\leq n+6$ listed in 7.3. Σ_7^8 decomposes into finitely many submanifolds of codimension $\geq n+7$, listed in 7.4 a),b), since the set Λ of 7.4 c) is empty.

If $n \geq 3$ then the open set $J \setminus \Sigma_6^8$ is the disjoint union of the finitely many orbits of 8-jets of germs of codimension ≤ 5, which are immersive submanifolds of codimension $\leq n+5$. Σ_6^8 decomposes into the orbits of 8-jets of germs of codimension = 6 and into the finitely many immersive submanifolds of codimension $\geq n+6$, listed in 7.4.

Remark: Mather has shown that an orbit like ours is indeed a proper submanifold, since the group actions here are algebraic actions of algebraic groups with special properties. We will not need this result, we will circumvent the arising difficulties in §9 with a simple trick (9.4). A proof of the result may be found in Mather, Stability of C^∞-mappings V, Advances in Mathematics 4.

§8. CATASTROPHE GERMS:

8.1. Let (r,f') be a universal unfolding of $f \in \mathcal{M}_n^2$ and let $\tilde{f}' : \mathbb{R}^{n+r} \longrightarrow \mathbb{R}$ represent f'. Put

$$M_{\tilde{f}'} := \{(x,y) \in \mathbb{R}^{n+r} : \partial\tilde{f}'/\partial x_1 (x,y) = \cdots = \partial\tilde{f}'/\partial x_n (x,y) = 0\}.$$

Denote by $\pi_r : \mathbb{R}^{n+r} \longrightarrow \mathbb{R}^r$ the natural projection and put

$$\chi_{\tilde{f}'} := \pi_r|_{M_{\tilde{f}'}} : M_{\tilde{f}'} \longrightarrow \mathbb{R}^r,$$

Then $0 \in M_{\tilde{f}'}$ since $f \in \mathcal{M}_n^2$.

$\chi_{f'}$ is defined to be the germ $[\chi_{\tilde{f}'}]_0$ of $\chi_{\tilde{f}'}$ at 0 and is called the catastrophe germ of f'. We can regard $\chi_{f'}$ as mapping $M_{f'}$ into $\mathbb{R}^r$ where $M_{f'}$ is the germ of a set. $\chi_{f'}$ depends only on f', not on $\tilde{f}'$.

8.2. Lemma: If f is an element of $\mathcal{M}_n^3$ with codim $f = c$, then there exists a (standard) universal unfolding (c,f') of f so that there is a diffeomorphism from $M_{f'}$ onto $\mathbb{R}^c$.

Proof: Since $f \in \mathcal{M}_n^3$, $\Delta(f) \leq \mathcal{M}_n^2$. Choose a basis $\{u_1, \cdots, u_c\}$ of $\mathcal{M}_n$ mod $\Delta(f)$ so that

$$u_j(x) = \begin{cases} x_j & (1 \leqslant j \leqslant n) \\ \text{a monomial of degree} \geqslant 2 & \text{otherwise.} \end{cases}$$

Then $f' : (x,y) \longrightarrow f(x) + \sum_{j=1}^{c} u_j(x) y_j$

is a universal unfolding (6.12) and

$$\partial f'/\partial x_i (x,y) = \partial f/\partial x_i (x) + y_i + \sum_{j=n+1}^{c} y_j \, \partial u_j/\partial x (x).$$

Consider the smooth mapping $\psi = (\psi_i)_{i=1}^n : \mathbb{R}^n \times \mathbb{R}^{c-n} \longrightarrow \mathbb{R}^n$

where $\psi_i : (x_1, \cdots, x_n, y_{n+1}, \cdots, y_c) \mapsto -\partial f/\partial x_i (x) - \sum_{j=n+1}^{c} y_j \, \partial u_j/\partial x_i (x)$.

Then $M_{f'} = \{(x,y) : y_i = \psi_i(x, y_{n+1}, \cdots, y_c), \ i=1,\cdots,n\}$

and so is the graph of ψ. The graph of a smooth mapping is a manifold which is diffeomorphic to the domain of definition, in this case $\mathbb{R}^n \times \mathbb{R}^{c-n} = \mathbb{R}^c$.

8.3. Consider the situation of 6.13.3), i.e. f is a germ in $\mathcal{M}_n^2$ with universal unfolding $f(x) + g(x,y)$ defined on $\mathbb{R}^k \times \mathbb{R}^r$ and q is a non degenerate quadratic form in separate variables $(x_{k+1}, \cdots, x_n)$. Then the universal unfolding of $f+q$ is $f+q+g$.

Lemma: $\chi_{f+g} = \chi_{f+g+q}$.

Proof: Without loss of generality, we can assume that q has the form $\pm x_{k+1}^2 \pm \cdots \pm x_n^2$.
If we write $x = (\bar{x}, \bar{\bar{x}}) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, then a simple calculation shows that

$$\frac{\partial(f+q+g)}{\partial x_i}(x,y) = \frac{\partial(f+g)}{\partial x_i}(x,y) \qquad (i \leq k);$$

$$\frac{\partial(f+q+g)}{\partial x_i}(x,y) = 2x_i \qquad (i > k).$$

Hence $M_{f+q+g} = M_{f+g} \times \{0\}$ and we have the following commutative diagram:

$$\begin{tikzcd}
\chi_{f+q+g} : M_{f+q+g} \arrow[r, hook] & \mathbb{R}^{n+r} \arrow[r, "\pi_r"] & \mathbb{R}^r \\
\chi_{f+g} : M_{f+g} \arrow[r, hook] \arrow[u, phantom, "\wr\|"] & \mathbb{R}^{k+r} \arrow[r, "\pi_r"] \arrow[u, hook] & \mathbb{R}^r
\end{tikzcd}$$

8.4. Lemma: Let (r,f') and (s,f'') be unfoldings of $f \in \mathcal{M}_n^2$ and let $(\phi, \bar{\phi}, \varepsilon)$ be a morphism from (s,f'') into (r,f'). Then $M_{f''} = \phi^{-1}(M_{f'})$ and $\chi_{f''}$ is the pullback of $\chi_{f'}$ with respect to $(\phi, \bar{\phi})$, i.e. we have the following commutative diagram:

$$\begin{tikzcd}
\mathbb{R}^{n+s} \arrow[rr, "\phi"] & & \mathbb{R}^{n+r} \\
M_{f''} \arrow[u, phantom, "\cup"] \arrow[rr, dashed] \arrow[d, "\chi_{f''}"'] & & M_{f'} \arrow[u, phantom, "\cup"] \arrow[d, "\chi_{f'}"] \\
\mathbb{R}^s \arrow[rr, "\bar{\phi}"] & & \mathbb{R}^r
\end{tikzcd}$$

Proof: $f''(x,z) = f'(\phi^1(x,z), \bar{\phi}(z)) + \varepsilon(z)$,

where $\phi(x,z) = (\phi^1(x,z), \bar{\phi}(z))$. Hence

$$\partial f''/\partial x_i (x,z) = \underbrace{{}^t(\partial \phi^1/\partial x_i (x,z))}_{\text{Jacobi matrix of } \phi} (\partial f'/\partial x_j (\phi(x,z))) .$$

Now the latter matrix is regular for z near 0 (since $D\phi(x,0) = Id$) and so

$$(x,z) \in M_{f''} \iff \frac{\partial f''}{\partial x_i}(x,z) = 0 \quad (i=1,\dots,n)$$
$$\iff \frac{\partial f'}{\partial x_i}(\phi(x,z)) = 0 \quad (i=1,\dots,n)$$
$$\iff \phi(x,z) \in M_{f'} ,$$

i.e. $M_{f''} = \phi^{-1}(M_{f'})$.

The equation $\chi_{f'} \circ \phi = \bar{\phi} \circ \chi_{f''}$ follows by restricting the equality $\pi_r \circ \phi = \bar{\phi} \circ \pi_s$ to the appropriate germ of sets.

8.5. If $g_i : (\mathbb{R}^n, p_i) \longrightarrow (\mathbb{R}^m, q_i)$ are smooth germs $(i=1,2)$, we say that g_1 is equivalent to g_2 (written $g_1 \sim g_2$) if there exist germs of diffeomorphisms $\phi : (\mathbb{R}^n, p_1) \longrightarrow (\mathbb{R}^n, p_2)$ and $\psi : (\mathbb{R}^m, q_1) \longrightarrow (\mathbb{R}^m, q_2)$ so that $\psi \circ g_1 = g_2 \circ \phi$

(there will be no confusion with the equivalence used in §2). This is the equivalence relation used in §0. Strictly speaking, we cannot apply it to $\chi_{f'}$ and $\chi_{f''}$ since $M_{f'}$ and $M_{f''}$ need not be manifolds. We shall therefore call the two germs $\chi_{f'}$ and $\chi_{f''}$, defined on the germs $M_{f'}$ and $M_{f''}$, equivalent if one can extend them to germs on open neighbourhoods in $\mathbb{R}^n$ and $\mathbb{R}^m$, so that they are smooth and equivalent in the above sense, using a ϕ which restricts to a bijection from $M_{f'}$ onto $M_{f''}$.

8.6. <u>Corollary</u>: a) If $f \in \mathcal{M}_n^2$ and $(\phi, \bar{\phi}, \varepsilon) : (r,f') \to (r,f'')$ is an isomorphism between unfoldings of f, then $\chi_{f'} \sim \chi_{f''}$ (in the sense of 8.5);

b) if (r,f') and (r,f'') are versal unfoldings of f, then $\chi_{f'} \sim \chi_{f''}$;

c) if (r,f'), (s,f'') are versal unfoldings of f with $r \leq s$, then $\chi_{f''} \sim \chi_{f'} \times \mathrm{Id}_{\mathbb{R}^{s-r}}$;

d) if $f,g \in \mathcal{M}_n$ and $f \overset{r}{\sim} g$ (i.e. f is right equivalent to g in the sense of 2.4) and if (r,f') and (r,g') are versal unfoldings of f and g resp., then $\chi_{f'} \sim \chi_{g'}$.

<u>Proof</u>: a) See 8.4 (ϕ and $\bar{\phi}$ provide the diffeomorphisms and π_r, π_s the extensions).

b) By 6.12 (r,f') and (r,f'') are isomorphic - now use a).

c) Let (s,f') be the trivial extension of f'. Then (s,f') is also versal (because there is a morphism from (r,f') into (s,f') - cf. 6.11 or 6.12.c)). By b), $(s,f') \sim (s,f'')$ and we have the following diagram:

$$\begin{tikzcd} M_{f''} \arrow[r] \arrow[d, "\chi_{f''}"'] & M_{(s,f')} \arrow[r, equal] \arrow[d, "\chi_{(s,f')}"'] & M_{f'} \times \mathbb{R}^{s-r} \arrow[d, "\chi_{f'} \times \mathrm{id}_{\mathbb{R}^{s-r}}"] \\ \mathbb{R}^s \arrow[r] & \mathbb{R}^s \arrow[r, equal] & \mathbb{R}^r \times \mathbb{R}^{s-r} \end{tikzcd}$$

d) Suppose that $f = g \circ \gamma$ $(\gamma \in L_n)$. Put $f'' := g' \circ (\gamma \times \mathrm{Id}_{\mathbb{R}^r})$. Then $f''|_{\mathbb{R}^n \times \{0\}} = g'|_{\mathbb{R}^n \times \{0\}} \circ \gamma = g \circ \gamma = f$ and so (r,f'') is an unfolding of f. The following diagram provides an equivalence between $\chi_{g'}$ and $\chi_{f''}$:

$$\begin{tikzcd} \mathbb{R}^{n+r} \arrow[r, "\gamma \times \mathrm{id}"] & \mathbb{R}^{n+r} \\ M_{f''} \arrow[u, phantom, "\cup" rotate=90] \arrow[r, dashed] \arrow[d, "\chi_{f''}"] & M_{g'} \arrow[u, phantom, "\cup" rotate=90] \arrow[d, "\chi_{g'}"'] \\ \mathbb{R}^r \arrow[r, equal] & \mathbb{R}^r \end{tikzcd}$$

Now (r,f'') is versal and so by b) $\chi_{f'} \sim \chi_{f''}$ - hence $\chi_{f'} \sim \chi_{f''} \sim \chi_{g'}$.
(Proof that (r,f'') is versal : let (s,f''') be an unfolding of f. Then (s,g''') is an unfolding of g where $g''' := f''' \circ (\gamma^{-1} \times \mathrm{Id}_{\mathbb{R}^s})$. Hence there is a morphism $(\phi, \bar{\phi}, \varepsilon) : (s,g''') \longrightarrow (r,g')$. Then

$$((\gamma \times \mathrm{Id}_{\mathbb{R}^r}) \circ \phi \circ (\gamma^{-1} \times \mathrm{Id}_{\mathbb{R}^r}), \bar{\phi}, \varepsilon)$$

is a morphism from (s,f''') into (r,f'')).

8.7. <u>Theorem</u>: If $f \in \mathcal{M}_n^2$ and $\chi_{f'}$ is the catastrophe germ of a universal unfolding of f then, up to equivalence in the sense of 8.5 and the addition of independent variable as in the proof of 8.6 c) $\chi_{f'}$ depends only on the right equivalence class of f. In addition, $\chi_{f'}$ is independent of the sign of f and of the addition of a non degenerate quadratic form in new variables. Hence we can write χ_f (instead of $\chi_{f'}$).

Proof: 8.6 and 8.3. For the statement about the sign of f note that $M_{-f'} = M_{f'}$ and so $\chi_{f'} = \chi_{-f'}$.

8.8. Theorem: If codim $f \leqslant 6$, then there are precisely 14 distinct catastrophe germs (which arise from the universal unfoldings listed in §7) up to equivalence and the addition of new variables.
These are called the elementary catastrophes.

§9. GLOBALISATION:

9.1. The Whitney topology: Denote by $C^\infty(\mathbb{R}^m, \mathbb{R})$ (resp. $C^k(\mathbb{R}^m, \mathbb{R})$) the space of smooth functions from $\mathbb{R}^m$ into $\mathbb{R}$ (resp. the space of k-times continuously differentiable functions). If $f \in C^k$, the function $j^k f : \mathbb{R}^m \longrightarrow J^k_m$ is defined by $j^k f(x) = j^k[f(\cdot - x)]$ where J^k_m is the space of polynomials of degree $\leqslant k$ in m variables. We provide the latter with a norm $\| \ \|$ (e.g. the ℓ^∞-norm on the coefficients). For every strictly positive continuous function $\varepsilon : \mathbb{R}^m \to \mathbb{R}$ and every $r \leqslant k$ we put

$$V^r_\varepsilon := \{ f \in C^k(\mathbb{R}^m, \mathbb{R}) : \| j^r f(x) \| < \varepsilon(x) \text{ for } x \in \mathbb{R}^m \}.$$

If ε runs through the family of all such functions and r through the family of integers less than k then we obtain a zero neighbourhood basis of a group topology on $C^k(\mathbb{R}^n, \mathbb{R})$ - the Whitney C^k-topology. Similarly, if we let r run through the positive integers we obtain a topology on $C^\infty(\mathbb{R}^m, \mathbb{R})$: the Whitney C^∞-topology. C^∞ and C^k are topological rings but not topological vector spaces since scalar multiplication is not continuous in the scalar variable (see the next Lemma).

9.2. Lemma: Let (f_n) be a sequence in C^∞, $f \in C^\infty$. Then f_n converges to f in the Whitney topology if and only if there is a compact subset K of $\mathbb{R}^m$ so that $f_n = f$ outside of K with the exception of at most finitely many n's and $f_n \to f$ uniformly on K together with all its derivatives. A similar statements holds for the C^k-topology.

Proof: Since the topology is, by definition, translation-invariant, we can assume that $f=0$.
Sufficiency: suppose that such a K exists and let V_ε^k be a neighbourhood of zero. Then $\varepsilon_0 := \inf_{x\in K} \varepsilon(x) > 0$. We can choose $N \in \mathbb{N}$ so that

$$\sup_{x\in K} \| j^k f_n(x) \| < \varepsilon_0 \qquad \text{and} \qquad f_n|_{\mathbb{R}^m \setminus K} = 0$$

for $n \geq N$. Then $f_n \in V_\varepsilon^k$ for $n \geq N$. Hence $f_n \to 0$.
Necessity: suppose that $f_n \to 0$. Then f_n and its derivatives converge uniformly to zero (choose a constant function ε). Hence it suffices to show the existence of a K with the required property with respect to the supports of the $\{f_n\}$. If no such K exists, then we could find a sequence (x_r) in $\mathbb{R}^m$ with $\|x_r\| \uparrow \infty$ and a subsequence (f_{n_r}) so that $|f_{n_r}(x_r)| > 0$. Choose $\varepsilon : \mathbb{R}^m \to \mathbb{R}_+$ continuous with $\varepsilon(x_r) < |f_{n_r}(x_r)|$. Then the subsequence lies outside of the neighbourhood V_ε^0 of zero - contradiction.

9.3. Proposition: $C^\infty(\mathbb{R}^m, \mathbb{R})$ is a Baire space in the Whitney C^∞-topology.

Proof: We must show that a countable intersection of open, dense subsets is dense. Let $U_1, U_2, \cdots$ be a sequence of open, dense subsets and let V be an open neighbourhood of $f \in C^\infty(\mathbb{R}^m, \mathbb{R})$. We have to show that $V \cap \bigcap_{j=1}^\infty U_j \neq \phi$. By translating, we can assume that $f=0$. Given $k \geq 0$ and continuous $\varepsilon : \mathbb{R}^m \to \mathbb{R}_+$ (strictly positive!) let

$$\bar{V}_\varepsilon^k := \{ g \in C^\infty(\mathbb{R}^m, \mathbb{R}) : \| j^k g(x) \| \leq \varepsilon(x) \}.$$

There exist k_0, ε_0 such that $V_{\varepsilon_0}^{k_0} \subseteq \bar{V}_{\varepsilon_0}^{k_0} \subseteq V$. We claim that there exist sequences (f_j) in $C^\infty(\mathbb{R}^m, \mathbb{R})$, (k_j) (of positive integers) and (ε_j) so that for each i the following hold:

(Ai) : $f_i \in V_{\varepsilon_0}^{k_0} \cap \bigcap_{j=1}^{i-1} (f_j + V_{\varepsilon_j}^{k_j}) \cap U_i$;

(Bi) : $f_i + \bar{V}_{\varepsilon_i}^{k_i} \subseteq U_i$;

(Ci) : $\| j^i f_i(x) - j^i f_{i-1}(x) \| \leq 2^{-i}$ $\qquad (i > 1)$.

We proceed by induction: since U_1 is dense we may find $f_1 \in V_{\varepsilon_0}^{k_0} \cap U_1$ Since U_1 is open, there exist k_1, ε_1 with $f_1 + \bar{V}_{\varepsilon_1}^{k_1} \subseteq U_1$. Hence A1 and B1 hold.

Having constructed the data for $j=1,\cdots,i-1$ we check first that $f_{i-1} \in V_{\varepsilon_0}^{k_0} \cap \bigcap_{j=1}^{i-1}(f_j + V_{\varepsilon_j}^{k_j}) \cap (f_{i-1} + V_{2^{-i}}^{i})$ and so this set is open and non empty. Hence there is an

$$f_i \in V_{\varepsilon_0}^{k_0} \cap \bigcap_{j=1}^{i-1}(f_j + V_{\varepsilon_j}^{k_j}) \cap (f_{i-1} + V_{2^{-i}}^{i}) \cap U_i$$

since U_i is dense. Clearly f_i satisfies Ai and Ci. Since U_i is open we can find k_i and ε_i with $f_i + \bar{V}_{\varepsilon_i}^{k_i} \subseteq U_i$. So Bi holds.

The sequence (f_j) converges uniformly in all derivatives on $\mathbb{R}^m$ and so $g = \lim_j f_j$ exists (but it need not converge in the Whitney topology) and, of course, $j^k g = \lim_j j^k f$ uniformly - hence $g \in C^\infty(\mathbb{R}^m, \mathbb{R})$.

Each $f_j \in V_{\varepsilon_0}^{k_0}$ by (A) and so $g \in \bar{V}_{\varepsilon_0}^{k_0}$ (by pointwise convergence!). Then $f_j \in (f_i + V_{\varepsilon_i}^{k_i})$ for all $j \geq i$ and so $g \in f_i + \bar{V}_{\varepsilon_i}^{k_i} \subseteq U_i$ by (B). Hence

$$g \in \bar{V}_{\varepsilon_0}^{k_0} \cap \bigcap_{j=1}^{\infty} U_j \subseteq V \cap \bigcap_{j=1}^{\infty} U_j . \qquad \text{q.e.d.}$$

9.4. Let X be a smooth manifold, $Y \subseteq X$ an immersive submanifold, i.e. Y is a manifold and the injection $i : Y \longrightarrow X$ is an immersion (but not necessarily a homeomorphism into). Then Y has the following property: each point $y \in Y$ has an open neighbourhood U_y (in Y) which is a proper submanifold of X (take $i(V_y)$ where V_y is a compact neighbourhood of y in Y so that $i|_{V_y}$ is a homeomorphism into. Then $U_y = i(V_y^\circ)$ has the stated property).

If, conversely, $Y \subseteq X$ is a subset and, for each $y \in Y$, there is a subset $U_y \subseteq Y$ $(y \in U_y)$ such that U_y is a proper submanifold of X of constant dimension for each y, then Y is an immersed submanifold of X. For we can provide Y with a manifold structure by pasting together the U_y's via the transition mappings $\mathrm{Id} : U_y \cap U_{y'} \longrightarrow U_{y'} \cap U_y$.

Then it is clear that the injection $i : Y \longrightarrow X$ is an immersion and the topology of Y is, in general, finer than that induced by X since the U_y's are now open in Y. The immersive submanifold is clearly uniquely determined by this property.

9.5. Lemma: Let X, Y be smooth manifolds with W a proper submanifold of Y. Let $f : X \longrightarrow Y$ be a smooth map and $x \in X$ be such that $f(x) \in W$ and $f \pitchfork W$ at x. Then there exists a neighbourhood N of x in X such $f \pitchfork W$ on N.

Proof: There is an open neighbourhood V of $f(x)$ in Y and a submersion $\pi : V \longrightarrow \mathbb{R}^{\operatorname{Codim} W}$ such that $V \cap W = \pi^{-1}(0)$. The fact that $f \pitchfork W$ at x is easily seen to be equivalent to $\pi \circ f$ being a submersion at x, i.e. in some local chart about x in X, the Jacobi matrix of $\pi \circ f$ has maximal rank at x. But then this is true on a neighbourhood of x so that $f \pitchfork W$ on this neighbourhood.

9.6. Lemma: Let X, Y be smooth manifolds with W a proper submanifold of Y. Let $f : X \rightarrow Y$ be a smooth map and assume that $f \pitchfork W$ on X. Then $f^{-1}(W)$ is a proper submanifold of X.

Proof: It suffices to show that for every point $x \in f^{-1}(W)$, there is an open neighbourhood U of x in X so that $U \cap f^{-1}(W)$ is a submanifold. Let V and π be as in the proof of 9.5. Then $\pi^{-1}(0) = V \cap W$ and $\pi \circ f$ is a submersion on $f^{-1}(V \cap W)$ (cf. 9.5). Now $(\pi \circ f)^{-1}(0) = f^{-1}(V) \cap f^{-1}(W)$ and so $f^{-1}(V) \cap f^{-1}(W)$ is a submanifold of $f^{-1}(V)$. Take $U = f^{-1}(V)$.

9.7. Lemma: Let X, B, Y be smooth manifolds, with W a proper submanifold of Y. Let $j : B \longrightarrow C^{\infty}(X, Y)$ be a mapping such that the mapping $\phi : X \times B \longrightarrow Y$, given by

$$\phi(x, b) = j(b)(x)$$

is smooth and $\phi \pitchfork W$. Then the set

$$\{b \in B : j(b) \pitchfork W\}$$

is dense in B.

Proof: Let $W_\phi := \phi^{-1}(W) \subset X \times B$. By 9.6 W_ϕ is a proper submanifod of $X \times B$. Let $\pi : W_\phi \longrightarrow B$ be the restriction to W_ϕ of the projection $X \times B \longrightarrow B$. We claim that if b is a regular value for π (i.e. either $b \notin \pi(W_\phi)$ of $b \in \pi(W_\phi)$ and π is a submersion on $\pi^{-1}(b)$), then $j(b) \pitchfork W$. Then by the theorem of Sard (cf. [4], §II.1 or [2]) the set of regular values is dense in B and so we are done. Thus let b be regular for π. If $\dim W_\phi < \dim B$, then $j(b)(X) \cap W = \phi$. If $x \in X$ with $(x,b) \in W_\phi$, then $\pi(x,b) = b$ but π cannot be submersive at (x,b). Hence b is not regular. In this case $j(b) \pitchfork W$.

Suppose that $\dim W_\phi \geqslant \dim B$. Take $x \in X$. If $(x,b) \notin W_\phi$, then $j(b)(x) \notin W$ and so $j(b) \pitchfork W$ at x. Thus we can assume that $(x,b) \in W_\phi$. Since $\pi(x,b) = b$ and π is submersive at (x,b) we have that

$$(T\pi)_{(x,b)} T_{(x,b)} W_\phi = T_b B$$

so
$$T_{(x,b)}(X \times B) = T_{(x,b)} W_\phi + T_{(x,b)}(X \times \{b\}).$$

Now apply $(T\phi)_{(x,b)}$:

$$(T\phi)_{(x,b)} T_{(x,b)}(X \times B) = (T\phi)_{(x,b)} T_{(x,b)} W_\phi + (Tj(b))_x T_x X$$
$$\subseteq T_{j(b)(x)} W + (Tj(b))_x T_x X.$$

By hypothesis we have $\phi \pitchfork W$ and so

$$T_{\phi(x,b)} Y = T_{\phi(x,b)} W + (T\phi)_{(x,b)} T_{(x,b)}(X \times B)$$

and so $T_{j(b)(x)} Y = T_{j(b)(x)} W + (Tj(b))_x T_x X$,

i.e. $j(b) \pitchfork W$ at x.

9.8 Put $J = j^8(\mathcal{M}_n) \subseteq J^8_n$ and denote by $P = \{Q_i\}$ the partition of J in the finite collection of immersive submanifolds as explained in 7.5. Remember that there are two groups of immersive submanifolds, divided up differently according to the cases n = 2 and n $\geqslant$ 3. One group consists of orbits alone all of which have codimension $\leqslant$ n+5 or n+6 respectively. The other is a mixed collection, but all members have codimension $\geqslant$ n+6 = 8 or n+7 respectively.
The case n = 1 is special. Here the theorem of §0 is true for any r and the following proofs may easily be adapted. We will not comment on this case any more.

9.9. If $f \in C^\infty(\mathbb{R}^{n+r}, \mathbb{R})$ we denote by $j_1^s f$ the mapping from $\mathbb{R}^{n+r}$ into $J = j^s(\mathcal{M}_n)$ defined by

$$j_1^s f(x,y) = j^s([x' \longrightarrow f(x+x',y) - f(x,y)]_0)$$

= the Taylor expansion of $f(\cdot,y)$ in $\mathbb{R}^n$ of order s without constant term.

Note that $[j_1^s f]_{(0,0)} = j_1^s [f]_{(0,0)}$ in the sense of 6.6.

9.10. If $X \subseteq \mathbb{R}^{n+r}$ and W is an immersive submanifold of J, $W' \subseteq J$, then put

$$\mathcal{F}^X_{W,W'} := \{f \in C^\infty(\mathbb{R}^{n+r}, \mathbb{R}) : j_1^s f \pitchfork W \text{ at each } x \in X \cap (j_1^s f)^{-1}(W')\}.$$

We denote $\mathcal{F}^X_{W,J}$ by $\mathcal{F}^X_W$ and $\mathcal{F}^{\mathbb{R}^{n+r}}_W$ by $\mathcal{F}_W$.
If $\varepsilon > 0$ is a constant, let

$$V^\ell_{\varepsilon,X} := \{g \in C^\infty(\mathbb{R}^{n+r}, \mathbb{R}) : \|j_1^\ell g(y)\| < \varepsilon \text{ for } y \in X\}.$$

Then this is a neighbourhood of zero since it contains V^ℓ_ε.

9.11. <u>Lemma</u>: Let W be an immersive submanifold of J, $X \subseteq \mathbb{R}^{n+r}$ be compact and $W' \subseteq W$ be a compact subset (so that W' is compact in J also). Then $\mathcal{F}^X_{W,W'}$ is open in $C^\infty(\mathbb{R}^{n+r}, \mathbb{R})$.
<u>Proof</u>: Choose f in $\mathcal{F}^X_{W,W'}$ and x in X. Then either $j_1^s f(x) \notin W'$ or $j_1^s f(x) \in W'$ and the matrix

$$[T(j_1^s f), B]$$

has rank $\dim J = \binom{n+s}{s} - 1$, where B is a matrix whose columns form a basis of $T_{j_1^s f(x)} W$. Since W' is closed, the above statement continues to hold for all $x' \in U_x \cap X$ where U_x is a neighbourhood of x and all $f' \in C^\infty(\mathbb{R}^{n+r}, \mathbb{R})$ so that $j_1^s f'(x')$ is near to $j_1^s f(x')$ for each $x' \in U_x \cap X$, i.e. for each $f' \in f + V^s_{\varepsilon_x, X}$ for some constant $\varepsilon_x > 0$ (cf. 9.5). Cover X by finitely many U_{x_j}. If ε is the minimum of the ε_{x_j} then $f + V^s_{\varepsilon,X} \subseteq \mathcal{F}^X_{W,W'}$.

9.12. <u>Proposition</u>: Let W be an immersive submanifold of J. Then $\mathcal{F}_W = \{f \in C^\infty(\mathbb{R}^{n+r}, \mathbb{R}) : j_1^s f \pitchfork W\}$ is a residual subset of $C^\infty(\mathbb{R}^{n+r}, \mathbb{R})$.

Proof: We have to show that $\mathcal{F}_W$ can be represented as a countable intersection of open dense subsets. Choose a cover of W by open (in W), relatively compact subsets $\{W_\iota\}$ as in 9.4. We can find a countable cover. Then each $\overline{W}_\iota \subseteq W$ is compact. Next choose a countable cover $\{X_j\}$ of $\mathbb{R}^{n+r}$ by compact sets. Then $\mathcal{F}_W = \bigcap_{\iota, j} \mathcal{F}^{X_j}_{W,\overline{W}_\iota}$ and, by 9.11, each $\mathcal{F}^{X_j}_{W,\overline{W}_\iota}$ is open. It remains to show that each $\mathcal{F}^{X_j}_{W,\overline{W}_\iota}$ is dense. To simplify the notation, we write X for X_j and W' for W_ι. We shall show that we can approximate an arbitrary $f \in C^\infty$ by functions in $\mathcal{F}^X_{W,W'}$. Take $J = j^s \mathcal{M}_n$ and consider it as a space of polynomial functions on $\mathbb{R}^{n+r}$, independent of the second variable and vanishing at 0. Let α be a C^∞-function on $\mathbb{R}^{n+r}$ with compact support and $\alpha = 1$ on a neighbourhood U of X in $\mathbb{R}^{n+r}$. For $b \in J$, let $f + \alpha b$ be the function

$$(f+\alpha b)(x,y) = f(x,y) + \alpha(x,y)b(x)\ ,\ (x,y) \in \mathbb{R}^{n+r}.$$

If $b_n \to 0$ in J (i.e. the coefficients of b_n converge to zero), then $f + \alpha b_n \to f$ in $C^\infty(\mathbb{R}^{n+r}, \mathbb{R})$ by 9.2. Now let $\phi : U \times J \longrightarrow J$ be the mapping

$$(x,y,b) \longmapsto j^s_1(f+\alpha b)(x,y) = j^s_1(f+b)(x,y)$$

$$= j^s_1 f(x,y) + j^s_1 b(x)$$

and let $j : J \longrightarrow C^\infty(U,J)$ be the mapping

$$b \longmapsto j^s_1(f+b)\ (: U \to J).$$

If W'' is a relatively compact open neighbourhood of W' in W, then W'' is a proper submanifold of J (cf. 9.4). We claim that $\phi \pitchfork W''$. For

$$j^s_1(f+b)(x,y) = j^s_1 f(x,y) + j^s_1 b(x)$$

and, for fixed x, $b \longmapsto j^s_1 b(x)$ is just the mapping of b onto its Taylor expansion at x without constant term. This mapping has an inverse (namely, the mapping $c \longmapsto j^s_1 c(-x)$, for $c \in J$) and so for any $(x,y) \in \mathbb{R}^{n+r}$ the mapping

$$b \longmapsto \phi(x,y,b) = j^s_1 f(x,y) + j^s_1 b(x)$$

is a diffeomorphism from J onto itself. Thus

$$\phi : U \times J \longrightarrow J$$

is a submersion and so clearly transversal to W'' in J.

The conditions of 9.7 are then fulfilled and we conclude that the set $\{b \in J : j(b) = j_1^8(f+b) \pitchfork W'' \text{ on } U\}$ is dense in B. Then we can find a sequence (b_n) in B converging to 0. Hence $f+\alpha b_n \to f$ in $C^\infty(\mathbb{R}^{n+r}, \mathbb{R})$ and $j_1^8(f+\alpha b_n) \pitchfork W''$ on U, i.e. $f+\alpha b_n \in \mathcal{F}^U_{W''} \subseteq \mathcal{F}^X_{W,\bar{W}'}$.

9.13. We would like to show that $\mathcal{F} = \bigcap_{Q\in P} \mathcal{F}_Q$, which is a residual subset of $C^\infty(\mathbb{R}^{n+r}, \mathbb{R})$ by 9.12, is, in fact, open. $\mathcal{F}_Q$ is not open in general (since Q is an immersive submanifold). Thus we have to choose another approach:

Let $\mathcal{F}_1 := \{ f \in C^\infty(\mathbb{R}^{n+r},\mathbb{R}) : j_1^8 f(\mathbb{R}^{n+r}) \cap \Sigma_7^8 = \emptyset\}$ in case n = 2,
$:= \{ f \in C^\infty(\mathbb{R}^{n+r},\mathbb{R}) : j_1^8 f(\mathbb{R}^{n+r}) \cap \Sigma_6^8 = \emptyset\}$ in case $n \geq 3$.
Since each $Q \in P$ with $Q \subseteq \Sigma^8_{6,7}$ (6,7 according to the two cases n = 2 or $n \geq 3$; we will stick to this notation from now on) has codimension $\geq n + 5,6$ in J, a mapping from $\mathbb{R}^{n+r}$ into J (for $r \leq 5,6$) which meets $\Sigma^8_{6,7}$ can never be transversal to all of the Q's in P. Thus $\mathcal{F} \subseteq \mathcal{F}_1$. This argument can easily be adapted to the case n = 1, r arbitrary.

9.14. <u>Lemma</u>: $\mathcal{F}_1$ is open in $C^\infty(\mathbb{R}^{n+r}, \mathbb{R})$.

<u>Proof</u>: Take f in $\mathcal{F}_1$. For $x \in \mathbb{R}^{n+r}$, $j_1^8 f(x) \notin \Sigma^8_{6,7}$ and since $\Sigma^8_{6,7}$ is closed in J (4.4) we have

$$\varepsilon(x) := \inf\{\|j_1^8 f(x) - b\| : b \in \Sigma^8_{6,7}\} > 0.$$

The mapping $\varepsilon : \mathbb{R}^{n+r} \to \mathbb{R}$ thus defined is strictly positive and continuous (since it has the form

$$\varepsilon(x) = d(j_1^8 f(x), \Sigma^8_{6,7})).$$

But then $f + V^8_{\varepsilon/2}$ is clearly a neighbourhood of f which is contained in $\mathcal{F}_1$.

9.15. Now put $J_1 := J \setminus \Sigma_7^8$ which is open in J and $P_1 := \{Q \in P : Q \subseteq J_1\}$. Then P_1 consists of the 21n-9 orbits listed in 9.8a). The following is the key Lemma for the openness property and uses heavily a special property of the decomposition P_1 which is normally subsumed under the name of stratification. Since we ~~can use~~ it directly, we will not dwell on the definitions.

Lemma: Let $f \in \mathcal{F}_1$, $x \in \mathbb{R}^{n+r}$, $j_1^8 f(x) \in Q_i$ for some $Q_i \in P_1$. Then if $j_1^8 \pitchfork Q_i$ at x, there is a neighbourhood U_x of x in $\mathbb{R}^{n+r}$, a neighbourhood V of f in $\mathcal{F}_1$ such that $j_1^8 f' \pitchfork Q_j$ on U_x for all $Q_j \in P_1$ and each $f' \in V$.

Proof: $j_1^8 f \pitchfork Q_i$ at x implies that

$$T_{j_1^8 f(x)}(J_1) = T_{j_1^8 f(x)}(J) = \mathrm{Im}\,(T(j_1^8 f)_x) + T_{j_1^8 f(x)} Q_i \,.$$

Now let $\rho : J_1 \times L_n^8 \longrightarrow J_1$ be the mapping $\rho(z, \phi) = z.\phi$, i.e. the action induced on J_1 by L_n^8. Now if $z \in J_1$, then $T_z(z.L_n^8) = \mathrm{Im}\,(T(\rho(z,\cdot))_e)$. Let $d(j_1^8 f)(x)$ and $d(\rho(z,.))(e)$ be matrix representations of $T(j_1^8 f)_x$ and $T(\rho(z,.))_e$ for some coordinates. Then the column vectors of $d(j_1^8 f)_x$ generate $\mathrm{Im}\,(T(j_1^8 f)_x)$ and those of $d(\rho(z,.))(e)$ generate $T_z(z.L_n^8)$ likewise. So, by assumption, the matrix

$$[d(j_1^8 f)(x) \mid d(\rho(j_1^8 f(x),.))(e)]$$

has maximal rank, namely $\dim J_1 = \dim J = \binom{n+8}{8} - 1$.

The mapping

$$(f', x', \phi) \longrightarrow [d(j_1^8 f')(x') \mid d(\rho(j_1^8 f'(x'),.)(\phi)]$$

is continuous on $\mathcal{F}_1 \times \mathbb{R}^{n+r} \times L_n^8$ since $d(j_1^8 f')(x')$ depends continuously on $j^9 f' : \mathbb{R}^{n+r} \longrightarrow J^9_{n+r}$. Thus the matrix

$$[d(j_1^8 f')(x') \mid d(\rho(j_1^8 f'(x'),.)(\phi))]$$

has maximal rank if x' is near x in $\mathbb{R}^{n+r}$, say $x' \in U_x$ a compact neighbourhood of x, and ϕ is near e in L_n^8, say $\phi \in W$, a neighbourhood of e in L_n^8, and if $d(j_1^8 f')(x')$ is near $d(j_1^8 f)(x')$ for all $x' \in U_x$, say $f' \in (f + V^9_{\varepsilon,U_x}) \cap \mathcal{F}_1$ where $\varepsilon > 0$ is a constant and the notation is from 9.10. But this means that $j_1^8 f' \pitchfork Q_j$ on U_x for all $Q_j \in P_1$ and all $f' \in (f + V^9_{\varepsilon,U_x}) \cap \mathcal{F}_1 =: V$.

9.16. Proposition: If $n = 1$ for all r, if $n = 2$ for $r \leq 6$, if $n \geq 3$ for $r \leq 5$ the set $\mathcal{F}$ is open in $C^\infty(\mathbb{R}^{n+r}, \mathbb{R})$.

Proof: Let $X \subseteq \mathbb{R}^{n+r}$ be compact. We show first that

$$\mathcal{F}_1^X := \{f \in \mathcal{F}_1 : j_1^8(f) \pitchfork Q_j \text{ on } X \text{ for all } Q_j \in P_1\}$$

is open. Choose f in $\mathcal{F}_1^X$. If $x \in X$ then $j_1^8 f(x) \in Q_j$ for some j so, by 9.15, there is a neighbourhood U_x of

x in $\mathbb{R}^{n+r}$ and a neighbourhood $(f + V^{9}_{\varepsilon_x, U_x}) \cap \mathcal{F}_1$ of f (see the proof of 9.15) so that $f' \pitchfork Q_i$ on U_x for all $Q_i \in P_1$ and all $f' \in (f + V^{9}_{\varepsilon_x, U_x}) \cap \mathcal{F}_1$. Cover X by finitely many $\{U_{x_k}\}$ of the U_x's and let $\varepsilon = \min \varepsilon_{x_k}$. Then

$$(f + V^{9}_{\varepsilon, X}) \cap \mathcal{F}_1 \subseteq \mathcal{F}_1^X .$$

Now let $X = \bigcup_{i=1}^{\infty} X_i$ be a disjoint union of compact subsets of $\mathbb{R}^{n+r}$ and suppose that the X_i's have pairwise disjoint neighbourhoods $\{Y_i\}$. We claim that $\mathcal{F}_1^X$ is again open. For let $\beta_i : \mathbb{R}^{n+r} \longrightarrow \mathbb{R}$ be smooth functions with $0 \leq \beta_i \leq 1$, $\beta_i = 1$ on X_i and $\beta_i = 0$ off Y_i for each i. Now if $f \in \mathcal{F}_1^X$, then $f \in \mathcal{F}_1^{X_i}$ for each i and so by the first part of the proof there is a constant $\varepsilon_i > 0$ so that $(f + V^{9}_{\varepsilon_i, X_i}) \cap \mathcal{F}_1 \subseteq \mathcal{F}_1^{X_i}$. Let

$$\mu := 1 - \Sigma \beta_i + \Sigma \varepsilon_i \beta_i .$$

Then μ is strictly positive and if $x \in X_i$ then $\mu(x) = \varepsilon_i$. Hence $(f + V^{9}_{\mu}) \cap \mathcal{F}_1 \subseteq \bigcap_i (f + V^{9}_{\varepsilon_i, X_i}) \cap \mathcal{F}_1 \subseteq \bigcap \mathcal{F}_1^{X_i} = \mathcal{F}_1^X$

We can now prove that $\mathcal{F}$ is open in $\mathcal{F}_1$. For we can choose subsets X, X' of $\mathbb{R}^{n+r}$ with the above property so that $X \cup X' = \mathbb{R}^{n+r}$ (e.g.

$$X := \bigcup_m \{x \in \mathbb{R}^{n+r} : 2m - \tfrac{1}{4} \leq \|x\| \leq 2m + \tfrac{5}{4}\}$$

$$X' := \bigcup_m \{x \in \mathbb{R}^{n+r} : 2m + \tfrac{3}{4} \leq \|x\| \leq 2m + \tfrac{9}{4}\} \quad).$$

Then $\mathcal{F} = \mathcal{F}_1^X \cap \mathcal{F}_1^{X'}$ and so is open.

9.17 <u>Theorem</u>: If $n=1$ and r is arbitrary, or $n=2$ and $r \leq 6$, or $n \geq 3$ and $r \leq 5$, then the open dense subset
$\mathcal{F} := \{ f \in C^\infty(\mathbb{R}^{n+r}, \mathbb{R}) : j_1^8 f \pitchfork Q_j \text{ for all } Q_j \text{ in } P \}$
has the following property: if $f \in \mathcal{F}$, then M_f is a manifold (notation from §8 or §0) and each singularity of $\mathcal{X}_f$ is equivalent to an elementary catastrophe.

<u>Proof</u>: Choose f in $\mathcal{F}$. Then $j_1^8 f \pitchfork j^8(\mathcal{M}_n^2)$ in $J = j^8(\mathcal{M}_n)$. For $j^8(\mathcal{M}_n^2)$ is a linear subspace of J and if $x \in \mathbb{R}^{n+r}$ with $j_1^8 f(x) \in j^8(\mathcal{M}_n^2)$, $j_1^8 f(x) \in Q$ for some $Q \in P$, $Q \subseteq j^8(\mathcal{M}_n^2)$. Then $j_1^8 f \pitchfork Q$ at x, that is

$$T_{j_1^8 f(x)} J = \operatorname{Im} T(j_1^8 f)_x + T_{j_1^1 f(x)} Q$$
$$\subsetneq \operatorname{Im} T(j_1^8 f)_x + T_{j_1^1 f(x)}(j^1 \mathcal{M}_n^2)$$

and so $j_1^8 f \pitchfork j^8(\mathcal{M}_n)$ at x.

By 9.6 $(j_1^8 f)^{-1}(j^8(\mathcal{M}_n^2))$ is a submanifold of $\mathbb{R}^{n+r}$ of codimension n since $j^8(\mathcal{M}_n^2)$ has codimension n.

Now $(j_1^8 f)^{-1}(j^8 \mathcal{M}_n^2) = \{(x,y) \in \mathbb{R}^{n+r} : j_1^8 f(x,y) \in j^8 \mathcal{M}_n^2\}$

$$= \{(x,y) \in \mathbb{R}^{n+r} : j^1([x' \longmapsto f(x'+x,y) - f(x,y)]_0) = 0\}$$
$$= \{(x,y) \in \mathbb{R}^{n+r} : \partial f/\partial x_1 (x,y) = \dots = \partial f/\partial x_n (x,y) = 0\}$$
$$= M_f .$$

Hence M_f is a submanifold of $\mathbb{R}^{n+r}$ of dimension r.
Let $\chi_f : M_f \longrightarrow \mathbb{R}^r$ be the restriction of the projection from $\mathbb{R}^{n+r}$ onto $\mathbb{R}^r$. Suppose that χ_f has a singularity at $(x,y) \in M_f$ and suppose w.l.o.g. that $(x,y) = (0,0)$.
Let $\hat{f} := f|_{\mathbb{R}^n \times \{0\}}$. Then $[\hat{f}]_0 \in \mathcal{M}_n^2$ since $(0,0) \in M_f$.
$j_1^8 [\hat{f}]_0 = [j_1^8 f]_{(0,0)} \pitchfork (j^8 \hat{f}).L_n^8$ in J since $f \in \mathcal{J}$ and so $[f]_0$ is an 8-transversal unfolding of $[\hat{f}]_0$ (cf. 6.6).
Thus by 6.7

$$\mathcal{M}_n = \Delta([\hat{f}]_0) + V_{[f](0,0)} + \mathcal{M}_n^9$$

and so $\dim(\mathcal{M}_n / \Delta([\hat{f}]_0) + \mathcal{M}_n^9) \leq \dim V_{[f](0,0)} \leq r < 6$.
Then $\tau(j^9[\hat{f}]_0) \leq 6$ (see the proof of 4.4) and so

$$\operatorname{codim} [\hat{f}]_0 = \tau(j^9[\hat{f}]_0) \leq 6 \quad (4.4)$$

which implies that

$$\det [\hat{f}]_0 \leq \operatorname{codim} [\hat{f}]_0 + 2 \leq 8 \quad (4.2),$$

i.e. $[\hat{f}]_0$ is 8-determined and $[f]_{(0,0)}$ is an 8-transversal unfolding of $[\hat{f}]_0$. Then $(r, [f]_{(0,0)})$ is a universal unfolding of $[\hat{f}]_0$ by 6.11. By 8.7, resp. 8.8, we conclude that $[\chi_f]_0 = \chi_{[f](0,0)}$ is equivalent to an elementary catastrophe.

9.18 <u>Theorem</u>: Under the assumptions of theorem 9.17 for any $f \in \mathcal{J}$ the mapping χ_f is locally stable on M_f.

(We say that χ_f is locally stable at $(x_o,y_o)\in M_f$ if the following holds: if N is a neighbourhood of (x_o,y_o) in $\mathbb{R}^{n+r}$ then there is a neighbourhood V of f in $\mathcal{F}$ such that for each $g\in V$ there exists an $(x_1,y_1)\in N\cap M_g$ with

$$[\chi_f]_{(x_o,y_o)} \sim [\chi_g]_{(x_1,y_1)}$$

in the sense of 8.5. Since M_f and M_g are manifolds, we can express the notion of equivalence of 8.5 in a simpler way:

there exist germs of diffeomorphisms

$$\phi : (M_f,(x_o,y_o)) \longrightarrow (M_g,(x_1,y_1))$$

$$\psi : (\mathbb{R}^r,y_o) \longrightarrow (\mathbb{R}^r,y_1).$$

so that $\psi\circ[\chi_f]_{(x_o,y_o)} = [\chi_g]_{(x_1,y_1)}\circ\phi$, *i.e.*

$$\begin{tikzcd} \cdot \arrow[r, "{[\chi_f]_{(x_o,y_o)}}"] \arrow[d, "\phi"'] & \cdot \arrow[d, "\psi"] \\ \cdot \arrow[r, "{[\chi_g]_{(x_1,y_1)}}"'] & \cdot \end{tikzcd}$$

<u>Proof</u>: Let $(x_0,y_0)\in M_f$. If $z_o := j_1^s f(x_o,y_o)\in J$ and N is a neighbourhood of (x_o,y_o) in $\mathbb{R}^{n+r}$, then

$$j_1^s f \pitchfork (z_o.L_n^s) \quad \text{at} \quad (x_o,y_o)$$

since $f\in\mathcal{F}$. Let $c = \operatorname{codim} z_o.L_n^s$ in J ($\leq n+6$). Then there exists a c-dimensional affine subspace C through (x_o,y_o) in $\mathbb{R}^{n+r}$ such that for $D = C\cap N$ we have $j_1^s f|_D \pitchfork z_o.L_n^s$ and thus $[j_1^s f|_D]_{(x_o,y_o)}$ is a germ of an embedding. There exists an open subset $D_1\subseteq D$, $(x_o,y_o)\in D_1$, such that $j_1^s f(D_1)\cap(z_o.L_n^s)$ is a one-point set. If g is near enough to f in $\mathcal{F}$, then $j_1^s g(D_1)\cap(z_o.L_n^s)$ is still a one-point set and $j_1^s g|_{D_1}\pitchfork(z_o.L_n^s)$ on D_1.

Let $V\subseteq\mathcal{F}$ be a neighbourhood of f such that each $g\in V$ has this property. Then for $g\in V$ we have $j_1^s g|_{D_1}\pitchfork(z_o.L_n^s)$ on D_1 and there exists $(x_1,y_1)\in D_1$ such that $j_1^s g(x_1,y_1) =: z_1 \in j_1^s g(D_1)\cap(z_o.L_n^s)$. Hence there exists $\phi\in L_n^s$ such that $z_1 = z_o.\phi$. Let

$$f_o(x,y) = f(x_o+x,y_o+y) - f(x_o,y_o),$$

$$g_1(x,y) = g(x_1+x,y_1+y) - g(x_1,y_1),$$

so that $f_r \cdot g_1 : (\mathbb{R}^{n+r}, 0) \longrightarrow (\mathbb{R}^r, 0)$ and

$$z_0 = j^8(f_0|_{\mathbb{R}^n \times \{0\}}), \quad z_1 = j^8(f_1|_{\mathbb{R}^n \times \{0\}}).$$

Now $j_1^8 f_0 = j_1^8 f.$(a translation) and so $j_1^8 f_0 \pitchfork (j^8(f_0|_{\mathbb{R}^n \times \{0\}}) \cdot L_n^8)$ and $(r, [f_0]_{(0,0)})$ is an 8-transversal unfolding of $[f_0|_{\mathbb{R}^n \times \{0\}}]_0$. Since $r \leqslant 6$ we may check as in the proof of 9.17 that $(r, [f_0]_{(0,0)})$ is a universal unfolding of $[f_0|_{\mathbb{R}^n \times \{0\}}]$ and that $[f_0|_{\mathbb{R}^n \times \{0\}}]_0$ is 8-determined, so that $[f_0|_{\mathbb{R}^n \times \{0\}}]$ is right-equivalent to its 8-jet z_0. Likewise g_1 and z_1. Now by construction $z_1 \overset{r}{\sim} z_0$ and so

$$[f_0|_{\mathbb{R}^n \times \{0\}}] \overset{r}{\sim} z_0 \overset{r}{\sim} z_1 \overset{r}{\sim} [g_1|_{\mathbb{R}^n \times \{0\}}]$$

and $(r, [f_0]_{(0,0)})$, $(r, [g_1]_{(0,0)})$ are universal unfoldings. Hence by 8.6(d) we may conclude that $\chi_{[f_0]} \sim \chi_{[g_1]}$, i.e. $[\chi_f]_{(x_0,y_0)} \sim \chi_{[f_0]_0} \sim \chi_{[g_1]_0} \sim [\chi_g]_{(x_1,y_1)}$, where the first equivalence is given by translation by $-(x_0, y_0)$ in $\mathbb{R}^{n+r}$ and by $-y_0$ in $\mathbb{R}^r$ and the last equivalence is given by translating by $-(x_1, y_1)$ in $\mathbb{R}^{n+r}$ and by $-y_1$ in $\mathbb{R}^r$.

§10 CATASTROPHES ON FOLIATED MANIFOLDS

10.1 Let M be a manifold, smooth without boundary. Our first aim is to treat the Whitney-C^∞-topology (sometimes also called fine topology) on $C^\infty(M)$, the algebra of smooth functions on M. Forthat we have to treat the jet bundles $J^k(M,\mathbb{R})$ over M.

Definition: A k-jet of functions on M is an equivalence class $[f,x]_k$ of pairs (f,x) where $f\in C^\infty(M)$ and $x\in M$. The equivalence relation is the following: $[f,x]_k = [g,y]_k$ if $x = y$ and f, g have the same Taylor expansion at 0 in some (hence any) chart of M centered at x.
A coordinate free version: $[f,x]_k = [g,y]_k$ if $x = y$ and $T^kf_x = T^kg_x$, where T^k is the k-times iterated tangent bundle functor.
We write $[f,x]_k = j^k_x f = j^kf(x)$ and call it the k-jet of f at x.
The set of all k-jets is called $J^k(M,\mathbb{R})$.

10.2. Now let $M = U$ be an open subset of $\mathbb{R}^m$. Then the k-jet at $x\in U$ of any function $f\in C^\infty(U)$ has a canonical representative, the Taylor-polynomial of f at x of order k :

$$(j^kf(x))(t) = f(x) + df(x)t + \frac{1}{2!}d^2f(x)(t,t) + \dots + \frac{1}{k!}d^kf(x)t^k.$$

So we have $J^k(U,\mathbb{R}) = U\times J^k_m$, where J^k_m is the space of k-jets at 0 of C^∞-functions on $\mathbb{R}^m$, treated in §1. Each $j^kf\colon U\to J^k(U,\mathbb{R})$ is a section of the trivial vector bundle $J^k(U,\mathbb{R})$ over U.
Now let $g\colon U\to U'$ be a diffeomorphism between open subsets of $\mathbb{R}^m$. Then for each $x\in U$ the k-jet $j^kg(x)$ is an invertible polynomial mapping from $(\mathbb{R}^m,x)$ to $(\mathbb{R}^m,g(x))$, and truncated composition with $j^kg(x)$ from the right hand side gives a linear isomorphism (even an algebra isomorphism, see §1) from $J^k_{g(x)}(U',\mathbb{R})$ to $J^k_x(U,\mathbb{R})$; where $J^k_y(U,\mathbb{R}) = J^k_m$ is the space of all jets with source y. In detail:

$$j^kf(g(x)) \longmapsto j^kf(g(x))\circ j^kg(x) = j^k(f\circ g)(x).$$

This gives a fibrewise linear (even fibrewise algebra-homomorphic) diffeomorphism $J^k(g,\mathbb{R})\colon J^k(U',\mathbb{R}) \longrightarrow J^k(U,\mathbb{R})$.

10.3. Now let M be again a manifold of dimension m. Let (U,u) be a chart, i.e. $u\colon U\longrightarrow u(U)\subseteq\mathbb{R}^m$ is a diffeomorphism from an open set U in M onto an open subset of $\mathbb{R}^m$.
For each k-jet $\sigma\in J^k(M,\mathbb{R})$ with source $x = \alpha(\sigma)\in U$, i.e. $\sigma = j^kf(x)$ for some f, and $\alpha\colon J^k(M,\mathbb{R})\to M$ is the source projection, we associate $j^k(f\circ u^{-1})(u(x))\in J^k(u(U),\mathbb{R})$ to σ.

This is a bijective mapping $J^k(u^{-1},\mathbb{R})$: $J^k_U(M,\mathbb{R}) = J^k(U,\mathbb{R}) \longrightarrow J^k(u(U),\mathbb{R}) = u(U)\times J^{'}_{m_k}$. All these mappings together for an atlas of M give an atlas of $J^k(M,\mathbb{R})$. By 10.2 the chart-change mappings are smooth and so $J^k(M,\mathbb{R})$ is a smooth manifold, $\alpha: J^k(M,\mathbb{R}) \to M$ is a smooth vector bundle projection (even an algebra bundle projection).

10.4 The Whitney-C^∞-topology on $C^\infty(M)$ is given by taking all sets of the form $U(k,V) = \{f \in C^\infty(M): j^k f(M) \subset V\}$, V an open set in $J^k(M,\mathbb{R})$, as a basis for the topology. It is easy to see that this is actually a basis of a topology. To prove that this topology is a Baire space we have to make the following constuction.

10.5. Let X, Y be arbitrary topological spaces. Let C(X,Y) be the space of all continuous functions $X \to Y$. The graph topology on C(X,Y) is given in the following way: Let $f \in C(X,Y)$, and $\Gamma_f = \{(x,f(x)),\ x \in X\} \subset X\times Y$ be the graph of f. Let W be an open neighbourhood of Γ_f in $X\times Y$. Then let $N(f,W) = \{g \in C(X,Y): \Gamma_g \subset W\}$, and take the filter N(f,W), W open in $X\times Y$ and containing Γ_f, as a base for the neighbourhoods of f in C(X,Y). If X is paracompact and Y is a metric space with metric d then the graph topology on C(X,Y) has a base consisting of all sets of the form $N(f,\varepsilon) = \{g \in C(X,Y): d(f(x),g(x)) < \varepsilon(x) \text{ for all } x \text{ in } X\}$, where $\varepsilon: X \to \mathbb{R}$ runs through all continuous strictly positive functions on X. Still another definition: A subset of C(X,Y) is called uniformly closed with respect to the metric d on Y, if it contains the limit of each uniformly convergent sequence in it. Any subset which is closed in the topology of pointwise convergence, is uniformly closed, as is a subset which is closed in the compact open topology.

Lemma: Let X be paracompact and let Y be a complete metric space. Then any uniformly closed subset Q of C(X,Y) is a Baire space in the graph topology.

Proof: Let (A_n) be sequence of subsets of Q which are open and dense in the graph topology. Let U be a non empty open set in Q.
We have to show that $U \cap \bigcap A_n \neq \emptyset$.
The set $A_0 \cap U$ is open and not empty, so there is some $f_0 \in A_0 \cap U$ and some $\varepsilon_0 \in C(X,(0,1))$ such that $Q \cap \bar{N}(f_0, 2\varepsilon_0) \subset A_0 \cap U$, where $\bar{N}(f_0,\varepsilon_0) = \{g \in C(X,Y): d(f_0(x),g(x)) \leq \varepsilon_0(x) \text{ for all } x \text{ in } X\}$.

By recursion we get sequences (f_n) in Q, (ε_n) in $C(X,(0,1))$ such that $\varepsilon_{n+1} \leq \varepsilon_n/2$ and $Q \cap \bar{N}(f_{n+1}, 2\varepsilon_{n+1}) \subseteq A_{n+1} \cap N(f_n, \varepsilon_n)$ for all n.
Then we have $d(f_{n+1}(x), f_n(x)) \leq 2^{-n}$, therefore (f_n) is uniformly convergent on X and $f := \lim f_n$ is in Q since Q is uniformly closed with respect to d. Also $d(f(x), f_n(x)) \leq \sum_{k>n} \varepsilon_n(x) \leq \varepsilon_n(x) \sum_{k\geq 0} 2^{-k} = 2\varepsilon_n(x)$.
So $f \in \bar{N}(f_n, 2\varepsilon_n) \cap Q \subseteq A_n \cap N(f_n, \varepsilon_n)$, $f \in A_n$ for each n.
Also $f \in \bar{N}(f_0, 2\varepsilon_0) \cap Q \subseteq A_0 \cap U$, thus $f \in U \cap \bigcap_n A_n$.

10.6 Theorem: $C^\infty(M)$ is a Baire space in the Whitney-C^∞-topology.

Proof: Let $J^\infty(M,\mathbb{R})$ be the projective limit of the sequence $\ldots \leftarrow J^k(M,\mathbb{R}) \leftarrow J^{k+1}(M,\mathbb{R}) \leftarrow J^{k+2}(M,\mathbb{R}) \leftarrow \ldots$, where the maps are the canonical truncations. Each $J^k(M,\mathbb{R})$ is a manifold, so it is complete metrizable, so $J^\infty(M,\mathbb{R})$ is a closed subset of $\prod_k J^k(M,\mathbb{R})$ and is thus also complete metrizable.
Let $j^\infty : C^\infty(M) \to C(M, J^\infty(M,\mathbb{R}))$ be the obvious mapping. Then j^∞ is injective (since truncation at order 0 of $j^\infty f$ gives back f), and the image is closed in the compact open topology (which induces on $C^\infty(M)$ the topology of uniform convergence on compact subsets, in each derivative separately, making it into a complete locally convex vector space).
So the image of j^∞ is uniformly closed und by the lemma it is a Baire space in the induced topology. Also the image is contained in the subspace of all continuous sections of the (topological) vector bundle $J^\infty(M,\mathbb{R}) \to M$, where the graph topology coincides with the topology given by the base $U(s,V) = \{s' : s'(X) \subseteq V\}$, V open in $J^\infty(M,\mathbb{R})$.
By well known properties of the topological projective limit, this last topology induces the Whitney-C^∞-topology on $C^\infty(M)$.

10.7. Let M be a manifold of dimension n+r. A foliation F of codimension r on M is given by a distinguished atlas on M, an atlas consisting of distinguished charts (U,u). These charts are maps $u: U \to \mathbb{R}^n \times \mathbb{R}^r$, $u(U) = Q_1 \times Q_2$, a product of two open cubes in $\mathbb{R}^n$, $\mathbb{R}^r$, respectively.
For any two distinguished charts (U,u) and (V,v) the chart-change map $u \circ v^{-1}: v(U\cap V) \to u(U \cap V)$ has the form $u \circ v^{-1}(x,y) = (f(x,y), g(y))$.
For any distinguished chart (U,u) the set $u^{-1}(Q_1 \times \{y\})$ is (a piece of) an n-dimensional submanifold in M. It is called a plaque. For any $x \in M$ let F(x) be the maximal n-dimensional connected immersive submanifold of M which conincides in ~~each distinguished chart with a plaque and~~ contains x. F(x) is called the leaf of the foliation F through x.

10.8. Let F be a foliation on M. To any x in M we may associate the tangent plane $T_xF(x)$ to the leaf through x, which we denote by T_xF for short. This gives a vector bundle TF over M, a subbundle of the tangent bundle TM. If X and Y are two vector fields on M taking values in TF (sections of $TF \to M$), then the Lie bracket $[X,Y]$ takes also values in TF.

<u>Theorem</u> (Frobenius): Let E be an n-plane bundle in TM over M. Then E is the tangent bundle of a foliation on M if and only if for any sections X,Y of E, the Lie bracket $[X,Y]$ in TM has also values in E.

This is a standard result of differential geometry.

10.9. Let M^{n+r} be a manifold with a foliation F of dimension n. We want to define the vector bundle (algebra bundle) $J^k_F(M,\mathbb{R})$ of k-jets along leafs of smooth functions on M.

For that let (U,u) be a distinguished chart on M, so $u(U) = Q_1 \times Q_2$ is a product of cubes in $\mathbb{R}^n$ and $\mathbb{R}^r$ respectively.
We define $J^k_F(u(U),\mathbb{R}) := Q_1 \times Q_2 \times J^k_n$ (here F indicates the trivial foliation $\mathbb{R}^{n+r} = \mathbb{R}^n \times \mathbb{R}^r$). For any $f \in C^\infty(u(U))$ we define $j^k_F f(x,y) := j^k(f(\,.\,,y))(x) \in J^k_n$.
Any chart-change mapping between distinguished charts (U,u) and (V,v) is of the form $u \circ v^{-1}(x,y) = (a(x,y),b(y))$, so it induces a fibrewise linear (and multiplicative) smooth mapping

$$J^k_F(u \circ v^{-1},\mathbb{R}) : J^k_F(u(U\cap V),\mathbb{R}) \longrightarrow J^k_F(v(U \cap V),\mathbb{R}) \quad \text{by}$$

$$J^k_F(u \circ v^{-1},\mathbb{R})(j^k_F f(u \circ v^{-1}(x,y))) := j^k_F f(a(x,y),b(y)) \circ j^k_F(u \circ v^{-1})(x,y)$$
$$= j^k_F(f \circ u \circ v^{-1})(x,y) = j^k(f \circ u \circ v^{-1}(\,.\,,y))(x)\ ,$$

and this depends only on $j^k_F f = j^k(f \mid F(a(x,y),b(y)) = j^k(f \mid Q_1 \times \{b(y)\})$.
So finally we have

$$J^k_F(u \circ v^{-1},\mathbb{R})(a(x,y),b(y),j^k_F f(a(x,y),b(y))) =$$
$$= (x,y,\ j^k_F f(a(x,y),\ b(y)) \circ j^k(a(\,.\,,y))(x).$$

Note that $j^k(a(\,.\,,y))(x)$ is a germ of a diffeomorphism $(Q_1,x) \to (Q_1',a(x,y))$.
So by glueing the sets $J^k_F(u(U),\mathbb{R})$ via the chart-change maps $J^k_F(u \circ v^{-1},\mathbb{R})$, we obtain the k-jet bundle $J^k_F(M,\mathbb{R})$ of k-jets along leafs of smooth functions on M.

10.10. The following considerations are parallel to 9.8 - 9.18. As in 9.8 we put $J = j^8(\mathcal{M}_n) \subset J^8_n$ and again let $P = \{V_i\}$ be the partition of J in the finite collection of immersive submanifolds as explained in 7.5. The reader is advised to look up 9.8 and 7.5 now.

The subbundle (in a fixed distinguished coordinate chart (U,u)) $Q_1 \times Q_2 \times J$ of $Q_1 \times Q_2 \times J^8_n$ is stable under all coordinate changes of distinguished charts, since these lie in the group L^8_n. Also all the members of the partition P are stable under these coordinate changes. So we see that the following facts hold:

1. There is a subbundle $J^8{}_F(M,\mathbb{R})_0$ of $J^8{}_F(M,\mathbb{R})$ of fibre codimension 1, consisting of all k-jets along leafs without constant terms. In any distinguished chart this bundle is mapped to $Q_1 \times Q_2 \times J$.
2. $J^8{}_F(M,\mathbb{R})_0$ is partitioned into a finite collection of immersive submanifolds W_i. Each W_i is a fibre bundle over M with structure group L^8_n and typical fibre V_i. Each W_i has either codimension $\leq n+5, n+6$ or codimension $\geq n+6$, $n+7$ (for $n \geq 3$ or $n = 2$ respectively; again we do not mention the simpler case $n = 1$ which is left to the reader). In a distinguished chart the immersive subbundle W_i is mapped to $Q_1 \times Q_2 \times V_i$, V_i the corresponding member of the partition P of J.

10.11. Let us fix some notation for the following:
M^{n+r} is a smooth manifold with a codimension r foliation F, where $r \leq 6$ for $n = 2$ and $r \leq 5$ for $n \geq 3$ and r arbitrary for $n = 1$.
We write $J(M) := J^8{}_F(M,\mathbb{R})_0$ for short.
If X is a subset of M, W is an immersive submanifold of J(M) and Y is a subset of J(M), we put
$\mathcal{F}^X_{W,Y} := \{f \in C^\infty(M): j^8{}_F f \pitchfork W$ at each $x \in X$ with $j^8{}_F f(x) \in Y\}$.
Furthermore put $\mathcal{F}_{W,Y} := \mathcal{F}^M_{W,Y}$, $\mathcal{F}^X_W := \mathcal{F}^X_{W,J(M)}$, $\mathcal{F}_W := \mathcal{F}^M_{W,J(M)}$.
We fix a metric d_k on each space $J^k{}_F(M,\mathbb{R})$ so that it becomes a complete metric space. For $f \in C^\infty(M)$, $\varepsilon \in C(M,(0,1))$, $X \subset M$ and $k \geq 0$ we put
$N^k_X(f,\varepsilon) := \{g \in C^\infty(M): d_k(j^8{}_F f(x), j^8{}_F g(x)) < \varepsilon(x)$ for each $x \in X\}$.
This is a neighbourhood of f in the Whitney-C^∞-topology since it contains the open neighbourhood $N^k_M(f,\varepsilon)$.

10.12. <u>Lemma:</u> Let W be an immersive submanifold of J(M), let X be a compact subset of M and let W' be compact in W. Then $\mathcal{F}^X_{W,W'}$ is open in $C^\infty(M)$.

Proof: Let $f \in \mathcal{F}^X_{W,W'}$. It suffices to show that for each $x \in X$ there is a compact neighbourhood U_x of x in M such that $\mathcal{F}^{U_x}_{W,W'}$ contains an open neighbourhood V_x of f in $C^\infty(M)$. For then we can cover the compact X by finitely many $U_{x_i} = U_i$ and then $\bigcap_i V_{x_i} =: V \subset \bigcap_i \mathcal{F}^{U_{x_i}}_{W,W'} \subset \mathcal{F}^X_{W,W'}$, and V is an open neighbourhood of f.
Now we have to control only near x, so via a distinguished chart we may assume that we are in $Q_1 \times Q_2 \subset \mathbb{R}^{n+r}$, and there we already proved lemma 9.11.

10.13. Proposition: Let W be an immersive submanifold of $J(M)$. Then $\mathcal{F}_W = \{f \in C^\infty(M) : j^8_F f \pitchfork W\}$ is a residual subset of $C^\infty(M)$.

Proof: We want to show that $\mathcal{F}_W$ can be represented as a countable intersection of open dense subsets. Choose a cover of W by open (in W), relatively compact (in W) subsets W_j as in 9.4. Then each W_j is an embedded submanifold of $J(M)$ and $\bar{W}_j$ is compact. Next choose a countable cover (X_k) of M by compact subsets such that each X_k is contained in a distinguished chart (U_k, u_k) of M.
Then $\mathcal{F}_W = \bigcap_{j,k} \mathcal{F}^{X_k}_{W,W_j}$ and by lemma 10.12 each $\mathcal{F}^{X_k}_{W,W_j}$ is open.
It remains to show that it is also dense. Since we need transversality only on $X_k \subset U_k$, we may assume that we are in $\mathbb{R}^{n+r}$ and finish the proof as in 9.12.

10.14. We want to show that the set $\mathcal{F} = \bigcap\{\mathcal{F}_{W_i} : W_i \in P$, the partition of $J(M)\}$, which is a residual subset by 10.13, is in fact open.
We repeat the procedure of §9 and put $\mathcal{F}_1 = \{f \in C^\infty(M) : j^8_F f(M) \cap W_i = \emptyset$ for those W_i in P which have codimension $\geq n+6,\ n+7\}$.
Σ, the union of those W_i with codim. $\geq n+6,\ n+7$, is closed in each distinguished trivialisation $Q_1 \times Q_2 \times J$ (it is $Q_1 \times Q_2 \times \Sigma^8_{6,7}$ there), and it is a locally trivial topological bundle over M, so Σ is closed in $J(M)$ and therefore $\mathcal{F}_1$ is open in $C^\infty(M)$, by the same argument as in lemma 9.14. Let $P_1 = \{W_i : \operatorname{codim} W_i \leq n+5,\ n+6\}$, a partition of the open set $J(M) \setminus \Sigma$.

10.15. Lemma: Let $f \in \mathcal{F}_1$, $x \in M$, $j^8_F f(x) \in W_i$ for some $W_i \in P_1$. If $j^8_F f \pitchfork W_i$ at x, then there is a neighbourhood U_x of x in M and a neighbourhood V of f in $\mathcal{F}_1$ such that $j^8_F g \pitchfork W_j$ on U_x for all $g \in V$ and for all $W_j \in P_1$.

Proof: The problem is local at $x \in M$, so we may choose a distinguished chart at $x \in M$ and use lemma 9.15 or better its proof.

10.16. Proposition: $\mathcal{F} \cap \mathcal{F}_1 = \bigcap \{\mathcal{F}_{W_i} : W_i \in P_1\} \cap \mathcal{F}_1$ is open in $C^\infty(M)$.

Proof: The same proof as for 9.16 applies here, since we did not use the spacial structure of $\mathbb{R}^{n+r}$-structure, For the last part of the proof, choose a complete metric on M.

10.17. Theorem: Let M^{n+r} be a smooth manifold with a foliation F of codimension r. If n = 1 let r be arbitrary. If n = 2 let $r \leq 6$. If $n \geq 3$ let $r \leq 5$.
Then the open dense subset $\mathcal{F} = \{f \in C^\infty(M) : j^8{}_F f \pitchfork W_i$ for all $W_i \in P\}$ $(j^{r+2}{}_F f \pitchfork \ldots$ if n = 1) has the following properties:

1. If $f \in \mathcal{F}$, then the set $M_f = \{x \in M : df(x)|T_x F = 0\}$ is a submanifold of M.
2. For $f \in \mathcal{F}$ and $z \in M_f$ there is a distinguished chart $(U, u: U \to Q_1 \times Q_2 \subseteq \mathbb{R}^n \times \mathbb{R}^r)$ centered at z and a mapping $v \in C^\infty(Q_2)$ such that $f \circ u^{-1}(x,y) - v(y)$ is a polynomial in the list of §7.

Proof: Using a distinguished neighbourhood we can assume that we are in $\mathbb{R}^{n+r}$ and we get from theorem 9.17:
M_f is locally a submanifold, so it is a submanifold.
The germ of f at z is a (uni)versal unfolding of the germ of f restricted to the leaf F(z) through z. This clearly implies 2 by §7.

10.18. Theorem: Under the assumptions of 10.17, for any $f \in \mathcal{F}$ the singularity type of f at z is locally stable: the polynomial in 10.17.2 stays the same if f is changed sufficiently few with respect to the Whitney-C^∞-topology , and the singularity point z stays near the original one.

This is seen by looking at the proof of 9.18.

10.19. Let $\pi : M \longrightarrow N$ be a submersion which we may assume surjective without loss of generality. Then the connected components of inverse images of points form the leafs of a foliation of M (with quotient structure N).

<u>Theorem:</u> Let $\pi : M^{n+r} \longrightarrow N^r$ be a submersion with induced foliation F on M. Suppose that r is arbitrary if $n = 1$, $r \leq 6$ if $n = 2$, $r \leq 5$ if $n \geq 3$.

Then there is an open dense set $\mathcal{F} \subseteq C^{\infty}(M)$ such that:

1. For $f \in \mathcal{F}$ the set $M_f = \{ x \in M: df(x) | T_x F = 0 \}$ is a submanifold of M.
2. For $f \in \mathcal{F}$ denote by $\chi_f : M_f \longrightarrow N$ the restriction of the projection π to $M_f \subseteq M$. Then each singularity of χ_f is equivalent to an elementary catastrophe.
3. The mapping χ_f is locally stable with respect to f: For any $x \in M_f$ there are open neighbourhoods U of x in M and V of f in $C^{\infty}(M)$ such that for any $g \in V$ there is some $y \in U$ with $y \in M_g$ and the property that the germ of χ_f at x is equivalent to the germ of χ_g at y.

This is a reformulation of 10.17 and 10.18 using 9.18 again.

REFERENCES

[1] V.I.Arnold, S.M.Gusein-Zade, A.N.Varchenko, Singularities of Differentiable Maps, Vol.I, Birkhäuser, Monographs in Mathematics 82, 1985.

[2] Th.Bröcker, Differentiable Germs and Catastrophes, London Mathematical Soc. Lecture Notes 17(1975).

[3] C.G.Gibson, K.Wirthmüller, A.A.du Plessis, E.J.N.Looijenga, Topological Stability of smooth mappings, Springer Lecture Notes 552, 1976.

[4] M.Golubitsky, V.Guillemin, Stable Mappings and their Singularities, Springer GTM 14(1974).

[5] H.I.Levine, Singularities of differentiable mappings, Liverpool Symp.on Singularities, Springer Lecture Notes 192(1971),1-89.

[6] Yung-Chen Lu, Singularity Theory/Catastrophe Theory, Springer Verlag 1976 (Unitext).

[7] B.Malgrange, Ideals of Differentiable Functions, Oxford Univ.Press (1966).

[8] J.Martinet, Singularités des fonctions et applications différentiables, Pontificia Universidad Católica, Rio de Janeiro (1974). London Math.Soc.Lecture Notes 58(1981).

[9] J.N.Mather, Stability of C^{∞}-mappings : the division theorem, Ann.Math.87(1968) 89-104.

[10] J.N.Mather, Stability of C^{∞}-mappings : finitely determined map germs, Publ.Math.IHES 35(1968),127-156.

[11] J.N.Mather, Right equivalence (Warwick Preprint, 1969).

[12] P.W.Michor, Manifolds of differentiable mappings. Shiva Mathematics Series 3, 1980.

[13] T.Poston, I.N.Stewart, Taylor expansions and catastrophes, Pitman Publishing, Research Notes in Mathematics 7, 1976.

[14] T.Poston, I.N.Stewart, Catastrophe Theory and Applications, Pitman Publishing,1978 (the Romanian translation has been published by Editura Tehnică,Bucureşti,1985).

[15] D.Siersma, The singularities of C^{∞}-functions of right codimension ≤ 8, Indagationes Math.35(1973),31-37.

[16] R.Thom, Les singularités des applications différentiables, Ann.Inst.Fourier (1956),17-86.

[17] R.Thom, Stabilité Structurelle et Morphogénèse, Benjamin, 1972.

[18] R.Thom, E.C.Zeeman, La théorie des catastrophes : etat present et perspectives, Reply and Answer, Warwick Dynamical Systems, Springer Lecture Notes, 1975.

[19] J.C.Tougeron, Ideaux de Fonctions Différentiables, Ergebnisse der Mathematik 71, Springer, 1972.

[20] C.T.C.Wall, Lectures on C^{∞}-stability and classification, Liverpool Symp.on Singularities, Springer Lecture Notes, 192(1971), 2o7-253.

[21] G.Wassermann, Stability of Unfoldings, Springer Lecture Notes, 393(1974).

[22] H.Whitney, Mappings of the plane into the plane, Ann. Math. 62(1955),374-47o.

[23] E.C.Zeeman, D.J.A.Trotman, The classification of the elementary catastrophes of codimension $\leqslant 5$ (Warwick Preprint,1974), published in Springer Lecture Notes, 525, and in : "Catastrophe Theory" selected papers by E.C.Zeeman, Addison Wesley, 1977.

[24] E.C.Zeeman, Catastrophe theory, selected papers 1972-1977, Addison-Wesley 1977.

Appendix. THE DIVISION THEOREM ON BANACH SPACES.[*]

The following is a detailed exposition of the proof of the division theorem for smooth functions, following Nirenberg [8][+] up to his extension lemma and Mather's proof [6] of the latter. This proof is developped in the context of Banach spaces - the necessary modifications are minor. The division theorem has been used in [5].

1. The division theorem. Let E,F be real Banach spaces and let $d : \mathbb{R}\times E \longrightarrow \mathbb{R}$ be a smooth function, defined near 0, such that $d(t,0) = \bar{d}(t)t^k$ for some $k \geqslant 0$, where $\bar{d}(0) \neq 0$, $\bar{d} : \mathbb{R} \longrightarrow \mathbb{R}$ is smooth, defined near 0.

Then given any smooth function near 0 $f : \mathbb{R}\times E \longrightarrow F$ there are smooth functions near 0 $q : \mathbb{R}\times E \longrightarrow F$, $r_i : E \longrightarrow F$, $i = 0,1,\ldots,k-1$, such that

$$f(t,x) = q(t,x)d(t,x) + \sum_{i=0}^{k-1} r_i(x)t^i.$$

2. Notation : Let $P_k : \mathbb{R}\times\mathbb{R}^k \longrightarrow \mathbb{R}$ be the Polynomial

$$P_k(t,\lambda) = t^k + \sum_{i=0}^{k-1}\lambda_i t^i, \quad \lambda = (\lambda_0,\ldots,\lambda_{k-1}).$$

3. The Polynomial Division Theorem : Let $f(t,x)$ be a smooth function $\mathbb{R}\times E \longrightarrow \mathbb{C}\otimes F$.

Then there are smooth functions, defined near 0 in $\mathbb{R}^k$, $q : \mathbb{R}\times E\times\mathbb{R}^k \longrightarrow \mathbb{C}\otimes F$, $r_i : E\times\mathbb{R}^k \longrightarrow \mathbb{C}\otimes F$, $i = 0,\ldots,k-1$, such that $f(t,x) = q(t,x,\lambda)P_k(t,\lambda) + \sum_{i=0}^{k-1} r_i(x,\lambda)t^i$. If f is realvalued in $\mathbb{C}\otimes F$ (i.e. takes its values in the real subspace $1\otimes F$), then q and r_i may be chosen realvalued too.

4. Remarks : a) $\mathbb{C}\otimes F = F \oplus iF$ is just the canonical complexification of the Banach space F, with some suitable norm.

[*] Reprinted from Sitzungsberichten der Österreichischen Akademie der Wissenschaften Mathem.-naturw. Klasse, Abteilung II, 189.Bd., 1.bis 3.Heft, 1980, with the permission of the Austrian Academy of Sciences

[+] References in parentheses refer to the bibliography at the end of this appendix.

b) The last assertion is trivial : just apply the projection $\mathbb{C}\otimes F \longrightarrow 1 \otimes F$ to q and r_i.

c) If f is in E defined near 0 only, then the polynomial division theorem remains valid for q and r_i defined near 0. Nothing in the proof to follow has to be changed. But the global version does not imply the local one in general, since there need not exist smooth partitions of unity on E (on C ([0,1]) e.g. there is no smooth function with bounded support, cf. Bonic and Frampton [1]).

d) Without loss of generality we may assume that $f(\cdot, x)$ has compact support in $\mathbb{R}$ for each $x \in E$ (or near 0). For suppose the theorem is valid in this case and f is arbitrary, let $g_j(t), h_j(t)$, $j \in \mathbb{N}$ be two locally finite families of smooth functions with compact support such that $(g_j(t)h_j(t))_j$ is a partition of unity. Then for each j we may write

$$h_j(t)f(t,x) = q_j(t,x,\lambda)P_k(t,\lambda) + \sum_{i=0}^{k-1} r_{ij}(x,\lambda)t^i,$$

but then clearly

$$f(t,x) = q(t,x,\lambda)P_k(t,\lambda) + \sum_{i=1}^{k-1} r_i(x,\lambda)t^i$$

for

$$q(t,x,\lambda) = \sum_j g_j(t)q_j(t,x,\lambda),\ r_i(x,\lambda) = \sum_j g_j(t)r_{ij}(x,\lambda).$$

5. <u>Proof</u> of the division theorem using the local form of the polynomial division theorem :

Given d as in 1. there are smooth functions defined near 0 $q : \mathbb{R}\times E\times\mathbb{R}^k \longrightarrow \mathbb{R}$, $r_i : E\times\mathbb{R}^k \longrightarrow \mathbb{R}$, $i = 0,\ldots,k-1$ so that

$$d(t,x) = q(t,x,\lambda)P_k(t,\lambda) + \sum_{i=0}^{k-1} r_i(x,\lambda)t^i. \qquad (6)$$

We claim that

$$q(0,0,0) \neq 0,\ r_i(0,0) = 0,\ \frac{\partial r_j}{\partial \lambda_j}(0,0) \neq 0 \quad \text{for all } j,$$

$$\frac{\partial r_j}{\partial \lambda_i}(0,0) = 0 \quad \text{for } j < i. \qquad (7)$$

By (6) we have

$$t^k\bar{d}(t)= d(t,0)= q(t,0,0)t^k + \sum_{i=0}^{k-1} r_i(0,0)t^i.$$

Looking at the Taylor expansions at 0 of both sides of this equation we see that $r_i(0,0) = 0$ for all i and $q(0,0,0)= = \bar{d}(0) \neq 0$. Now differentiate (6) at $x = 0$, $\lambda = 0$ with respect to λ_i, to obtain

$$0 = q(t,0,0)t^i + \frac{\partial q}{\partial \lambda_i}(t,0,0)t^k + \sum_{j=0}^{k-1} \frac{\partial r_j}{\partial \lambda_i}(0,0)t^j.$$

Again Taylor expansion at 0 tells us that for $j < i$ we have $\frac{\partial r_j}{\partial \lambda_i}(0,0)= 0$ and $\frac{\partial r_i}{\partial \lambda_i}(0,0)= -q(0,0,0) \neq 0$. So (7) follows. Let now $R = (r_0,\dots,r_{k-1}): E\times \mathbb{R}^k \to \mathbb{R}^k$, then $D_2R(0,0)= = (\frac{\partial r_i}{\partial \lambda_j}(0,0)): \mathbb{R}^k \to \mathbb{R}^k$ and this matrix is invertible by (7).

Now consider the mapping $(x,\lambda)\mapsto(x,R(x,\lambda))$ from $E\times\mathbb{R}^k$ into itself, defined near 0. Its derivative at 0 has the form

$$\begin{pmatrix} Id_E & 0 \\ D_1R(0,0) & D_2R(0,0) \end{pmatrix}$$

and so is invertible too. By the inverse function theorem on Banach spaces (cf. S. Lang [4], I, §5, this mapping is locally invertible at $(0,0)$, its inverse (again fibered over E) being of the form $(x,\lambda)\mapsto(x,s(x,\lambda))$. Then of course $R(x,s(x,\lambda)) = \lambda$.

Let now $\bar{P},\bar{q} : \mathbb{R}\times E \to \mathbb{R}$ be given by $\bar{P}(t,x)= P_k(t,s(x,0))$, $\bar{q}(t,x)= q(t,x,s(x,0))$. Using (6) again for $\lambda = s(x,0)$ we have

$$\begin{aligned} d(t,x) &= q(t,x,s(x,0))P_k(t,s(x,0))+\sum_{i=1}^{k-1} r_i(x,s(x,0))t^i \\ &= \bar{q}(t,x)\bar{P}(t,x). \end{aligned}$$

$1/\bar{q}(t,x)$ exists and is smooth near 0 since $\bar{q}(0,0) = = Q(0,0,0) \neq 0$, so $\bar{P}(t,x) = d(t,x)/\bar{q}(t,x)$ near 0.

Now if some f is given as in 1. then by 3. again there are functions $m : \mathbb{R}\times E\times\mathbb{R}^k \to F$, $n_i : E\times\mathbb{R}^k \to F$, $i=0,\dots,k-1$, defined nea 0, such that (for $\lambda = s(x,0)$)

$$f(t,x) = m(t,x,s(x,0))P_k(t,s(x,0)) + \sum_{i=0}^{k-1} n_i(x,s(x,0))t^i$$

$$= \frac{m(t,x,s(x,0))}{\bar{q}(t,x)} d(t,x) + \sum_{i=0}^{k-1} n_i(x,s(x,0))t^i$$

$$= \tilde{q}(t,x)d(t,x) + \sum_{i=0}^{k-1} \tilde{r}_i(x)t^i.$$

qed.

8. For the proof of 3. we will need two lemmas. Before proving the first one, some notation :

Let $f : \mathbf{C} \to \mathbf{C}$ be smooth as a real function. If $z = x + iy$, then $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}$,

where $\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$, $\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$.

$d(fdz) = df \wedge dz = 0 + \frac{\partial f}{\partial \bar{z}} d\bar{z} \wedge dz.$

9. <u>Lemma</u> : Let $f : \mathbf{C} \to \mathbf{C} \otimes F$ be smooth. Let γ be a simple closed curve in $\mathbf{C}$ whose interiour is U. Then for $w \in U$ we have

$$f(w) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(z)}{z-w} dz + \frac{1}{2\pi i} \iint_U \frac{\partial f(z)}{\partial \bar{z}} \frac{dz \wedge d\bar{z}}{z-w}.$$

(If f is holomorphic $\mathbf{C} \to \mathbf{C} \otimes F$, i.e. $\frac{\partial f}{\partial \bar{z}} = 0$, this reduces to the Banach space valued Cauchy formula. The integrals in this lemma are meant to be Bochner integrals : Riemannian sums will converge in the Banach space $\mathbf{C} \otimes F$. See Dunford-Schwartz I [3] for a discussion of vector valued integration.)

<u>Proof</u> : First we reduce the lemma to the one dimensional case : The first integral exists in $\mathbf{C} \otimes F$ since γ is compact and $f(z)/(z-w)$ is continuous on γ. The second one exists, since $\partial f/\partial \bar{z}$ is continuous on $\bar{U}$ and $\frac{dz \wedge d\bar{z}}{z-w}$ defines a finite Radon measure on $\bar{U}$.

Now we use duality. Take any continuous $\mathbf{C}$ - linear functional φ on $\mathbf{C} \otimes F$. That commutes with integration (with the limits of Riemannian sums by continuity and with those sums by linearity) and with $\frac{\partial}{\partial \bar{z}}$ by the chain rule, since it is its own derivative. So we may compute :

$$\varphi\left(\frac{1}{2\pi i}\int_{\gamma}\frac{f(z)}{z-w}\,dz + \frac{1}{2\pi i}\iint_U \frac{\partial}{\partial \bar{z}} f(z)\,\frac{dz\wedge d\bar{z}}{z-w}\right)$$
$$= \frac{1}{2\pi i}\int_{\gamma}\frac{\varphi(f(z))}{z-w}\,dz + \frac{1}{2\pi i}\iint_U \frac{\partial}{\partial \bar{z}}\varphi(f(z))\,\frac{dz\wedge d\bar{z}}{z-w}$$

$=\varphi(f(w))$ by the one dimensional formula. So by the theorem of Hahn Banach the formula holds in $\mathbb{C}\otimes F$.

Now we prove the one dimensional case. Let $w\in U$, choose $\varepsilon < \min\{|w-z| : z\in\gamma\}$. Let $U_\varepsilon = U\setminus$(disc of radius ε about w) and $\gamma_\varepsilon = \partial U_\varepsilon$.

We apply 8. and Stokes'theorem to the function $f(z)/(z-w)$, which is smooth on a neighbourhood of U_ε .

$$\iint_{U_\varepsilon}\frac{\partial f}{\partial \bar{z}}\,\frac{dz\wedge d\bar{z}}{z-w} = -\iint_{U_\varepsilon} d\left(\frac{f(z)}{z-w}\,dz\right) = -\int_{\partial U_\varepsilon}\frac{f(z)}{z-w}\,dz$$
$$= -\int_{\gamma}\frac{f(z)}{z-w}\,dz + \int_0^{2\pi}\frac{f(w+\varepsilon\exp(i\vartheta))}{\varepsilon\exp(i\vartheta)}\,i\,\varepsilon\exp(i\vartheta)\,d\vartheta.$$

As $\varepsilon\longrightarrow 0$, the last integral converges to $2\pi i\, f(w)$ by form continuity of f, and the integral on the left-hand-side converges to $\iint_U \frac{\partial f}{\partial \bar{z}}\,\frac{dz\wedge d\bar{z}}{z-w}$, since $\frac{\partial f}{\partial \bar{z}}$ is bounded, $\frac{dz\wedge d\bar{z}}{z-w}$ induces a finite Radon measure which, applied to the difference set $U\setminus U_\varepsilon$, converges to 0.

qed.

1o. <u>The Nirenberg Extension Lemma</u> : Let $f : \mathbb{R}\times E\longrightarrow \mathbb{C}\otimes F$ be a smooth function with support contained in $K\times E$ for some compact K in $\mathbb{R}$. Then there exists a smooth function $\tilde{f} : \mathbb{C}\times E\times \mathbb{C}^k\longrightarrow \mathbb{C}\otimes F$ such that

$$\tilde{f}(t,x,\lambda) = f(t,x) \quad \text{for } t\in\mathbb{R} \text{ and all } \lambda\in\mathbb{C}^k. \quad (11)$$

$$\frac{\partial \tilde{f}}{\partial \bar{z}}(z,x,\lambda) \text{ vanishes to infinite order for } \{\operatorname{Im} z = 0\} \quad (12)$$

and on $\{(z,\lambda) : P_k(z,\lambda) = 0\}$ for all $x\in E$.

13. <u>Proof</u> of the polynomial division theorem 3., using

the Nirenberg extension lemma 10.

Let be as in 10, and let $\widetilde{f}$ be its extension. It suffices to prove the theorem for such an f, cf. 4.d).

Let γ be a smooth simple closed curve near 0 in $\mathbb{C}$, U the interiour of γ, $0 \in U$.

For $P_k(z,\lambda) = z^k + \sum_{i=0}^{k-1} \lambda_i z^i$, $\lambda = (\lambda_0, \dots, \lambda_{k-1}) \in \mathbb{C}^k$, $z \in \mathbb{C}$ we have

$$
\begin{aligned}
P_k(z,\lambda) - P_k(w,\lambda) &= z^k - w^k + \sum_{i=1}^{k-1} \lambda_i (z^i - w^i) \\
&= (z-w)(z^{k-1} + z^{k-2}w + \dots + zw^{k-2} + w^{k-1} + \\
&\quad + \sum_{i=1}^{k-1} \lambda_i (z^{i-1} + z^{i-2}w + \dots + zw^{i-2} + w^{i-1})) \\
&= (z-w) \sum_{i=0}^{k-1} p_i(z,\lambda) w^i \quad \text{for polynomials } p_i \text{ in } z, \lambda.
\end{aligned}
$$

So $$\frac{P_k(z,\lambda)}{z-w} = \frac{P_k(w,\lambda)}{z-w} + \sum_{i=0}^{k-1} p_i(z,\lambda) w^i.$$

Now by 9. we can compute

$$
\begin{aligned}
\widetilde{f}(w,x,\lambda) &= \frac{1}{2\pi i} \int_\gamma \frac{\widetilde{f}(z,x,\lambda)}{z-w}\, dz + \frac{1}{2\pi i} \iint_U \frac{\partial \widetilde{f}(z,x,\lambda)}{\partial \bar{z}} \frac{dz \wedge d\bar{z}}{z-w} \\
&= \frac{1}{2\pi i} \int_\gamma \frac{\widetilde{f}(z,x,\lambda)}{z-w} \frac{P_k(z,\lambda)}{P_k(z,\lambda)}\, dz + \frac{1}{2\pi i} \iint_U \frac{\partial \widetilde{f}(z,x,\lambda)}{\partial \bar{z}} \frac{P_k(z,\lambda)}{P_k(z,\lambda)} \frac{dz \wedge d\bar{z}}{z-w} \\
&= \Big[\frac{1}{2\pi i} \int_\gamma \frac{\widetilde{f}(z,x,\lambda)}{(z-w)} \frac{1}{P_k(z,\lambda)}\, dz + \\
&\quad + \frac{1}{2\pi i} \iint_U \frac{\partial \widetilde{f}(z,x,\lambda)}{\partial \bar{z}} \frac{1}{P_k(z,\lambda)} \frac{dz \wedge d\bar{z}}{z-w}\Big] P_k(w,\lambda) + \\
&\quad + \sum_{i=0}^{k-1} \Big[\frac{1}{2\pi i} \int_\gamma \widetilde{f}(z,x,\lambda) \frac{p_i(z,\lambda)}{P_k(z,\lambda)}\, dz + \\
&\quad + \frac{1}{2\pi i} \iint_U \frac{\partial \widetilde{f}(z,x,\lambda)}{\partial \bar{z}} \frac{p_i(z,\lambda)}{P_k(z,\lambda)}\, dz \wedge d\bar{z}\Big] \cdot w^i \\
&= q(w,x,\lambda) P_k(w,\lambda) + \sum_{i=0}^{k-1} r_i(x,\lambda) w^i, \quad \text{where}
\end{aligned}
$$

$$q(w,x,\lambda) = \frac{1}{2\pi i} \int_\gamma \frac{\widetilde{f}(z,x,\lambda)}{(z-w)} \frac{1}{P_k(z,\lambda)}\, dz +$$

$$+\frac{1}{2\pi i}\iint_U \frac{\partial \tilde{f}(z,x,\lambda)}{\partial \bar{z}}\,\frac{1}{P_k(z,\lambda)}\,\frac{dz\wedge d\bar{z}}{z-w}\,,\quad \text{and}$$

$$r_i(x,\lambda)=\frac{1}{2\pi i}\int_\gamma \tilde{f}(z,x,\lambda)\,\frac{p_i(z,\lambda)}{P_k(z,\lambda)}\,dz\;+$$

$$+\frac{1}{2\pi i}\iint_U \frac{\partial \tilde{f}(z,x,\lambda)}{\partial \bar{z}}\,\frac{p_i(z,\lambda)}{P_k(z,\lambda)}\,dz\wedge d\bar{z}\,.$$

All these integrals are Bochner integrals in $\mathbb{C}\otimes F$. We have to check, that they are defined and yield smooth functions. The (formal) computation above is valid, since we used only linearity of the integral. Now a Bochner integral is defined, if the function is continuous and the domain (or its closure) is compact. The result is smooth in the remaining variables, if all derivatives of the integrand are continuous (we may interchange differentiation and integration).

The first integrals in the definition of both q and r_i are defined and smooth as long as the zeros of $P_k(z,\lambda)$ in z do not occur on the curve γ if λ is small enough.

Let us check this : Assume that for $z\in\gamma$ we have $0<\eta_1<|z|<\eta_2<1$, $\max|\lambda_i|<\varepsilon$. Then

$$|z^k+\sum_{i=0}^{k-1}\lambda_i z^i|\geq |z^k|-\sum_{i=0}^{k-1}|\lambda_i||z^i|>\eta_1^k-k\varepsilon\eta_2.$$

For ε small enough the last number will be positive.

The second integrals in the definitions exist and are smooth, since $\dfrac{\partial\tilde{f}}{\partial\bar{z}}$ vanishes to infinite order on the zeros of P_k and for real z (we need w real in the theorem) : this takes care of $1/(z-w)$.

qed.

14. To prove the Nirenberg extension lemma we need another lemma first. We denote

$$\delta(y,\lambda)=\inf\{|y-\operatorname{Im} z| : z\in\mathbb{C},\ P_k(z,\lambda)=0\}\quad\text{for}\quad y\in\mathbb{R}$$

and $\lambda\in\mathbb{C}^k$.

15. <u>Lemma</u> (Mather) : There exists a continuous function $\rho: \mathbb{R}\times\mathbb{C}^k\times\mathbb{R}\longrightarrow[0,1]$ such that

$$\rho(\xi,\lambda,y)=0 \text{ in a neighbourhood of } y=0. \qquad (16)$$

$$\rho(\xi,\lambda,y)=0 \quad \text{when} \quad |\xi y|\geqslant 1. \tag{17}$$

$$\frac{\partial}{\partial y}\rho(\xi,\lambda,y)=0 \quad \text{in a neighbourhood of } \delta(y,\lambda)=0 \tag{18}$$

(19) The function $\rho(\xi,\lambda,y)$ is infinitely often differentiable with respect to λ, y, and its derivatives are continuous with respect to all variables and satisfy

$$\left|\frac{\partial^{|\alpha|}}{\partial\lambda^{\alpha}}\frac{\partial^{|\beta|}}{\partial\bar\lambda^{\beta}}\frac{\partial^{\gamma}}{\partial y^{\gamma}}\rho(\xi,\lambda,y)\right|\leqslant C(\alpha,\beta,\gamma,K)(1+|\xi|^{1+|\alpha|+|\beta|+\gamma})$$

for all multiindices α,β and all $\gamma\in\mathbb{N}$, and all $\lambda\in K$ where K is compact in $\mathbb{C}^k$ and C is a constant depending as indicated.

20. Proof of the Nirenberg extension lemma 10, using lemma 15.

Given a smooth function $f: \mathbb{R}\times E\longrightarrow C\otimes F$, in $\mathbb{R}$ compactly supported, we consider the Fouriertransform

$$\hat f(\xi,x)=\int_{-\infty}^{\infty} f(t,x)e^{-2\pi i t\xi}dt.$$

This integral exists in $C\otimes F$ since $f(.,x)$ is compactly supported, and $\hat f(\xi,x)$ is smooth (compare the last arguments in 13.). Furthermore $\|\hat f(\xi,x)\|\leqslant\int_{-\infty}^{\infty}\|f(t,x)\|dt$, so is uniformly bounded in ξ for each $x\in E$. If $p(\xi)$ is a polynomial, then

$$\begin{aligned}
p(\xi)\hat f(\xi,x) &= \int_{-\infty}^{\infty} f(t,x)p(\xi)e^{-2\pi i t\xi}dt\\
&= \int_{-\infty}^{\infty} f(t,x)p\left(-\frac{1}{2\pi i}\frac{\partial}{\partial t}\right)(e^{-2\pi i t\xi})dt\\
&= \int_{-\infty}^{\infty} p\left(-\frac{1}{2\pi i}\frac{\partial}{\partial t}\right)(f(t,x))e^{-2\pi i t\xi}dt,
\end{aligned}$$

the last equation holds, since $-\frac{1}{2\pi i}\frac{\partial}{\partial t}$ is formally selfadjoint (use integration by parts, after reducing to $F=\mathbb{R}$ by duality as in the proof of 9.).

So $|p(\xi)|\|\hat f(\xi,x)\|$ is uniformly bounded too and $\|\hat f(\xi,x)\|$

is rapidly decreasing in ξ, and each derivative of $\hat{f}$ has the same property (use the same argument for the derivative). We define now the extension $\tilde{f}$ of f. For $(z,x,\lambda)\in \mathbb{C}\times E\times \mathbb{C}^k$ we put

$$f(z,x,\lambda)=\int_{-\infty}^{\infty}\rho(\xi,\lambda,\mathrm{Im}\ z)e^{2\pi i\xi z}\hat{f}(\xi,x)d\xi\ ,$$

where ρ is the function of lemma 15.

We claim that this integral is uniformly absolutely convergent in $\mathbb{C}\otimes F$ and that we can differentiate under the integral sign, i.e. for any multiindices α, β and $\gamma,\delta\in \mathbb{N}$ the following integral is again uniformly absolutely convergent:

$$\int_{-\infty}^{\infty}\frac{\partial^{|\alpha|}}{\partial\lambda^{\alpha}}\frac{\partial^{|\beta|}}{\partial\bar{\lambda}^{\beta}}\frac{\partial^{\gamma}}{\partial z^{\gamma}}\frac{\partial^{\delta}}{\partial\bar{z}^{\delta}}(\rho(\xi,\lambda,\mathrm{Im}\ z)e^{2\pi iz\xi})f(\xi,x)d\xi\ .$$

For, by (17) and (19),

$$\left|\frac{\partial^{|\alpha|}}{\partial\lambda^{\alpha}}\frac{\partial^{|\beta|}}{\partial\bar{\lambda}^{\beta}}\frac{\partial^{\gamma}}{\partial z^{\gamma}}\frac{\partial^{\delta}}{\partial\bar{z}^{\delta}}(\rho(\xi,\lambda,\mathrm{Im}\ z)e^{2\pi iz\xi})\right|$$

is uniformly bounded by a polynomial in $|\xi|$, and $\|\hat{f}(\xi,x)\|$ is rapidly decreasing. So the integral exists in $\mathbb{C}\otimes F$ (it does so on each compact in $\mathbb{R}$, and if we piece together compacts in an appropriate manner the integrals of the norm of the function over these compacts will converge). An even simpler argument applies to each derivative of $\hat{f}$ with respect to x. So $\tilde{f}$ exists and is smooth.

By (16) and the Fourier inversion formula (which holds $\mathbb{C}\otimes F$ too : use duality to reduce it to the case $F=\mathbb{R}$ as in the proof of 9.) $\tilde{f}$ is an extension of f :

$$f(t,x)=\int_{-\infty}^{\infty}\hat{f}(\xi,x)e^{2\pi i\xi t}d\xi=f(t,x,\lambda),\ t\in\mathbb{R}.$$

So (11) holds.

By (16) again $\dfrac{\partial f}{\partial\bar{z}}$ vanishes to infinite order on $\{\mathrm{Im}\ z=0\}$ and by (18) to infinite order on $\{(z,\lambda):P_k(z,\lambda)=0\}$, so (12) holds.

qed.

21. Proof of lemma 15. Let

$$\sigma(\eta,\lambda)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{d}{dx}\log P_k(x+\eta i,\lambda)\right|^2 dx, \text{ so } \sigma: \mathbb{R}\times\mathbb{C}^k\to\mathbb{R}. \quad (22)$$

We claim that (δ is defined in 14.)

$$1/2\,\delta(\eta,\lambda)\le\sigma(\eta,\lambda)\le k^2/2\,\delta(\eta,\lambda) \quad \text{if } \delta(\eta,\lambda)\ne 0. \quad (23)$$

To show this we integrate by residues.

Fix $\eta\in\mathbb{R}$ and $\lambda\in\mathbb{C}^k$, let $z_1,\dots,z_k$ be the zeros of $z\mapsto P_k(z,\lambda)$. Then $P_k(z,\lambda)=\prod_i(z-z_i)$.

$$\left|\frac{d}{dx}\log P_k(x+\eta i,\lambda)\right|^2=\left|\frac{d}{dx}\log\prod_j(x+\eta i-z_j)\right|^2$$

$$=\left|\sum_{j=1}^{k}1/(x+\eta i-z_j)\right|^2=\left(\sum_j\frac{1}{x+\eta i-z_j}\right)\left(\sum_j\frac{1}{x-\eta i-\bar z_j}\right).$$

Let $Q(z)=\frac{1}{2\pi}\left(\sum_j\frac{1}{z+\eta i-z_j}\right)\left(\sum_j\frac{1}{z-\eta i-\bar z_j}\right)$, $z\in\mathbb{C}$, so that for $x\in\mathbb{R}$

$$Q(x)=\frac{1}{2\pi}\left|\frac{d}{dx}\log P_k(x+\eta i,\lambda)\right|^2.$$

Clearly $Q(z)$ is meromorphic and $z^2\,Q(z)$ is bounded outside a suitable compact set. If $Q(x)$ has no real poles, i.e. if $\delta(\eta,\lambda)>0$, then by the method of residues it follows that

$\sigma(\eta,\lambda)=2\pi\int_{-\infty}^{\infty}Q(x)dx=i$ (sum of all residues of $Q(z)$ in the upper half plane)

$$=i\left(\sum_{j\in A}\sum_{\ell=1}^{k}\frac{1}{z_j-\bar z_\ell-2\eta i}+\sum_{j\in B}\sum_{\ell=1}^{k}\frac{1}{\bar z_j-z_\ell+2\eta i}\right),$$

where A denotes the set of all j such that $\operatorname{Im} z_j>\eta$ and B denotes the set of all j such that $\operatorname{Im} z_j<\eta$; we suppose furthermore that $z_j-\eta i\ne\bar z_k+\eta i$ for all j,k (this is a condition on λ), so the last equation holds.

Let now $b_{jk}=1$ if $j,k\in A$, $b_{jk}=-1$ if $j,k\in B$, and $b_{jk}=0$ otherwise. Then the above is equal to

$$\sum_{1\leq j,\ell\leq k} b_{j\ell}\,\frac{\operatorname{Im} z_j + \operatorname{Im} z_\ell - 2\eta}{|z_j - \bar{z}_\ell - 2\eta i|^2}\,.$$

This is the sum of k^2 nonnegative quantities each of which is $\leq 1/2\,\delta(\eta,\lambda)$ and at least one of them is $= 1/2\,\delta(\eta,\lambda)$. Hence (23) follows in case that $z_j - \eta i \neq \bar{z}_\ell + \eta i$ for all j, ℓ. By continuity, (23) holds in general.

Now we want to estimate the partial derivatives of σ. We claim that

$$\left|\frac{\partial^{|\alpha|}}{\partial\lambda^\alpha}\frac{\partial^{|\beta|}}{\partial\bar{\lambda}^\beta}\frac{\partial^\gamma}{\partial\eta^\gamma}\sigma(\eta,\lambda)\right| \leq C(\alpha,\beta,\gamma,K)(1+\delta(\eta,\lambda)^{-2k(1+|\alpha|+|\beta|+\gamma)}) \quad (24)$$

for all multiindices α, β, all $\gamma\in\mathbb{N}$ and all $\lambda\in K$, a compact subset of $\mathbb{C}^k$, whenever $\delta(\eta,\lambda)\neq 0$.

We have

$$\left|\frac{d}{dx}\log P_k(x+\eta i,\lambda)\right|^2 = R(x,\eta,\lambda)/|P_k(x+\eta i,\lambda)|^2, \quad (25)$$

where $R(x,\eta,\lambda)$ is a polynomial in $(x,\lambda,\eta)\in\mathbb{R}\times\mathbb{C}^k\times\mathbb{R}$ of degree $2k-2$ in x.

Any first partial derivative of (25) is of the form $R_1(x,\eta,\lambda)/|P_k(x+\eta i,\lambda)|^4$, where R_1 is a polynomial of degree at most $2k-2+2k = 4k-2$ in x.

Any j-th partial derivative of (25) is of the form $R_j(x,\eta,\lambda)/|P_k(x+\eta i,\lambda)|^{2(1+j)}$, where R_j is a polynomial of degree at most $2k(1+j-1)-2+2(1+j-1)k = 2kj-2$ in x by induction. $|P_k(x+\eta i,\lambda)|^{2(1+j)}$ is a polynomial in $x\in\mathbb{R}$, $\eta\in\mathbb{R}$, $\lambda\in\mathbb{C}^k$, with leading coefficient 1 in x, of degree $2k(1+j)$ in x, this is $2k+2$ higher than the degree of R_j in x. The same argument applies to η, if $\delta(\eta,\lambda)\neq 0$, i.e. if there are no poles on the line $x+\eta i$, $x\in\mathbb{R}$. So the dominating factor is the distance to the next pole, in the appropriate power, and $\delta(\eta,\lambda)$ measures the "vertical" distance to the next pole. So we obtain the following : For any compact subset K of $\mathbb{C}^k$ there exists a constant $C(K,j)$ such that

$$\frac{R_j(x,\eta,\lambda)}{P_k(x+\eta i,\lambda)^{2(1+j)}} \leq C(K,j)\,\frac{1}{(1+|x|^{2k+2})(1+|\eta|^{2k+2})}\,.$$

$$\cdot(1+\delta(\eta,\lambda))^{-2k(1+|\alpha|+|\beta|+\gamma)}.$$

$1/(1+|x|^{2k+?})$ is integrable along $\mathbb{R}$, so we may integrate the above estimate with respect to x to obtain (24).

We will construct the function ρ.

Let g be a smooth function $[0,\infty)\longrightarrow\mathbb{R}$ satisfying

$$\begin{aligned} g(t) &= 1 && \text{if } 0 \leqslant t \leqslant 4k^3, \\ 0 \leqslant g(t) &\leqslant 1 && \text{if } 4k^3 \leqslant t \leqslant 8k^3, \\ g(t) &= 0 && \text{if } 8k^3 \leqslant t. \end{aligned} \tag{26}$$

Let h be a second smooth function $[0,\infty)\longrightarrow\mathbb{R}$ satisfying

$$\begin{aligned} h(t) &= 0 && \text{if } t \leqslant \varepsilon \text{ for some } 0<\varepsilon<1/2, \\ 0 \leqslant h(t) &\leqslant 1 && \text{if } \varepsilon \leqslant t \leqslant 1-\varepsilon, \\ h(t) &= 1 && \text{if } 1-\varepsilon \leqslant t. \end{aligned} \tag{27}$$

Then define $\rho(\xi,\lambda,y)$ for $(\xi,\lambda,y)\in\mathbb{R}\times\mathbb{C}^k\times\mathbb{R}$ as follows :

$$\begin{aligned} \rho(\xi,\lambda,y) &= 0 && \text{if } 1/(1+|\xi|) \leqslant |y|, \\ &= 1 && \text{if } |y| \leqslant 1/2(1+|\xi|), \\ &= h\Big(4(1+|\xi|)\int_y^{\frac{1}{1+|\xi|}} g(\sigma(\eta,\lambda)/(1+|\xi|))\,d\eta\Big) && \text{if } 1/2(1+|\xi|) \leqslant y \leqslant 1/(1+|\xi|), \\ &= h\Big(4(1+|\xi|)\int_{-\frac{1}{1+|\xi|}}^{y} g(\sigma(\eta,\lambda)/(1+|\xi|))\,d\eta\Big) && \text{if } -1/(1+|\xi|) \leqslant y \leqslant -1/2(1+|\xi|). \end{aligned}$$

First we claim that

$$\int_{\frac{1}{2(1+|\xi|)}}^{\frac{1}{1+|\xi|}} g(\sigma(\eta,\lambda)/(1+|\xi|))\,d\eta \geqslant 1/4(1+|\xi|) \tag{29}$$

$$\int_{-\frac{1}{1+|\xi|}}^{-\frac{1}{2(1+|\xi|)}} g(\sigma(\eta,\lambda)/(1+|\xi|))\,d\eta \geqslant 1/4(1+|\xi|). \tag{30}$$

For that remember the definition of δ (14) and (23).

Let m be Lebesgue measure on $\mathbb{R}$ and $I(\xi)=[1/2(1+|\xi|), 1/(1+|\xi|)]$. Then a simple geometrical argument (there are at most k different zeros of $P_k(\cdot,\lambda)$ for fixed λ) gives

$$\begin{aligned} m(\{\eta\in I(\xi): \delta(\eta,\lambda)\geqslant r\}) &= m(I(\xi)) - m(\{\eta\in I(\xi): \delta(\eta,\lambda)< r\}) \\ &\geqslant 1/2(1+|\xi|) - 2rk. \end{aligned}$$

We are interested in the set of those η, for which $g(\sigma(\eta,\lambda)/(1+|\xi|))=1$. Sufficient for that is $\sigma(\eta,\lambda)/(1+|\xi|) \leqslant 4k^2$ by (26). By (23) $\sigma(\eta,\lambda) \leqslant k^2/2\,\delta(\eta,\lambda)$, so we obtain the sufficient condition $\delta(\eta,\lambda) \geqslant 1/8k(1+|\xi|)$. But now $m(\{\eta \in I(\xi):\, \delta(\eta,\lambda) \geqslant 1/8k(1+|\xi|)\}) \geqslant 1/4(1+|\xi|)$ as we computed above. (29) follows since on this set $g=1$.

A similar argument proves (30).

Using (29) and (30) we see that the definitions of $\rho(\xi,\lambda,y)$ coincide on overlapping intervalls, so $\rho(\xi,\lambda,y)$ is smooth in λ and y for fixed ξ.

Let us check now whether the conditions (16)-(19) of lemma 15 are satisfied:

(16) On a neighbourhood of $y=0$, exactly for $|y| \leqslant 1/2(1+|\xi|)$, we have $\rho(\xi,\lambda,y)=1$ by definition.

(17) If $|\xi y| \geqslant 1$, then $|y| \geqslant 1/|\xi| > 1/(1+|\xi|)$, so $\rho(\xi,\lambda,y)= = 0$ by definition.

(18) We want that $\frac{\partial}{\partial y}\rho(\xi,\lambda,y)=0$ in a neighbourhood of $\delta(y,\lambda)=0$.

$$\frac{\partial}{\partial y}\rho(\xi,\lambda,y)= h'(4(1+|\xi|)\int_y^{\frac{1}{1+|\xi|}} g(\sigma(\eta,\lambda)/(1+|\xi|))d\eta).$$
$$\cdot\ (-4(1+|\xi|)g(\sigma(y,\lambda)/(1+|\xi|))),\ \text{if } y \in I(\xi).$$

If y is so near at $\{\eta : \delta(\eta,\lambda)=0\}$ that

$$\delta(y,\lambda) \leqslant 1/16k^3(1+|\xi|),\ \text{then}\ \sigma(y,\lambda)/(1+|\xi|) \geqslant$$
$$\geqslant 1/2\,\delta(y,\lambda)(1+|\xi|) \geqslant 8k^3,$$

using (23), so $g(\sigma(y,\lambda)/(1+|\xi|))=0$ by (26). If $\delta(y,\lambda)=0$, then (23) does not hold but the conclusion holds by continuity. So $\frac{\partial}{\partial y}\rho(\xi,\lambda,y)=0$. Exactly the same argument applies, if $y \in -I(\xi)$.

(19) We already know that $\rho(\xi,\lambda,y)$ is smooth with respect to λ, y. So we have only to estimate

$$\left|\frac{\partial^{|\alpha|}}{\partial\lambda^\alpha}\frac{\partial^{|\beta|}}{\partial\bar\lambda^\beta}\frac{\partial^\gamma}{\partial y^\gamma}\rho(\xi,\lambda,y)\right|.$$

For fixed ξ we know that $\rho(\xi,\lambda,y)$ is constant outside

$I(\xi)\cup(-I(\xi))$, in particular for $|y|\geqslant 1/(1+|\xi|)$.

Let now $K\subseteq\mathbb{C}^k$ be compact and consider $\lambda\in\mathbb{C}^k$. We want to estimate for $\lambda\in K$ the expression

$$\frac{\partial^{|\alpha|}}{\partial\lambda^{\alpha}}\frac{\partial^{|\beta|}}{\partial\bar{\lambda}^{\beta}}\frac{\partial^{\tau}}{\partial y^{\tau}}\,h(4(1+|\xi|)\int_y^{\frac{1}{1+|\xi|}} g(\sigma(\eta,\lambda)/(1+|\xi|))d\eta\,) \qquad (31)$$

and the similar expression for $y\in(-I(\xi))$.

The partial derivative (31) is a polynomial in the partial derivatives of h and g (which are uniformly bounded since both are constant outside a compact set) and in $1+|\xi|$, $1/(1+|\xi|)$ and the partial derivatives of σ.

The latter are bounded by an expression

$$C(\alpha,\beta,\tau,K)(1+\delta(y,\lambda)^{-2k(1+|\alpha|+|\beta|+\tau)}),$$

using (24). Recall that $\delta(y,\lambda)$ is the "vertical" distance from y to the next zero of $P_k(.,\lambda)$. If λ remains in K then the set of all these zeros is bounded, so this expression above becomes big only in a compact set, where we can bound it uniformly. So we can disregard all partial derivatives and of course $1/(1+|\xi|)$ in (31). So (31) is bounded by a polynomial in $1+|\xi|$, of order $|\alpha|+|\beta|+\tau$, i.e. just the order partial derivative (31). So finally we obtain a bound of the form $C(\alpha,\beta,\tau,K)(1+|\xi|^{1+|\alpha|+|\beta|+\tau})$.

qed.

REFERENCES

[1] R.Bonic, J.Frampton, Smooth functions on Banach manifolds, Journ.Math.and Mech. 15(1966), 877-898.

[2] Th.Bröcker, Differentiable Germs and Catastrophes, London Math.Soc.Lecture Notes 17(1975).

[3] N.Dunford, J.T.Schwarz, Linear Operators I, Interscience Publishers, New York, London 1958.

[4] S.Lang, Differential Manifolds, Addison Wesley, 1972.

[5] R.J.Magnus, Universal unfoldings in Banach spaces : reduction and stability, Math.Proc.Cambridge Phil.Soc. 86(1979), 41-55.

[6] J.N.Mather, On Nirenberg's proof of Malgrange's preparation theorem, Proceedings of Liverpool Singularities Symposium I, 116-120, Springer Lecture Notes 192 (1971).

[7] J.N.Mather, Stability of C^{∞}-mappings I : The division theorem, Ann.Math. 87(1968), 89-104.

[8] L.Nirenberg, A proof of the Malgrange preparation theorem, Proceedings of Liverpool Singularities Symposium I, 97-105, Springer Lecture Notes 192(1971).

[9] V.Poenaru, Analyse différentielle, Springer Lecture Notes 371 (1974).