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Characters on Algebras of Smooth Functions

A. Kriegl, P. Michor and W. Schachermayer

Abstract

For a huge class of spaces it is shown that the real characters on the algebra of differentiable functions are exactly the evaluations at points.

Introduction

"Milnor and Stasheff's exercise" [17] says that for a smooth manifold M each algebra homomorphism $C^\infty(M) \to \mathbf{R}$ is given by evaluation at some point of M. Although the classical proof of this statement depends heavily on locally compactness arguments we were able to extend it to a wide class of spaces M. In fact, the existence of partitions of unity is sufficient, even in the infinite dimensional situation. We use a setting which hopefully encompasses all existing notions of "differentiable spaces" and within which we identify the statement above as a completeness property (smoothly real-compactness) of M.

1. General Setting

1.1 Definition: A *smooth space* is a pair $(M, \mathscr{S})$ where M is a set and $\mathscr{S}$ a set of real valued functions on M which separates points and has the following properties:

1. The set $\mathscr{S}$ is closed under composition with C^∞-functions: For any $f_1, \ldots, f_n$ in $\mathscr{S}$ and F in $C^\infty(U, \mathbf{R})$, where U is open in $\mathbf{R}^n$, such that $(f_1, \ldots, f_n)(M) \subseteq U$, we have $F \circ (f_1, \ldots, f_n) \in \mathscr{S}$.

2. If $\mathscr{F}$ is a subset of $\mathscr{S}$ such that the family of carriers $\operatorname{carr}(f) := \{x \in M : f(x) \neq 0\}$ of functions $f \in \mathscr{F}$ is locally finite in the initial topology induced by $\mathscr{S}$, then $\sum_{f \in \mathscr{F}} f$ is in $\mathscr{S}$.

Note that condition 1 implies in particular that $\mathscr{S}$ is a C^∞-algebra in the sense of [18]. The converse is not true: let M be the open unit intervall and let $\mathscr{S}$ be the set of restrictions of global smooth functions; then $\mathscr{S}$ is a C^∞-algebra but does not satisfy 1.

In the following we will always equip M with the initial topology with respect to $\mathscr{S}$ without further notice, and we will call it the $\mathscr{S}$-topology.

1.2 We may also remark that an important class of smooth spaces may be described in the following manner: A Hausdorff topological space M together with a sheaf of C^∞-algebras consisting of continuous functions such that the topology is initial for the global sections.

Given $(M, \mathscr{S})$ as in 1.1 and U open in M let $\mathscr{S}(U)$ be the set of all real functions f on U such that for each $x \in U$ there is an open neighborhood U_x of x and an $f_x \in \mathscr{S}$ with $f \mid U_x = f_x \mid U_x$. Then $\mathscr{S}(M)$ contains $\mathscr{S}$ and we have equality in the most important cases.

1.3 Examples of smooth spaces:

1. Any completely regular topological space M, with the algebra of continuous functions $\mathscr{S}$.
2. Finite dimensional manifolds of class C^k, $k < \omega$, with C^k-functions.
3. Locally convex vector spaces with all standard notions of C^k-functions on them. In particular we consider C_c^∞ in the sense of [7] and C^∞ in the sense of [3], [9,10] (attention: the topologies generated by these smooth maps are in general not the locally convex ones, in the latter case it might be even incompatible, since C^∞-maps need not be continuous with respect to the locally convex topology).
4. Manifolds with charts modelled on open subsets of locally convex vector spaces using one of the C^k notions mentioned in 3 (in particular manifolds of mappings, see [14].
5. Vector sets of [11], smooth structures in the sense of [3], differential spaces in the sense of [19], and "manifolds" in the sense of [16].

2. Smoothly Real Compact Spaces

2.1 Definition: A smooth space $(M, \mathscr{S})$ is called *smoothly real-compact* iff any algebra homomorphism $\mathscr{S} \to \mathbf{R}$ is an evaluation at a point of M, i.e. iff the canonial map $\mathrm{ev}: M \to \mathrm{Alg}(\mathscr{S}, \mathbf{R})$, $\mathrm{ev}(x): f \mapsto f(x)$ is a bijection onto the set $\mathrm{Alg}(\mathscr{S}, \mathbf{R})$ of algebra homomorphisms.

So we require that the conclusion of "Milnor and Stasheff's exercise" is true.

If M is a completely regular topological space, then the smooth space $(M, C(M))$ is smoothly real-compact if and only if it is real-compact in the usual sense. See [6].

2.2 Lemma: *Let $(M, \mathscr{S})$ be a smooth space and consider the mappings $\iota: M \to \prod_{f \in \mathscr{S}} \mathbf{R}$, $\iota(x)_f := f(x)$ and $\mathrm{ev}: M \to \mathrm{Alg}(\mathscr{S}, \mathbf{R})$, $\mathrm{ev}(x): f \mapsto f(x)$.*

The set $\mathrm{Alg}(\mathscr{S}, \mathbf{R})$ can be considered as a subset of $\prod_{f \in \mathscr{S}} \mathbf{R}$ and the map ev composed with this inclusion gives ι. The map ι is topological embedding and the closure of its image $\iota(M)$ is $\mathrm{Alg}(\mathscr{S}, \mathbf{R})$.

Proof: That ev composed with the inclusion of $\mathrm{Alg}(\mathscr{S}, \mathbf{R})$ into $\prod_{\mathscr{S}} \mathbf{R}$ gives ι is obvious.

This map ι is an embedding, since the topology of M is by definition initial with respect to the maps $f \in \mathscr{S}$ and that of $\prod_{\mathscr{S}} \mathbf{R}$ is initial with respect to pr_f for $f \in \mathscr{S}$ and $\mathrm{pr}_f \circ \iota = f$.

Let $\varphi: \mathscr{S} \to \mathbf{R}$ be an algebra homomorphism. We claim that φ considered as the point $x_\varphi \in \prod_{\mathscr{S}} \mathbf{R}$ with coordinates $(x_\varphi)_f = \varphi(f)$ is in the closure of $\iota(M)$. For $f \in \mathscr{S}$ let Z_f be the set $\{x \in M: f(x) - \varphi(f) = 0\}$. The sets Z_f are not empty for otherwise

$f - \varphi(f) \cdot 1$ is invertible in $\mathscr{S}$ but $\varphi(f - \varphi(f) \cdot 1) = 0$. Since $Z_f \cap Z_g = Z_{(f-\varphi(f))^2 + (g-\varphi(g))^2}$, the family $\{Z_f: f \in \mathscr{S}\}$ has the finite intersection property. For any finite subset $\mathscr{F} \subseteq \mathscr{S}$ and $x_{\mathscr{F}} \in \bigcap_{f \in \mathscr{F}} Z_f$ we have $\iota(x_{\mathscr{F}})_f = (x_\varphi)_f$ for all $f \in \mathscr{F}$. So $\iota(x_{\mathscr{F}})$ converges to x_φ in $\prod_{\mathscr{S}} \mathbf{R}$.

Let conversely $\iota(x_\alpha)$ be a net that converges to x_∞ in $\prod_{\mathscr{S}} \mathbf{R}$. We have to show that the map $\varphi: \mathscr{S} \to \mathbf{R}$ which corresponds to x_∞ is an algebra homomorphism. Since $\iota(x_\alpha)$ corresponds to the algebra homomorphism ev (x_α) and the net ev (x_α) converges pointwise to φ on f for all $f \in \mathscr{S}$, the limit point φ is also an algebra homomorphism. □

2.3 Corollary: *Let $(M, \mathscr{S})$ be a smooth space. Then M is smoothly real-compact if and only if $\iota(M)$ is closed in $\prod_{\mathscr{S}} \mathbf{R}$. Furthermore, the algebra* Alg $(\mathscr{S}, \mathbf{R})$ *of algebra homomorphisms can be made into a smooth space which is smoothly real-compact and is the universal solution for extending smooth functions.*

Proof: The space M is by definition smoothly real-compact iff the map ev is onto, and this corresponds via lemma (2.2) to the statement that the map ι has closed image.

Every $f \in \mathscr{S}$ defines a map $f^\sim$: Alg $(\mathscr{S}, \mathbf{R}) \to \mathbf{R}$ by $\varphi \mapsto \varphi(f)$. We consider as structure on Alg $(\mathscr{S}, \mathbf{R})$ the family $\{f^\sim: f \in \mathscr{S}\} =: \mathscr{S}^\sim$. $\mathscr{S}^\sim$ is point separating since different $\varphi \in$ Alg $(\mathscr{S}, \mathbf{R})$ differ at least at one $f \in \mathscr{S}$.

The initial topology induced on Alg $(\mathscr{S}, \mathbf{R})$ by the family $\mathscr{S}^\sim$ is just the trace topology inherited as a subset of $\Pi_{\mathscr{S}}\mathbf{R}$. In particular M is dense in Alg $(\mathscr{S}, \mathbf{R})$.

$\mathscr{S}^\sim$ satisfies condition 1: Let $f^\sim \in \mathscr{S}^\sim$, $(f_1^\sim, \ldots, f_n^\sim)$ (Alg $(\mathscr{S}, \mathbf{R})) \subseteq U$, $F: U \to \mathbf{R}$ smooth. Then $(f_1, \ldots, f_n)(M) \subseteq U$ hence $F \circ (f_1, \ldots, f_n) \in \mathscr{S}$, and since M is dense in Alg $(\mathscr{S}, \mathbf{R})$, we have $F \circ (f_1^\sim, \ldots, f_n^\sim) = (F \circ (f_1, \ldots, f_n))^\sim$.

$\mathscr{S}^\sim$ satisfies condition 2: Let $\mathscr{F}^\sim \subseteq \mathscr{S}$ such that $\{\text{carrier } f^\sim: f^\sim \in \mathscr{F}^\sim\}$ is locally finite, then $\{\text{carrier } f: f \in \mathscr{F}\}$ is locally finite and hence $\sum f \in \mathscr{S}$, i.e. $(\sum f)^\sim \in \mathscr{S}^\sim$ and by density of M in Alg $(\mathscr{S}, \mathbf{R})$ we have $(\sum f)^\sim = \sum f^\sim$.

(Alg $(\mathscr{S}, \mathbf{R})$, $\mathscr{S}^\sim$) is smoothly real compact: Let $\varphi^\sim: \mathscr{S}^\sim \to \mathbf{R}$ be an algebra homomorphism, then $\varphi: \mathscr{S} \to \mathbf{R}$ defined by $\varphi f := \varphi^\sim f^\sim$ is an algebra homomorphism, hence $\varphi \in$ Alg $(\mathscr{S}, \mathbf{R})$ and $\varphi^\sim f^\sim = \varphi f = f^\sim \varphi$. □

2.4 Corollary: *If a smooth space $(M, \mathscr{S})$ is smoothly real-compact, then the $\mathscr{S}$-topology on M is real-compact.*

Proof: By the previous corollary a smoothly real-compact space is embedded as a closed subspace of $\prod_{\mathscr{S}} \mathbf{R}$, hence it is real-compact, see [6] or [2, p. 154].

2.5 Remark: If one defines a map $\varphi: M_0 \to M_1$ between smooth spaces $(M_0, \mathscr{S}_0)$ and $(M_1, \mathscr{S}_1)$ to be smooth iff $f \circ \varphi \in \mathscr{S}_0$ for all $f \in \mathscr{S}_1$, then the following can be said:

1. The smooth space Alg $(\mathscr{S}, \mathbf{R})$ with the structure $\{\text{ev}_f: f \in \mathscr{S}\}$ defined in (2.3) is the universal solution for extending smooth maps into smoothly real-compact spaces.

2. Smoothly real-compact spaces are completely determined by the algebra Alg $(\mathscr{S}, \mathbf{R})$, since ev: $M \to$ Alg $(\mathscr{S}, \mathbf{R})$ is for these spaces a diffeomorphism.

3. If for two smoothly real-compact spaces $(M_0, \mathscr{S}_0)$ and $(M_1, \mathscr{S}_1)$ the algebras $\mathrm{Alg}(\mathscr{S}_0, \mathbf{R})$ and $\mathrm{Alg}(\mathscr{S}_1, \mathbf{R})$ are isomorphic, then the smooth spaces are diffeomorphic.

3. The Main Theorem

3.1 Lemma: *For a smooth space $(M, \mathscr{S})$ the following four conditions are equivalent:*

1. Let $f: M \to \mathbf{R}$ be continuous in the $\mathscr{S}$-topology and $a < b$. Then there is some $g \in \mathscr{S}$ *with* $g|_{\{x: f(x) \leqq a\}} = 0$, $g|_{\{x: f(x) \geqq b\}} = 1$.

2. *For any continuous function f and $a < b$ there is some $g \in \mathscr{S}$ such that $\{x \in M: f(x) \leqq a\} \subseteq \{x \in M: g(x) = 0\} \subseteq \{x \in M: f(x) < b\}$.*

3. *The algebra $\mathscr{S}$ is dense in the set of all continuous functions in the topology of uniform convergence.*

4. *The bounded functions in $\mathscr{S}$ are dense in the space of all bounded continuous functions on M with respect to the sup-norm.*

Proof: $(2 \Rightarrow 4)$ We want to apply the Stone-Weierstrass theorem to the Stone-Čech compactification βM of M and the algebra of bounded functions in $\mathscr{S}$. So let $x, y \in \beta M$. Then there is a bounded continuous f with $f(x) < f(y)$. Choose a smooth g according to (2) for $a := f(x)$ and $b := f(y)$. Make it bounded and non-negative by composing with a suitable real function. Then $g(x) = 0$ and $g(y) > 0$. Thus the algebra of bounded functions in $\mathscr{S}$ separates points in βM and hence is by the Stone-Weierstrass theorem dense in the algebra $C(\beta M)$ of continuous functions on βM. But $C(\beta M)$ is the algebra of continuous bounded functions on M.

$(4 \Rightarrow 1)$ Choose a $g \in \mathscr{S}$ with $|g - f| < \frac{b-a}{3}$. Then $g(x) \leqq \frac{2a+b}{3}$ for $f(x) \leqq a$ and $g(x) \geqq \frac{a+2b}{3}$ for $f(x) \geqq b$. By composing with a smooth function one obtains everything needed.

$(1 \Rightarrow 3)$ Let f be continuous, without loss of generality we may assume $f \geqq 0$ (decompose $f = f_+ - f_-$). Let $\varepsilon > 0$ and choose smooth g_k with image in $[0, 1]$ and $g_k(x) = 0$ for x with $f(x) \leqq k\varepsilon$ and $g_k(x) = 1$ for x with $f(x) \geqq (k+1)\varepsilon$. Then the sum $g := \sum_{k \in \mathbf{N}} \varepsilon g_k$ is locally finite and $|f - g| < 2\varepsilon$.

$(3 \Rightarrow 2)$ Choose a $g \in \mathscr{S}$ with $|g - f| < \frac{b-a}{2}$ and an appropriate map $\varrho \in C^\infty(\mathbf{R}, \mathbf{R})$. Then $\varrho \circ \left(g - \frac{a+b}{2}\right)$ satisfies (2). □

3.2 Theorem: *Let $(M, \mathscr{S})$ be a smooth space such that:*

1. *M is real-compact in the $\mathscr{S}$-topology.*
2. *The (equivalent) properties of* 3.1 *hold.*

Then $(M, \mathscr{S})$ is smoothly real-compact.

First Proof: Let φ be an algebra homomorphism. Then $I := \ker \varphi$ is an ideal in $\mathscr{S}$.

Step 1: If $f_1, \ldots, f_n \in I$ and $g \in C^\infty(\mathbf{R}^n, \mathbf{R})$ with $g(0) = 0$, then $g \circ (f_1, \ldots, f_n) \in I$, because $g(x) = \int_0^1 \sum_{i=1}^n \frac{\partial g}{\partial x^i}(tx)\, dt \cdot x^i =: \sum h_i \cdot x^i$ and $g \circ (f_1, \ldots, f_n) = \sum h_i(f_1, \ldots, f_n) \cdot f_i \in I$.

Step 2: For $f \in \mathcal{S}$ let again $Z_f := \{x: f(x) = \varphi f\}$. Then $\mathcal{Z} := \{Z_f : f \in \mathcal{S}\} = \{Z_f : f \in I\}$, since $Z_f = Z_{f-\varphi f \cdot 1}$ and $f - \varphi f \cdot 1 \in I$.

$\mathcal{Z}$ has the finite intersection property (see the proof of 2.2). We claim that it has the countable intersection property. If not there is a sequence $(f_n)_{n \in \mathbf{N}}$ with $\bigcap Z_{f_n} = \emptyset$ and we may assume that $Z_{f_n} \supseteq Z_{f_{n+1}}$ and $f_n \in I$ for all n.

Step 3: Put $U_n := \{x \in M : |f_i(x)| < \frac{1}{n}$ for $i < n$ and $f_n(x) \neq 0\}$. We claim that $\{U_n : n \in \mathbf{N}\}$ is a locally finite cover of M: Let $x \in M$. There is a minimal n with $f_n(x) \neq 0$, so $x \in U_n$. Let $V := \{y \in M : |f_n(y) - f_n(x)| < \frac{1}{2}|f_n(x)|\}$. Then $V \cap U_m = \emptyset$ if $m > n$ and $\frac{1}{m} \leqq \frac{1}{2}|f_n(x)|$.

Step 4: Choose $\varrho_n \in C^\infty(\mathbf{R}^n, [0, 1])$ such that $\varrho_n(t_1, \ldots, t_n) > 0$ iff $|t_i| < \frac{1}{n}$ for $i < n$ and $t_n \neq 0$.

By step 1 we have that $\varrho_n \circ (f_1, \ldots, f_n) =: g_n \in I$ and carr $(g_n) = U_n$. Let $g := \sum_{n=1}^{\infty} \frac{1}{2^n} g_n$. Then $g \in \mathcal{S}$ by 1.1.2, and $g(x) > 0$ for all $x \in M$. Hence $\frac{1}{g} \in \mathcal{S}$ and $\alpha := \varphi\left(\frac{1}{g}\right) > 0$. Let $2^n > \alpha$. Hence there exists a point $x_0 \in Z_{g_1} \cap \ldots \cap Z_{g_n} \cap Z_{g^{-1}-\alpha} \neq \emptyset$. Therefore $0 = g_1(x_0) = \ldots = g_n(x_0)$ and $\frac{1}{\alpha} = g(x_0) = \sum_{k>n} \frac{1}{2^k} g_k(x_0) \leqq \frac{1}{2^n}$ yields a contradiction. Hence $\mathcal{Z}$ has the countable intersection property.

Step 5: Let now $x_\infty \in \bigcap_{Z \in \mathcal{Z}} Z^{\beta M}$, where βM denotes the Stone-Čech compactification. We claim that $x_\infty \in M$: Otherwise the real-compactness of M implies the existence of a function $f \in C(\beta M, I)$ with $f|_M > 0$ and $f(x_\infty) = 0$ (vide [2, p. 152]). Since M is assumed to have the property of the lemma above there exists a smooth $f_i \in \mathcal{S}$ with $\left\{x \in M : f(x) \leqq \frac{1}{i+1}\right\} \subseteq \{x \in M : f_i(x) = 0\} \subseteq \left\{x \in M : f(x) < \frac{1}{i}\right\}$. Consider $Z_i := \{x \in M: f_i(x) = 0\}$. $Z_i \in \mathcal{Z}$ since $f_i - \varphi f_i \cdot 1 \in \operatorname{Ker} \varphi$ and $Z_{f_i - \varphi f_i \cdot 1} \cap Z_i \neq \emptyset$ because $\left\{x \in \beta M: f(x) < \frac{1}{i+1}\right\} \subseteq Z_i$ has non empty intersection with $Z_{f_i - \varphi f_i \cdot 1}$ as neighborhood of x_∞. But this implies that $\varphi(f_i) = 0$, i.e. $Z_{f_i} = Z_i$. Hence by the countable intersection property $\emptyset \neq \bigcap Z_i$. Which is contradicted by the fact that $x \in Z_i$ implies $0 < f(x) < \frac{1}{i}$.

Step 6: The point x_∞ is in $\bigcap_{Z \in \mathcal{Z}} Z$. Therefore $x_\infty \in Z$ for all $Z \in \mathcal{Z}$, i.e. for all $f \in \mathcal{S}$ we have $f x_\infty - \varphi f \cdot 1 = 0$ or $\varphi f = f x_\infty$. □

Second Proof: Condition 3.1.3 implies that the uniformity generated by the continuous maps and that generated by the smooth maps is equal. (For a continuous f and an $\varepsilon > 0$ choose a smooth g with $|g - f| < \varepsilon$. Then $\{(x, y): |g(x) - g(y)| < \varepsilon\} \subseteq \{(x, y): |f(x) - f(y)| < 3\varepsilon\}$.)

M real-compact implies that the uniformity generated by the family of continuous mappings is complete, hence the uniformity generated by $\mathscr{S}$ is complete, i.e. $\iota(M)$ is closed in $\Pi_{\mathscr{S}} \mathbf{R}$ and hence M is smoothly real-compact. □

3.3 Corollary: *Let $(M, \mathscr{S})$ be a smooth space with smooth partitions of unity (i.e. to every open covering U of M there is a family $\mathscr{F} \subseteq \mathscr{S}$ such that all $f \in \mathscr{F}$ are non negative and the family $\{x: f(x) > 0\}_{f \in \mathscr{F}}$ is a locally finite covering subordinated to U and $\sum_{f \in \mathscr{F}} f = 1$), then M is smoothly real-compact.*

Proof: Since M admits smooth partitions of unity, M is paracompact and therefore real-compact (see [2, p. 337]). This corollary depends on the set-theory: beware of measurable cardinals!). Furthermore M has the property (1) of the lemma, since $A_0 := \{x: f(x) \leqq a\}$ and $A_1 := \{x: f(x) \geqq b\}$ are disjoint closed subsets hence by partition of unity there is an $f_0 \in \mathscr{S}$ with $f_0|_{A_0} = 0$, $f_0|_{A_1} = 1$. □

3.4 Remark:

1. For finite dimensional paracompact manifolds this gives the classical "Exercise of Milnor and Stasheff".

2. Every paracompact manifold modelled on a locally convex space with smooth partitions of unity has itself smooth partitions of unity, hence is smoothly real-compact. This applies especially to the NLF-manifolds considered by [15], as well as to paracompact manifolds modelled on arbitrary Hilbertspaces or $c_0(\Gamma)$ with any set Γ (see [20]).

3. Results for finite order differentiability can be obtained along similar lines. We are content with the arche-typical C^∞-case.

3.5 Proposition: *Any product of real lines with the smooth functions in the sense C_c^∞ of* [7] *and C^∞ of* [3] *and* [9, 10] *is smoothly real-compact.*

Proof: First for C^∞: Any continuous map $f: \prod \mathbf{R} \to \mathbf{R}$ factorizes over $\prod_A \mathbf{R}$ with A countable (see [2]) this is even true for sequentially continuous maps provided the index set of the product has a non-real-measurable cardinal (see [13]) and for Mackey sequentially continuous maps by a similar proof. Since every smooth map in the sense of C^∞ is continuous with respect to Mackey converging sequences (cf. [10]) it is thus continuous with respect to the product topology, and hence the initial topology induced by the smooth maps is just the product topology. Obviously $\prod \boldsymbol{R}$ is real-compact. So it remains to verify condition (2) of the lemma. Let $f: \prod \mathbf{R} \to \mathbf{R}$ be continuous. Then f can be factorized into $f \circ \mathrm{pr}_A$ for some countable A. Thus we have to verify the property (2) only for smooth functions on $\mathbf{R}^{\mathbf{N}}$, but this is obvious, since this space is a nuclear Frechet space and hence has smooth partitions of unity.

Now for C_c^∞: We proceed directly, so let $\varphi: C_c^\infty(\mathbf{R}^\Gamma) \to \mathbf{R}$ be an algebra homomorphism.

Step 1: Consider the restriction of φ to the linear subspace $C_c^\infty(\mathbf{R}^\Gamma) \supseteq \mathbf{R}^{(\Gamma)} = \bigoplus_\Gamma \mathbf{R}$. Being a linear functional this restriction is an element of $(\mathbf{R}^{(\Gamma)})' = \mathbf{R}^\Gamma$. Call this point x_φ and we have $\varphi(g) = \langle g, x_\varphi \rangle$ for every continuous linear g on $\mathbf{R}^\Gamma$.

Step 2: Let $f \in C^\infty(\mathbf{R}^\Gamma, \mathbf{R})$ be such that $f|_U = 0$ for some neighborhood U of x_φ. We claim that $\varphi(f) = 0$:

Without loss of generality let us assume that $x_\varphi = 0$. Then U contains a neighborhood $\{x = (x^i) \in \mathbf{R}^\Gamma : |x^i| < \frac{1}{n}$ for all $i \in F\}$ where F is a finite subset of Γ. Let $g \in C^\infty(\mathbf{R}^F)$ be such that $g(0) = 0$ and $g(t) = 1$ for all t with some coordinate $t_i \geqq \frac{1}{n}$. $g(u) = \sum_j \int_0^1 \frac{\partial g}{\partial y^j}(ty) \cdot y^j \, dt = \sum_j h_j(y)\, y^j$. Then $h := g \circ \mathrm{pr}_F$ has the property $h(0) = 0$ and $h(x) = 1$ for $x \notin U$. Thus $h \cdot f = f$. So $\varphi(f) = \sum_j \varphi(h_j \circ \mathrm{pr}_F)\, \varphi(\mathrm{pr}_j)\, \varphi(f) = 0$, since pr_j is linear and thus $\varphi(\mathrm{pr}_j) = 0$.

Step 3: Now let $f \in C_c^\infty(\mathbf{R}^\Gamma, \mathbf{R})$ be arbitrary. We claim that $\varphi(f) = f(x_\varphi)$:

By step 2, $\varphi(f)$ depends only on $f|_U$ for some neighborhood U of x_φ in $\mathbf{R}^\Gamma$ and we take U so small that $f|_U$ depends only on finitely many coordinates $(x^i)_{i \in F}$. Then

$$f(x) = f(x_\varphi) + \int_0^1 \sum_i \frac{\partial f}{\partial x^i}(x_\varphi + t(x - x_\varphi))\,(x^i - x^i_\varphi)\,dt \qquad \text{for } x \in U$$

and so:

$$f|_U = f(x_\varphi) \cdot 1 + \sum_i h_i\, \mathrm{pr}_i(\cdot - x_\varphi)$$

$$\varphi(f) = \varphi(f|_U) = f(x_\varphi) \cdot 1 + \sum_i \varphi(h_i)\, \varphi(\mathrm{pr}_i(\cdot - x_\varphi)) = f(x_\varphi)\,. \quad \square$$

3.6 Remarks:

1. For a measurable cardinal Γ the smooth functions on $\mathbf{R}^\Gamma$ in the sense of C_c^∞ and C^∞ are different.

2. The subspace of an uncountable product of **R**'s given by all vectors with countable support is not smoothly real-compact if structured with the C^∞-functions, because it is not real-compact [2, p. 148, 153] although it is a convenient vector space in the sense of [3], [9].

3. An uncountable product $\Pi\, \mathbf{R}$ does not satisfy the following property stronger than the one in Lemma 3.1: For two closed disjoint subsets $A_i \subseteq \Pi\, \mathbf{R}$ there is a continuous function $f\colon \Pi\, \mathbf{R} \to \mathbf{R}$ with $f(A_1) \cap f(A_2) = \emptyset$.

Hence such a product is neither paracompact nor normal, although the smooth maps do generate the topology.

Proof: Let $A_i := \{x \in \prod_\Gamma \mathbf{N} : \text{for every } j \neq i \text{ there is at most one } s \in \Gamma \text{ with } x_s = j\}$. Clearly the sets A_i are closed and disjoint and $\mathrm{pr}_A(A_1) \cap \mathrm{pr}_A(A_2) \neq \emptyset$ for any countable subset A of Γ. Since any continuous function f depends only on countably many coordinates it cannot separate these two sets. $\square$

3.7 Open problems:

1. Is (l^1, C^∞) not smoothly real-compact? The C^∞-topology on l^1 is coarser than the norm topology. More generally, is any Banach space with rough norm [12] not smoothly real-compact?

2. We suspect that for any smooth space real-compactness and smoothly real-compactness are equivalent.

References

[1] R. Bonic, J. Frampton: Smooth functions on Banach manifolds, J. Math. Mech. **15** (1966), 877–898.

[2] R. Engelking: Outline of general topology, North Holland 1968.

[3] A. Frölicher: Smooth structures, in: Springer Lecture Notes **962** (1982), 69–81.

[4] A. Frölicher, B. Gisin, A. Kriegl: General differentiation theory, in: Category theoretic methods in geometry Proc., Aarhus (1983), p. 125–153.

[5] A. Frölicher, A. Kriegl: Linear spaces and Differentiation Theory, John Wiley & Sons, Chichester 1988.

[6] L. Gilman, M. Jerison: Rings of continuous functions, van Nostrand 1960.

[7] H. H. Keller: Differential calculus in locally convex spaces. Springer Lecture Notes **417,** 1974.

[8] A. Kock: Synthetic differential geometry, London Math. Soc. Lect. Notes Series **51,** 1981.

[9] A. Kriegl: Die richtigen Räume für Analysis im Unendlich-Dimensionalen, Monatshefte f. Math. **94** (1982), 109–124.

[10] A. Kriegl: Eine kartesisch abgeschlossene Kategorie glatter Abbildungen zwischen beliebigen lokalkonvexen Vektorräumen, Monatshefte f. Math. **95** (1983), 287–309.

[11] A. Kriegl: A cartesian closed extension of the category of Banach manifolds, in: Categorical Topology, Proc. Conference Toledo, Ohio, 1983, (1984), p. 323–336.

[12] E. B. Leach, J. H. M. Whitfield: Differentiable functions and rough norms on Banach spaces, Proc. AMS **33** (1972), 120–126.

[13] S. Mazur: On continuous mappings on cartesian products, Fund. Math. **39** (1952), 229–238.

[14] P. Michor: Manifolds of differentiable mappings, Shiva Math. Series **3**, 1980 Orpington.

[15] P. Michor: Manifolds of smooth mappings IV, Cahiers Top. Geom. Diff. **24** (1983), 57–86.

[16] P. Michor: A convenient setting for differential geometry and global analysis, Cahiers Top. Geom. Diff. **25** (1984), 63–112.

[17] J. W. Milnor, J. D. Stasheff: Characteristic Classes, Ann. of Math. Stud., Princeton Univ. Press, 1974 Princeton.

[18] I. Moerdijk, G. E. Reyes: C^∞-Rings, preprint 1984 Montreal.

[19] R. Sikorski: Differential Modules, Colloq. M. **24** (1971), 45–79.

[20] H. Torunzcyk: Smooth partitions of unity on some non-separable Banach spaces, Stud. Math. **46** (1973), 43–51.

A. Kriegl
P. Michor
W. Schachmayer
Mathematisches Institut
der Universität Wien
Strudlhofgasse 4
A-1090 Vienna

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