

# THE HOMOGENEOUS SOBOLEV METRIC OF ORDER ONE ON DIFFEOMORPHISM GROUPS ON THE REAL LINE

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ABSTRACT. In this article we study Sobolev metrics of order one on diffeomorphism groups on the real line. We prove that the space  $\text{Diff}_1(\mathbb{R})$  equipped with the homogeneous Sobolev metric of order one is a flat space in the sense of Riemannian geometry, as it is isometric to an open subset of a mapping space equipped with the flat  $L^2$ -metric. Here  $\text{Diff}_1(\mathbb{R})$  denotes the extension of the group of all either compactly supported, rapidly decreasing, or  $W^{\infty,1}$ -diffeomorphisms, which allows for a shift toward infinity. Surprisingly, on the non-extended group the Levi-Civita connection does not exist. In particular, this result provides an analytic solution formula for the corresponding geodesic equation, the non-periodic Hunter–Saxton (HS) equation. In addition, we show that one can obtain a similar result for the two-component HS equation and discuss the case of the non-homogeneous Sobolev one metric, which is related to the Camassa–Holm equation.

## 1. INTRODUCTION

In recent decades it has been shown that various prominent partial differential equations (PDEs) arise as geodesic equations on certain infinite-dimensional manifolds. This phenomenon was first observed in the groundbreaking paper [1] for the incompressible Euler equation, which is the geodesic equation on the group of volume preserving diffeomorphisms with respect to the right-invariant  $L^2$ -metric. It was shown that this geometric approach could be extended to a whole variety of other PDEs used in hydrodynamics: the Camassa–Holm equation [6, 17], the Constantin–Lax–Majda equation [40, 4] and the Korteweg–de Vries equation [39, 2] to name but a few examples.

It was later realized that the geometric interpretation could be used to prove results about the behaviour of the PDEs. The first such result was [9], where the researchers showed the local well-posedness of Euler equations. Similar techniques were then applied to other PDEs, which arise as geodesic equations, see e.g. [7, 8, 10, 13, 5].

The analysis in this paper is mainly concerned with the Hunter–Saxton (HS) equation. For the periodic case it was shown in [16] that the HS equation is the geodesic equation on the homogeneous space  $\text{Diff}(S^1)/S^1$  with respect to the homogeneous Sobolev  $\dot{H}^1$ -metric of order one. Lenells used this geometric interpretation in [27, 28] to construct an analytic solution formula for the equation. In fact, he showed that the Riemannian manifold  $(\text{Diff}(S^1)/S^1, \dot{H}^1)$  was isometric to an open

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subset of an  $L^2$ -sphere in the space  $C^\infty(S^1, \mathbb{R})$  of periodic functions and thereby obtained an explicit formula for the corresponding geodesics on  $\text{Diff}(S^1)/S^1$ .

The aim of this article is twofold. First, we extend the results of [27] to groups of real-analytic and ultra-differentiable diffeomorphisms and show that the solutions of the HS equation are analytic in time and space. Second we consider the  $\dot{H}^1$ -metric and the HS equation on the real line. Our main result can be paraphrased as follows (see Sect. 4).

**Theorem.** *The non-periodic HS equation is the geodesic equation on the space  $\text{Diff}_{\mathcal{A}_1}(\mathbb{R})$  with respect to the right-invariant  $\dot{H}^1$ -metric. Furthermore, the space  $(\text{Diff}_{\mathcal{A}_1}(\mathbb{R}), \dot{H}^1)$  is isometric to an open subset in  $(\mathcal{A}(\mathbb{R}), L^2)$  and is thus a flat space in the sense of Riemannian geometry.*

Here  $\mathcal{A}(\mathbb{R})$  denotes one of the function spaces  $C_c^\infty(\mathbb{R})$ ,  $\mathcal{S}(\mathbb{R})$  or  $W^{\infty,1}(\mathbb{R})$  and  $\text{Diff}_{\mathcal{A}_1}(\mathbb{R})$  is an extension over the diffeomorphism group including shifts near  $+\infty$  (Sect. 2).

The first surprising fact is that the normal subgroups  $\text{Diff}_{\mathcal{A}}(\mathbb{R})$  do not admit the geodesic equation (or the Levi-Civita covariant derivative) for this right-invariant metric. The extended group  $\text{Diff}_{\mathcal{A}_1}(\mathbb{R})$  admits it but in a weaker sense than realized in [1] and follow-up papers (Sect. 3.2). We also sketch Arnold's curvature formula in this weaker setting.

The second surprising fact is that  $\text{Diff}_{\mathcal{A}_1}(\mathbb{R})$  with the  $\dot{H}^1$ -metric is a flat Riemannian manifold, as opposed to  $\text{Diff}(S^1)$ , which is a positively curved one.

The main ingredient for the proof of this result is the  $R$ -map, which allows us to isometrically embed  $\text{Diff}_{\mathcal{A}_1}(\mathbb{R})$  with the right-invariant  $\dot{H}^1$ -metric as an open subset of a (flat) pre-Hilbert space. This phenomenon has also been observed for the space  $\text{Imm}(S^1, \mathbb{R}^2)/\mathbb{R}^2$  of plane curves modulo translations (see. e.g., [26, 42, 3]), and on the semi-direct product space  $\text{Diff}(S^1) \ltimes C^\infty(S^1, \mathbb{R})$  (the corresponding geodesic equation is the periodic two-component HS equation; see [29]).

In the periodic case we extend the results to groups of real-analytic diffeomorphisms and ultra-differentiable diffeomorphisms of certain types and show that the HS equation has solutions which are real-analytic or ultra-differentiable if the initial diffeomorphism is.

In Sect. 5 we apply the same techniques to treat the two-component HS equation on the real line. We discuss the existence of the geodesic equation and construct an isometry between the configuration space  $\text{Diff}_{\mathcal{A}}(\mathbb{R}) \ltimes \mathcal{A}(\mathbb{R})$  and an open subset of a pre-Hilbert space.

Finally, we generalize the constructions to the case of the right-invariant non-homogeneous  $H^1$ -metric on  $\text{Diff}(S^1)$ , whose geodesic equation is the dispersion-free Camassa–Holm equation. In this case we define an  $R$ -map, whose image is a subspace of a pre-Hilbert space, no longer open.

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## 2. SOME DIFFEOMORPHISM GROUPS ON THE REAL LINE AND THE CIRCLE

The group of all orientation-preserving diffeomorphisms  $\text{Diff}(\mathbb{R})$  is not an open subset of  $C^\infty(\mathbb{R}, \mathbb{R})$  endowed with the compact  $C^\infty$ -topology and so it is not a smooth manifold with charts in the usual sense. One option is to consider it as a Lie group in the cartesian closed category of Frölicher spaces (see [21, Sect. 23]) with the structure induced by the injection  $f \mapsto (f, f^{-1}) \in C^\infty(\mathbb{R}, \mathbb{R}) \times C^\infty(\mathbb{R}, \mathbb{R})$ . Alternatively, one can use the theory of smooth manifolds based on smooth curves instead of charts from [34], [35], which agrees with the usual theory up to Banach manifolds. In this paper we will restrict our attention to subgroups of the whole diffeomorphism group, which are smooth Fréchet manifolds.

Let us first briefly recall the definition of a regular Lie group in the sense of [19]; see also [21, Sect. 38.4]. A smooth Lie group  $G$  with Lie algebra  $\mathfrak{g} = T_e G$  is called regular if the following conditions hold:

- For each smooth curve  $X \in C^\infty(\mathbb{R}, \mathfrak{g})$  there exists a smooth curve  $g \in C^\infty(\mathbb{R}, G)$  whose right logarithmic derivative is  $X$ , i.e.

$$(1) \quad \begin{cases} g(0) &= e \\ \partial_t g(t) &= T_e(\mu^{g(t)})X(t) = X(t).g(t). \end{cases}$$

The curve  $g$ , if it exists, is uniquely determined by its initial value  $g(0)$ .

- The map  $\text{evol}_G^r(X) = g(1)$  where  $g$  is the unique solution of (1), considered as a map  $\text{evol}_G^r : C^\infty(\mathbb{R}, \mathfrak{g}) \rightarrow G$  is  $C^\infty$ -smooth.

**2.1. The Group  $\text{Diff}_{\mathcal{B}}(\mathbb{R})$ .** The ‘largest’ regular Lie group in  $\text{Diff}(\mathbb{R})$  with charts is the group of all diffeomorphisms  $\varphi = \text{Id}_{\mathbb{R}} + f$  with  $f \in \mathcal{B}(\mathbb{R})$  such that  $f' > -1$ .  $\mathcal{B}(\mathbb{R})$  is the space of functions which have all derivatives (separately) bounded. It is a reflexive nuclear Fréchet space.

The space  $C^\infty(\mathbb{R}, \mathcal{B}(\mathbb{R}))$  of smooth curves  $t \mapsto f(t, \cdot)$  in  $\mathcal{B}(\mathbb{R})$  consists of all functions  $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$  satisfying the following property:

- For all  $k \in \mathbb{N}_{\geq 0}$  and  $n \in \mathbb{N}_{\geq 0}$  the expression  $\partial_t^k \partial_x^n f(t, x)$  is uniformly bounded in  $x \in \mathbb{R}$ , locally in  $t$ .

We can specify other regular Lie groups by requiring that  $g$  lies in certain spaces of smooth functions. Now we will discuss these spaces, describe the smooth curves in them, and describe the corresponding groups, specializing the results from [38], where most of these groups are treated on  $\mathbb{R}^n$  in full detail.

**2.2. Groups Related to  $\text{Diff}_c(\mathbb{R})$ .** The reflexive nuclear (LF) space  $C_c^\infty(\mathbb{R})$  of smooth functions with compact support leads to the well-known regular Lie group  $\text{Diff}_c(\mathbb{R})$ , see [21, Sect. 43.1].

We will now define an extension of this group which will play a major role in subsequent parts of this article.

Define  $C_{c,2}^\infty(\mathbb{R}) = \{f : f' \in C_c^\infty(\mathbb{R})\}$  as the space of antiderivatives of smooth functions with compact support. It is a reflexive nuclear (LF) space. We also define the space  $C_{c,1}^\infty(\mathbb{R}) = \{f \in C_{c,2}^\infty(\mathbb{R}) : f(-\infty) = 0\}$  of antiderivatives of the form  $x \mapsto \int_{-\infty}^x g \, dy$  with  $g \in C_c^\infty(\mathbb{R})$ .

$\text{Diff}_{c,2}(\mathbb{R}) = \{\varphi = \text{Id} + f : f \in C_{c,2}^\infty(\mathbb{R}), f' > -1\}$  is the corresponding group.

Define the two functionals  $\text{Shift}_\ell, \text{Shift}_r : \text{Diff}_{c,2}(\mathbb{R}) \rightarrow \mathbb{R}$  by

$$\text{Shift}_\ell(\varphi) = \text{ev}_{-\infty}(f) = \lim_{x \rightarrow -\infty} f(x), \quad \text{Shift}_r(\varphi) = \text{ev}_\infty(f) = \lim_{x \rightarrow \infty} f(x)$$

for  $\varphi(x) = x + f(x)$ . Then the short exact sequence of smooth homomorphisms of Lie groups

$$\text{Diff}_c(\mathbb{R}) \longrightarrow \text{Diff}_{c,2}(\mathbb{R}) \xrightarrow{(\text{Shift}_\ell, \text{Shift}_r)} (\mathbb{R}^2, +)$$

describes a semi-direct product, where a smooth homomorphic section  $s : \mathbb{R}^2 \rightarrow \text{Diff}_{c,2}(\mathbb{R})$  is given by the composition of flows  $s(a, b) = \text{Fl}_a^{X_\ell} \circ \text{Fl}_b^{X_r}$  for the vector fields  $X_\ell = f_\ell \partial_x, X_r = f_r \partial_x$  with  $[X_\ell, X_r] = 0$  where  $f_\ell, f_r \in C^\infty(\mathbb{R}, [0, 1])$  satisfy

$$(2) \quad f_\ell(x) = \begin{cases} 1 & \text{for } x \leq -1 \\ 0 & \text{for } x \geq 0, \end{cases} \quad f_r(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ 1 & \text{for } x \geq 1. \end{cases}$$

The normal subgroup  $\text{Diff}_{c,1}(\mathbb{R}) = \ker(\text{Shift}_\ell) = \{\varphi = \text{Id} + f : f \in C_{c,1}^\infty(\mathbb{R}), f' > -1\}$  of diffeomorphisms which have no shift at  $-\infty$  will play an important role subsequently.

**2.3. Groups Related to  $\text{Diff}_S(\mathbb{R})$ .** The regular Lie group  $\text{Diff}_S(\mathbb{R})$  was treated in [36, Sect. 6.4]. Let us summarize the most important results: the space  $\mathcal{S}(\mathbb{R})$  consisting of all rapidly decreasing functions is a reflexive nuclear Fréchet space.

*The space  $C^\infty(\mathbb{R}, \mathcal{S}(\mathbb{R}))$  of smooth curves in  $\mathcal{S}(\mathbb{R})$  consists of all functions  $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$  satisfying the following property:*

- For all  $k, m \in \mathbb{N}_{\geq 0}$  and  $n \in \mathbb{N}_{\geq 0}$ , the expression  $(1 + |x|^2)^m \partial_t^k \partial_x^n f(t, x)$  is uniformly bounded in  $x \in \mathbb{R}$ , locally uniformly bounded in  $t \in \mathbb{R}$ .

$\text{Diff}_S(\mathbb{R}) = \{\varphi = \text{Id} + f : f \in \mathcal{S}(\mathbb{R}), f' > -1\}$  is the corresponding regular Lie group.

We again define an extended space:

$\mathcal{S}_2(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : f' \in \mathcal{S}(\mathbb{R})\}$ , the space of antiderivatives of functions in  $\mathcal{S}(\mathbb{R})$ . It is isomorphic to  $\mathbb{R} \times \mathcal{S}(\mathbb{R})$  via  $f \mapsto (f(0), f')$ . It is again a reflexive nuclear Fréchet space, contained in  $\mathcal{B}(\mathbb{R})$ .

*The space  $C^\infty(\mathbb{R}, \mathcal{S}_2(\mathbb{R}))$  of smooth curves in  $\mathcal{S}_2(\mathbb{R})$  consists of all functions  $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$  satisfying the following property:*

- For all  $k, m \in \mathbb{N}_{\geq 0}$ , and  $n \in \mathbb{N}_{> 0}$ , the expression  $(1 + |x|^2)^m \partial_t^k \partial_x^n f(t, x)$  is uniformly bounded in  $x$  and locally uniformly bounded in  $t$ .

We also define the space  $\mathcal{S}_1(\mathbb{R}) = \{f \in \mathcal{S}_2(\mathbb{R}) : f(-\infty) = 0\}$  of antiderivatives of the form  $x \mapsto \int_{-\infty}^x g \, dy$  with  $g \in \mathcal{S}(\mathbb{R})$ .

$\text{Diff}_{\mathcal{S}_2}(\mathbb{R}) = \{\varphi = \text{Id} + f : f \in \mathcal{S}_2(\mathbb{R}), f' > -1\}$  is the corresponding regular Lie group. We have again the short exact sequence of smooth homomorphisms of Lie groups

$$\text{Diff}_S(\mathbb{R}) \longrightarrow \text{Diff}_{\mathcal{S}_2}(\mathbb{R}) \xrightarrow{(\text{Shift}_\ell, \text{Shift}_r)} (\mathbb{R}^2, +),$$

which splits via the same smooth homomorphic section  $s : \mathbb{R}^2 \rightarrow \text{Diff}_{\mathcal{S}_2}(\mathbb{R})$  as in 2.2 and, thus, describes a semi-direct product. The normal Lie subgroup  $\text{Diff}_{\mathcal{S}_1}(\mathbb{R}) = \ker(\text{Shift}_\ell)$  of diffeomorphisms which have no shift at  $-\infty$  will also play an important role later on.

**2.4. Groups Related to  $\text{Diff}_{W^{\infty,1}}(\mathbb{R})$ .** The space  $W^{\infty,1}(\mathbb{R}) = \bigcap_{k \geq 0} W^{k,1}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : f^{(k)} \in L^1(\mathbb{R}) \text{ for } k = 0, 1, 2, \dots\}$  is the intersection of all Besov spaces of type  $L^1$ . It is a reflexive Fréchet space. By the Sobolev inequality we have  $W^{\infty,1}(\mathbb{R}) \subset \mathcal{B}(\mathbb{R})$ ; thus also  $W^{\infty,1}(\mathbb{R}) \subset W^{\infty,p}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : f^{(k)} \in L^p(\mathbb{R}) \text{ for } p = 0, 1, 2, \dots\}$ . For any  $f \in W^{\infty,1}(\mathbb{R})$  each derivative  $f^{(k)}$  is smooth and converges to 0 for  $x \rightarrow \pm\infty$  since it is in  $\mathcal{B}(\mathbb{R})$  and since  $C_c^\infty(\mathbb{R})$  is dense in  $W^{\infty,1}(\mathbb{R})$ .

The space  $C^\infty(\mathbb{R}, W^{\infty,1}(\mathbb{R}))$  of smooth curves  $t \mapsto f(t, \cdot)$  in  $W^{\infty,1}(\mathbb{R})$  consists of all functions  $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$  satisfying the following property:

- For all  $k \in \mathbb{N}_{\geq 0}$  and  $n \in \mathbb{N}_{\geq 0}$  the expression  $\|\partial_t^k \partial_x^n f(t, \cdot)\|_{L^1(\mathbb{R})}$  is locally bounded in  $t$ .

$\text{Diff}_{W^{\infty,1}}(\mathbb{R}) = \{\varphi = \text{Id} + f : f \in W^{\infty,1}(\mathbb{R}), f' > -1\}$  denotes the corresponding regular Lie group.

We again consider an extended space:

$W_2^{\infty,1}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) : f' \in W^{\infty,1}(\mathbb{R})\}$  is the space of bounded antiderivatives of functions in  $W^{\infty,1}(\mathbb{R})$ . It is isomorphic to  $\mathbb{R} \times W^{\infty,1}(\mathbb{R})$  via  $f \mapsto (f(0), f')$ . The space  $C^\infty(\mathbb{R}, W_2^{\infty,1}(\mathbb{R}))$  of smooth curves in  $W_2^{\infty,1}(\mathbb{R})$  consists of all functions  $f \in C^\infty(\mathbb{R}^2, \mathbb{R})$  satisfying the following property:

- For all  $k \in \mathbb{N}_{\geq 0}$ ,  $n \in \mathbb{N}_{> 0}$  and  $t \in \mathbb{R}$  the expression  $\|\partial_t^k \partial_x^n f(t, \cdot)\|_{L^1(\mathbb{R})}$  is locally bounded in  $t$ .

We also define the space  $W_1^{\infty,1}(\mathbb{R}) = \{f \in W_2^{\infty,1}(\mathbb{R}) : f(-\infty) = 0\}$  of antiderivatives of the form  $x \mapsto \int_{-\infty}^x g \, dy$  for  $g \in W^{\infty,1}(\mathbb{R})$ .

$\text{Diff}_{W_2^{\infty,1}}(\mathbb{R}) = \{\varphi = \text{Id} + f : f \in W^{\infty,1}(\mathbb{R}), f' > -1\}$  denotes the corresponding regular Lie group.

We have again the following exact sequence of smooth homomorphisms of regular Lie groups:

$$\text{Diff}_{W^{\infty,1}}(\mathbb{R}) \longrightarrow \text{Diff}_{W_2^{\infty,1}}(\mathbb{R}) \xrightarrow{(\text{Shift}_\ell, \text{Shift}_r)} (\mathbb{R}^2, +),$$

which splits with the same section as for  $\text{Diff}_{c,2}(\mathbb{R})$ . The group  $\text{Diff}_{W_1^{\infty,1}}(\mathbb{R}) = \ker(\text{Shift}_\ell)$  of diffeomorphisms, which have no shift at  $-\infty$ , will play an important role later on.

**Remark on the  $H^\infty = W^{\infty,2}$ -case.** One may wonder why we use the groups related to  $\text{Diff}_{W^{\infty,1}}(\mathbb{R})$  instead of those modelled on the more usual intersection  $H^\infty$  of all Sobolev spaces. The reason is that  $H^\infty \not\subset L^1$ ; thus, for  $f \in H^\infty(\mathbb{R})$  the antiderivative  $x \mapsto \int_{-\infty}^x f(y) \, dy$  is not bounded in general, and the extended groups are not contained in  $\text{Diff}_{\mathcal{B}}(\mathbb{R})$  and thus do not admit charts. If we model the groups on  $H^\infty \cap L^1$ , then they are **not** Lie groups: right translations are smooth, but left translations are not. See [25, Sect. 14] for this surprising fact.

**Theorem.** *All the groups  $\text{Diff}_c(\mathbb{R})$ ,  $\text{Diff}_{c,1}(\mathbb{R})$ ,  $\text{Diff}_{c,2}(\mathbb{R})$ ,  $\text{Diff}_{\mathcal{S}}(\mathbb{R})$ ,  $\text{Diff}_{\mathcal{S}_1}(\mathbb{R})$ ,  $\text{Diff}_{\mathcal{S}_2}(\mathbb{R})$ ,  $\text{Diff}_{W^{\infty,1}}(\mathbb{R})$ ,  $\text{Diff}_{W_1^{\infty,1}}(\mathbb{R})$ ,  $\text{Diff}_{W_2^{\infty,1}}(\mathbb{R})$ , and  $\text{Diff}_{\mathcal{B}}(\mathbb{R})$  are smooth regular Lie groups. We have the following smooth injective group homomorphisms:*

$$\begin{array}{ccccccc}
\text{Diff}_c(\mathbb{R}) & \longrightarrow & \text{Diff}_{\mathcal{S}}(\mathbb{R}) & \longrightarrow & \text{Diff}_{W^{\infty,1}}(\mathbb{R}) & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{Diff}_{c,1}(\mathbb{R}) & \longrightarrow & \text{Diff}_{\mathcal{S}_1}(\mathbb{R}) & \longrightarrow & \text{Diff}_{W_1^{\infty,1}}(\mathbb{R}) & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{Diff}_{c,2}(\mathbb{R}) & \longrightarrow & \text{Diff}_{\mathcal{S}_2}(\mathbb{R}) & \longrightarrow & \text{Diff}_{W_2^{\infty,1}}(\mathbb{R}) & \longrightarrow & \text{Diff}_{\mathcal{B}}(\mathbb{R})
\end{array}$$

*Each group is a normal subgroup in any other in which it is contained, in particular in  $\text{Diff}_{\mathcal{B}}(\mathbb{R})$ .*

*Proof.* That the groups  $\text{Diff}_c(\mathbb{R})$ ,  $\text{Diff}_{\mathcal{S}}(\mathbb{R})$ , and  $\text{Diff}_{\mathcal{B}}(\mathbb{R})$  are regular Lie groups is proved (for  $\mathbb{R}^n$  instead of  $\mathbb{R}$  in [38], and in [21, Sect. 43.1] for  $\text{Diff}_c(\mathbb{R})$ ). Moreover, the group  $\text{Diff}_{H^{\infty}}(\mathbb{R}^n)$  (where  $H^{\infty} = W^{\infty,2}$ ) is treated in [38]; the proof for  $W^{\infty,1}$  is the same. See also [25, Theorem 8.1]. The extension to the semi-direct products is easy and is proved in [21, Sect. 38.9]. That each group is normal in the largest one is also proved in [38].  $\square$

**2.5. A Remark on the Existence of Normal Subgroups.** This section will not be used in the remainder of the paper. It is a well-known result that  $\text{Diff}_c(\mathbb{R})$  is a simple group (see [30], [31], [32]). We want to discuss some effects of this result for the diffeomorphism groups introduced in the previous sections.

*Existence of normal subgroups in  $\text{Diff}_{c,2}(\mathbb{R})$ :* We first claim that any non-trivial normal subgroup  $N$  of  $\text{Diff}_{c,2}(\mathbb{R})$  intersects  $\text{Diff}_c(\mathbb{R})$  non-trivially: let  $\text{Id} \neq \varphi \in N$ . If  $\varphi$  has compact support, we are done. If  $\varphi$  does not have compact support, without loss of generality assume that  $\text{Shift}_r(\varphi) = a > 0$ . Thus, for some  $x_0$  we have  $\varphi(x) = x + a$  for  $x \geq x_0$ . Choose  $x_0 + 2a < x_1 < x_2 < x_0 + 3a$  and  $\psi \in \text{Diff}_c(\mathbb{R})$  with support in the interval  $[x_1, x_2]$ , so that  $\psi(x) = x$  for  $x \notin [x_1, x_2]$ . For  $x \in [x_1, x_2]$  we then have  $(\psi^{-1} \circ \varphi \circ \psi)(x) = \psi^{-1}(\psi(x) + a) = \psi(x) + a$ . Thus,  $\psi^{-1} \circ \varphi \circ \psi \in N$  differs from  $\varphi$  just on the compact interval  $[x_1, x_2]$ . But then  $\psi^{-1} \circ \varphi \circ \psi \circ \varphi^{-1} \in N$  has compact support, and we are done.

By simplicity of  $\text{Diff}_c(\mathbb{R})$  we obtain  $N \supseteq \text{Diff}_c(\mathbb{R})$ . Therefore, the lattice of normal subgroups of  $\text{Diff}_{c,2}(\mathbb{R})$  has a  $\text{Diff}_c(\mathbb{R})$  as minimal element and, thus, is isomorphic to the lattice of subgroups of  $(\mathbb{R}^2, +)$ , which is quite large (see [11], [12]).

*Existence of normal subgroups  $N$  with  $\text{Diff}_c(\mathbb{R}) \rightarrow N \rightarrow \text{Diff}_{\mathcal{S}}(\mathbb{R})$ :* By conjugating with  $x \mapsto 1/x$  we see that the quotient group  $\text{Diff}_{\mathcal{S}}(\mathbb{R})/\text{Diff}_c(\mathbb{R})$  is isomorphic to the group of germs at 0 of smooth diffeomorphisms  $\varphi : (\mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)$  such that  $\varphi(x) - x$  is flat at 0:  $\varphi(x) - x = o(|x|^N)$  for each  $N$ . This group contains infinitely many normal subgroups:  $\varphi(x) - x = o(e^{-1/|x|})$  or  $o(\exp(-\exp(1/|x|)))$ , and so on.

We have not looked for normal subgroups  $N$  with

$$\text{Diff}_{W^{\infty,1}}(\mathbb{R}) \rightarrow N \rightarrow \text{Diff}_{W_2^{\infty,1}}(\mathbb{R}).$$

**2.6. Groups of Real-Analytic Diffeomorphisms.** Since the HS equation will turn out to have solutions in smaller groups of diffeomorphisms, we give here a description of them. For simplicity's sake, we restrict our attention to the periodic case. Let  $\text{Diff}^\omega(S^1)$  be the real-analytic regular Lie group of all real-analytic diffeomorphisms of  $S^1$ , with the real-analytic structure described in [18, Sect. 8.11], see also [21, theorem 43.4]. Recall that a mapping between  $c^\infty$ -open sets of convenient vector spaces is real-analytic if preserves smooth curves and also real-analytic curves.

**2.7. Groups of Ultra-Differentiable Diffeomorphisms.** Let us now describe the Denjoy–Carleman ultra-differentiable function classes which admit convenient calculus, following [22], [23], and [24]. We consider a sequence  $M = (M_k)$  of positive real numbers serving as weights for derivatives. For a smooth function  $f$  on an open subset  $U$  in  $\mathbb{R}^n$ , a compact set  $K \subset U$ , and for  $\rho > 0$  consider the set

$$(3) \quad \left\{ \frac{d^k f(x)}{\rho^k k! M_k} : x \in K, k \in \mathbb{N} \right\}.$$

We define the *Denjoy–Carleman classes*

$$\begin{aligned} C^{(M)}(U) &:= \{f \in C^\infty(U) : \forall \text{ compact } K \subseteq U \forall \rho > 0 : (3) \text{ is bounded}\}, \\ C^{\{M\}}(U) &:= \{f \in C^\infty(U) : \forall \text{ compact } K \subseteq U \exists \rho > 0 : (3) \text{ is bounded}\}. \end{aligned}$$

The elements of  $C^{(M)}(U)$  are said to be of *Beurling type*; those of  $C^{\{M\}}(U)$  of *Roumieu type*. If  $M_k = 1$ , for all  $k$ , then  $C^{(M)}(\mathbb{R})$  consists of the restrictions to  $U$  of the real and imaginary parts of all entire functions, while  $C^{\{M\}}(\mathbb{R})$  coincides with the ring  $C^\omega(\mathbb{R})$  of real-analytic functions. We shall also write  $C^{[M]}$  to mean either  $C^{(M)}$  or  $C^{\{M\}}$ . We shall assume that the sequence  $M = (M_k)$  has the following properties:

- $M$  is log-convex:  $k \mapsto \log(M_k)$  is convex, i.e.  $M_k^2 \leq M_{k-1} M_{k+1}$  for all  $k$ .
- $M$  has moderate growth, i.e.  $\sup_{j,k \in \mathbb{N}_{>0}} \left( \frac{M_{j+k}}{M_j M_k} \right)^{\frac{1}{j+k}} < \infty$ .
- In the Beurling case  $C^{[M]} = C^{(M)}$  we also require that  $C^\omega \subseteq C^{(M)}$ , or equivalently  $M_k^{1/k} \rightarrow \infty$ .

Then, both classes  $C^{[M]}$  are closed under composition and differentiation, can be extended to convenient vector spaces, and form monoidally closed categories (i.e. admit convenient settings). Moreover, on open sets in  $\mathbb{R}^n$ ,  $C^{[M]}$ -vector fields have  $C^{[M]}$ -flows, and between Banach spaces, the  $C^{[M]}$  implicit function theorem holds.

For mappings between  $c^\infty$ -open subsets of convenient vector spaces we have the following statements:

- For non-quasi-analytic  $M$ , the mapping  $f$  is  $C^{\{M\}}$  if it maps  $C^{\{M\}}$ -curves to  $C^{\{M\}}$ -curves, by [22].
- For certain quasi-analytic  $M$ , the mapping  $f$  is  $C^{\{M\}}$  if it maps  $C^{\{N\}}$ -curves to  $C^{\{N\}}$ -curves, for all non-quasi-analytic  $N$  which are larger than  $M$  and have the aforementioned properties, by [23].
- For any  $M$ , the mapping  $f$  is  $C^{[M]}$  if it respects  $C^{[M]}$ -maps from open balls in Banach spaces, by [24].

For every  $M$  with the aforementioned properties we have the regular Lie groups  $\text{Diff}^{\{M\}}(S^1)$  and  $\text{Diff}^{[M]}(S^1)$  (we write  $\text{Diff}^{[M]}(S^1)$  if we mean any of the two)

of  $C^{[M]}$ -diffeomorphisms of  $S^1$  which is a  $C^{[M]}$ -group (but not better), by [22, Sect. 6.5], [23, Sect. 5.6], and [24, Sect. 9.8].

### 3. RIGHT-INVARIANT RIEMANNIAN METRICS ON LIE GROUPS

**3.1. Notation on Lie Groups.** Let  $G$  be a regular Lie group, which may be infinite-dimensional, with Lie algebra  $\mathfrak{g}$ . Let  $\mu : G \times G \rightarrow G$  be the group multiplication,  $\mu_x$  the left translation and  $\mu^y$  the right translation, given by  $\mu_x(y) = \mu^y(x) = xy = \mu(x, y)$ .

Let  $L, R : \mathfrak{g} \rightarrow \mathfrak{X}(G)$  be the left- and right-invariant vector field mappings, given by  $L_X(g) = T_e(\mu_g).X$  and  $R_X = T_e(\mu^g).X$  respectively. They are related by  $L_X(g) = R_{\text{Ad}(g)X}(g)$ . Their flows are given by

$$\text{Fl}_t^{L_X}(g) = g \cdot \exp(tX) = \mu^{\exp(tX)}(g), \quad \text{Fl}_t^{R_X}(g) = \exp(tX).g = \mu_{\exp(tX)}(g).$$

We also need the right Maurer–Cartan form  $\kappa = \kappa^r \in \Omega^1(G, \mathfrak{g})$ , given by  $\kappa_x(\xi) := T_x(\mu^{x^{-1}}) \cdot \xi$ . It satisfies the left Maurer–Cartan equation  $d\kappa - \frac{1}{2}[\kappa, \kappa]_\wedge = 0$ , where  $[\ , \ ]_\wedge$  denotes the wedge product of  $\mathfrak{g}$ -valued forms on  $G$  induced by the Lie bracket. Note that  $\frac{1}{2}[\kappa, \kappa]_\wedge(\xi, \eta) = [\kappa(\xi), \kappa(\eta)]$ . The (exterior) derivative of the function  $\text{Ad} : G \rightarrow GL(\mathfrak{g})$  can be expressed by

$$d\text{Ad} = \text{Ad} \cdot (\text{ad} \circ \kappa^l) = (\text{ad} \circ \kappa^r) \cdot \text{Ad}$$

since we have  $d\text{Ad}(T\mu_g.X) = \frac{d}{dt}|_0 \text{Ad}(g \cdot \exp(tX)) = \text{Ad}(g) \cdot \text{ad}(\kappa^l(T\mu_g.X))$ .

**3.2. Geodesics of a Right-Invariant Metric on a Lie Group.** Let  $\gamma = \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$  be a positive-definite bounded (weak) inner product. Then

$$(4) \quad \gamma_x(\xi, \eta) = \gamma(T(\mu^{x^{-1}}) \cdot \xi, T(\mu^{x^{-1}}) \cdot \eta) = \gamma(\kappa(\xi), \kappa(\eta))$$

is a right-invariant (weak) Riemannian metric on  $G$ , and any (weak) right-invariant bounded Riemannian metric is of this form, for suitable  $\gamma$ . We shall denote by  $\tilde{\gamma} : \mathfrak{g} \rightarrow \mathfrak{g}^*$  the mapping induced by  $\gamma$  from the Lie algebra into its dual (of bounded linear functionals) and by  $\langle \alpha, X \rangle_{\mathfrak{g}}$  the duality evaluation between  $\alpha \in \mathfrak{g}^*$  and  $X \in \mathfrak{g}$ .

Let  $g : [a, b] \rightarrow G$  be a smooth curve. The velocity field of  $g$ , viewed in the right trivializations, coincides with the right logarithmic derivative

$$\delta^r(g) = T(\mu^{g^{-1}}) \cdot \partial_t g = \kappa(\partial_t g) = (g^* \kappa)(\partial_t), \quad \text{where } \partial_t = \frac{\partial}{\partial t}.$$

The energy of the curve  $g(t)$  is given by

$$(5) \quad E(g) = \frac{1}{2} \int_a^b \gamma_g(g', g') dt = \frac{1}{2} \int_a^b \gamma((g^* \kappa)(\partial_t), (g^* \kappa)(\partial_t)) dt.$$

For a variation  $g(s, t)$  with fixed endpoints we then use that

$$d(g^* \kappa)(\partial_t, \partial_s) = \partial_t(g^* \kappa(\partial_s)) - \partial_s(g^* \kappa(\partial_t)) - 0,$$

partial integration and the left Maurer–Cartan equation to obtain

$$\begin{aligned}
\partial_s E(g) &= \frac{1}{2} \int_a^b 2\gamma(\partial_s(g^* \kappa)(\partial_t), (g^* \kappa)(\partial_t)) dt \\
&= \int_a^b \gamma(\partial_t(g^* \kappa)(\partial_s) - d(g^* \kappa)(\partial_t, \partial_s), (g^* \kappa)(\partial_t)) dt \\
&= - \int_a^b \gamma((g^* \kappa)(\partial_s), \partial_t(g^* \kappa)(\partial_t)) dt - \int_a^b \gamma([(g^* \kappa)(\partial_t), (g^* \kappa)(\partial_s)], (g^* \kappa)(\partial_t)) dt \\
&= - \int_a^b \langle \check{\gamma}(\partial_t(g^* \kappa)(\partial_t)), (g^* \kappa)(\partial_s) \rangle_{\mathfrak{g}} dt \\
&\quad - \int_a^b \langle \check{\gamma}((g^* \kappa)(\partial_t)), \text{ad}_{(g^* \kappa)(\partial_t)}(g^* \kappa)(\partial_s) \rangle_{\mathfrak{g}} dt \\
&= - \int_a^b \langle \check{\gamma}(\partial_t(g^* \kappa)(\partial_t)) + (\text{ad}_{(g^* \kappa)(\partial_t)})^* \check{\gamma}((g^* \kappa)(\partial_t)), (g^* \kappa)(\partial_s) \rangle_{\mathfrak{g}} dt.
\end{aligned}$$

Thus the curve  $g(0, t)$  is critical for the energy (5) if and only if

$$\check{\gamma}(\partial_t(g^* \kappa)(\partial_t)) + (\text{ad}_{(g^* \kappa)(\partial_t)})^* \check{\gamma}((g^* \kappa)(\partial_t)) = 0.$$

In terms of the right logarithmic derivative  $u : [a, b] \rightarrow \mathfrak{g}$  of  $g : [a, b] \rightarrow G$ , given by  $u(t) := g^* \kappa(\partial_t) = T_{g(t)}(\mu^{g(t)^{-1}}) \cdot g'(t)$ , the geodesic equation has the expression

$$(6) \quad \boxed{\partial_t u = -\check{\gamma}^{-1} \text{ad}(u)^* \check{\gamma}(u)}$$

Thus the geodesic equation exists in general if and only if  $\text{ad}(X)^* \check{\gamma}(X)$  is in the image of  $\check{\gamma} : \mathfrak{g} \rightarrow \mathfrak{g}^*$ , i.e.

$$(7) \quad \text{ad}(X)^* \check{\gamma}(X) \in \check{\gamma}(\mathfrak{g})$$

for every  $X \in \mathfrak{X}$ . Condition (7) then leads to the existence of the so-called Christoffel symbols. Interestingly, it is not necessary for the more restrictive condition  $\text{ad}(X)^* \check{\gamma}(Y) \in \check{\gamma} \in \mathfrak{g}$  to be satisfied in order to obtain the geodesic equation, Christoffel symbols and the curvature; compare with [36, Lemma 3.3]. Note here the appearance of the geodesic equation for the *momentum*  $p := \gamma(u)$ :

$$p_t = -\text{ad}(\check{\gamma}^{-1}(p))^* p.$$

Subsequently, we shall encounter situations where (7) is satisfied but where the usual transpose  $\text{ad}^\top(X)$  of  $\text{ad}(X)$ ,

$$(8) \quad \text{ad}^\top(X) := \check{\gamma}^{-1} \circ \text{ad}_X^* \circ \check{\gamma}$$

does not exist for all  $X$ .

**3.3. Covariant Derivative.** The right trivialization  $(\pi_G, \kappa^r) : TG \rightarrow G \times \mathfrak{g}$  induces the isomorphism  $R : C^\infty(G, \mathfrak{g}) \rightarrow \mathfrak{X}(G)$ , given by  $R(X)(x) := R_X(x) := T_e(\mu^x) \cdot X(x)$ , for  $X \in C^\infty(G, \mathfrak{g})$  and  $x \in G$ . Here  $\mathfrak{X}(G) := \Gamma(TG)$  denotes the Lie algebra of all vector fields. For the Lie bracket and the Riemannian metric we have

$$\begin{aligned}
[R_X, R_Y] &= R(-[X, Y]_{\mathfrak{g}} + dY \cdot R_X - dX \cdot R_Y), \\
R^{-1}[R_X, R_Y] &= -[X, Y]_{\mathfrak{g}} + R_X(Y) - R_Y(X), \\
\gamma_x(R_X(x), R_Y(x)) &= \gamma(X(x), Y(x)), \quad x \in G.
\end{aligned}$$

In what follows, we shall perform all computations in  $C^\infty(G, \mathfrak{g})$  instead of  $\mathfrak{X}(G)$ . In particular, we shall use the convention

$$\nabla_X Y := R^{-1}(\nabla_{R_X} R_Y) \quad \text{for } X, Y \in C^\infty(G, \mathfrak{g}).$$

to express the Levi-Civita covariant derivative.

**Lemma.** *Assume that for all  $\xi \in \mathfrak{g}$  the element  $\text{ad}(\xi)^* \tilde{\gamma}(\xi) \in \mathfrak{g}^*$  is in the image of  $\tilde{\gamma} : \mathfrak{g} \rightarrow \mathfrak{g}^*$  and that  $\xi \mapsto \tilde{\gamma}^{-1} \text{ad}(\xi)^* \tilde{\gamma}(\xi)$  is bounded quadratic (or, equivalently, smooth). Then the Levi-Civita covariant derivative of the metric  $\gamma$  exists and is given for any  $X, Y \in C^\infty(G, \mathfrak{g})$  in terms of the isomorphism  $R$  by*

$$\nabla_X Y = dY.R_X + \rho(X)Y - \frac{1}{2} \text{ad}(X)Y,$$

where

$$\rho(\xi)\eta = \frac{1}{4} \tilde{\gamma}^{-1} (\text{ad}_{\xi+\eta}^* \tilde{\gamma}(\xi+\eta) - \text{ad}_{\xi-\eta}^* \tilde{\gamma}(\xi-\eta)) = \frac{1}{2} \tilde{\gamma}^{-1} (\text{ad}_\xi^* \tilde{\gamma}(\eta) + \text{ad}_\eta^* \tilde{\gamma}(\xi))$$

is the polarized version.  $\rho : \mathfrak{g} \rightarrow L(\mathfrak{g}, \mathfrak{g})$  is bounded, and we have  $\rho(\xi)\eta = \rho(\eta)\xi$ . We also have

$$\begin{aligned} \gamma(\rho(\xi)\eta, \zeta) &= \frac{1}{2} \gamma(\xi, \text{ad}(\eta)\zeta) + \frac{1}{2} \gamma(\eta, \text{ad}(\xi)\zeta), \\ \gamma(\rho(\xi)\eta, \zeta) + \gamma(\rho(\eta)\zeta, \xi) + \gamma(\rho(\zeta)\xi, \eta) &= 0. \end{aligned}$$

*Proof.* It is easily checked that  $\nabla$  is a covariant derivative. The Riemannian metric is covariantly constant since

$$R_X \gamma(Y, Z) = \gamma(dY.R_X, Z) + \gamma(Y, dZ.R_X) = \gamma(\nabla_X Y, Z) + \gamma(Y, \nabla_X Z).$$

Since  $\rho$  is symmetric, the connection is also torsion-free:

$$\nabla_X Y - \nabla_Y X + [X, Y]_{\mathfrak{g}} - dY.R_X + dX.R_Y = 0.$$

□

**3.4. Curvature.** For  $X, Y \in C^\infty(G, \mathfrak{g})$  we have

$$[R_X, \text{ad}(Y)] = \text{ad}(R_X(Y)) \quad \text{and} \quad [R_X, \rho(Y)] = \rho(R_X(Y)).$$

The Riemannian curvature is then computed by

$$\begin{aligned} \mathcal{R}(X, Y) &= [\nabla_X, \nabla_Y] - \nabla_{-[X, Y]_{\mathfrak{g}} + R_X(Y) - R_Y(X)} \\ &= [R_X + \rho_X - \frac{1}{2} \text{ad}_X, R_Y + \rho_Y - \frac{1}{2} \text{ad}_Y] \\ &\quad - R(-[X, Y]_{\mathfrak{g}} + R_X(Y) - R_Y(X)) - \rho(-[X, Y]_{\mathfrak{g}} + R_X(Y) - R_Y(X)) \\ &\quad + \frac{1}{2} \text{ad}(-[X, Y]_{\mathfrak{g}} + R_X(Y) - R_Y(X)) \\ &= [\rho_X, \rho_Y] + \rho_{[X, Y]_{\mathfrak{g}}} - \frac{1}{2} [\rho_X, \text{ad}_Y] + \frac{1}{2} [\rho_Y, \text{ad}_X] - \frac{1}{4} \text{ad}_{[X, Y]_{\mathfrak{g}}}. \end{aligned}$$

which is visibly a tensor field.

**3.5. Sectional Curvature.** For the linear two-dimensional subspace  $P \subseteq \mathfrak{g}$ , that is spanned by linearly independent  $X, Y \in \mathfrak{g}$ , the sectional curvature is defined as

$$k(P) = - \frac{\gamma(\mathcal{R}(X, Y)X, Y)}{\|X\|_\gamma^2 \|Y\|_\gamma^2 - \gamma(X, Y)^2}.$$

For the numerator we obtain

$$\begin{aligned}
\gamma(\mathcal{R}(X, Y)X, Y) &= \gamma(\rho_X \rho_Y X, Y) - \gamma(\rho_Y \rho_X X, Y) + \gamma(\rho_{[X, Y]} X, Y) \\
&\quad - \frac{1}{2} \gamma(\rho_X [Y, X], Y) + \frac{1}{2} \gamma([Y, \rho_X X], Y) \\
&\quad + 0 - \frac{1}{2} \gamma([X, \rho_Y X], Y) - \frac{1}{4} \gamma([[X, Y], X], Y) \\
&= \gamma(\rho_X X, \rho_Y Y) - \|\rho_X Y\|_\gamma^2 + \frac{3}{4} \|[X, Y]\|_\gamma^2 \\
&\quad - \frac{1}{2} \gamma(X, [Y, [X, Y]]) + \frac{1}{2} \gamma(Y, [X, [X, Y]]) \\
&= \gamma(\rho_X X, \rho_Y Y) - \|\rho_X Y\|_\gamma^2 + \frac{3}{4} \|[X, Y]\|_\gamma^2 \\
&\quad - \gamma(\rho_X Y, [X, Y]) + \gamma(Y, [X, [X, Y]]).
\end{aligned}$$

If the adjoint  $\text{ad}(X)^\top : \mathfrak{g} \rightarrow \mathfrak{g}$  exists, this is easily seen to coincide with Arnold's original formula [1],

$$\begin{aligned}
\gamma(\mathcal{R}(X, Y)X, Y) &= -\frac{1}{4} \|\text{ad}(X)^\top Y + \text{ad}(Y)^\top X\|_\gamma^2 + \gamma(\text{ad}(X)^\top X, \text{ad}(Y)^\top Y) \\
&\quad + \frac{1}{2} \gamma(\text{ad}(X)^\top Y - \text{ad}(Y)^\top X, \text{ad}(X)Y) + \frac{3}{4} \|[X, Y]\|_\gamma^2.
\end{aligned}$$

#### 4. HOMOGENEOUS $H^1$ -METRIC ON $\text{Diff}(\mathbb{R})$ AND THE HUNTER–SAXTON EQUATION

In this section we will study the homogeneous  $H^1$  or  $\dot{H}^1$ -metric on the various diffeomorphism groups of  $\mathbb{R}$  defined in Sect. 2. It was shown in [16] that the geodesic equation of the  $\dot{H}^1$ -metric on  $\text{Diff}(S^1)$  is the HS equation. We will show that suitable diffeomorphism groups on the real line also have the HS equation as geodesic equation. In [27] a way was found to map  $\text{Diff}(S^1)$  isometrically to an open subset of an  $L^2$ -sphere in  $C^\infty(S^1, \mathbb{R})$ . We will generalize this representation to the non-periodic case.

In the situation studied here — diffeomorphism groups on the real line — the resulting geometry will be different from the periodic case. Some of the diffeomorphism groups will be flat in the sense of Riemannian geometry, while others will be submanifolds of a flat space (see [19, Sect. 27.11] for the definition of (splitting) submanifolds in an infinite dimensional setting).

**4.1. Setting.** In this section, we shall use any of the following regular Lie groups:

- (1) We will denote by  $\mathcal{A}(\mathbb{R})$  any of the spaces  $C_c^\infty(\mathbb{R})$ ,  $\mathcal{S}(\mathbb{R})$ ,  $W^{\infty,1}(\mathbb{R})$ , or  $H_0^\infty(\mathbb{R})$ . By  $\text{Diff}_{\mathcal{A}}(\mathbb{R})$  we will denote the corresponding groups  $\text{Diff}_c(\mathbb{R})$ ,  $\text{Diff}_{\mathcal{S}}(\mathbb{R})$ , and  $\text{Diff}_{W^{\infty,1}}(\mathbb{R})$ . as defined in Sects. 2.2, 2.3 and 2.4 respectively.
- (2) Similarly  $\mathcal{A}_1(\mathbb{R})$  will denote any of the spaces  $C_{c,1}^\infty(\mathbb{R})$ ,  $\mathcal{S}_1(\mathbb{R})$ , or  $W_1^{\infty,1}(\mathbb{R})$ . By  $\text{Diff}_{\mathcal{A}_1}(\mathbb{R})$  we will denote the corresponding groups  $\text{Diff}_{c,1}(\mathbb{R})$ ,  $\text{Diff}_{\mathcal{S}_1}(\mathbb{R})$ , or  $\text{Diff}_{W_1^{\infty,1}}(\mathbb{R})$ . as defined in Sects. 2.2, 2.3 and 2.4 respectively.

**4.2.  $\dot{H}^1$ -Metric.** For  $\text{Diff}_{\mathcal{A}}(\mathbb{R})$  and  $\text{Diff}_{\mathcal{A}_1}(\mathbb{R})$  the homogeneous  $H^1$ -metric is given by

$$G_\varphi(X \circ \varphi, Y \circ \varphi) = G_{\text{Id}}(X, Y) = \int_{\mathbb{R}} X'(x)Y'(x) dx,$$

where  $X, Y$  are elements of the Lie algebra  $\mathcal{A}(\mathbb{R})$  or  $\mathcal{A}_1(\mathbb{R})$ . We shall also use the notation

$$\langle \cdot, \cdot \rangle_{\dot{H}^1} := G(\cdot, \cdot).$$

**Theorem.** *On  $\text{Diff}_{\mathcal{A}_1}(\mathbb{R})$  the geodesic equation is the HS equation*

$$(9) \quad \boxed{(\varphi_t) \circ \varphi^{-1} = u \quad u_t = -uu_x + \frac{1}{2} \int_{-\infty}^x (u_x(z))^2 dz,}$$

and the induced geodesic distance is positive.

On the other hand, the geodesic equation does not exist on the subgroup  $\text{Diff}_{\mathcal{A}}(\mathbb{R})$ , since the adjoint  $\text{ad}(X)^* \check{G}_{\text{Id}}(X)$  does not lie in  $\check{G}_{\text{Id}}(\mathcal{A}(\mathbb{R}))$  for all  $X \in \mathcal{A}(\mathbb{R})$ .

Note that this is a natural example of a non-robust Riemannian manifold in the sense of [37, Sect. 2.4].

*Proof.* Note that  $\check{G}_{\text{Id}} : \mathcal{A}_1(\mathbb{R}) \rightarrow \mathcal{A}_1(\mathbb{R})^*$  is given by  $\check{G}_{\text{Id}}(X) = -X''$  if we use the  $L^2$ -pairing  $X \mapsto (Y \mapsto \int XY dx)$  to embed functions into the space of distributions. We now compute the adjoint of the operator  $\text{ad}(X)$  as defined in Sect. 3.2.

$$\begin{aligned} \langle \text{ad}(X)^* \check{G}_{\text{Id}}(Y), Z \rangle &= \check{G}_{\text{Id}}(Y, \text{ad}(X)Z) = G_{\text{Id}}(Y, -[X, Z]) \\ &= \int_{\mathbb{R}} Y'(x)(X'(x)Z(x) - X(x)Z'(x))' dx \\ &= \int_{\mathbb{R}} Z(x)(X''(x)Y'(x) - (X(x)Y'(x))'') dx. \end{aligned}$$

Therefore, the adjoint as an element of  $\mathcal{A}_1^*$  is given by

$$\text{ad}(X)^* \check{G}_{\text{Id}}(Y) = X''Y' - (XY')''.$$

For  $X = Y$  we can rewrite this as

$$\begin{aligned} \text{ad}(X)^* \check{G}_{\text{Id}}(X) &= \frac{1}{2}((X'^2)' - (X^2)''') = \frac{1}{2} \left( \int_{-\infty}^x X'(y)^2 dy - (X^2)' \right)'' \\ &= \frac{1}{2} \check{G}_{\text{Id}} \left( - \int_{-\infty}^x X'(y)^2 dy + (X^2)' \right). \end{aligned}$$

If  $X \in \mathcal{A}_1(\mathbb{R})$  then the function  $-\frac{1}{2} \int_{-\infty}^x X'(y)^2 dy + \frac{1}{2}(X^2)'$  is again an element of  $\mathcal{A}_1(\mathbb{R})$ . This follows immediately from the definition of  $\mathcal{A}_1(\mathbb{R})$ . Therefore, the geodesic equation exists on  $\text{Diff}_{\mathcal{A}_1}(\mathbb{R})$  and is given by (9).

However, if  $X \in \mathcal{A}(\mathbb{R})$ , a necessary condition for  $\int_{-\infty}^x (X'(y))^2 dy \in \mathcal{A}(\mathbb{R})$  would be  $\int_{-\infty}^{\infty} X'(y)^2 dy = 0$ , which would imply  $X' = 0$ . Thus, the geodesic equation does not exist on  $\mathcal{A}(\mathbb{R})$ .

The positivity of geodesic distance will follow from the explicit formula given in Corollary 4.3.  $\square$

**Remark.** One obtains the classical form of the HS equation

$$u_{tx} = -uu_{xx} - \frac{1}{2}u_x^2,$$

by differentiating the preceding geodesic Eq. (9). In Sect. 4.3 we will use a geometric argument to give an explicit solution formula, which will also imply the well-posedness of the equation. For  $\mathcal{A}_1(\mathbb{R}) = W_1^{\infty,1}(\mathbb{R})$  an analytic proof of well-posedness could also be carried out similarly to that in [5, Sect. 10] by adapting the arguments to  $\mathbb{R}$ . Furthermore, using this geometric trick, we will conclude that

the curvature of the  $\dot{H}^1$  metric vanishes. One can also show this statement directly using the adaption of Arnold's formula presented in Sect. 3.5. From the foregoing proof one can easily deduce the formula for the mapping  $\rho$ :

$$\begin{aligned} \rho(X)Y &= \frac{1}{2}\check{G}^{-1}(\text{ad}(X)^*G(Y) + \text{ad}(X)^*G(Y)) \\ &= \frac{1}{2}\check{G}^{-1}(X''Y' - (XY')'' + Y''X' - (YX')'') \\ &= \frac{1}{2}\check{G}^{-1}((X'Y')' + (XY)''') \\ &= \frac{1}{2}\left(-\int_{-\infty}^x (X'Y') dx + (XY)'\right) \end{aligned}$$

Using this formula, the desired formula for the curvature is a straightforward calculation.

**Remark.** For general  $X \neq Y \in \mathcal{A}_1(\mathbb{R})$  we will have  $\text{ad}(X)^*\check{G}_{\text{Id}}(Y) \notin \check{G}_{\text{Id}}(\mathcal{A}_1(\mathbb{R}))$ . If there were a function  $Z \in \mathcal{A}_1(\mathbb{R})$  such that  $-Z'' = X''Y' - (XY')''$  then a necessary condition would be  $0 = Z'(-\infty) = -\int_{-\infty}^{\infty} X''Y' dx$ , which would in general not be satisfied. Thus, the transpose of  $\text{ad}(X)$  as defined in (8) does not exist; only the symmetric version  $X \mapsto \check{G}_{\text{Id}}^{-1}(\text{ad}(X)^*\check{G}_{\text{Id}}(X))$  exists.

**4.3. The Square-Root Representation on  $\text{Diff}_{\mathcal{A}_1}(\mathbb{R})$ .** We will define a map  $R$  from  $\text{Diff}_{\mathcal{A}_1}(\mathbb{R})$  to the space

$$\mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2}) = \{f \in \mathcal{A}(\mathbb{R}) : f(x) > -2\},$$

such that the pullback of the  $L^2$ -metric on  $\mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2})$  is the  $\dot{H}^1$ -metric on the space  $\text{Diff}_{\mathcal{A}_1}(\mathbb{R})$ . Since  $\mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2})$  is an open subset of  $C^\infty(\mathbb{R})$  this implies that  $\text{Diff}_{\mathcal{A}_1}(\mathbb{R})$  with the  $\dot{H}^1$ -metric is a flat space in the sense of Riemannian geometry. This is an adaptation of the square-root representation of  $\text{Diff}(S^1)/S^1$  used in [27]; see also Sect. 6, where we review this construction.

**Theorem.** *We define the  $R$ -map by*

$$R : \begin{cases} \text{Diff}_{\mathcal{A}_1}(\mathbb{R}) \rightarrow \mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2}) \subset \mathcal{A}(\mathbb{R}, \mathbb{R}) \\ \varphi \mapsto 2((\varphi')^{1/2} - 1). \end{cases}$$

*The  $R$ -map is invertible with inverse*

$$R^{-1} : \begin{cases} \mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2}) \rightarrow \text{Diff}_{\mathcal{A}_1}(\mathbb{R}) \\ \gamma \mapsto x + \frac{1}{4} \int_{-\infty}^x (\gamma^2 + 4\gamma) dx. \end{cases}$$

*The pullback of the flat  $L^2$ -metric via  $R$  is the  $\dot{H}^1$ -metric on  $\text{Diff}_{\mathcal{A}_1}(\mathbb{R})$ , i.e.*

$$R^*\langle \cdot, \cdot \rangle_{L^2} = \langle \cdot, \cdot \rangle_{\dot{H}^1}.$$

*Thus the space  $(\text{Diff}_{\mathcal{A}_1}(\mathbb{R}), \dot{H}^1)$  is a flat space in the sense of Riemannian geometry.*

Here  $\langle \cdot, \cdot \rangle_{L^2}$  denotes the  $L^2$ -inner product on  $\mathcal{A}(\mathbb{R})$  interpreted as a Riemannian metric on  $\mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2})$ , which does not depend on the basepoint, i.e.

$$G_\gamma^{L^2}(h, k) = \langle h, k \rangle_{L^2} = \int_{\mathbb{R}} h(x)k(x) dx,$$

for  $h, k \in \mathcal{A}(\mathbb{R}) \cong T_\gamma \mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2})$ .

*Proof.* We will first prove that for  $\varphi \in \text{Diff}_{\mathcal{A}_1}(\mathbb{R})$  the image  $R(\varphi)$  is an element of  $\mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2})$ . To do so, we write  $\varphi(x) = x + f(x)$ , with  $f \in \mathcal{A}_1$ . Using a Taylor expansion of  $\sqrt{1+x}$  around  $x=0$ ,

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{4} \int_0^1 \frac{1-t}{(1+tx)^{\frac{3}{2}}} dt x^2,$$

we obtain

$$\begin{aligned} R(\varphi) &= 2((\varphi')^{1/2} - 1) = 2\sqrt{1+f'} - 2 \\ &= f' - \frac{1}{2} \int_0^1 \frac{1-t}{(1+tf')^{\frac{3}{2}}} f' dt f' \\ &=: f' + F(f')f', \end{aligned}$$

with  $F \in C^\omega(\mathbb{R}_{>-1}, \mathbb{R})$  satisfying  $F(0) = 0$ .

Because  $\varphi = \text{Id} + f$  is a diffeomorphism, we have  $f' > -1$  and since  $f' \in \mathcal{A}(\mathbb{R})$  implies that  $f'$  vanishes at  $-\infty$  and at  $\infty$  we can even conclude that  $f' > -1 + \varepsilon$  for some  $\varepsilon > 0$ . Therefore,  $F(f')$  is a bounded function for each  $f' \in \text{Diff}_{\mathcal{A}_1}(\mathbb{R})$ . Using that all the spaces  $\mathcal{A}(\mathbb{R})$  are  $\mathcal{B}(\mathbb{R})$ -modules we conclude that  $F(f')f'$  and, hence,  $R(\varphi)$  are elements of  $\mathcal{A}(\mathbb{R})$ .

To check that the mapping  $R : \text{Diff}_{\mathcal{A}_1}(\mathbb{R}) \rightarrow \mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2})$  is bijective, we use the identity  $\frac{1}{4}(\gamma(x) + 2)^2 - 1 = f'(x)$  with  $\gamma = R(\varphi) = R(\text{Id} + f)$ . Using this identity it is straightforward to calculate that

$$R \circ R^{-1} = \text{Id}_{\mathcal{A}}, \quad R^{-1} \circ R = \text{Id}_{\text{Diff}}.$$

To compute the pullback of the  $L^2$ -metric via the  $R$ -map we first need to calculate its tangent mapping. To do this, let  $h = X \circ \varphi \in T_\varphi \text{Diff}_{\mathcal{A}_1}(\mathbb{R})$  and let  $t \mapsto \psi(t)$  be a smooth curve in  $\text{Diff}_{\mathcal{A}_1}(\mathbb{R})$  with  $\psi(0) = \text{Id}$  and  $\partial_t|_0 \psi(t) = X$ . We have:

$$\begin{aligned} T_\varphi R.h &= \partial_t|_0 R(\psi(t) \circ \varphi) = \partial_t|_0 2 \left( ((\psi(t) \circ \varphi)_x)^{1/2} - 1 \right) = \partial_t|_0 2 \left( (\psi(t)_x \circ \varphi) \varphi_x \right)^{1/2} \\ &= 2(\varphi_x)^{1/2} \partial_t|_0 \left( (\psi(t)_x)^{1/2} \circ \varphi \right) = (\varphi_x)^{1/2} \left( \frac{\psi_{tx}(0)}{(\psi(0)_x)^{-1/2}} \circ \varphi \right) \\ &= (\varphi_x)^{1/2} (X' \circ \varphi) = (\varphi')^{1/2} (X' \circ \varphi). \end{aligned}$$

Using this formula we have for  $h = X_1 \circ \varphi, k = X_2 \circ \varphi$ :

$$R^* \langle h, k \rangle_{L^2} = \langle T_\varphi R.h, T_\varphi R.k \rangle_{L^2} = \int_{\mathbb{R}} X_1'(x) X_2'(x) dx = \langle h, k \rangle_{\dot{H}^1}. \quad \square$$

**Corollary.** *Given  $\varphi_0, \varphi_1 \in \text{Diff}_{\mathcal{A}_1}(\mathbb{R})$  the geodesic  $\varphi(t, x)$  connecting them is given by*

$$(10) \quad \varphi(t, x) = R^{-1} \left( (1-t)R(\varphi_0) + tR(\varphi_1) \right) (x)$$

and their geodesic distance is

$$(11) \quad d(\varphi_0, \varphi_1)^2 = 4 \int_{\mathbb{R}} \left( (\varphi_1')^{1/2} - (\varphi_0')^{1/2} \right)^2 dx.$$

Furthermore, the support of the geodesic is localized in the following sense: if  $\varphi(t, x) = x + f(t, x)$  with  $f(t) \in \mathcal{A}_1(\mathbb{R})$  and similarly for  $\varphi_0, \varphi_1$ , then  $\text{supp}(\partial_x f(t))$  is contained in  $\text{supp}(\partial_x f_0) \cup \text{supp}(\partial_x f_1)$ .

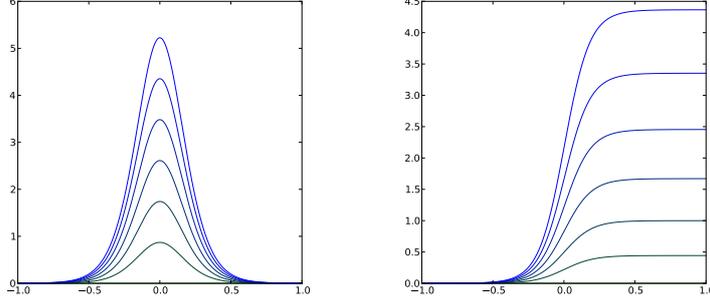


FIGURE 1. A complete geodesic  $\varphi(t, x) \in \text{Diff}_{\mathcal{A}_1}(\mathbb{R})$  sampled at time points  $t = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3$ . Left image: The geodesic in the  $R$ -map space. Right image: The geodesic in the original space, visualized as  $\varphi(t, x) - x$ .

*Proof.* The formula for the geodesic  $\varphi(t, x)$  is clear. The geodesic distance between  $\varphi_0, \varphi_1$  is given as the  $L^2$  difference between their  $R$ -maps:

$$\begin{aligned} d^{\text{Diff}}(\varphi_0, \varphi_1) &= d^{\mathcal{A}}(R(\varphi_0), R(\varphi_1)) = \int_0^1 \sqrt{\int_{\mathbb{R}} (R(\varphi_1) - R(\varphi_0))^2 dx} dt \\ &= 2 \sqrt{\int_{\mathbb{R}} ((\varphi_1')^{1/2} - (\varphi_0')^{1/2})^2 dx}. \end{aligned}$$

To prove the statement regarding the support of the geodesic, we use the inversion formula of  $R$  to obtain

$$f'(t, x) = \varphi'(t, x) - 1 = \frac{1}{4} \gamma(t, x) (\gamma(t, x) + 4),$$

where  $\gamma(t, x) = (1-t)R(\varphi_0)(x) + tR(\varphi_1)(x)$  is the image of the geodesic under the  $R$ -map. Next we note that at the points where  $f'_i(x) = 0$  we have  $\varphi'_i(x) = 1$  and  $R(\varphi_i)(x) = 0$ . Hence, at the points where both  $f'_0(x) = f'_1(x) = 0$  we also have  $f'(t, x) = 0$  for all  $t \in [0, 1]$ .  $\square$

**Example.** A geodesic connecting the identity to the diffeomorphism  $\varphi_1(x) = x + e^{-1/(x+1)^2} e^{-1/(x-1)^2}$  can be seen in Fig. 1. In all the examples presented in this article, we consider diffeomorphisms  $\varphi$  with  $\text{supp}(\varphi' - 1) \subset [-1, 1]$ . We approximated the diffeomorphisms with 1000 points on this interval. In the following lemma it is shown that this behaviour does not hold in general.

**Lemma.** *The metric space  $(\text{Diff}_{\mathcal{A}_1}(\mathbb{R}), \dot{H}^1)$  is path-connected and geodesically convex but not geodesically complete.*

*In particular, for every  $\varphi_0 \in \text{Diff}_{\mathcal{A}_1}(\mathbb{R})$  and  $h \in T_{\varphi_0} \text{Diff}_{\mathcal{A}_1}(\mathbb{R})$ ,  $h \neq 0$  there exists a time  $T \in \mathbb{R}$  such that  $\varphi(t, \cdot)$  is a geodesic for  $|t| < |T|$  starting at  $\varphi_0$  with  $\varphi_t(0) = h$ , but  $\varphi_x(T, x) = 0$  for some  $x \in \mathbb{R}$ .*

*Proof.* Set  $\gamma_0 = R(\varphi_0)$  and  $k = T_{\varphi_0} R.h$ . Then  $R$  maps the geodesic  $\varphi(t)$  to

$$R(\varphi(t))(x) = \gamma(t, x) = \gamma_0(x) + tk(x)$$

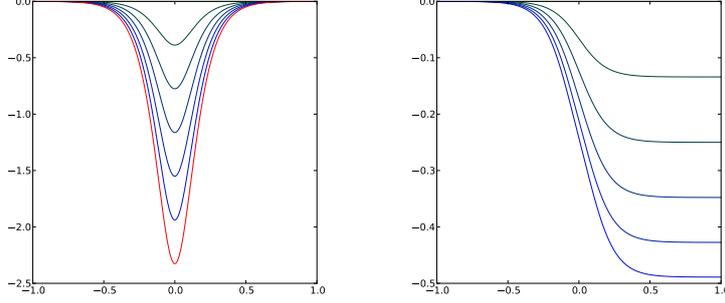


FIGURE 2. An incomplete geodesic  $\varphi(t, x) \in \text{Diff}_{\mathcal{A}_1}(\mathbb{R})$ . Left image: The geodesic in  $R$ -map space at time points  $t = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3$ . At time  $t = 3$  (red line) the geodesic has already left the space of  $R$ -maps. Right image: The geodesic in original space visualized as  $\varphi(t, x) - x$  sampled at time points  $t = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}$ .

and the geodesic  $\varphi(t)$  ceases to exist when  $\gamma(t)$  leaves the image of the  $R$ -map, i.e. when  $\gamma(t, x) = -2$  for some pair  $(t, x)$ . Consider the function  $g(x) = |2 + \gamma_0(x)|/|k(x)|$  and set  $g(x) = \infty$  where  $k(x) = 0$ . Since we assumed  $h \neq 0$ , there is at least one  $x \in \mathbb{R}$ , such that  $g(x)$  is finite. Since  $k(x) \rightarrow 0$  as  $|x| \rightarrow \infty$  we have  $g(x) \rightarrow \infty$  for  $x$  large, and  $g$  attains the minimum at some point. Let this point be  $x_0$  and define  $T = -(2 + \gamma_0(x_0))/k(x_0)$ . Then  $|T|$  is the time when  $\gamma(t, x)$  first reaches  $-2$ . So for  $|t| < |T|$  the geodesic  $\gamma(t)$  lies in  $\mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2})$  and  $\gamma(T, x_0) = -2$ . Then we have

$$\varphi_x(T, x_0) = 1 + \frac{1}{4}\gamma(T, x_0)(\gamma(T, x_0) + 4) = 1 - 1 = 0,$$

as required. This proves that the space is not geodesically complete.

The statement regarding path connectedness and geodesic convexity are direct consequences of the path connectedness and convexity of  $\mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2})$ .  $\square$

**Example.** This behaviour is illustrated in Fig. 2, where we have again chosen  $\varphi_0 = \text{Id}$  and we have solved the geodesic equation in the direction

$$h(x) = R^{-1}\left(\frac{-1}{4 + 4e^{-10x}}\right)$$

until the geodesic leaves the space of diffeomorphisms — which happens approximately at time  $t = 2.58$ .

**4.4. The Square-Root Representation on  $\text{Diff}_{\mathcal{A}}(\mathbb{R})$ .** We will now study the homogeneous  $H^1$ -metric on diffeomorphism groups which do not allow a shift towards infinity. We can still use the same square-root representation as in the previous section, but now the image of this map will be a splitting submanifold in the image space  $\mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2})$ .

**Theorem.** *The square-root representation on the diffeomorphism group  $\text{Diff}_{\mathcal{A}}(\mathbb{R})$  is a bijective mapping given by*

$$R : \begin{cases} \text{Diff}_{\mathcal{A}}(\mathbb{R}) \rightarrow (\text{Im}(R), \|\cdot\|_{L^2}) \subset (\mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2}), \|\cdot\|_{L^2}) \\ \varphi \mapsto 2((\varphi')^{1/2} - 1). \end{cases}$$

The pullback of the restriction of the flat  $L^2$ -metric to  $\text{Im}(R)$  via  $R$  is again the homogeneous Sobolev metric of order one. The image of the  $R$ -map is the splitting submanifold (in the sense of [21, Sect. 27.11]) of  $\mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2})$  given by:

$$\text{Im}(R) = \left\{ \gamma \in \mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2}) : F(\gamma) := \int_{\mathbb{R}} \gamma(\gamma + 4) dx = 0 \right\}.$$

*Proof.* The statement regarding the image of  $R$ , follows from the fact that a diffeomorphism  $\varphi = \text{Id} + g \in \text{Diff}_{\mathcal{A}_1}(\mathbb{R})$  is also an element of  $\text{Diff}_{\mathcal{A}}(\mathbb{R})$  iff

$$\int_{\mathbb{R}} (\varphi'(x) - 1) dx = \int_{\mathbb{R}} g'(x) dx = 0.$$

Using that for  $\gamma = R(\varphi)$  we have  $\frac{1}{4}(\gamma(x) + 2)^2 - 1 = g'(x)$  we obtain the desired result. The mapping  $\mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2}) \ni f \mapsto 2((f + 1)^{1/2} - 1) \in \mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2})$  is a diffeomorphism, and it maps  $\mathcal{A}(\mathbb{R}, \mathbb{R}_{>-1}) \cap \{f : \int f dx = 0\}$  diffeomorphically onto  $\text{Im}(R)$ , which is therefore a splitting submanifold.  $\square$

**Remark.** Note that we have  $dF(\gamma)(\delta) = \int_{\mathbb{R}} (2\gamma + 4)\delta dx$ , but  $2\gamma + 4$  is not in  $\mathcal{A}$ , only in the dual  $\mathcal{A}^*$ . So the so-called normal field along the codimension 1 submanifold  $\text{Im}(\mathbb{R})$  has no length; methods like the Gauss formula or the Gauss equation do not make sense. This is a diffeomorphic translation of the nonexistence of the geodesic equation in  $\text{Diff}_{\mathcal{A}}(\mathbb{R})$ .

**4.5. Geodesic Distance on  $\text{Diff}_{\mathcal{A}}(\mathbb{R})$ .** We have seen that on the space  $\text{Diff}_{\mathcal{A}}(\mathbb{R})$  the geodesic equation does not exist. It is, however, possible to define the geodesic distance between two diffeomorphisms  $\varphi_0, \varphi_1 \in \text{Diff}_{\mathcal{A}}(\mathbb{R})$  as the infimum over all paths  $\varphi : [0, 1] \rightarrow \text{Diff}_{\mathcal{A}}(\mathbb{R})$  connecting these:

$$d^{\mathcal{A}}(\varphi_0, \varphi_1) = \inf_{\substack{\varphi(0)=\varphi_0 \\ \varphi(1)=\varphi_1}} \int_0^1 \sqrt{G_{\varphi(t)}(\partial_t \varphi(t), \partial_t \varphi(t))} dt.$$

In Corollary 4.3 we gave an explicit formula for the geodesic distance  $d^{\mathcal{A}_1}$  on the space  $\text{Diff}_{\mathcal{A}_1}(\mathbb{R})$ . It turns out that the geodesic distance  $d^{\mathcal{A}}$  on  $\text{Diff}_{\mathcal{A}}(\mathbb{R})$  is the same as the restriction of  $d^{\mathcal{A}_1}$  to  $\text{Diff}_{\mathcal{A}}(\mathbb{R})$ . Note that this is not a trivial statement, since in the definition of  $d^{\mathcal{A}}$  the infimum is taken over all paths lying in the smaller space  $\text{Diff}_{\mathcal{A}}(\mathbb{R})$ .

**Theorem.** *The geodesic distance  $d^{\mathcal{A}}$  on  $\text{Diff}_{\mathcal{A}}(\mathbb{R})$  coincides with the restriction of  $d^{\mathcal{A}_1}$  to  $\text{Diff}_{\mathcal{A}}(\mathbb{R})$ , i.e. for  $\varphi_0, \varphi_1 \in \text{Diff}_{\mathcal{A}}(\mathbb{R})$  we have*

$$d^{\mathcal{A}}(\varphi_0, \varphi_1) = d^{\mathcal{A}_1}(\varphi_0, \varphi_1).$$

*Proof.* Because the space  $\text{Diff}_{\mathcal{A}}(\mathbb{R})$  is smaller than  $\text{Diff}_{\mathcal{A}_1}(\mathbb{R})$ , we have the inequality  $d_{\mathcal{A}_1} \leq d_{\mathcal{A}}$ . The following argument will establish the other inequality  $d_{\mathcal{A}} \leq d_{\mathcal{A}_1}$  and hence with both of these inequalities together we will be able to conclude that  $d_{\mathcal{A}_1} = d_{\mathcal{A}}$ .

Take  $\varphi_0, \varphi_1 \in \text{Diff}_{\mathcal{A}}(\mathbb{R})$  and let  $\varphi(t)$  be a path connecting them in the larger space  $\text{Diff}_{\mathcal{A}_1}(\mathbb{R})$ . Then  $\gamma(t) = R(\varphi(t))$  is a path in  $\mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2})$  and its length is measured by

$$L(\gamma) = \int_0^1 \sqrt{\int_{\mathbb{R}} \partial_t \gamma(t, x)^2 dx} dt.$$

We also have the functional

$$F(\gamma) = \int_{\mathbb{R}} \gamma(x)^2 + 4\gamma(x) dx,$$

which measures whether  $\gamma$  lies in the image of  $\text{Diff}_{\mathcal{A}}(\mathbb{R})$  under the  $R$ -map.

We will construct a sequence  $\tilde{\gamma}_n$  of paths with  $n \rightarrow \infty$ , such that these paths satisfy  $F(\tilde{\gamma}_n) = 0$  and  $L(\tilde{\gamma}_n) \rightarrow L(\tilde{\gamma})$ . This will show that each path in  $\text{Diff}_{\mathcal{A}_1}(\mathbb{R})$  can be approximated arbitrarily well by path in the smaller space  $\text{Diff}_{\mathcal{A}}(\mathbb{R})$  and hence we will have established the other inequality  $d_{\mathcal{A}} \leq d_{\mathcal{A}_1}$ .

Since  $\gamma(t)$  and  $\partial_t \gamma(t)$  decay to 0 as  $x \rightarrow \infty$  there exists for each  $n > 0$  some  $x_n \in \mathbb{R}$  such that

$$\begin{aligned} |\gamma(t, x)| &< \frac{1}{n} \\ |\partial_t \gamma(t, x)| &< \frac{1}{n} \end{aligned} \quad x > x_n.$$

We also define  $\varepsilon_n = \frac{1}{n}$ . Let  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  be a smooth function with support in  $[-1, 1]$ ,  $\psi(x) \geq 0$  and  $\int \psi(x) dx = 1$  as well as  $\int \psi(x)^2 dx = 1$ . Now define the new path

$$\tilde{\gamma}_n(t, x) = \gamma(t, x) + \alpha_n(t) \varepsilon_n \psi(\varepsilon_n(x - x_n - \varepsilon_n^{-1}))$$

with the function  $\alpha_n(t)$  determined by  $F(\tilde{\gamma}_n(t)) = 0$ . To make the calculations easier, we set  $\psi_n(x) = \psi(\varepsilon_n(x - x_n - \varepsilon_n^{-1}))$  and we note that

$$\varepsilon_n \int_{\mathbb{R}} \psi_n(x) dx = 1, \quad \varepsilon_n \int_{\mathbb{R}} \psi_n(x)^2 dx = 1.$$

Writing the condition  $F(\tilde{\gamma}_n(t)) = 0$  more explicitly we obtain

$$\begin{aligned} F(\tilde{\gamma}_n(t)) &= \int_{\mathbb{R}} \gamma(t)^2 + 2\alpha_n(t) \varepsilon_n \psi_n \gamma(t) + \alpha_n(t)^2 \varepsilon_n^2 \psi_n^2 + 4(\gamma(t) + \alpha_n(t) \varepsilon_n \psi_n) dx \\ &= \varepsilon_n \alpha_n(t)^2 + \left(4 + 2\varepsilon_n \int_{\mathbb{R}} \psi_n \gamma(t) dx\right) \alpha_n(t) + F(\gamma(t)). \end{aligned}$$

We need to estimate the integral

$$C_n(t) := \varepsilon_n \int_{\mathbb{R}} \psi_n(x) \gamma(t, x) dx$$

to be able to control  $\alpha(t)$ . We have

$$\begin{aligned} |C_n(t)| &= \left| \varepsilon_n \int_{\mathbb{R}} \psi(\varepsilon_n(x - x_n - \varepsilon_n^{-1})) \gamma(t, x) dx \right| \\ &\leq \varepsilon_n \int_{x_n}^{x_n + 2\varepsilon_n^{-1}} \psi(\varepsilon_n(x - x_n - \varepsilon_n^{-1})) |\gamma(t, x)| dx \\ &\leq \varepsilon_n \int_{x_n}^{x_n + 2\varepsilon_n^{-1}} \psi(\varepsilon_n(x - x_n - \varepsilon_n^{-1})) \frac{1}{n} dx \\ &\leq \frac{1}{n}. \end{aligned}$$

Hence we see that for large  $n$  we will have  $\alpha_n(t) \rightarrow -F(\gamma(t))/4$  and the convergence is uniform in  $t$ . To see that  $\alpha(t)$  is smooth in  $t$  we can use the explicit formula

$$\alpha_n(t) = -\varepsilon_n^{-1} (2 + C_n(t)) - \sqrt{\varepsilon_n^{-2} (2 + C_n(t))^2 - \varepsilon_n^{-1} F(\gamma(t))}$$

and note that for  $\varepsilon_n$  sufficiently small the term under the square root will always be positive.

Now we look at the length of the path  $\tilde{\gamma}_n$ . Let us consider only the inner integral  $\int_{\mathbb{R}} \partial_t \tilde{\gamma}_n(t, x) dx$ . We will show the convergence

$$\int_{\mathbb{R}} (\partial_t \tilde{\gamma}_n(t, x))^2 dx \rightarrow \int_{\mathbb{R}} (\partial_t \gamma(t, x))^2 dx$$

uniformly in  $t$ , which will imply the convergence  $L(\tilde{\gamma}_n) \rightarrow L(\gamma)$ . To do so, we look at

$$\begin{aligned} \int_{\mathbb{R}} (\partial_t \tilde{\gamma}_n(t))^2 dx &= \int_{\mathbb{R}} (\partial_t \gamma(t))^2 + 2\partial_t \alpha_n(t) \varepsilon_n \psi_n \partial_t \gamma(t) + (\partial_t \alpha_n(t))^2 \varepsilon_n^2 \psi_n(x)^2 dx \\ &= \int_{\mathbb{R}} (\partial_t \gamma(t))^2 dx + 2\partial_t \alpha_n(t) \varepsilon_n \int_{\mathbb{R}} \psi_n \partial_t \gamma(t) dx + (\partial_t \alpha_n(t))^2 \varepsilon_n. \end{aligned}$$

The integral

$$D_n(t) := \varepsilon_n \int_{\mathbb{R}} \psi_n(x) \partial_t \gamma(t, x) dx$$

can be estimated in the same way as  $C_n(t)$  previously to obtain  $D_n(t) \leq \frac{1}{n}$ . It remains to bound  $\partial_t \alpha(t)$  uniformly in  $t$  to show convergence. We differentiate the equation

$$\varepsilon_n \alpha_n(t)^2 + \left( 4 + 2\varepsilon_n \int_{\mathbb{R}} \psi_n \gamma(t) dx \right) \alpha_n(t) + F(\gamma(t)) = 0$$

that defines  $\alpha(t)$ , which gives us

$$\partial_t \alpha_n(t) (2\varepsilon_n \alpha_n(t) + 4 + 2C_n(t)) + T_{\gamma(t)} F \cdot \partial_t \gamma(t) = 0$$

and thus

$$\partial_t \alpha_n(t) = -\frac{T_{\gamma(t)} F \cdot \partial_t \gamma(t)}{4 + 2\varepsilon_n \alpha_n(t) + 2C_n(t)}.$$

We see that  $\partial_t \alpha_n(t) \rightarrow -T_{\gamma(t)} F \cdot \partial_t \gamma(t) / 4$  and the convergence is uniform in  $t$ . Thus we have shown that convergence  $L(\tilde{\gamma}_n) \rightarrow L(\gamma)$  of the length functional.  $\square$

**4.6. Submanifold  $\text{Diff}_{\mathcal{A}}(\mathbb{R})$  Inside  $\text{Diff}_{\mathcal{A}_1}(\mathbb{R})$ .** The following theorem deals with the question how  $\text{Diff}_{\mathcal{A}}(\mathbb{R})$  lies inside the extension  $\text{Diff}_{\mathcal{A}_1}(\mathbb{R})$ . We give an upper bound for how often a geodesic in  $\text{Diff}_{\mathcal{A}_1}(\mathbb{R})$  might intersect or be tangent to  $\text{Diff}_{\mathcal{A}}(\mathbb{R})$ . It is only an upper bound because the geodesic might leave the group of diffeomorphisms before intersecting  $\text{Diff}_{\mathcal{A}}(\mathbb{R})$ .

**Theorem.** *Consider a geodesic  $\varphi(t)$  in  $\text{Diff}_{\mathcal{A}_1}(\mathbb{R})$  starting at  $\varphi(0) = \varphi_0$  with initial velocity  $\partial_t \varphi(0) = u_0 \circ \varphi_0$  and denote by  $u(t) = \partial_t \varphi(t) \circ \varphi(t)^{-1}$  the right-trivialized velocity. Then the size of the shift at infinity is given by*

$$\begin{aligned} \text{Shift}(\varphi(t)) &= \text{Shift}(\varphi_0) + t u_0(\infty) + \frac{t^2}{4} \int_{\mathbb{R}} (u'_0)^2 dx \\ u(t, \infty) &= u_0(\infty) + t \int_{\mathbb{R}} (u'_0)^2 dx. \end{aligned}$$

*This means that every geodesic in  $\text{Diff}_{\mathcal{A}_1}(\mathbb{R})$  intersects  $\text{Diff}_{\mathcal{A}}(\mathbb{R})$  at most twice, and every geodesic is tangent to a right coset of  $\text{Diff}_{\mathcal{A}}(\mathbb{R})$  at most once.*

*For  $\varphi_0, \varphi_1 \in \text{Diff}_{\mathcal{A}}(\mathbb{R})$  we can give the following formula for the size of the shift along the connecting minimal geodesic:*

$$\text{Shift}(\varphi(t)) = \frac{t^2 - t}{4} \|R(\varphi_0) - R(\varphi_1)\|_{L^2}^2 = (t^2 - t) \|\sqrt{\varphi'_0} - \sqrt{\varphi'_1}\|_{L^2}^2.$$

*Proof.* To make the computations easier, define the following variables:

$$\begin{aligned}\gamma_0 &= R(\varphi_0) \\ k_0 &= T_{\varphi_0}R.(u_0 \circ \varphi_0) = \sqrt{\varphi_0'}u_0' \circ \varphi_0 \\ \gamma(t) &= R(\varphi(t)) = \gamma_0 + tk_0.\end{aligned}$$

For a diffeomorphism  $\varphi \in \text{Diff}_{\mathcal{A}_1}(\mathbb{R})$  the size of the shift at  $\infty$  is given by

$$\text{Shift}(\varphi) = \lim_{x \rightarrow \infty} \varphi(x) - x = \int_{-\infty}^{\infty} \varphi'(x) - 1 \, dx = \frac{1}{4} \int_{-\infty}^{\infty} R(\varphi)^2 + 4R(\varphi) \, dx.$$

Similarly, the value of a function  $u \in \mathcal{A}_1$  at  $\infty$  can be computed by

$$\begin{aligned}u(\infty) &= \int_{-\infty}^{\infty} u'(x) dx = \int_{-\infty}^{\infty} (u' \circ \varphi)(x) \varphi'(x) \, dx \\ &= \frac{1}{2} \int_{-\infty}^{\infty} (R(\varphi) + 2) T_{\varphi}R.(u \circ \varphi)(x) \, dx.\end{aligned}$$

Note that this holds for any  $\varphi \in \text{Diff}_{\mathcal{A}_1}(\mathbb{R})$ . Since the  $R$ -map pulls back the  $L^2$ -metric to the  $\dot{H}^1$ -metric we also have the identity

$$\int_{-\infty}^{\infty} u'(x) dx = \int_{-\infty}^{\infty} (T_{\varphi}R.(u \circ \varphi))^2 dx.$$

For the sake of convenience let us rewrite the last two equations using the variables  $u_0, \gamma_0$  and  $k_0$ .

$$u_0(\infty) = \frac{1}{2} \int_{-\infty}^{\infty} (\gamma_0 + 2)k_0 \, dx \quad \int_{-\infty}^{\infty} (u_0')^2 \, dx = \int_{-\infty}^{\infty} k_0^2 \, dx$$

Now computing the shift of  $\varphi(t)$  at  $\infty$  is easy:

$$\begin{aligned}\text{Shift}(\varphi(t)) &= \frac{1}{4} \int_{-\infty}^{\infty} (\gamma_0 + tk_0)^2 + 4(\gamma_0 + tk_0) \, dx \\ &= \frac{1}{4} \int_{-\infty}^{\infty} \gamma_0^2 + 4\gamma_0 + 2t\gamma_0k_0 + 4tk_0 + t^2k_0^2 \, dx \\ &= \text{Shift}(\varphi_0) + tu_0(\infty) + \frac{t^2}{4} \int_{-\infty}^{\infty} (u_0')^2 dx.\end{aligned}$$

Computing the value of  $u(t)$  at  $\infty$  is just as simple:

$$\begin{aligned}u(t, \infty) &= \frac{1}{2} \int_{-\infty}^{\infty} (\gamma_0 + tk_0 + 2)k_0 \, dx \\ &= u_0(\infty) + t \int_{\mathbb{R}} (u_0')^2 dx.\end{aligned}$$

If we start with  $\varphi_0, \varphi_1 \in \text{Diff}_{\mathcal{A}}(\mathbb{R})$ , then the geodesic connecting them has  $k_0 = \gamma_1 - \gamma_0$  with  $\gamma_1 = R(\varphi_1)$ . Some algebraic manipulations, keeping in mind that  $\text{Shift}(\varphi_i) = \frac{1}{4} \int \gamma_i^2 + 4\gamma_i = 0$ , give us

$$\begin{aligned}u_0(\infty) &= \frac{1}{2} \int_{-\infty}^{\infty} (\gamma_0 + 2)(\gamma_1 - \gamma_0) \, dx = \frac{1}{2} \int_{-\infty}^{\infty} \gamma_0\gamma_1 - \gamma_0^2 - 2\gamma_0 + 2\gamma_1 \, dx \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \gamma_0\gamma_1 - \gamma_0^2 + \frac{1}{2}\gamma_0^2 - \frac{1}{2}\gamma_1^2 \, dx = -\frac{1}{4} \int_{-\infty}^{\infty} (\gamma_1 - \gamma_0)^2 \, dx,\end{aligned}$$

which in turn leads to

$$\text{Shift}(\varphi(t)) = tu_0(\infty) + \frac{t^2}{4} \int_{-\infty}^{\infty} (u_0')^2 \, dx = \frac{t^2 - t}{4} \int_{-\infty}^{\infty} (\gamma_1 - \gamma_0)^2 \, dx.$$

This completes the proof.  $\square$

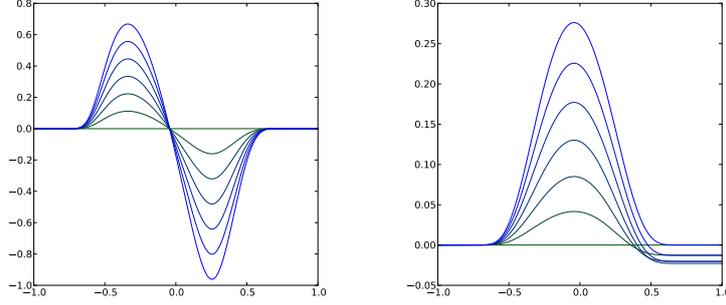


FIGURE 3. Geodesic  $\varphi(t, x) \in \text{Diff}_{\mathcal{A}_1}(\mathbb{R})$  between two diffeomorphisms in  $\text{Diff}_{\mathcal{A}}(\mathbb{R})$  sampled at times  $t = 0, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, 1$ . Left image: The geodesic in  $R$ -map space. Right image: The geodesic in original space, visualized as  $\varphi(t, x) - x$ .

An example of a geodesic illustrating the behaviour described in the lemma can be seen in Fig. 3.

**4.7. Solving the Hunter–Saxton Equation.** The theory described in the preceding sections allows us to construct an analytic solution formula for the HS equation on  $\mathcal{A}_1(\mathbb{R})$ . Here  $\mathcal{A}_1(\mathbb{R})$  denotes one of the function spaces  $C_{c,1}^\infty(\mathbb{R})$ ,  $\mathcal{S}_1(\mathbb{R})$  or  $W_1^{\infty,1}(\mathbb{R})$ , as defined in Sects. 2.2–2.4.

**Theorem** (Solutions to the HS equation). *Given an initial value  $u_0$  in  $\mathcal{A}_1(\mathbb{R})$  the solution to the HS equation is given by*

$$u(t, x) = \varphi_t(t, \varphi^{-1}(t, x)), \quad \text{with} \quad \varphi(t, x) = R^{-1}(tu'_0)(x).$$

*In particular, this means that a solution with initial condition in one of the spaces  $C_c^\infty(\mathbb{R})$ ,  $\mathcal{S}(\mathbb{R})$  or  $W^{\infty,1}(\mathbb{R})$  exists for all time  $t > 0$ , if and only if  $u'_0(x) \geq 0$  for all  $x \in \mathbb{R}$ . All solutions are real-analytic in time in the sense of [21, Section 9].*

*Proof.* By the theory of the previous sections, we know that the path

$$\varphi(t, x) = R^{-1}(t\gamma)(x),$$

with  $\varphi(0, x) = x$  is a solution to the geodesic equation for every  $\gamma \in \mathcal{A}_1(\mathbb{R})$ . It remains to choose  $\gamma$  such that the initial condition

$$\varphi_t(0, \varphi^{-1}(0, x)) = \varphi_t(0, x) = u_0(x)$$

is satisfied. This can be achieved by choosing  $\gamma = T_{\text{Id}}R.u_0$ , since

$$\gamma = \partial_t|_0(t\gamma) = \partial_t|_0R(\varphi(t)) = T_{\varphi(t)}R.\varphi_t(t)|_0 = T_{\text{Id}}R.u_0.$$

Using the formula  $T_\varphi R.h = (\varphi')^{-1/2}h'$  from the proof of Theorem 4.3 yields  $T_{\text{Id}}R.u_0 = u'_0$ .

The solution is real-analytic in  $t$  since  $t \mapsto t.u_0$  is a real-analytic curve in  $\mathcal{A}_1(\mathbb{R})$  and since  $R^{-1}$  respects real-analytic curves; see [21, Section 9].

Given  $x_0, u_0$  such that  $u'_0(x_0) < 0$  there exist  $t_0 \in \mathbb{R}$  with  $t_0 u'_0(x_0) < -2$ . Thus the geodesic at time  $t_0$  has left the  $R$ -map space, and the solution of the HS equation leaves the space  $\mathcal{A}_1$ .  $\square$

A more explicit formula for the solution is given by

$$\begin{aligned} u(t, x) &= u_0(\varphi^{-1}(t, x)) + \frac{t}{2} \int_{-\infty}^{\varphi^{-1}(t, x)} u_0'(y)^2 dy \\ \varphi(t, x) &= x + \frac{1}{4} \int_{-\infty}^x t^2 (u_0'(y))^2 + 4t u_0'(y) dy. \end{aligned}$$

**Remark.** The HS equation on the real line also provides an example of how geometry and PDE behaviour influence each other. It was shown in Sect. 4.2 that the geodesic equation on  $\text{Diff}_{\mathcal{A}}(\mathbb{R})$  does not exist because the condition  $\text{ad}^*(u)u \in \check{G}_{\text{Id}}(u)$  is not satisfied. From a naive point of view we could start with the energy

$$(12) \quad E(u) = \int_0^1 \int_{-\infty}^{\infty} u_x^2 dx dt$$

defined on functions  $u \in C^\infty([0, 1], \mathcal{A}(\mathbb{R}))$  and take variations of the form  $\delta u = \eta_t + \eta_x u - \eta u_x$  with fixed endpoints  $\eta(0, x) = \varphi_0(x)$  and  $\eta(1, x) = \varphi_1(x)$ . This would lead, after some integration by parts, to

$$\langle DE(u), \delta u \rangle = \int_0^1 \int_{-\infty}^{\infty} (u_{txx} + (uu_x)_{xx} - u_x u_{xx}) \eta dx dt$$

and we could now declare

$$u_{txx} + (uu_x)_{xx} - u_x u_{xx} = 0$$

to be the geodesic equation. It is in any case the equation which the critical points of the energy functional (12) must satisfy. But this equation has no solutions in  $\mathcal{A}$ . It is shown in Theorem 4.6 that a solution  $u \in C^\infty([0, 1], \mathcal{A}_1(\mathbb{R}))$  intersects  $\mathcal{A}(\mathbb{R})$  at most once. To find solutions, we must enlarge the space to  $\mathcal{A}_1(\mathbb{R})$  and then the preceding theorem via the  $R$ -map gives us the existence of solutions.

**4.8. Continuing Geodesics Beyond the Group, or How Solutions of the Hunter–Saxton Equation Blow Up.** Consider a straight line  $\gamma(t) = \gamma_0 + t\gamma_1$  in  $\mathcal{A}(\mathbb{R}, \mathbb{R})$ . Then  $\gamma(t) \in \mathcal{A}(\mathbb{R}, \mathbb{R}_{>-2})$  precisely for  $t$  in an open interval  $(t_0, t_1)$  which is finite at least on one side, say, at  $t_1 < \infty$ . Note that

$$\varphi(t)(x) := R^{-1}(\gamma(t))(x) = x + \frac{1}{4} \int_{-\infty}^x \gamma^2(t)(u) + 4\gamma(t)(u) du$$

makes sense for all  $t$ , that  $\varphi(t) : \mathbb{R} \rightarrow \mathbb{R}$  is smooth and that  $\varphi(t)'(x) \geq 0$  for all  $x$  and  $t$ ; thus,  $\varphi(t)$  is monotone non-decreasing. Moreover,  $\varphi(t)$  is proper and surjective since  $\gamma(t)$  vanishes at  $-\infty$  and  $\infty$ . Let

$$\text{Mon}_{\mathcal{A}_1}(\mathbb{R}) := \{ \text{Id} + f : f \in \mathcal{A}_1(\mathbb{R}, \mathbb{R}), f' \geq -1 \}$$

be the monoid (under composition) of all such functions.

For  $\gamma \in \mathcal{A}(\mathbb{R}, \mathbb{R})$  let  $x(\gamma) := \min\{x \in \mathbb{R} \cup \{\infty\} : \gamma(x) = -2\}$ . Then for the line  $\gamma(t)$  from above we see that  $x(\gamma(t)) < \infty$  for all  $t > t_1$ . Thus, if the ‘geodesic’  $\varphi(t)$  leaves the diffeomorphism group at  $t_1$ , it never comes back but stays inside  $\text{Mon}_{\mathcal{A}_1}(\mathbb{R}) \setminus \text{Diff}_{\mathcal{A}_1}(\mathbb{R})$  for the rest of its life. In this sense,  $\text{Mon}_{\mathcal{A}_1}(\mathbb{R})$  is a *geodesic completion* of  $\text{Diff}_{\mathcal{A}_1}(\mathbb{R})$ , and  $\text{Mon}_{\mathcal{A}_1}(\mathbb{R}) \setminus \text{Diff}_{\mathcal{A}_1}(\mathbb{R})$  is the *boundary*.

What happens to the corresponding solution  $u(t, x) = \varphi_t(t, \varphi(t)^{-1}(x))$  of the HS equation? In certain points it has infinite derivative, it may be multivalued, or its graph can contain whole vertical intervals. If we replace an element  $\varphi \in \text{Mon}_{\mathcal{A}_1}(\mathbb{R})$  by its graph  $\{(x, \varphi(x)) : x \in \mathbb{R}\} \subset \mathbb{R}$  we get a smooth ‘monotone’ submanifold, a smooth monotone relation. The inverse  $\varphi^{-1}$  is then also a smooth monotone

relation. Then  $t \mapsto \{(x, u(t, x)) : x \in \mathbb{R}\}$  is a (smooth) curve of relations. Checking that it satisfies the HS equation is an exercise left for the interested reader. What we have described here is the *flow completion* of the HS equation in the spirit of [15].

**4.9. Soliton-Like Solutions.** For a right-invariant metric on a diffeomorphism group one can ask whether (generalized) solutions  $u(t) = \varphi_t(t) \circ \varphi(t)^{-1}$  exist such that the momenta  $\check{G}(u(t)) =: p(t)$  are distributions with finite support. Here the geodesic  $\varphi(t)$  may exist only in some suitable Sobolev completion of the diffeomorphism group. By the general theory (see in particular [36, Sect. 4.4], or [33]) the momentum  $\text{Ad}(\varphi(t))^* p(t) = \varphi(t)^* p(t) = p(0)$  is constant. In other words,  $p(t) = (\varphi(t)^{-1})^* p(0) = \varphi(t)_* p(0)$ , i.e., the momentum is carried forward by the flow and remains in the space of distributions with finite support. The infinitesimal version (take  $\partial_t$  of the last expression) is  $p_t(t) = -\mathcal{L}_{u(t)} p(t) = -\text{ad}_{u(t)}^* p(t)$ ; c.f. Sect. 3.2. The space of  $N$ -solitons of order 0 consists of momenta of the form  $p_{y,a} = \sum_{i=1}^N a_i \delta_{y_i}$  with  $(y, a) \in \mathbb{R}^{2N}$ . Consider an initial soliton  $p_0 = \check{G}(u_0) = -u_0'' = \sum_{i=1}^N a_i \delta_{y_i}$  with  $y_1 < y_2 < \dots < y_N$ . Let  $H$  be the Heaviside function

$$H(x) = \begin{cases} 0, & x < 0, \\ \frac{1}{2}, & x = 0, \\ 1, & x > 0, \end{cases}$$

and  $D(x) = 0$  for  $x \leq 0$  and  $D(x) = x$  for  $x > 0$ . We will see later why the choice  $H(0) = \frac{1}{2}$  is the most natural one; note that the behavior is called the Gibbs phenomenon. With these functions we can write

$$\begin{aligned} u_0''(x) &= -\sum_{i=1}^N a_i \delta_{y_i}(x) \\ u_0'(x) &= -\sum_{i=1}^N a_i H(x - y_i) \\ u_0(x) &= -\sum_{i=1}^N a_i D(x - y_i). \end{aligned}$$

We will assume henceforth that  $\sum_{i=1}^N a_i = 0$ . Then  $u_0(x)$  is constant for  $x > y_N$  and thus  $u_0 \in H_1^1(\mathbb{R})$ ; with a slight abuse of notation we assume that  $H_1^1(\mathbb{R})$  is defined similarly to  $H_1^\infty(\mathbb{R})$ . Defining  $S_i = \sum_{j=1}^i a_j$  we can write

$$u_0'(x) = -\sum_{i=1}^N S_i (H(x - y_i) - H(x - y_{i+1})).$$

This formula will be useful because  $\text{supp}(H(\cdot - y_i) - H(\cdot - y_{i+1})) = [y_i, y_{i+1}]$ .

The evolution of the geodesic  $u(t)$  with initial value  $u(0) = u_0$  can be described by a system of ordinary differential equations (ODEs) for the variables  $(y, a)$ . We cite the following result.

**Theorem** ([14]). *The map  $(y, a) \mapsto \sum_{i=1}^N a_i \delta_{y_i}$  is a Poisson map between the canonical symplectic structure on  $\mathbb{R}^{2N}$  and the Lie-Poisson structure on the dual  $T_{\text{Id}}^* \text{Diff}_{\mathcal{A}}(\mathbb{R})$  of the Lie algebra.*

In particular, this means that the ODEs for  $(y, a)$  are Hamilton's equations for the pullback Hamiltonian

$$E(y, a) = \frac{1}{2} G_{\text{Id}}(u_{(y,a)}, u_{(y,a)}),$$

with  $u_{(y,a)} = \check{G}^{-1}(\sum_{i=1}^N a_i \delta_{y_i}) = -\sum_{i=1}^N a_i D(\cdot - y_i)$ . We can obtain the more explicit expression

$$\begin{aligned} E(y, a) &= \frac{1}{2} \int_{\mathbb{R}} (u_{(y,a)}(x)')^2 dx = \frac{1}{2} \int_{\mathbb{R}} \left( \sum_{i=1}^N S_i \mathbb{1}_{[y_i, y_{i+1}]} \right)^2 dx \\ &= \frac{1}{2} \sum_{i=1}^N S_i^2 (y_{i+1} - y_i). \end{aligned}$$

Hamilton's equations  $\dot{y}_i = \partial E / \partial a_i$ ,  $\dot{a}_i = -\partial E / \partial y_i$  are in this case

$$\begin{aligned} \dot{y}_i(t) &= \sum_{j=i}^{N-1} S_j(t) (y_{j+1}(t) - y_j(t)), \\ \dot{a}_i(t) &= \frac{1}{2} (S_i(t)^2 - S_{i-1}(t)^2). \end{aligned}$$

Using the  $R$ -map we can find explicit solutions for these equations as follows. Let us write  $a_i(0) = a_i$  and  $y_i(0) = y_i$ . By Theorem 4.7 the geodesic with initial velocity  $u_0$  is given by

$$\begin{aligned} \varphi(t, x) &= x + \frac{1}{4} \int_{-\infty}^x t^2 (u'_0(y))^2 + 4t u'_0(y) dy \\ u(t, x) &= u_0(\varphi^{-1}(t, x)) + \frac{t}{2} \int_{-\infty}^{\varphi^{-1}(t, x)} u'_0(y)^2 dy. \end{aligned}$$

First note that

$$\begin{aligned} \varphi'(t, x) &= \left( 1 + \frac{t}{2} u'_0(x) \right)^2 \\ u'(t, z) &= \frac{u'_0(\varphi^{-1}(t, z))}{1 + \frac{t}{2} u'_0(\varphi^{-1}(t, z))}. \end{aligned}$$

Using the identity  $H(\varphi^{-1}(t, z) - y_i) = H(z - \varphi(t, y_i))$  we obtain

$$u'_0(\varphi^{-1}(t, z)) = -\sum_{i=1}^N a_i H(z - \varphi(t, y_i)),$$

and thus

$$(u'_0(\varphi^{-1}(t, z)))' = -\sum_{i=1}^N a_i \delta_{\varphi(t, y_i)}(z).$$

Combining these we obtain

$$\begin{aligned} u''(t, z) &= \frac{1}{\left( 1 + \frac{t}{2} u'_0(\varphi^{-1}(t, z)) \right)^2} \left( -\sum_{i=1}^N a_i \delta_{\varphi(t, y_i)}(z) \right) \\ &= \sum_{i=1}^N \frac{-a_i}{\left( 1 + \frac{t}{2} u'_0(y_i) \right)^2} \delta_{\varphi(t, y_i)}(z). \end{aligned}$$

From here we can read off the solution of Hamilton's equations

$$\begin{aligned} y_i(t) &= \varphi(t, y_i) \\ a_i(t) &= -a_i \left(1 + \frac{t}{2} u'_0(y_i)\right)^{-2}. \end{aligned}$$

When trying to evaluate  $u'_0(y_i)$ ,

$$u'_0(y_i) = a_i H(0) - S_i,$$

we see that  $u'_0$  is discontinuous at  $y_i$  and it is here that we seem to have the freedom to choose the value  $H(0)$ . However, it turns out that we observe the Gibbs phenomenon, i.e., only the choice  $H(0) = \frac{1}{2}$  leads to solutions of Hamilton's equations. Also, the regularized theory of multiplications of distributions (see the discussion in [20, Sect. 1.1]) leads to this choice. Thus we obtain

$$\begin{aligned} y_i(t) &= y_i + \sum_{j=1}^{i-1} \left( \frac{t^2}{4} S_j^2 - t S_j \right) (y_{j+1} - y_j) \\ a_i(t) &= \frac{-a_i}{\left(1 + \frac{t}{2} \left(\frac{a_i}{2} - S_i\right)\right)^2} = - \left( \frac{S_i}{1 - \frac{t}{2} S_i} - \frac{S_{i-1}}{1 - \frac{t}{2} S_{i-1}} \right). \end{aligned}$$

It can be checked by direct computation that these functions indeed solve Hamilton's equations.

## 5. TWO-COMPONENT HUNTER-SAXTON EQUATION ON REAL LINE

In this section we will show, similarly to the previous section, that one can adapt the work of Lenells on the periodic two-component HS equation [29] to obtain results for the non-periodic case. On the real line this system has been studied from an analytical viewpoint in Sect. [41, Section 4].

**Theorem.** *Let  $\mathcal{M} = \text{Diff}_{\mathcal{A}}(\mathbb{R}) \ltimes \mathcal{A}(\mathbb{R}, \mathbb{R})$  and  $\widetilde{\mathcal{M}} = \text{Diff}_{\mathcal{A}_1}(\mathbb{R}) \ltimes \mathcal{A}(\mathbb{R}, \mathbb{R})$  be the semi-direct product Lie groups with multiplication*

$$(\varphi, \alpha)(\psi, \beta) = (\varphi \circ \psi, \beta + \alpha \circ \psi),$$

where  $\mathcal{A}$  and  $\mathcal{A}_1$  are as defined in 4.1. Consider the following weak Riemannian metric  $G$  on  $M$  and  $\widetilde{M}$ :

$$G_{(\text{Id}, 0)}((X, a), (Y, b)) = \int X'(x)Y'(x) + a(x)b(x) dx,$$

where  $(X, a)$  and  $(Y, b)$  are elements of the corresponding Lie algebra.

Then the geodesic equation on  $\widetilde{\mathcal{M}}$  is the two-component non-periodic HS equation given by

$$\begin{aligned} (u, \rho) &= (\varphi_t \circ \varphi^{-1}, \alpha_t \circ \varphi^{-1}) \\ u_t &= -uu_x + \frac{1}{2} \int_{-\infty}^x u_x(z)^2 + \rho^2(z) dz \\ \rho_t &= -(\rho u)_x \end{aligned}$$

The geodesic equation does not exist on  $\mathcal{M}$ , since the adjoint  $\text{ad}((X, a))^* G_{(\text{Id}, 0)}$  is not in  $G_{(\text{Id}, 0)}$  (Lie algebra) for all  $(X, a)$ . These are not robust Riemannian manifolds in the sense of [37, Sect. 2.4].

**Remark.** Note that one obtains more so-called classical forms of the HS equation by differentiating the equation for  $u_t$ :

$$\begin{aligned} u_{tx} &= -uu_{xx} + \frac{1}{2}(-u_x^2 + \rho^2); \\ u_{txx} &= \left(-uu_{xx} + \frac{1}{2}(-u_x^2 + \rho^2)\right)_x = -2u_x u_{xx} - uu_{xxx} + \rho\rho_x. \end{aligned}$$

The second of the preceding equations is the version which Lenells called the two-component HS equation in [29].

*Proof.* To prove the formula for the geodesic equation we need to calculate the adjoint as defined in Sect. 3.2. For vector fields  $(X, a)$  and  $(Y, b)$  the Lie bracket is given by

$$[(X, a), (Y, b)] = (-[X, Y], -(\mathcal{L}_X b - \mathcal{L}_Y a)).$$

We calculate

$$\begin{aligned} &\langle \text{ad}((X, a))^* G((Y, b)), (Z, c) \rangle \\ &= G((Y, b), \text{ad}((X, a))(Z, c)) \\ &= G\left((Y, b), (-[X, Z], -\mathcal{L}_X c + \mathcal{L}_Z a)\right) \\ &= \int_{\mathbb{R}} Y'(x)(X'(x)Z(x) - X(x)Z'(x))' + b(x)(-\mathcal{L}_X c(x) + \mathcal{L}_Z a(x)) dx \\ &= \int_{\mathbb{R}} Y'(x)(X''(x)Z(x) + X'(x)Z'(x) - X'(x)Z'(x) - X(x)Z''(x)) dx \\ &\quad + \int_{\mathbb{R}} b(x)(-c'(x)X(x) + a'(x)Z(x)) dx \\ &= \int_{\mathbb{R}} Z(x)(X''(x)Y(x) + (X(x)Y'(x))'' + b(x)a'(x)) dx \\ &\quad + \int_{\mathbb{R}} c(x)(b'(x)X(x) + b(x)X'(x)) dx \\ &= \left\langle (X''Y + (XY')'' + ba', b'X + bX'), (Z, c) \right\rangle. \end{aligned}$$

Therefore, the adjoint is given by

$$\text{ad}(X, a)^* G(Y, b)(x) = (X''Y + (XY')'' + ba', b'X + bX').$$

Note that for general  $(X, a), (Y, b) \in \mathcal{A} \times \mathcal{A}$  the adjoint is not an element of  $G(\mathcal{A} \times \mathcal{A})$ ; the same statement is true for  $\mathcal{A}_1 \times \mathcal{A}$ . But for  $(X, a)$  equal to  $(Y, b)$  we can rewrite the preceding equation in a form similar to that in Sect. 4.2 to obtain

$$\begin{aligned} \text{ad}(X, a)^* G(X, a)(x) &= \left(\frac{1}{2}(X'^2)' + \frac{1}{2}(X^2)''' + \frac{1}{2}(a^2)', a'X + aX'\right) \\ &= \left(\frac{1}{2}\left(\int_{-\infty}^x X'^2 + a^2 dx + (X^2)'\right)'', a'X + aX'\right) \\ &= \check{G}\left(\frac{1}{2}\left(\int_{-\infty}^x X'^2 + a^2 dx + (X^2)'\right), a'X + aX'\right). \end{aligned}$$

An argument similar to that in Sect. 4.2 proves the existence of the adjoint and, thus, of the geodesic equation on  $\widetilde{\mathcal{M}}$  and the non-existence on  $\mathcal{M}$ .  $\square$

**Theorem** (*R*-map for 2HS equation). *Define the map*

$$R : \begin{cases} \widetilde{\mathcal{M}} \rightarrow (\mathcal{A}(\mathbb{R}, \mathbb{C}/\{-2\}), \|\cdot\|_{L^2}) \\ (\varphi, \alpha) \mapsto 2\varphi^{1/2}e^{i\alpha/2} - 2. \end{cases}$$

The  $R$ -map is invertible with inverse

$$R^{-1} : \begin{cases} \mathcal{A}(\mathbb{R}, \mathbb{C}/\{-2\}) \rightarrow \widetilde{\mathcal{M}} \\ \gamma \mapsto \left( x + \frac{1}{4} \int_{-\infty}^x (|\gamma + 2|^2 - 4) dx, 2 \arg(\gamma(x) + 2) \right). \end{cases}$$

The pullback of the flat  $L^2$ -metric via  $R$  is the metric  $G$  as defined in the previous theorem. Thus the space  $(\widetilde{\mathcal{M}}, G)$  is a flat space in the sense of Riemannian geometry.

*Proof.* An argument similar to that in Sect. 4.3 shows that the image  $R(\varphi, \alpha)$  is an element of  $\mathcal{A}(\mathbb{R}, \mathbb{C}/\{-2\})$ . The bijectivity follows from a straightforward calculation using that for  $\gamma = R(\varphi, \alpha) = R(\text{Id} + f, \alpha)$  we have

$$\frac{1}{4} |\gamma(x) + 2|^2 - 1 = f'(x),$$

which proves the identities  $R \circ R^{-1} = R \circ R^{-1} = \text{Id}$ .

Since the mapping  $R$  is bijective, the pullback via  $R$  yields a well-defined metric on  $\widetilde{\mathcal{M}}$ . To obtain its formula, we must calculate the tangent mapping of  $R$ . Let  $(h, U) = (X \circ \varphi, U) \in T_{\varphi, \alpha} \widetilde{\mathcal{M}}$ . We have

$$T_{\varphi, \alpha} R(h, U) = \varphi_x^{-1/2} h' e^{i\alpha/2} + i \varphi_x^{1/2} e^{i\alpha/2} U.$$

Using this formula we have for  $h = X_1 \circ \varphi, k = X_2 \circ \varphi$ :

$$\begin{aligned} R^* \langle (h, U), (k, V) \rangle_{L^2} &= \text{Re} \int_{\mathbb{R}} \langle T_{\varphi, \alpha} R(h, U), \overline{T_{\varphi, \alpha} R(k, V)} \rangle dx \\ &= \int_{\mathbb{R}} X_1'(x) X_2'(x) + \alpha(x) \beta(x) dx \\ &= G_{\varphi, \alpha} \langle (h, U), (k, V) \rangle. \end{aligned} \quad \square$$

We can now again use this result to construct an analytic solution formula for the corresponding geodesic equation — the two-component HS equation.

**Theorem** (Solutions to 2HS -equation). *Given an initial value  $(u_0, \rho_0)$  in  $\mathcal{A}_1(\mathbb{R}) \times \mathcal{A}(\mathbb{R})$  the solution to the HS equation is given by*

$$(u, \rho) = (\varphi_t \circ \varphi^{-1}, -\alpha \circ \varphi + \alpha_t \circ \varphi^{-1}) \quad \text{with} \quad (\varphi, \alpha) = R^{-1}(t(u'_0 + i\rho_0)).$$

*In particular, this means that a solution breaks in finite time  $T$ , if and only if there exists a  $x \in \mathbb{R}$  such that  $u'_0(x) < 0$  and  $\rho_0(x) = 0$ .*

*Proof.* By the previous theorem we know that the path

$$(\varphi(t, x), \rho(t, x)) = R^{-1}(t \gamma_0)(x)$$

is a solution to the geodesic equation for every  $\gamma_0 \in \mathcal{A}_1(\mathbb{R}, \mathbb{C}/\{-2\})$ . It remains to choose  $\gamma$  such that the initial conditions are satisfied. This can be achieved exactly by choosing  $\gamma_0 = T_{\text{Id}, 0} R(u_0, \rho_0) = (u'_0 + i\rho_0)$ .  $\square$

**Remark.** This theorem holds also in more general situations, i.e., for the spaces  $\mathcal{A}_1(\mathbb{R}) \times \mathcal{C}(\mathbb{R})$  with  $\mathcal{A} \neq \mathcal{C}$ , e.g.,  $W_1^{\infty, 1}(\mathbb{R}) \times \mathcal{S}(\mathbb{R})$ . The result holds in this situation since the diffeomorphism group  $\text{Diff}_{\mathcal{A}_1}(\mathbb{R})$  acts on  $\mathcal{C}(\mathbb{R})$  for all choices of  $\mathcal{A}$  and  $\mathcal{C}$  among  $C_c^\infty(\mathbb{R})$ ,  $\mathcal{S}(\mathbb{R})$  and  $W^{\infty, 1}(\mathbb{R})$ .

## 6. REMARKS ON PERIODIC CASE

In this section we will briefly review the results of [27] and extend them to the case of real-analytic or ultra-differentiable functions on the circle. In this section,  $\text{Diff}^\circ(S^1)/S^1$  denotes one of the following homogeneous spaces:

- (1)  $\text{Diff}(S^1)/S^1$  the space of smooth diffeomorphisms on the circle modulo rotations.
- (2)  $\text{Diff}^\omega(S^1)/S^1$  the space of real-analytic diffeomorphisms on the circle modulo rotations; c.f. Sect 2.6.
- (3)  $\text{Diff}^{[M]}(S^1)/S^1$  the space of ultra-differentiable diffeomorphisms of Beurling type or Roumieu type on the circle modulo rotations; c.f. Sect. 2.7.

A diffeomorphism  $\varphi \in \text{Diff}^\circ(S^1)$  is related to its universal covering diffeomorphism  $\tilde{\varphi}$  by  $\varphi(e^{ix}) = e^{i\tilde{\varphi}(x)}$ . Then  $\tilde{\varphi}(x) = x + f(x)$  where  $f$  is a  $2\pi$ -periodic real-valued function. Rotations correspond to constant functions  $f$ . Let  $\widetilde{\text{Diff}}^\circ(S^1)$  denote the regular Lie group of lift to the universal cover of diffeomorphisms. The corresponding homogeneous space is then  $\widetilde{\text{Diff}}^\circ(S^1)/\mathbb{R}$ , factoring out all translations.

**Theorem** ([27]). *On the homogeneous space  $\widetilde{\text{Diff}}^\circ(S^1)/\mathbb{R}$  the square root representation is a bijective mapping given as follows:*

$$R : \begin{cases} \widetilde{\text{Diff}}^\circ(S^1)/\mathbb{R} \rightarrow (\text{Im}(R), \|\cdot\|_{L^2([0,2\pi])}) \subset (C_{2\pi\text{-per}}^\circ(\mathbb{R}, \mathbb{R}_{>0}), \|\cdot\|_{L^2}) \\ \tilde{\varphi} \mapsto 2(\tilde{\varphi}')^{1/2}. \end{cases}$$

The image of the  $R$ -map is the sphere of radius  $\sqrt{8\pi}$ , i.e.,

$$\text{Im}(R) = \left\{ \gamma \in C_{2\pi\text{-per}}^\circ(\mathbb{R}, \mathbb{R}_{>0}) : \|\gamma\|_{L^2}^2 = \int_0^{2\pi} \gamma^2 d\theta = 8\pi \right\}.$$

The pullback of the restriction of the  $L^2$ -metric to  $\text{Im}(R)$  via  $R$  is the homogeneous Sobolev metric of order one, i.e.,

$$R^*\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\dot{H}^1}.$$

Thus the spaces  $(\text{Diff}^\circ(S^1)/S^1, \dot{H}^1)$  have constant positive sectional curvature.

Here  $C_{2\pi\text{-per}}^\circ(\mathbb{R}, \mathbb{R}_{>0})$  denotes the corresponding space of sufficiently smooth functions, i.e., either the space of  $C^\infty$ , real-analytic or ultra-differentiable functions.

*Proof.* The case of  $C^\infty$ -functions is proven in the work of Lenells [27]. The statement about the image of  $R$  follows from

$$\|R(\tilde{\varphi})\|^2 = 4\|(\tilde{\varphi}')^{1/2}\|^2 = 4\int_0^{2\pi} \tilde{\varphi}'(x) dx = 4\int_0^{2\pi} 1 + f'(x) dx = 8\pi$$

The remaining cases follow similarly using that  $\text{Diff}^\omega(S^1)$  and  $\text{Diff}^{[M]}(S^1)$  are Lie subgroups of  $\text{Diff}(S^1)$ , c.f. 2.6 and 2.7.  $\square$

As a direct consequence we obtain the following result:

**Theorem** (Solutions to the periodic HS equation). *Given an initial value  $u_0$  in  $C^\circ(S^1, \mathbb{R}_{>0})$  the solution to the HS equation stays locally in the same space. A solution exists for all time  $t$ , if and only if  $u'_0(\theta) \geq 0$  for all  $\theta \in S^1$ .*

**Remark.** From our setup in Sects. 2.6 and 2.7 it is obvious that the results of [29] for the two-component HS equation extend to the cases of real-analytic and ultra-differentiable functions.

## 7. SIMILAR REPRESENTATION FOR CAMASSA-HOLM EQUATION

In this article we have shown that certain non-trivial Riemannian spaces that have flat or constant curvature can be represented as a simple submanifold of the flat manifold of all sufficiently smooth functions equipped with the  $L^2$ -metric. In this section we will present a natural example of a metric space with non-trivial curvature which can also be represented as a (complicated) subspace of the flat manifold of all smooth functions, namely the Lie group  $\text{Diff}(S^1)$  equipped with the right-invariant non-homogeneous  $H^1$ -metric.

**Theorem** ([17]). *The right-invariant  $H^1$ -metric on the Lie group  $\text{Diff}(S^1)$  is given by*

$$G_\varphi(X \circ \varphi, Y \circ \varphi) = \int X(x)Y(x) + X'(x)Y'(x) dx,$$

where  $X, Y$  are vector fields in the Lie algebra  $\mathfrak{X}(S^1)$ . The induced geodesic distance is positive, and the corresponding geodesic equation is the Camassa-Holm Eq. [6] given by

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}.$$

The geodesic equation is well posed and the exponential map is a local diffeomorphism.

Using the ideas of [29] we can introduce an  $R$ -map for this space. We use again  $\varphi(e^{ix}) = e^{i\tilde{\varphi}(x)}$  with  $\tilde{\varphi}(x) = x + f(x)$  for periodic  $f$ . Again let  $\widetilde{\text{Diff}}(S^1)$  denote the regular Lie group of lifts to the universal cover of diffeomorphisms. For curves we obtain  $\partial_t|_0 \varphi(t, e^{ix}) = i\partial_t|_0 \tilde{\varphi}(t, x) \cdot e^{i\tilde{\varphi}(x)} = i\partial_t|_0 f(t, x) \cdot e^{i\tilde{\varphi}(x)}$ . Thus, for tangent vectors we obtain  $\delta\varphi = i \cdot \delta\tilde{\varphi} \cdot \varphi = i \cdot \delta f \cdot \varphi$ .

**Theorem.** *The  $R$ -map is defined by*

$$R(\tilde{\varphi}) := 2\tilde{\varphi}'^{\frac{1}{2}} e^{i(\tilde{\varphi} - \text{Id})/2} = 2(1 + f')^{\frac{1}{2}} e^{if/2},$$

$$R : \widetilde{\text{Diff}}(S^1) \rightarrow C_{2\pi\text{-per}}^\infty(\mathbb{R}, \mathbb{C}).$$

The image under the  $R$ -map of the diffeomorphism group is the space  $\mathcal{S}$  given by

$$\mathcal{S} := R(\widetilde{\text{Diff}}(S^1)) = \left\{ \gamma \in C_{2\pi\text{-per}}^\infty(\mathbb{R}, \mathbb{C} \setminus \{0\}) : F(\gamma) = (F_1(\gamma), F_2(\gamma)) = 0 \right\},$$

$$\text{where } F_1(\gamma) := \int_0^{2\pi} (|\gamma|^2 - 1) d\theta,$$

$$\text{and } F_2(\gamma) := 8 \arg(\gamma)' - |\gamma|^2.$$

The  $R$  map is injective, and for any curve in  $\mathcal{S}$  the inverse of  $R$  is given by

$$R^{-1}(\gamma) = 2 \arg(\gamma) + \text{Id}_{\mathbb{R}} \quad R^{-1} : \mathcal{S} \mapsto \text{Diff}(S^1).$$

Furthermore, the pullback of the  $L^2$ -inner product on  $C^\infty(S^1, \mathbb{C})$  to the diffeomorphism group by the  $R$ -map is the right-invariant Sobolev metric of order one.

*Proof.* To prove the characterization for the image of  $R$  we observe that for  $\gamma \in C^\infty(S^1, \mathbb{C} \setminus \{0\})$  the function  $R^{-1}(\gamma) \in C^\infty([0, 2\pi], [0, 2\pi))$  is periodic if and only if  $F_1(\gamma) = 0$ . Furthermore, we have that  $R(R^{-1}(\gamma)) = \gamma$  if and only if  $F_2(\gamma) = 0$ .

To calculate the formula for the pullback metric, we need to calculate the tangent of the  $R$ -map where  $h$  is tangent to  $\tilde{\varphi}$ , i.e., to  $f$ .

$$T_{\tilde{\varphi}}Rh = \tilde{\varphi}'^{-\frac{1}{2}}h'e^{if/2} + i\tilde{\varphi}'^{\frac{1}{2}}e^{if/2}h.$$

Thus the pullback of the  $L^2$  inner product on  $C^\infty(S^1, \mathbb{C})$  is given by

$$\begin{aligned} (R^*\langle \cdot, \cdot \rangle_{L^2})_{\tilde{\varphi}}(h, h) &= \int_0^{2\pi} T_{\tilde{\varphi}}R(h) \cdot \overline{T_{\tilde{\varphi}}R(h)} dx \\ &= \int_0^{2\pi} \frac{h'^2}{\tilde{\varphi}'} + h\tilde{\varphi}' dx = \int_{S^1} X^2 + X'^2 dx, \end{aligned}$$

with  $h = X \circ \tilde{\varphi}$ . □

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