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GRADED DERIVATIONS OF THE ALGEBRA OF DIFFERENTIAL FORMS ASSOCIATED WITH A CONNECTION

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Introduction

The central part of calculus on manifolds is usually the calculus of differential forms and the best known operators are exterior derivative, Lie derivatives, pullback and insertion operators. Differential forms are a graded commutative algebra and one may ask for the space of graded derivations of it. It was described by Frölicher and Nijenhuis in [1], who found that any such derivation is the sum of a Lie derivation $\Theta(K)$ and an insertion operator $i(L)$ for tangent bundle valued differential forms $K, L \in \Omega^k(M; TM)$. The Lie derivations give rise to the famous Frölicher-Nijenhuis bracket, an extension of the Lie bracket for vector fields to a graded Lie algebra structure on the space $\Omega(M; TM)$ of vector valued differential forms. The space of graded derivations is a graded Lie algebra with the graded commutator as bracket, and this is the natural living ground for even the usual formulas of calculus of differential forms. In [8] derivations of even degree were integrated to local flows of automorphisms of the algebra of differential forms.

In [6] we have investigated the space of all graded derivations of the graded $\Omega(M)$-module $\Omega(M; E)$ of all vector bundle valued differential forms. We found that any such derivation, if a covariant derivative ∇ is fixed, may uniquely be written as $\Theta_\nabla(K) + i(L) + \mu(\Xi)$ and that this space of derivations is a very convenient setup for covariant derivatives, curvature etc. and that one can get the characteristic classes of the vector bundle in a very straightforward and simple manner. But the question arose of how all these nice formulas may be lifted to the linear frame bundle of the vector bundle. This paper gives an answer.

In [7] we have shown that differential geometry of principal bundles carries over nicely to principal bundles with structure group the diffeomorphism group of a fixed manifold S, and that it may be expressed completely in terms of finite dimensional manifolds, namely as (generalized) connections on fiber bundles with standard fiber S, where the structure group is the whole diffeomorphism group. But some of the properties of connections remain true for still more general situations: in the main part of this paper a connection will be just a fiber projection onto a (not necessarily integrable) distribution or sub vector

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX bundle of the tangent bundle. Here curvature is complemented by cocurvature and the Bianchi identity still holds. In this situation we determine the graded Lie algebra of all graded derivations over the horizontal projection of a connection and we determine their commutation relations. Finally, for a principal connection on a principal bundle and the induced connection on an associated bundle we show how one may pass from one to the other. The final results relate derivations on vector bundle valued forms and derivations over the horizontal projection of the algebra of forms on the principal bundle with values in the standard vector space.

We use [6] and [7] as standard references: even the notation is the same. Some formulas of [6, section 1] can also be found in the original paper [1].

1. Connections

1.1. Let (E, p, M, S) be a fiber bundle: So E, M and S are smooth manifolds, $p : E \to M$ is smooth and each $x \in M$ has an open neighborhood U such that $p^{-1}(x) =: E|U$ is fiber respectingly diffeomorphic to $U \times S$. S is called the *standard fiber*. The *vertical bundle*, called VE or $V(E)$, is the kernel of $Tp : TE \to TM$. It is a sub vector bundle of $TE \to E$.

A *connection* on (E, p, M, S) is just a fiber linear projection $\phi : TE \to VE$; so ϕ is a 1-form on E with values in the vertical bundle, $\phi \in \Omega^1(E; VE)$. The kernel of ϕ is called the *horizontal subspace* $ker\phi$. Clearly for each $u \in E$ the mapping $T_u p : ker\phi \to T_{p(u)}M$ is a linear isomorphism, so $(Tp, \pi_E) : ker\phi \to TM \times_M E$ is a diffeomorphism, fiber linear in the first coordinate, whose inverse $\chi : TM \times_M E \to ker\phi \to TE$ is called the *horizontal lifting*. Clearly the connection ϕ can equivalently be described by giving a horizontal sub vector bundle of $TE \to E$ or by specifying the horizontal lifting χ satisfying $(Tp, \pi_E) \circ \chi = Id_{TM\times_M E}$.

The notion of connection described here is thoroughly treated in [7]. There one can find parallel transport, which is not globally defined in general, holonomy groups, holonomy Lie algebras, a method for recognizing G-connections on associated fiber bundles, classifying spaces for fiber bundles with fixed standard fiber S and universal connections for this. Here we want to treat a more general concept of connection.

1.2. Let M be a smooth manifold and let F be a sub vector bundle of its tangent bundle TM. Bear in mind the vertical bundle over the total space of a principal fiber bundle.

Definition. A *connection* for F is just a smooth fiber projection $\phi : TM \to F$ which we view as a 1-form $\phi \in \Omega^1(M; TM)$.

So $\phi_x : T_x M \to F_x$ is a linear projection for all $x \in M$. $ker\phi =: H$ is a sub vector bundle of constant rank of TM. We call F the *vertical bundle* and H the *horizontal bundle*. $h := Id_{TM} - \phi$ is then the complementary projection, called the *horizontal projection*. A connection ϕ as defined here has been called an almost product structure by Guggenheim and Spencer.

Let $\Omega_{\text{hor}}(M)$ denote the space of all *horizontal* differential forms $\{\omega : i_X\omega = 0 \text{ for } X \in F\}$. Despite its name, this space depends only on F, not on the choice of the connection ϕ. Likewise, let $\Omega_{\text{ver}}(M)$ be the space of *vertical* differential forms $\{\omega : i_X\omega = 0 \text{ for } X \in H\}$. This space depends on the connection.

1.3. Curvature. Let

$$[\ \ ,\ \ ] : \Omega^k(M; TM) \times \Omega^l(M; TM) \longrightarrow \Omega^{k+l}(M; TM)$$

be the Frölicher-Nijenhuis-bracket as explained in [6] or in the original [1]. It induces a graded Lie algebra structure on $\Omega(M;TM) := \bigoplus \Omega^k(M;TM)$ with one-dimensional center generated by $Id \in \Omega^1(M;TM)$. For $K, L \in \Omega^1(M;TM)$ we have (see [1] or [6, 1.9])

$$\begin{aligned}[K,L](X,Y) = {} & [K(X),L(Y)] - [K(Y),L(X)] \\ & - L([K(X),Y] - [K(Y),X]) \\ & - K([L(X),Y] - [L(Y),X]) \\ & + (L\circ K + K \circ L)([X,Y]).\end{aligned}$$

From this formula it follows immediately that

$$[\phi,\phi] = [h,h] = -[\phi,h] = 2(R+\bar{R}),$$

where $R \in \Omega^2_{\text{hor}}(M;TM)$ and $\bar{R} \in \Omega^2_{\text{ver}}(M;TM)$ are given by $R(X,Y) = \phi[hX,hY]$ and $\bar{R}(X,Y) = h[\phi X, \phi Y]$, respectively. Thus R is *the* obstruction against integrability of the horizontal bundle H; R is called the *curvature* of the connection ϕ. Likewise $\bar{R}$ is *the* obstruction against integrability of the vertical bundle F; we call $\bar{R}$ the *cocurvature* of ϕ.

$[\phi,\phi] = 2(R+\bar{R})$ has been called the torsion of ϕ by Frölicher and Nijenhuis.

1.4. Lemma. *(Bianchi-Identity)* $[R+\bar{R},\phi] = 0$ *and* $[R,\phi] = i(R)\bar{R} + i(\bar{R})R$

Proof:. We have $2[R+\bar{R},\phi] = [[\phi,\phi],\phi] = 0$ by the graded Jacobi identity. For the second equation we use the Frölicher-Nijenhuis operators as explained in [6,section1]. $2R = \phi.[\phi,\phi] = i([\phi,\phi])\phi$, and by [6,1.10.2] we have $i([\phi,\phi])[\phi,\phi] = [i([\phi,\phi])\phi,\phi] - [\phi, i([\phi,\phi])\phi] + i([\phi,[\phi,\phi]])\phi + i([\phi,[\phi,\phi]])\phi = 2[i([\phi,\phi])\phi,\phi] = 4[R,\phi]$. So $[R,\phi] = \frac{1}{4} i([\phi,\phi])[\phi,\phi] = i(R+\bar{R})(R+\bar{R}) = i(R)\bar{R} + i(\bar{R})R$, since R has vertical values and kills vertical values, so $i(R)R = 0$; likewise for $\bar{R}$. □

2. Graded derivations for a connection

2.1.. We begin with some algebraic preliminaries. Let $A = \bigoplus_{k\in\mathbb{Z}} A_k$ be a graded commutative algebra and let $I : A \to A$ be an idempotent homomorphism of graded algebras, so we have $I(A_k) \subset A_k$, $I(a.b) = I(a).I(b)$ and $I \circ I = I$. A linear mapping $D : A \to A$ is called a *graded derivation over I of degree k*, if $D : A_q \to A_{q+k}$ and $D(a.b) = D(a).I(b) + (-1)^{k.|a|} I(a).D(b)$, where $\mid a \mid$ denotes the degree of a.

Lemma. *If D_k and D_l are derivations over I of degree k and l respectively, and if furthermore D_k and D_l both commute with I, then the graded commutator $[D_k, D_l] := D_k \circ D_l - (-1)^{k.l} D_l \circ D_k$ is again a derivation over I of degree $k+l$.*

The space $Der^I(A) = \bigoplus Der^I_k(A)$ of derivations over I which commute with I is a graded Lie subalgebra of $(End(A), [\quad,\quad])$.

The proof is a straightforward computation.

2.2.. Let M be a smooth manifold and let F be a sub vector bundle of TM as considered in section 1. Let ϕ be a connection for F and consider its horizontal projection $h : TM \to H$. We define $h^* : \Omega(M) \to \Omega_{\text{hor}}(M)$ by

$$(h^*\omega)(X_1, \dots, X_p) := \omega(hX_1, \dots, hX_p).$$

Then h^* is a surjective graded algebra homomorphism and $h^*|\Omega_{\text{hor}}(M) = Id$. Thus $h^* : \Omega(M) \to \Omega_{\text{hor}}(M) \to \Omega(M)$ is an idempotent graded algebra homomorphism.

2.3. Let now D be a derivation over h^* of $\Omega(M)$ of degree k. Then $D|\Omega^0(M)$ might be nonzero, e.g. $h^* \circ d$. We call D *algebraic*, if $D|\Omega^0(M) = 0$. Then $D(f.\omega) = 0 + f.D(\omega)$, so D is of tensorial character. Since D is a derivation, it is uniquely determined by $D|\Omega^1(M)$. This is given by a vector bundle homomorphism $T^*M \to \Lambda^{k+1}T^*M$, which we view as $K \in \Gamma(\Lambda^{k+1} \otimes TM) = \Omega^{k+1}(M;TM)$. We write $D = i^h(K)$ to express this dependence.

Lemma. 1. *We have*

$$(i^h(K)\omega)(X_1,\dots,X_{k+p}) = \\ = \tfrac{1}{(k+1)!\,(p-1)!}\sum_\sigma \varepsilon(\sigma).\omega(K(X_{\sigma 1},\dots,X_{\sigma(k+1)}), hX_{\sigma(k+2)},\dots,hX_{\sigma(k+p)}),$$

and for any $K \in \Omega^{k+1}(M;TM)$ this formula defines a derivation over h^*.

2. *We have $[i^h(K), h^*] = i^h(K)\circ h^* - h^*\circ i^h(K) = 0$ if and only if $h\circ K = K \circ \Lambda^{k+1}h : \Lambda^{k+1}TM \to TM$. We write $\Omega^{k+1}(M;TM)^h$ for the space of all h-equivariant forms like that.*

3. *For $K \in \Omega^{k+1}(M;TM)^h$ and $L \in \Omega^{l+1}(M;TM)^h$ we have: $[K,L]^{\wedge,h} := i^h(K)L - (-1)^{kl}\, i^h(L)K$ is an element of $\Omega^{k+l+1}(M;TM)^h$ and this bracket is a graded Lie algebra structure on $\Omega^{*+1}(M;TM)^h$ such that $i^h([K,L]^{\wedge,h}) = [i^h(K), i^h(L)]$ in $Der^h\Omega(M)$.*

4. *For $K \in \Omega^{k+1}(M;TM)$ and the usual insertion operator* [6, 1.2] *we have:*

$$\begin{aligned} &h^*\circ i(K) = i^h(K\circ \Lambda^{k+1}h) = h^*\circ i^h(K)\\ &i(K)\circ h^* = i^h(h\circ K) = i^h(K)\circ h^*\\ &[i(K),h^*] = [i^h(K),h^*] = i^h(h\circ K - K\circ\Lambda^{k+1}h)\\ &h^*\circ i(K)\circ h^* = h^*\circ i^h(K)\circ h^* = i^h(h\circ K\circ \Lambda^{k+1}h). \end{aligned}$$

Proof. **1.** We first need the following assertion: For $\omega_j \in \Omega^1(M)$ we have

$$(1)\qquad i^h(K)(\omega_1\wedge\dots\wedge\omega_p) = \sum_{j=1}^p (-1)^{(j+1).k}(\omega_1\circ h)\wedge\dots\wedge(\omega_j\circ K)\wedge\dots\wedge(\omega_p\circ h).$$

This follows by induction on p from the derivation property. From (1) we get

$$\begin{aligned} (2)\qquad & i^h(K)(\omega_1\wedge\dots\wedge\omega_p)(X_1,\dots,X_{k+p}) = \\ &= \sum_{j=1}^p (-1)^{(j-1)k}\big((\omega_1\circ h)\wedge\dots\wedge(\omega_j\circ K)\wedge\dots\wedge(\omega_p\circ h)\big)(X_1,\dots,X_{k+p}) = \\ &= \sum_j (-1)^{(j-1)k}\tfrac{1}{(k+1)!}\sum_\sigma \varepsilon(\sigma)\omega_1(hX_{\sigma 1})\dots\omega_j\big(K(X_{\sigma j},\dots,X_{\sigma(j+k)})\big)\dots \\ &\qquad\qquad \dots\omega_p(hX_{\sigma(k+p)}). \end{aligned}$$

Now we consider the following expression:

$$\begin{aligned} (3)\qquad & \sum_{\pi\in\mathcal{S}_{k+p}} \varepsilon(\pi).(\omega_1\wedge\dots\wedge\omega_p)\big(K(X_{\pi 1},\dots,X_{\pi(k+1)}), hX_{\pi(k+2)},\dots,hX_{\pi(k+p)}\big) = \\ &= \sum_\pi \varepsilon(\pi)\sum_{\rho\in\mathcal{S}_p}\varepsilon(\rho)\omega_{\rho 1}\big(K(X_{\pi 1},\dots,X_{\pi(k+1)})\big)\omega_{\rho 2}(hX_{\pi(k+2)})\dots\omega_{\rho p}(hX_{\pi(k+p)}), \end{aligned}$$

where we sum over $\pi \in \mathcal{S}_{k+p}$ and $\rho \in \mathcal{S}_p$. Reshuffling these permutations one may check that $\frac{1}{(k+1)!\,(p-1)!}$ times expression (3) equals expression (2). By linearity assertion 1 follows.

2. For $\omega \in \Omega^1(M)$ we have $\omega \circ h \circ K = i^h(K)h^*\omega = h^* i^h(K)\omega = h^*(\omega \circ K) = \omega \circ K \circ \Lambda^{k+1} h$.

3. We have $[i^h(K), i^h(L)] = i^h([K,L]^{\wedge,h})$ for a unique $[K,L]^{\wedge,h} \in \Omega^{k+l+1}(M;TM)^h$ by Lemma 2.1. For $\omega \in \Omega^1(M)$ we get then

$$\begin{aligned}\omega \circ [K,L]^{\wedge,h} &= i^h([K,L]^{\wedge,h})\omega = [i^h(K), i^h(L)]\omega = \\ &= i^h(K)i^h(L)\omega - (-1)^{kl} i^h(L) i^h(K)\omega = i^h(K)(\omega \circ L) - (-1)^{kl} i^h(L)(\omega \circ K).\end{aligned}$$

So by 1 we get

$$\begin{aligned}& i^h(K)(\omega \circ L)(X_1, \dots, X_{k+l+1}) = \\ &= \tfrac{1}{(k+1)!\,l!} \sum_{\sigma \in \mathcal{S}_{k+l+1}} \varepsilon(\sigma)(\omega \circ L)\big(K(X_{\sigma 1}, \dots, X_{\sigma(k+l+1)}), hX_{\sigma(k+2)}, \dots, hX_{\sigma(k+l+1)})\big)\end{aligned}$$

and similarly for the second term.

4. All these mappings are derivations over h^* of $\Omega(M)$. So it suffices to check that they coincide on $\Omega^1(M)$ which is easy for the first two assertions. The latter two assertions are formal consequences thereof. □

2.4. For $K \in \Omega^k(M;TM)$ we have the *Lie derivation* $\Theta(K) = [i(K), d] \in \mathrm{Der}_k \Omega(M)$, where d denotes exterior derivative. See [6, 1.3].

Proposition. 1. *Let D be a derivation over h^* of $\Omega(M)$ of degree k. Then there are unique elements $K \in \Omega^k(M;TM)$ and $L \in \Omega^{k+1}(M;TM)$ such that*

$$D = \Theta(K) \circ h^* + i^h(L).$$

D is algebraic if and only if $K = 0$.

2. *If D is in $Der^h_k\Omega(M)$ (so $[D, h^*] = 0$) then*

$$D = h^* \circ \Theta(K) \circ h^* + i^h(\tilde{L})$$

for unique $K \in \Omega^k_{\mathrm{hor}}(M;TM)$ and $\tilde{L} \in \Omega^{k+1}(M;TM)^h$. K is the same as in 1.

Define $\Theta^h(K) := h^* \circ \Theta(K) \circ h^* \in \mathrm{Der}^h_k \Omega(M)$, then in 2 we can write $D = \Theta^h(K) + i^h(\tilde{L})$.

Proof. Let $X_j \in \mathcal{X}(M)$ be vector fields and consider the mapping $ev_{(X_1,\dots,X_k)} \circ D : C^\infty(M) = \Omega^0(M) \to \Omega^k(M) \to \Omega^0(M) = C^\infty(M)$, given by $f \mapsto (Df)(X_1, \dots, X_k)$. This map is a derivation of the algebra $C^\infty(M)$, since we have $D(f.g)(X_1, \dots, X_k) = (Df.g + f.Dg)(X_1, \dots, X_k) = (Df)(X_1, \dots, X_k).g + f.(Dg)(X_1, \dots, X_k)$. So it is given by the action of a unique vector field $K(X_1, \dots, X_k)$, which clearly is an alternating and $C^\infty(M)$-multilinear expression of the X_j; we may thus view K as an element of $\Omega^k(M;TM)$. The defining equation for K is $Df = df \circ K$ for $f \in C^\infty(M)$. Now we consider $D - [i(K), d] \circ h^*$. This is a derivation over h^* and vanishes on $\Omega^0(M)$, since $[i(K),d]h^*f = [i(K),d]f = df \circ K = Df$. It is algebraic and by 2.3 we have $D - [i(K),d] \circ h^* = i^h(L)$ for some unique $L \in \Omega^{k+1}(M;TM)$.

Now suppose that D commutes with h^*, i. e. $D \in \operatorname{Der}^h_k(M)$. Then for all $f \in C^\infty(M)$ we have $df \circ K \circ \Lambda^k h = h^*(df \circ K) = h^*Df = Dh^*f = Df = df \circ K$, so $K \circ \Lambda^k h = K$ or $h^*K = K$ and K is in $\Omega^k_{\text{hor}}(M;TM)$. Now let us consider $D - h^*[i(K),d]h^* \in \operatorname{Der}^h_k\Omega(M)$, which is algebraic, since $h^*[i(K),d]h^*f = h^*(df \circ K) = df \circ (h^*K) = df \circ K = Df$. So $D - h^*[i(K),d]h^* = i^h(\tilde{L})$ for some unique $\tilde{L} \in \Omega^{k+1}(M;TM)^h$ by 2.3 again. □

2.5.. For the connection ϕ (respectively. the horizontal projection H) we define the *classical covariant derivative* $D^h := h^* \circ d$. Then clearly $D^h : \Omega(M) \to \Omega_{\text{hor}}(M)$ and D^h is a derivation over h^*, but D^h does not commute with h^*, so it is not an element of $\operatorname{Der}^h_1\Omega(M)$. Therefore and guided by 2.4 we define the (new) *covariant derivative* as $d^h := h^* \circ d \circ h^* = D^h \circ h^* = \Theta^h(Id)$. We will consider d^h as the most important element in $\operatorname{Der}^h_1\Omega(M)$.

Proposition.

1. *$d^h - D^h = i^h(R)$, where R is the curvature.*
2. *$[d,h^*] = \Theta(\phi) \circ h^* + i^h(R + \bar{R})$.*
3. *$d \circ h^* - d^h = \Theta(\phi) \circ h^* + i^h(\bar{R})$.*
4. *$D^h \circ D^h = i^h(R) \circ d$.*
5. *$[d^h, d^h] = 2d^h \circ d^h = 2i^h(R) \circ d \circ h^* = 2h^* \circ i(R) \circ d \circ h^*$.*
6. *$D^h|\Omega_{\text{hor}}(M) = d^h|\Omega_{\text{hor}}(M)$.*

Assertion 6 shows that not a lot of differential geometry, on principal fiber bundles e.g., is changed if we use d^h instead of D^h. In this paper we focus our attention on d^h.

Proof. 1 is 2 minus 3. Both sides of 2 are derivations over h^* and it is straightforward to check that they agree on $\Omega^0(M) = C^\infty(M)$ and $\Omega^1(M)$. So they agree on the whole of $\Omega(M)$. The same method proves equation 3. The rest is easy. □

2.6. Theorem. *Let K be in $\Omega^k(M;TM)$. Then we have:*

1. *If $K \in \Omega^k(M;TM)^h$, then $\Theta^h(K) = [i^h(K), d^h]$.*
2. *$h^* \circ \Theta(K) = h^* \circ \Theta(K \circ \Lambda^k h) + (-1)^{k-1} i^h(i^h(R)K)$.*
3. *$\Theta^h(K) = \Theta^h(h^*K) + (-1)^{k-1} i^h(i^h(R)(h \circ K))$.*
4. *If $K \in \Omega^k(M;TM)^h$, then $\Theta^h(K) = \Theta^h(h^*K) = \Theta^h(h \circ K)$.*

Now let $K \in \Omega^k_{\text{hor}}(M;TM)$. Then we have:

5. *$h^* \circ \Theta(K) - \Theta^h(K) = i^h(\phi \circ [K,\phi] \circ \Lambda h)$.*
6. *$\Theta(K) \circ h^* - \Theta^h(K) = i^h((-1)^k i(\bar{R})(h \circ K) - h \circ [K,\phi])$.*
7. *$[h^*, \Theta(K)] = i^h(\phi \circ [K,\phi] \circ \Lambda h + h \circ [K,\phi] - (-1)^k i(\bar{R})(h \circ K))$.*
8. *Let $K_j \in \Omega^{k_j}_{\text{hor}}(M;TM), j = 1,2$. Then*

$$[\Theta^h(K_1), \Theta^h(K_2)] = \Theta^h([K_1,K_2]) - \\ - i^h\big(h \circ [K_1, \phi \circ [K_2,\phi]] \circ \Lambda h - (-1)^{k_1k_2} h \circ [K_2, \phi \circ [K_1,\phi]] \circ \Lambda h\big).$$

Note. The third term in 8 should play some rôle in the study of deformations of graded Lie algebras.

Proof: 1. Plug in the definitions.

2. Check that $h^*\Theta(K)f = h^*\Theta(h^*K)f$ for $f \in C^\infty(M)$. For $\omega \in \Omega^1(M)$ we get $h^*\Theta(K)\omega = h^*i^h(h^*K)d\omega - (-1)^{k-1}h^*di(K)\omega$, using 2.3.4; also we have $h^*\Theta(h^*K)\omega = h^*i^h(h^*K)d\omega - (-1)^{k-1}h^*dh^*i(K)\omega$. Collecting terms and using 2.5.1 one obtains the result.

3 follows from 2 and 2.3.4. **4** is easy.

5. It is easily checked that the left hand side is an algebraic derivation over h^*. So it suffices to show that both sides coincide if applied to any $\omega \in \Omega^1(M)$. This will be done by induction on k. The case $k = 0$ is easy. Suppose $k \geq 1$ and choose $X \in \mathfrak{X}(M)$. Then we have in turn:

$$\begin{aligned}
i(X)h^*\Theta(K)\omega &= i^h(hX)\Theta(K)\omega = h^*i(hX)\Theta(K)\omega \\
&= h^*[i(hX), \Theta(K)]\omega + (-1)^k h^*\Theta(K)i(hX)\omega \\
&= h^*\Theta(i(hX)K)\omega + (-1)^k h^*i([hX, K])\omega + (-1)^k h^*\Theta(K)(\omega(hX)) \\
i(X)h^*\Theta(K)h^*\omega &= i(X)h^*\Theta(K)(\omega \circ h) \\
&= h^*\Theta(i(hX)K)(\omega \circ h) + (-1)^k h^*i([hX, K])(\omega \circ h) + (-1)^k h^*\Theta(K)(\omega(hX)) \\
i(X)\big(h^*\Theta(K) &- h^*\Theta(K)h^*\big)\omega \\
&= \big(h^*\Theta(i(hX)K) - h^*\Theta(i(hX)K)h^*\big)\omega + (-1)^k h^*i([hX, K])(\omega \circ \phi) \\
&= i^h\big(\phi \circ [i(hX)K, \phi] \circ \Lambda h\big)\omega + (-1)^k h^*i([hX, K])(\omega \circ \phi) \qquad \text{by induction on } k. \\
i(X)i^h\big(\phi \circ [K, \phi] \circ \Lambda h\big)\omega &= i(X)\big(\omega \circ \phi \circ [K, \phi] \circ \Lambda h\big) = (\omega \circ \phi)\big(i(hX)[K, \phi]\big) \circ \Lambda h\big) \\
&= (\omega \circ \phi)\big([i(hX)K, \phi] + (-1)^k[K, i(hX)\phi = 0] - (-1)^k i([K, hX])\phi + \\
&\qquad + (-1)^{k-1} i([\phi, hX])K\big) \circ \Lambda h, \qquad \text{by [6, 1.10.2]} \\
&= i^h\big(\phi \circ [i(hX)K, \phi] \circ \Lambda h\big) + (-1)^k h^*i([hX, K])(\omega \circ \phi) + \\
&\qquad + (-1)^k \omega \circ \phi \circ (i([\phi, hX])K) \circ \Lambda h.
\end{aligned}$$

Now $[\phi, hX](Y) = [\phi Y, hX] - \phi([Y, hX])$ by [6,1.9]. Therefore $(i([\phi, hX])K) \circ \Lambda h = i(h \circ [\phi, hX])K \circ \Lambda h = 0$, and $i(X)\big(h^* \circ \Theta(K) - h^* \circ \Theta(K) \circ h^*\big)\omega = i(X)i^h(\phi \circ [K, \phi] \circ \Lambda h)\omega$ for all $X \in \mathfrak{X}(M)$. Since the $i(X)$ jointly separate points, the equation follows.

6. K horizontal implies that $\Theta(K) \circ h^* - h^* \circ \Theta(K) \circ h^*$ is algebraic. So again it suffices to check that both sides of equation 6 agree when applied to an arbitrary 1-form $\omega \in \Omega^1(M)$. This will again be proved by induction on k. We start with the case $k = 0$: Let $K = X$ and $Y \in \mathfrak{X}(M)$.

$$\begin{aligned}
&\big(\Theta(X)h^*\omega\big)(Y) - \big(h^*\Theta(X)\, h^*\omega\big)(Y) = \\
&\qquad = \big(\Theta(X)(\omega \circ h)\big)(Y - hY) = X.\omega(h\phi Y) - (\omega \circ h)([X, \phi Y]) = 0 - \omega(h[X, \phi Y])
\end{aligned}$$

Note that $[X, \phi](Y) = [X, \phi Y] - \phi[X, Y]$, so $h[X, \phi](Y) = h[X, \phi Y]$, and therefore $\big(i^h(0 - h[X, \phi])\omega\big)(Y) = -\omega(h[X, \phi Y])$.

Now we treat the case $k > 0$. For $X \in \mathfrak{X}(M)$ we have in turn:

$$\begin{aligned}
i(X)\Theta(K)h^*\omega &= \big((-1)^k\Theta(K)i(X) + \Theta(i(X)K) + (-1)^k i([X, K])\big)(h^*\omega) \qquad \text{by [6,1.6]} \\
&= (-1)^k\Theta(K)(\omega(hX)) + \Theta(i(X)K)h^*\omega + (-1)^k\omega(h[X, K]).
\end{aligned}$$

We noticed above that $\Theta(K)f = h^*\Theta(K)f$ for $f \in C^\infty(M)$ since K is horizontal.

$$\begin{aligned}
&i(X)h^*\Theta(K)h^*\omega = i^h(hX)\Theta(K)h^*\omega = h^*i(hX)\Theta(K)h^*\omega, \quad \text{by 2.3.4.}\\
&= (-1)^k h^*\Theta(K)(\omega(hX)) + h^*\Theta(i(hX)K)h^*\omega + (-1)^k h^*(\omega(h[X,K])), \quad \text{see above.}
\end{aligned}$$

$$\begin{aligned}
&i(X)\big(\Theta(K)h^*\omega - h^*\Theta(K)h^*\omega\big) =\\
&= \Theta(i(X)K)h^*\omega - h^*\Theta(i(hX)K)h^*\omega + (-1)^k\big(\omega(h[X,K]) - h^*(\omega\circ h\circ [hX,K])\big) =\\
&\qquad = -i^h(h[i(X)K,\phi])\omega + (-1)^{k-1}i^h(h\circ i(\bar R)i(X)K)\omega +\\
&\qquad\qquad + (-1)^k\omega(h\circ [X,K]) - (-1)^k\omega(h\circ (h^*[hX,K])),
\end{aligned}$$

by induction, where we also used $i(X)K = i(hX)K$, which holds for horizontal K.

$$\begin{aligned}
&-i(X)i^h(h\circ [K,\phi])\omega = -i(X)(\omega\circ h\circ [K,\phi]) = \omega\circ h\circ (i(X)[K,\phi]) =\\
&= \omega\circ h\circ \big([i(X)K,\phi] + (-1)^k[K,i(X)\phi] - (-1)^k i([K,X])\phi + (-1)^{k-1}i([\phi,X])K\big) =\\
&\quad = -i^h(h\circ [i(X)K,\phi])\omega + (-1)^k\omega(h\circ [\phi X,K]) + 0 + (-1)^k\omega(h\circ (i([\phi,X])K),
\end{aligned}$$

where we used [6, 1.10.2] and $h\circ\phi = 0$.

$$\begin{aligned}
&i(X)i^h(h\circ i(\bar R)K)\omega = i(X)(\omega\circ h\circ i(\bar R)K) = \omega\circ h\circ (i(X)i(\bar R)K).\\
&i(X)(\text{left hand side} - \text{right hand side})\omega =\\
&\quad = \omega\big((-1)^{k-1}h\circ i(\bar R)\circ i(X)\circ K + (-1)^k h\circ [X,K] - (-1)^k h\circ (h^*[hX,K]) -\\
&\qquad - (-1)^k h\circ [\phi X,K] - (-1)^k h\circ i([\phi,X])K - (-1)^k h\circ (i(X)i(\bar R)K)\big)\\
&\quad = (-1)^k(\omega\circ h)\big(-i([\bar R,X]^\wedge)K + [X-\phi X,K] - h^*[hX,K] - i([\phi,X])K\big) =\\
&\quad = (-1)^k(\omega\circ h)\big(i(0 - i(X)\bar R)K + [hX,K] + h^*[hX,K] - i([\phi,X])K\big)
\end{aligned}$$

$$\begin{aligned}
&-h\big(i(i(X)\bar R)K\big)(Y_1,\dots,Y_k) = -\sum_{j=1}^k hK(Y_1,\dots,\bar R(X,Y_j),\dots,Y_k) =\\
&\quad = -\sum_{j=1}^k hK(Y_1,\dots,[\phi X,\phi Y_j],\dots,Y_k), \text{ since } K \text{ is horizontal.}
\end{aligned}$$

$$\begin{aligned}
&-h[K,hX]((Y_1,\dots,Y_k) = -h[K(Y_1,\dots,Y_k),hX] + \sum_{j=1}^k hK(Y_1,\dots,[Y_j,hX],\dots,Y_k).\\
&h(h^*[K,hX])(Y_1,\dots,Y_k) = h[K,hX](hY_1,\dots,hY_k) =\\
&\quad = h[K(hY_1,\dots,hY_k),hX] - \sum hK(hY_1,\dots,[hY_j,hX],\dots,hY_k) =\\
&\quad = h[K(Y_1,\dots,Y_k),hX] - \sum hK(Y_1,\dots,[hY_j,hX],\dots,Y_k),
\end{aligned}$$

since K is horizontal. We also have $h[\phi,X](Y) = h([\phi Y,X] - \phi[Y,X]) = h[\phi Y,X]$, and thus we finally get

$$\begin{aligned}
&-h(i([\phi,X]K)(Y_1,\dots,Y_k) = -h(i(h[\phi,X])K)(Y_1,\dots,Y_k) =\\
&\quad = -\sum hK(Y_1,\dots,h[\phi,X](Y_j),\dots,Y_k) = -\sum hK(Y_1,\dots,[\phi Y_j,X],\dots,Y_k).
\end{aligned}$$

All these sum to zero and equation 6 follows.

7. This is equation 5 minus equation 6.

8. We compute as follows:

$$\begin{aligned}
&[\Theta^h(K_1),\Theta^h(K_2)] = \\
&\quad = h^*\Theta(K_1)h^*\Theta(K_2)h^* - (-1)^{k_1k_2}h^*\Theta(K_2)h^*\Theta(K_1)h^* = \\
&\quad = \big(h^*\Theta(K_1) - h^*i(\phi\circ[K_1,\phi])\big)\theta(K_2)h^* - \\
&\qquad\quad - (-1)^{k_1k_2}\big(h^*\Theta(K_2) - h^*i(\phi\circ[K_2,\phi])\big)\Theta(K_1)h^*, \quad \text{by 5,} \\
&\quad = h^*\big(\Theta(K_1)\Theta(K_2) - (-1)^{k_1k_2}\Theta(K_2)\Theta(K_1)\big)h^* - \\
&\qquad\quad - h^*\big(i(\phi\circ[K_1,\phi])\Theta(K_2) - (-1)^{k_1k_2}i(\phi\circ[K_2,\phi])\Theta(K_1)\big)h^* = \\
&\quad = h^*\Theta([K_1,K_2])h^* + \text{Remainder} = \Theta^h([K_1,K_2]) + \text{Remainder}.
\end{aligned}$$

For the following note that $i(\phi\circ[K_1,\phi])h^* = i^h(h\circ\phi\circ[K_1,\phi]) = 0$, by 2.3.4.

$$\begin{aligned}
\text{Remainder} &= -h^*i(\phi\circ[K_1,\phi])\Theta(K_2)h^* + (-1)^{k_1k_2}i(\phi[K_2,\phi])\Theta(K_1)h^* + \\
&\quad + (-1)^{k_1k_2}h^*\Theta(K_2)i(\phi\circ[K_1,\phi])h^* - h^*\Theta(K_1)i(\phi\circ[K_2,\phi])h^*, \quad \text{which are 0,} \\
&= -h^*[i(\phi\circ[K_1,\phi]),\Theta(K_2)]h^* + (-1)^{k_1k_2}h^*[i(\phi\circ[K_2,\phi]),\Theta(K_1)]h^* = \\
&= h^*\big(\Theta(i(\phi\circ[K_1,\phi])K_2 = 0) + (-1)^{k_2}i([\phi\circ[K_1,\phi],K_2])\big)h^* + \\
&\quad + (-1)^{k_1k_2}h^*\big(\Theta(i(\phi\circ[K_2,\phi])K_1 = 0) + (-1)^{k_1}i([\phi\circ[K_2,\phi],K_1])\big)h^* = \\
&= -i^h\big(h\circ[K_1,\phi\circ[K_2,\phi]]\circ\Lambda h - (-1)^{k_1k_2}h\circ[K_2,\phi\circ[K_1,\phi]]\circ\Lambda h\big), \quad \text{by 2.3.4.}
\end{aligned}$$

2.7. Corollary.

(1) $h^*\circ\Theta(\phi) = i^h(R)$.
(2) $\Theta^h(\phi) = 0$.
(3) $[h^*,\Theta(h)] = -2i^h(R) - i^h(\bar R)$.
(4) $h^*\circ\Theta(h) = \Theta^h(h) - 2i^h(R)$.
(5) $\Theta^h(h) + i^h(\bar R) = \Theta(h)\circ h^*$.
(6) $[\Theta^h(h),\Theta^h(h)] = 2\Theta^h(R) = 2h^*\circ i(R)\circ d\circ h^*$.
(7) $\Theta^h(\bar R) = 0$.

Proof. 1 follows from 2.6.2. 2 follows from 2.6.3. 3 follows from 2.6.7. 4 follows from 2.6.5. 5 is 4 minus 3. 6 follows from 2.6.8 and some further computation and 7 is similar. □

2.8. Theorem: 1. *For $K\in\Omega^k(M;TM)$ and $L\in\Omega^{l+1}(M;TM)^h$ we have*

$$[i^h(L),\Theta^h(K)] = \Theta^h(i(L)K) + (-1)ki^h(h\circ[L,K]\circ\Lambda h).$$

2. *For $L\in\Omega^{l+1}(M;TM)^h$ and $K_i\in\Omega^{k_i}(M;TM)$ we have*

$$\begin{aligned}
i^h(L)(h^*[K_1,K_2]) &= h^*[i(L)K_1,K_2] + (-1)^{k_1l}h^*[K_1,i(L)K_2] - \\
&\quad - (-1)^{k_1l}i^h(h^*[K_1,L])K_2 + (-1)^{(k_1+l)k_2}i^h(h^*[K_2,L])K_1.
\end{aligned}$$

3. *For $L \in \Omega^{l+1}(M;TM)^h$ and $K_i \in \Omega^{k_i}_{\text{hor}}(M;TM)$ we have*

$$i^h(L)(h^*[K_1,K_2]) = h^*[i^h(L)K_1,K_2] + (-1)^{k_1 l}h^*[K_1,i^h(L)K_2] - \\ - (-1)^{k_1 l}i^h(h^*[K_1,L])K_2 + (-1)^{(k_1+l)k_2}i^h(h^*[K_2,L])K_1.$$

4. *For $K \in \Omega^k_{\text{hor}}(M;TM)$ and $L_i \in \Omega^{l_i+1}(M;TM)^h$ we have*

$$h\circ[K,[L_1,L_2]^{\wedge,h}]\circ\Lambda h = \\ = [h\circ[K,L_1]\circ\Lambda h, L_2]^{\wedge,h} + (-1)^{kl_1}[L_1, h\circ[K,L_2]\circ\Lambda h]^{\wedge,h} - \\ - (-1)^{kl_1}h\circ[i^h(L_1)K,L_2]\circ\Lambda h - (-1)^{(l_1+k)l_2}h\circ[i^h(L_2)K,L_1]\circ\Lambda h.$$

5. *Finally for $K_i \in \Omega^{k_i}(M;TM)$ and $L_i \in \Omega^{k_i+1}(M;TM)^h$ we have*

$$[\Theta^h(K_1)+i^h(L_1),\Theta^h(K_2)+i^h(L_2)] = \\ = \Theta^h\big([K_1,K_2]+i(L_1)K_2-(-1)^{k_1k_2}i(L_2)K_1\big) + \\ + i^h\big([L_1,L_2]^{\wedge,h} + (-1)^{k_2}h\circ[L_1,K_2]\circ\Lambda h - (-1)^{k_1(k_2+1)}h\circ[L_,K_1]\circ\Lambda h - \\ - h\circ[K_1,\phi\circ[K_2,\phi]]\circ\Lambda h + (-1)^{k_1k_2}h\circ[K_2,\phi\circ[K_1,\phi]]\circ\Lambda h\big).$$

Proof: 1.

$$\begin{aligned}[i^h(L),\Theta^h(K)] &= i^h(L)h^*\Theta(K)h^* - (-1)^{kl}h^*\Theta(K)h^*i^h(L) = \\ &= h^*\big(i(L)\Theta(K) - (-1)^{kl}\Theta(K)i(L)\big)h^*, && \text{by 2.3.4.}\\ &= h^*\big(\Theta(i(L)K) + (-1)^k i([L,K])\big)h^*, && \text{by [6,1.6]}\\ &= \Theta^h(i(L)K) + (-1)^k i^h(h\circ[L,K]\circ\Lambda h), && \text{by 2.3.4 and 2.4.}\end{aligned}$$

2. $i^h(L)(h^*[K_1,K_2]) = h^*i(L)[K_1,K_2]$ by 2.3.4 and the rest follows from [6, 1.10].

3. Note that for horizontal K we have $i(L)K = i^h(L)K$ and use this in formula 2.

4. This follows from 1 by writing out the graded Jacobi identity for the graded commutators, uses horizontality of K and 2.3 several times.

5. Collect all terms from 2.6.8 and 1. □

2.9. The space of derivations over h^* of $\Omega(M)$ is a graded module over the graded algebra $\Omega(M)$ with the action $(\omega\wedge D)\psi = \omega\wedge D\psi$. The subspace $\operatorname{Der}^h\Omega(M)$ is stable under $\omega\wedge\cdot$ if and only if $\omega\in\Omega_{\text{hor}}(M)$.

Theorem. *1. For derivations D_1, D_2 over h^* of degree k_1, k_2, respectively, and $\omega \in \Omega^q_{\text{hor}}(M)$ we have $[\omega\wedge D_1, D_2] = \omega\wedge[D_1,D_2] - (-1)^{(q+k_1)k_2}D_2\omega\wedge h^*D_1$.*

2. For $L\in\Omega(M;TM)$ and $\omega\in\Omega(M)$ we have $\omega\wedge i^h(L) = i^h(\omega\wedge L)$.

3. For $K\in\Omega^k(M)$ and $\omega\in\Omega^q(M)$ we have

$$(\omega\wedge\Theta(K))\circ h^* = \Theta^h(\omega\wedge K)\circ h^* + (-1)^{q+k-1}i^h(d\omega\wedge(h\circ K)).$$

4. For $K\in\Omega^k_{\text{hor}}(M;TM)$ and $\omega\in\Omega^q_{\text{hor}}(M)$ we have

$$\omega\wedge\Theta^h(K) = \Theta^h(\omega\wedge K) + (-1)^{q+k-1}i^h(d^h\omega\wedge(h\circ K)).$$

5. $\Omega(M;TM)^h$ is stable under multiplication with $\omega\wedge$. if and only if $\omega\in\Omega_{\text{hor}}(M)$. For $L_j\in\Omega^{l_j+1}(M;TM)^h$ and $\omega\in\Omega^q_{\text{hor}}(M)$ we have

$$[\omega\wedge L_1, L_2]^{\wedge,h} = \omega\wedge[L_1,L_2]^{\wedge,h} - (-1)^{(q+l_1)l_2}\,i^h(L_2)\omega\wedge(h\circ L_1).$$

The proof consists of straightforward computations.

3. Derivations on Principal Fiber Bundles and Associated Bundles. Liftings.

3.1. Let (E,p,M,S) be a fiber bundle and let $\phi \in \Omega^1(M;TM)$ be a connection for it as described in 1.1. The cocurvature $\bar{R}$ is then zero. We consider the horizontal lifting $\chi : TM \times_M E \to TE$ and use it to define the mapping $\chi_* : \Omega^k(M;TM) \to \Omega^k(E;TE)$ by

$$(\chi_* K)_u(X_1,\dots,X_k) := \chi\big(K(T_u p.X_1,\dots,T_u p.X_k),u\big).$$

Then $\chi_* K$ is horizontal with horizontal values: $h \circ \chi_* K = \chi_* K = \chi_* K \circ \Lambda h$. For $\omega \in \Omega^q(M)$ we have $\chi_*(\omega \wedge K) = p^*\omega \wedge \chi_* K$, so $\chi_* : \Omega(M;TM) \to \Omega(E;TE)^h$ is a module homomorphism of degree 0 over the algebra homomorphism $p^* : \Omega(M) \to \Omega_{\text{hor}}(E)$.

Theorem. *In this setting we have for $K, K_i \in \Omega(M;TM)$:*

(1) $p^* \circ i(K) = i(\chi_* K) \circ p^* = i^h(\chi_* K) \circ p^* : \Omega(M) \to \Omega_{\text{hor}}(E)$.
(2) $p^* \circ \Theta(K) = \Theta(\chi_* K) \circ p^* = \Theta^h(\chi_* K) \circ p^* : \Omega(M) \to \Omega_{\text{hor}}(E)$.
(3) $i(\chi_* K_1)\chi_* K_2 = i^h(\chi_* K_1)\chi_* K_2 = \chi_*(i(K_1)K_2)$.
(4) *The following is a homomorphism of graded Lie algebras:*

$$\chi_* : (\Omega(M;TM),[\quad,\quad]^\wedge) \longrightarrow (\Omega(E;TE)^h,[\quad,\quad]^{\wedge,h}) \subset (\Omega(E;TE),[\quad,\quad]^\wedge)$$

(5) $\chi_*[K_1,K_2] = h \circ [\chi_* K_1, \chi_* K_2] = h \circ [\chi_* K_1, \chi_* K_2] \circ \Lambda h$.

Proof. (1). $p^* \circ i(K)$ and $i(\chi_* K) \circ p^*$ are graded module derivations : $\Omega(M) \to \Omega(E)$ over the algebra homomorphism p^* in a sense similar to 2.1: $p^* i(K)(\omega \wedge \psi) = p^* i(K)\omega \wedge p^*\psi + (-1)^{(k-1)|\omega|} p^*\omega \wedge p^* i(K)\psi$ and similarly for the other expression. Both are zero on $C^\infty(M)$ and it suffices to show that they agree on $\omega \in \Omega^1(M)$. This is an easy computation. Furthermore $i^h(\chi_* K)|\Omega_{\text{hor}}(E) = i(\chi_* K)|\Omega_{\text{hor}}(E)$, so the rest of 1 follows. (2) follows from (1) by expanding the graded commutators. (3). Plug in the definitions and use 2.3.1. (4) follows from 2.3.3 and [6,1.2.2]. (5) follows from 2 and some further considerations. □

3.2. Let (P,p,M,G) be a principal fiber bundle with structure group G, and write $r : P \times G \to P$ for the principal right action. A connection $\phi^P : TP \to VP$ in the sense of 1.1 is a *principal connection* if and only if it is G-equivariant i.e., $T(r^g) \circ \phi^P = \phi^P \circ T(r^g)$ for all $g \in G$. Then $\phi^P_u = T_e(r_u).\varphi_u$, where φ, a 1-form on P with values in the Lie algebra of G, is the usual description of a principal connection. Here we used the convention $r(u,g) = r^g(u) = r_u(g)$ for $g \in G$ and $u \in P$. The curvature R of 1.3 corresponds to the negative of the usual curvature $d\varphi + \frac{1}{2}[\varphi,\varphi]$ and the Bianchi identity of 1.4 corresponds to the usual Bianchi identity.

The G-equivariant graded derivations of $\Omega(P)$ are exactly the $\Theta(K) + i(L)$ with K and L G-equivariant i.e., $T(r^g) \circ K = K \circ \Lambda T(r^g)$ for all $g \in G$. This follows from [6, 2.1]. The G-equivariant derivations in $\mathrm{Der}^h\Omega(P)$ are exactly the $\Theta^h(K) + i^h(L)$ with $K \in \Omega_{\text{hor}}(P;TP)$, $L \in \Omega(P;TP)^h$ and K,L G-equivariant. Theorem 3.1 can be applied and can be complemented by G-equivariance.

3.3. In the situation of 2.2 let us suppose furthermore, that we have a smooth left action of the structure group G on a manifold S, $\ell : G \times S \to S$. We consider the associated bundle $(P[S] = P \times_G S, p, M, S)$ with its G-structure and induced connection $\phi^{P[S]}$. Let

$q : P \times S \to P[S]$ be the quotient mapping, which is also the projection of a principal G-bundle. Then $\phi^{P[S]} Tq(X_u, Y_s) = Tq(\phi^P X_u, Y_s)$ by [7, 2.4].

Now we want to analyze $q^* \circ (\Theta^h(K) + i^h(L)) : \Omega(P[S]) \to \Omega(P \times S)$. For that we consider the associated bundle $(P \times S = P \times_G (G \times S), p \circ pr_1, M, G \times S, G)$, where the structure group G acts on $G \times S$ by left translation on G alone. Then the induced connection is $\phi^{P \times S} = \phi^P \times Id_S : T(P \times S) = TP \times TS \to VP \times TS$ and we have the following

Lemma. *$Tq \circ h^{P \times S} = h^{P[S]} \circ Tq$ and $D^{h^{P \times S}} \circ q^* = q^* \circ D^{h^{P[S]}}$ for the classical covariant derivatives.*

The proof is obvious from the description of induced connections given in [7, 2.4].

3.4. Theorem. *In the situation of 3.3 we have:*
1. $q^* \circ (h^{P[S]})^* = (h^{P \times S})^* \circ q^* : \Omega(P[S]) \to \Omega(P \times S)$.
2. For $K \in \Omega^k(P[S]; T(P[S]))$ we have

$$q^* \circ i^{h^{P[S]}}(K) = i^{h^{P \times S}}({\chi^{P \times S}}_* K) \circ q^* : \Omega(P[S]) \to \Omega_{\text{hor}}(P \times S)$$
$$q^* \circ \Theta^{h^{P[S]}}(K) = \Theta^{h^{P \times S}}({\chi^{P \times S}}_* K) \circ q^* : \Omega(P[S]) \to \Omega_{\text{hor}}(P \times S).$$

Proof. 1. Use lemma 3.3 and the definition of h^* in 2.2. For 2 use also 2.3.1 and $Tq \circ \chi_* K = K \circ \Lambda Tq$. The last equation is easy. □

3.5. For a principal bundle (P, p, M, G) let $\rho : G \to GL(V)$ be a linear representation in a finite dimensional vector space V. We consider the associated vector bundle $(E = P \times_G V = P[V], p, M, V)$, a principal connection ϕ^P on P and the induced G-connection on E, which gives rise to the covariant exterior derivative $\nabla^E \in \text{Der}_1 \Omega(M; E)$ as investigated in [6, section 3]. We also have the mapping $q^\sharp : \Omega^k(M; E) \to \Omega^k(P, V)$, given by

$$(q^\sharp \Psi)_u(X_1, \dots, X_k) = {q_u}^{-1}(\Psi_{p(u)}(T_u p.X_1, \dots, T_u p.X_k).$$

Recall that $q_u : \{u\} \times V \to E_{p(u)}$ is a linear isomorphism. Then $q^\sharp$ is an isomorphism of $\Omega(M; E)$ onto the subspace $\Omega_{\text{hor}}(P, V)^G$ of all horizontal and G-equivariant forms.

Theorem. *1. $q^\sharp \circ \nabla^E = D^{h^P} \circ q^\sharp = d^{h^P} \circ q^\sharp$.*
2. For $K \in \Omega^k(M; TM)$ we have:

$$q^\sharp \circ i(K) = i({\chi^P}_* K) \circ q^\sharp = i^{h^P}({\chi^P}_* K) \circ q^\sharp$$
$$q^\sharp \circ \Theta_{\nabla^E}(K) = \Theta^{h^P}({\chi^P}_* K) \circ q^\sharp : \Omega(M; E) \to \Omega_{\text{hor}}(P, V)^G.$$

Proof. 1. A proof of this is buried in [3, p 76, p 115]. A global proof is possible, but it needs a more detailed description of the passage from ϕ^P to ∇^E, e.g. the connector $K : TE \to E$. We will not go into that here.

2. $q^\sharp \circ i(K)$ and $i({\chi^P}_* P) \circ q^\sharp$ are both derivations from the $\omega(M)$-module $\Omega(M; E)$ into the $\Omega(P)$-module $\Omega(P, V)$ over the algebra homomorphism $q^* : \Omega(M) \to \Omega(P)$. Both vanish on 0-forms. Thus it suffices to check that they coincide on $\Psi \in \Omega^1(M; E)$, which may be done by plugging in the definitions. $i({\chi^P}_* K)$ and $i^{h^P}({\chi^P}_* K)$ coincide on horizontal forms, so the first assertion follows. The second assertion follows from the first one, lemma 3.3, and [6, 3.7]. Note finally that $q^\sharp \circ \Theta_{\nabla^E}(K) \neq \Theta({\chi^P}_* K) \circ q^\sharp$ in general. □

3.6. In the setting of 3.6, for $\Xi \in \Omega^k(M; L(E,E))$, let $q^\sharp\Xi \in \Omega(P; L(V,V))$ be given by

$$(q^\sharp\Xi)_u(X_1, \dots, X_k) = {q_u}^{-1} \circ \Xi_{p(u)}(T_u p.X_1, \dots, T_u p.X_k) \circ q_u : V \to V,$$

where $L(E,E) = P[L(V,V), Ad \circ \rho]$ and $q^\sharp\Xi$ defined here coincides with that of 3.5.

Furthermore recall from [6, section 3] that any graded $\Omega(M)$-module homomorphism $\Omega(M;E) \to \Omega(M;E)$ of degree k is of the form $\mu(\Xi)$ for unique $\Xi \in \Omega^k(M; L(E,E))$, where

$$(\mu(\Xi)\Psi)(X_1, \dots, X_{k+q}) = \tfrac{1}{k!\,q!} \sum_\sigma \varepsilon(\sigma).\Xi(X_{\sigma 1}, \dots, X_{\sigma k}).\Psi(X_{\sigma(k+1)}, \dots, X_{\sigma(k+q)}).$$

Then any graded derivation D of the $\Omega(M)$-module $\Omega(M;E)$ can be uniquely written in the form $D = \Theta_{\nabla^E}(K) + i(L) + \mu(\Xi)$.

Theorem. *For $\Xi \in \Omega(M; L(E,E))$ we have*

$$q^\sharp \circ \mu(\Xi) = \mu(q^\sharp\Xi) \circ q^\sharp = \mu^{h^P}(q^\sharp\Xi) \circ q^\sharp : \Omega(M;E) \to \Omega_{\mathrm{hor}}(P,V)^G,$$

where for $\xi \in \Omega^k(P, L(V,V))$ the module homomorphism $\mu(\xi)$ of $\Omega(P,V)$ is as above for the trivial vector bundle $P \times V \to P$ and $\mu^h(\xi)$ is given by

$$(\mu^h(\xi)\omega)(X_1, \dots, X_{k+q}) = \\ = \frac{1}{k!\,q!} \sum_{\sigma \in \mathcal{S}_{k+q}} \varepsilon(\sigma)\xi(X_{\sigma 1}, \dots, X_{\sigma k})\omega(h^P X_{\sigma(k+1)}, \dots, h^P X_{\sigma(k+q)}).$$

The proof is straightforward.

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