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REMARKS ON THE FRÖLICHER-NIJENHUIS BRACKET

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Abstract. After reviewing graded derivations on the algebra of differential forms and the basic properties of the Frölicher-Nijenhuis bracket we show, that this bracket is well-behaved with respect to f-related vector valued forms. Then graded derivations on the graded module of vector bundle valued differential forms are investigated, and also graded derivations from differential forms to vector bundle valued differential forms. Finally we start to look for all natural concomitants of Frölicher-Nijenhuis-type and we give some preliminary results in this direction.

Key words. Frölicher-Nijenhuis bracket, graded derivations, (vector valued) differential forms, natural concomitants.

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The Frölicher-Nijenhuis bracket is an extension of the Lie bracket of vector fields to a graded Lie bracket of tangent bundle valued differential forms. It's component of degree 1 expresses obstructions to integrability in various contexts (connections, almost complex structures). Recently it has been used by M. Modugno to give a new foundation for the theory of connections and curvatures on fibred manifolds.

This paper is in final form and no version of it will appear elsewhere.

1. Derivations on the algebra of differential forms and the Frölicher Nijenhuis bracket - a review.

This introductory section is meant as a reference for the following. The results are due to [4] and [9], see also [2],[6],[7],[8],[13].

1.1. Let M be a smooth second countable manifold, let $\Omega(M) = \bigoplus_k \Omega^k(M)$ be the graded commutative algebra of differential forms. An $\mathbb{R}$-linear mapping $D: \Omega(M) \to \Omega(M)$ is said to be <u>of degree</u> k if $D(\Omega^h(M)) \subset \Omega^{h+k}(M)$; and D is said to be a <u>(graded) derivation</u> of degree k if furthermore

$$D(\phi\wedge\psi) = D\phi\wedge\psi + (-1)^{hk}\phi\wedge D\psi \quad \text{for } \phi \in \Omega^h(M),\ \psi \in \Omega(M).$$

Let $\mathrm{Der}_k\Omega(M)$ be the linear space of all derivations of degree k and let $\mathrm{Der}\,\Omega(M) = \bigoplus_k \mathrm{Der}_k\Omega(M)$ be the space of all derivations.

<u>Proposition</u>: Der $\Omega(M)$ becomes a graded Lie algebra with the <u>graded commutator</u>

$$[D_1,D_2] = D_1 \circ D_2 - (-1)^{k_1k_2} D_2 \circ D_1,\quad D_i \in \mathrm{Der}_{k_i}\Omega(M).$$

This means that the the bracket is <u>graded anticommutative</u>,

$[D_1, D_2] = -(-1)^{k_1k_2}[D_1, D_2]$, and satisfies the <u>graded Jacobi identity</u>:

$[D_1,[D_2,D_3]] = [[D_1,D_2],D_3] + (-1)^{k_1k_2}[D_2,[D_1,D_3]]$ (so $\mathrm{ad}(D_1) = [D_1,.]$ is itself a derivation).

The proof is by computation.

1.2. A derivation $D \in \mathrm{Der}_k\Omega(M)$ is called <u>algebraic</u>, if $D|\Omega^0(M) = 0$. Then $D(f.\phi) = f.D\phi$ for $f \in C^\infty(M)$ and D is tensorial.

Furthermore D is uniquely determined by $D|\Omega^1(M)$: $\Omega^1(M) \to \Omega^{k+1}(M)$, which is induced by a vector bundle mapping $K: T^*M \to \Lambda^{k+1}T^*M$, which we view as an element of $\Omega^{k+1}(M;TM)$, the space of all TM-valued (k+1)-forms on M.

We write $D = i(K)$ and note the defining equation $D\phi = \phi \circ K$ for $\phi \in \Omega^1(M)$.

Proposition: 1. For $\phi \in \Omega^h(M)$ and $X_j \in \mathfrak{X}(M)$ (the space of vector fields) we have:

$$(i(K)\phi)(X_1,\dots,X_{k+h}) = = \frac{1}{(k+1)!\,(h-1)!}\sum_{\sigma\in S_{k+h}} \operatorname{sign}\sigma\ \phi(K(X_{\sigma 1},\dots,X_{\sigma(k+1)}),X_{\sigma(k+2)},\dots,X_{\sigma(k+h)}).$$

Note that this formula makes sense also if $\phi \in \Omega^h(M;E)$ is a vector bundle valued differential form.

2. For $K_j \in \Omega^{k_j+1}(M;TM)$ the derivation $[i(K_1),i(K_2)]$ is again algebraic, so it is of the form $i([K_1,K_2]^\wedge)$ for some unique $[K_1,K_2]^\wedge \in \Omega^{k_1+k_2+1}(M;TM)$. With the bracket $[\ ,\ ]^\wedge$, $\Omega^{*+1}(M;TM)$ becomes also a graded Lie algebra. We have $[K_1,K_2]^\wedge = i(K_1)K_2 - (-1)^{k_1 k_2} i(K_2)K_1$ (see 1.)

In [4] the expression $i(K)\phi$ is denoted by $\phi \,\bar{\wedge}\, K$. If $X \in \Omega^0(M;TM)$ is a vector field, then $i(X)$ is the usual insertion operator of degree -1 on $\Omega(M)$.

1.3. The exteriour derivative d is also a derivation of degree 1, which is not algebraic. In view of the well known equation $\Theta(X) = i(X)d + di(X)$ ($\Theta(X)$ the Lie derivation, X a vector field) we define the derivation $\Theta(K) := [i(K),d] \in \mathrm{Der}_k\Omega(M)$ for $K \in \Omega^k(M;TM)$ and call it the Lie derivation along K. Note that $\Theta(\mathrm{Id}_{TM}) = d$.

Proposition: Any derivation $D \in \mathrm{Der}_k\Omega(M)$ can uniquely be written in the form $D = \Theta(K) + i(L)$ for $K \in \Omega^k(M;TM)$ and $L \in \Omega^{k+1}(M;TM)$. D is algebraic if and only if $K = 0$. $[D,d] = 0$ if and only if $L = 0$.

Scetch of proof: Let $X_j \in \mathfrak{X}(M)$ be vector fields. Then $f \mapsto (Df)(X_1,\dots,X_k)$ is a derivation (of degree 0) of $C^\infty(M) = \Omega^0(M)$, so it is given by the action of a vector field $K(X_1,\dots,X_k)$, which is skew and $C^\infty(M)$-linear in the X_j, so $K \in \Omega^k(M;TM)$. Then $D - \Theta(K)$ is algebraic, so equals $i(L)$ for some L. Note that $[\Theta(K),d] = 0$ by the graded Jacobi identity.

1.4. Definition: Let $K_j \in \Omega^{k_j}(M;TM)$. Then clearly $[[\Theta(K_1),\Theta(K_2)],d] = 0$. So by 1.3 $[\Theta(K_1),\Theta(K_2)] = \Theta([K_1,K_2])$ for some unique $[K_1,K_2] \in \Omega^{k_1+k_2}(M;TM)$, which is called the Frölicher Nijenhuis bracket of K_1,K_2.

1.5. Proposition: 1. With the Frölicher Nijenhuis bracket the space $\Omega(M;TM)$ becomes a graded Lie algebra.

2. For vector fields X,Y the bracket $[X,Y]$ is the usual Lie bracket of vector fields.

1.6. Proposition: For $K \in \Omega^k(M;TM)$ and $L \in \Omega^{h+1}(M;TM)$ we have

$$[\Theta(K),i(L)] = i([K,L]) - (-1)^{kh}\Theta(i(L)K).$$

Proof: $[\Theta(K),i(L)] + (-1)^{hk}\Theta(i(L)K)$ vanishes on $\Omega^0(M)$, so is algebraic. By the graded Jacoby identity we get $[[\Theta(K),i(L)],d] = [i([K,L]),d]$, and since $[.,d]$ is injective on algebraic derivations the formula follows. qed.

1.7. Proposition: 1. The space $\mathrm{Der}\,\Omega(M)$ is a graded module over the graded commutative algebra $\Omega(M)$ with the action $(\phi\wedge D)\psi = \phi\wedge D\psi$.

2. For $D_i \in \mathrm{Der}_{k_i}\Omega(M)$ and $\phi \in \Omega^q(M)$ we have

$$[\phi\wedge D_1, D_2] = \phi\wedge[D_1,D_2] - (-1)^{(q+k_1)k_2} D_2\phi\wedge D_1 .$$

3. For $L \in \Omega(M;TM)$ we have $i(\phi\wedge L) = \phi\wedge i(L)$.

4. For $K \in \Omega^k(M;TM)$ we have $\Theta(\phi\wedge K) = \phi\wedge\Theta(K) + (-1)^{q+k} d\phi\wedge i(K)$.

5. For $L_i \in \Omega^{h_i+1}(M;TM)$ we have

$$[\phi\wedge L_1, L_2]^\wedge = \phi\wedge[L_1,L_2]^\wedge - (-1)^{(q+h_1)h_2} i(L_2)\phi\wedge L_1 .$$

6. For $K_i \in \Omega^{k_i}(M;TM)$ we have

$$[\phi\wedge K_1,K_2] = \phi\wedge[K_1,K_2] - (-1)^{(q+k_1)k_2}\Theta(K_2)\phi\wedge K_1 + (-1)^{q+k_1} d\phi\wedge i(K_1)K_2 .$$

7. For $X,Y \in \mathfrak{X}(M)$, $\phi \in \Omega^q(M)$, $\psi \in \Omega(M)$ we have:

$$[\phi\otimes X,\psi\otimes Y] = \phi\wedge\psi\otimes[X,Y] + \phi\wedge\Theta(X)\psi\otimes Y - \Theta(Y)\phi\wedge\psi\otimes X$$
$$+(-1)^q(d\phi\wedge i(X)\psi\otimes Y + i(Y)\phi\wedge d\psi\otimes X).$$

Proof: For 2,3,4 just compute. For 5 compute $i([\phi\wedge L_1,L_2]^\wedge)$. For 6 compute $\Theta([\phi\wedge K_1,K_2])$. For 7 use 6. qed.

1.8. Proposition: For $K \in \Omega^k(M;TM)$, $\phi \in \Omega^h(M)$, $X_i \in \mathfrak{X}(M)$ we have

$$(\Theta(K)\phi)(X_1,\dots,X_{k+h}) =$$
$$= \frac{1}{k!\,h!}\sum_{\sigma \in S_{h+k}} \operatorname{sign}\sigma\ \Theta(K(X_{\sigma 1},\dots,X_{\sigma k}))(\phi(X_{\sigma(k+1)},\dots,X_{\sigma(k+h)}))$$
$$+ \frac{-1}{k!\,(h-1)!}\sum_{\sigma} \operatorname{sign}\sigma\ \phi([K(X_{\sigma 1},\dots,X_{\sigma k}),X_{\sigma(k+1)}],X_{\sigma(k+2)},\dots,X_{\sigma(k+h)})$$
$$+ \frac{(-1)^{k-1}}{(k-1)!\,(h-1)!\,2!}\sum_{\sigma} \operatorname{sign}\sigma\ \phi(K([X_{\sigma 1},X_{\sigma 2}],X_{\sigma 3},\dots,X_{\sigma(k+1)}),X_{\sigma(k+2)},\dots,X_{\sigma(k+h)}).$$

This can be proved by combinatorics starting from the formula in 1.2.1 (difficult), or by putting $K = \psi\otimes X$ and using 1.7.4.

1.9. Proposition: For $K \in \Omega^k(M;TM)$ and $L \in \Omega^h(M;TM)$ we have

$$[K,L]\,(X_1,\dots,X_{k+h}) =$$
$$= \frac{1}{k!\,h!}\sum_{\sigma \in S_{k+h}} \operatorname{sign}\sigma\ [K(X_{\sigma 1},\dots,X_{\sigma k}),L(X_{\sigma(k+1)},\dots,X_{\sigma(k+h)})]$$
$$+ \frac{-1}{k!\,(h-1)!}\sum_{\sigma} \operatorname{sign}\sigma\ L([K(X_{\sigma 1},\dots,X_{\sigma k}),X_{\sigma(k+1)}],X_{\sigma(k+2)},\dots,X_{\sigma(k+h)})$$
$$+ \frac{(-1)^{kh}}{(k-1)!\,h!}\sum_{\sigma} \operatorname{sign}\sigma\ K([L(X_{\sigma 1},\dots,X_{\sigma h}),X_{\sigma(h+1)}],X_{\sigma(h+2)},\dots,X_{\sigma(h+k)})$$
$$+ \frac{(-1)^{k-1}}{(k-1)!\,(h-1)!\,2!}\sum \operatorname{sign}\sigma\ L(K([X_{\sigma 1},X_{\sigma 2}],X_{\sigma 3},\dots,X_{\sigma(k+1)}),X_{\sigma(k+2)},\dots,X_{\sigma(k+h)})$$
$$+ \frac{(-1)^{(k-1)h}}{(k-1)!\,(h-1)!\,2!}\sum \operatorname{sign}\sigma\ K(L([X_{\sigma 1},X_{\sigma 2}],X_{\sigma 3},\dots,X_{\sigma(h+1)}),X_{\sigma(h+2)},\dots,X_{\sigma(h+k)}).$$

This formula has been found independently by [6] and [7]. The proof of it will be given in section 3.17.

1.10. Proposition: 1. For K_i in $\Omega^{k_i}(M;TM)$ and L_i in $\Omega^{k_i+1}(M;TM)$ we have

$$[\Theta(K_1) + i(L_1), \Theta(K_2) + i(L_2)] = \Theta([K_1, K_2] + i(L_1)K_2 - (-1)^{k_1k_2} i(L_2)K_1)$$
$$+ i([L_1, L_2]^\wedge + [K_1, L_2] - (-1)^{k_1k_2}[K_2, L_1]).$$

Each side in this formula looks like a semidirect product, but the mappings $i\colon \Omega(M;TM) \to End(\Omega(M;TM), [\ ,\ ])$ and $ad\colon \Omega(M;TM) \to End(\Omega(M;TM), [\ ,\ ]^\wedge)$

do not take values in the subspaces of graded derivations. We have instead:

2. For L in $\Omega^{q+1}(M;TM)$ and K_i in $\Omega^{k_i}(M;TM)$ we have

$$i(L)[K_1, K_2] = [i(L)K_1, K_2] + (-1)^{q k_1}[K_1, i(L)K_2] - $$
$$-((-1)^{q k_1} i([K_1, L])K_2 - (-1)^{k_1 k_2 + q k_2} i([K_2, L])K_1).$$

3. For K in $\Omega^{k}(M;TM)$ and L_i in $\Omega^{q_i}(M;TM)$ we have

$$[K, [L_1, L_2]^\wedge] = [[K, L_1], L_2]^\wedge + (-1)^{k q_1}[L_1, [K, L_2]]^\wedge - $$
$$-((-1)^{k q_1}[i(L_1)K, L_1] - (-1)^{q_1 q_2 + k q_2}[i(L_2)K, L_1]).$$

Proof: Equation 1 follows easily from 1.6. Equations 2 and 3 follow from 1 by writing out the graded Jacobi identity for for the graded commutator, or as follows: Consider $\theta(i(L)[K_1, K_2])$ and use 1.6 repeatedly to obtain θ of the right hand side of formula 2. Then consider $i([K, [L_1, L_2]^\wedge])$ and use again 1.6 repeatedly to obtain i of the right hand side of formula 3. qed.

1.11. Remark: The formulas of 1.10 lead to the concept of knit product of two graded Lie algebras, described by a derivatively knitted pair of representations (here the mappings i and ad mentioned in 1.10.1). This makes also sense for ordinary Lie algebras. The "integrated" version of the knit product for Lie groups (and groups in general) is the "Zappa-Szep"-product of groups. I will explain this in some other paper. Here I just want to remark that the knit product describes the general situation: whenever a (graded) Lie algebra is the direct sum of two subalgebras, it is a knit product of them.

2. Naturality of the Frölicher Nijenhuis bracket

2.1. Let $f: M \to N$ be a smooth mapping between smooth second countable manifolds. Two vector valued differential forms K in $\Omega^k(M;TM)$ and K' in $\Omega^k(N;TN)$ will be called f-related or f-dependent, if for all X_i in T_xM we have $K'_{f(x)}(T_xf.X_1,\dots, T_xf.X_k) = T_xf.K_x(X_1,\dots,X_k)$.

2.2. Theorem: 1. If K and K' as above are f-related, then

$$i(K)\circ f^* = f^* \circ i(K'): \Omega(N) \to \Omega(M).$$

2. If $i(K)\circ f^*|B^1(N) = f^*\circ i(K')|B^1(N)$, then K and K' are f-related. Here B^1 is the space of exact 1-forms.

3. If K_j and K'_j are f-related for $j = 1, 2$, then their algebraic brackets $[K_1, K_2]^\wedge$ and $[K'_1, K'_2]^\wedge$ are also f-related.

4. If K and K' are f-related, then $\Theta(K)\circ f^* = f^*\circ\Theta(K')$.

5. If $\Theta(K)\circ f^*|\Omega^0(N) = f^*\circ\Theta(K')|\Omega^0(N)$, then K and K' are f-related.

6. If K_j and K'_j are f-related for $j = 1, 2$, then their Frölicher Nijenhuis brackets $[K_1, K_2]$ and $[K'_1, K'_2]$ are also f-related.

Proof: 1. Let ϕ be in $\Omega^p(N)$ and X_i in T_xM. Then we have by formula 1.2.1:

$$(i(K)\, f^*\phi)_x(X_1,\dots,X_{p+k-1}) =$$
$$= \frac{1}{k!\,(p-1)!}\sum_\sigma \text{sign}\,\sigma\ (f^*\phi)_x(K_x(X_{\sigma 1},\dots,X_{\sigma k}),X_{\sigma(k+1)},\dots,X_{\sigma(k+p-1)})$$
$$= \frac{1}{k!\,(p-1)!}\sum \text{sign}\,\sigma\ \phi_{f(x)}(T_xf.K_x(X_{\sigma 1},\dots,X_{\sigma k}),T_xf.X_{\sigma(k+1)},\dots)$$
$$= \frac{1}{k!\,(p-1)!}\sum \text{sign}\,\sigma\ \phi_{f(x)}(K'_{f(x)}(T_xf.X_{\sigma 1},\dots,T_xf.X_{\sigma k}),T_xf.X_{\sigma(k+1)},\dots)$$
$$= (f^*\, i(K')\,\phi)_x(X_1,\dots,X_{p+k-1}).$$

2. For all ϕ in $B^1(N)$ we have

$$(i(K)\, f^*\,\phi)_x(X_1,\dots,X_k) = ((f^*\phi)\circ K)_x(X_1,\dots,X_k) = \phi_{f(x)}.T_xf.K_x(X_1,\dots,X_k),$$

$(f^* \, i(K') \, \phi)_x(X_1,\dots,X_k) = (f^*(\phi \circ K'))_x(X_1,\dots) = \phi_{f(x)}.K'_{f(x)}(T_x f.X_1,\dots).$
Since elements in $B^1(N)$ separate points in TN, K and K' are f-related.

3. The operator f^* intertwines $i(K_j)$ and $i(K'_j)$, so it also intertwines their graded commutators, which in turn equal $i([K_1, K_2]^\wedge)$ and $i([K'_1, K'_2]^\wedge)$ respectively. Now use 2.

4. $\theta(K) \, f^* = [i(K),d] \, f^* = i(K) \, d \, f^* - (-1)^{k-1} \, d \, i(K) \, f^* = f^* \, \theta(K')$.

5. For g in $\Omega^0(N) = C^\infty(N)$ we have $\theta(K) \, f^* g = i(K) \, d \, f^* g = i(K) \, f^* dg$ and $f^* \theta(K') \, g = f^* \, i(K') \, dg$. By 2 the result follows.

6. The operator f^* intertwines $\theta(K_j)$ and $\theta(K'_j)$, so also their graded commutators which equal $\theta([K_1, K_2])$ and $\theta([K'_1, K'_2])$, respectively. Now use 5. qed.

2.3. Proposition: If K_j and K'_j as in 2.1 are f-dependent for $j = 1, 2$, then also $i(K_1)K_2$ and $i(K'_1)K'_2$ are f-dependent.

Proof: $T_x f.(i(K_1)K_2)_x(X_1,\dots,X_{k_1+k_2-1}) =$

$$= \frac{1}{k_1! \, (k_2-1)!} \sum_\sigma \operatorname{sign}\sigma \; T_x f.(K_2)_x((K_1)_x(X_{\sigma 1},\dots,X_{\sigma k_1}),X_{\sigma(k_1+1)},\dots)$$
$$= \frac{1}{k_1! \, (k_2-1)!} \sum \operatorname{sign}\sigma \; (K'_2)_{f(x)}(T_x f.(K_1)_x(X_{\sigma 1},\dots,X_{\sigma k_1}),T_x f.X_{\sigma(k_1+1)},\dots)$$
$$= \frac{1}{k_1! \, (k_2-1)!} \sum \operatorname{sign}\sigma \; (K'_2)_{f(x)}((K'_1)_{f(x)}(T_x f.X_{\sigma 1},\dots),T_x f.X_{\sigma(k_1+1)},\dots)$$
$$= (i(K'_1)K'_2)_{f(x)}(T_x f.X_1,\dots,T_x f.X_{k_1+k_2-1}).$$
qed.

2.4. Let $f: M \to M$ be a diffeomorphism. We define the "pullback-operator" $f^*: \Omega(M;TM) \to \Omega(M;TM)$ by the formula:

$$(f^* K)_x(X_1,\dots,X_k) := (T_x f)^{-1} K_{f(x)}(T_x f.X_1,\dots,T_x f.X_k).$$

Thus $f^* K$ and K are f-related. Clearly this concept makes sense for locally defined diffeomorphisms and open embeddings.

2.5. Corollary: If f is a diffeomorphism or open embedding then
$f^*[K,L] = [f^* K, f^* L]$, $f^*[K, L]^\wedge = [f^* K, f^* L]^\wedge$ and
$f^*(i(K)L) = i(f^* K)(f^* L)$ for all vector valued forms K, L.

This is immediate from 2.2 and 2.3. We may thus say that the Frölicher Nijenhuis bracket, the algebraic bracket and the insertion operation are natural bilinear (algebraic or differential) concomitants.

2.6. Let X be a vector field on M and let Fl^X_t denote it's local flow. We define the operator $\Theta(X): \Omega(M;TM) \to \Omega(M;TM)$, the <u>Lie derivation along</u> X of vector valued forms, by $\Theta(X)\,K = \frac{d}{dt}|_0\,(Fl^X_t)^*K$.

<u>Lemma</u>: $\Theta(X)\,K = [X, K]$, the Frölicher Nijenhuis bracket.

<u>Proof</u>: Obviously $\Theta(X)$ is $\mathbb{R}$-linear, so it suffices to check this for vector valued forms $K = \psi \otimes Y$, ψ in $\Omega^k(M)$, Y a vector field. But then

$$\Theta(X)(\psi\otimes Y) = \frac{d}{dt}|_0\,(Fl^X_t)^*\,(\psi\otimes Y) = \frac{d}{dt}|_0((Fl^X_t)^*\psi \otimes (Fl^X_t)^*Y)$$

$$= \Theta(X)\psi\otimes Y + \psi\otimes\Theta(X)Y = \Theta(X)\psi\,\otimes Y + \psi\otimes[X,Y]$$

$$= [X,\,\psi\,\otimes Y]$$ by formula 1.7.7. qed.

2.7. <u>Remark</u>: If we put $f = Fl^X_t$ in 2.5 and differentiate with respect to t we get $\Theta(X)[K, L] = [\Theta(X)\,K, L] + [K, \Theta(X)\,L]$. In view of lemma 2.6 this is a special case of the graded Jacobi identity. For the other two concomitants we get $[X, [K, L]^\wedge] = [[X, K], L]^\wedge + [K, [X, L]]^\wedge$ and $[X, i(K)L] = i([X, K])L + i(K)[X, L]$. Both eqations become wrong if we insert a vector valued form of higher degree for the zero form X.
We may say that all these concomitants commute with Lie derivation along vector fields. In a later section we will see how this property determines these concomitants and some others.

3. Derivations on the module of vector bundle valued differential forms and the global formula for the Frölicher Nijenhuis bracket.

3.1. Let (E,p,M) be a smooth vector bundle with projection $p: E \to M$. Let $\Omega^k(M;E)$ denote the space of all E-valued differential forms of degree k on M, i.e. sections of the bundle $\Lambda^k M \times E$. Then $\Omega(M;E)$, the space of all E-valued forms, is a graded module over the graded commutative algebra $\Omega(M)$ of forms on M. The module action is given by

$$(\omega \wedge \Phi)(X_1,\dots,X_{p+q}) = \frac{1}{p!\,q!}\sum_\sigma \operatorname{sign}\sigma\ \omega(X_{\sigma 1},\dots,X_{\sigma p})\ \Phi(X_{\sigma(p+1)},\dots,X_{\sigma(p+q)}).$$

Definition: A derivation of degree k on $\Omega(M;E)$ is a linear mapping

$D: \Omega(M;E) \to \Omega(M;E)$ with $D(\Omega^p(M;E)) \subseteq \Omega^{p+k}(M;E)$ such that

$D(\omega \wedge \Phi) = \bar{D}(\omega)\wedge\Phi + (-1)^{kp}\omega\wedge D(\Phi)$ for ω in $\Omega^p(M)$ and Φ in $\Omega^q(M;E)$,

where $\bar{D}: \Omega(M) \to \Omega(M)$ is some mapping.

Lemma: $\bar{D}$ is uniquely determined by D and is a derivation of degree k on $\Omega(M)$.

Proof: The action of $\Omega(M)$ on $\Omega(M;E)$ is effective: $\omega\wedge\Phi = \psi\wedge\Phi$ for all Φ in $\Omega(M;E)$ implies $\omega = \psi$. Now the assertion follows easily from the derivation property of D.

3.2. A derivation D of $\Omega(M;E)$ is called algebraic if the uniquely associated derivation $\bar{D}$ of $\Omega(M)$ is algebraic (i.e. vanishes on $\Omega^0(M)$ and is therefore of tensorial character). This is the case if and only if $D(f.\Phi) = f.D(\Phi)$ for all f in $C^\infty(M)$ and Φ in $\Omega(M;E)$. So D itself is of tensorial character.

Lemma: Let D be a derivation of degree k on $\Omega(M;E)$ with $\bar{D} = 0$. Then there is a unique P in $\Omega^k(M;L(E,E))$ such that $D = \mu(P)$, where $L(E,E)$ is the bundle of linear endomorphisms of the fibres of E and

$$(\mu(P)\ \Phi)_x(X_1,\dots,X_{q+k}) = \frac{1}{q!\,k!}\sum_\sigma \operatorname{sign}\sigma\ P_x(X_{\sigma 1},\dots,X_{\sigma k})\ \Phi_x(X_{\sigma(q+k)},\dots).$$

So the the graded $\Omega(M)$-module homomorphisms are exactly of the form $\mu(P)$.

3.3. <u>Corollary</u>: If D is an algebraic derivation on $\Omega(M;E)$ of degree k, then there are unique forms L in $\Omega^{k+1}(M;TM)$ and P in $\Omega^{k}(M;L(E,E))$ such that $D = i(L) + \mu(P)$, where $i(L)$ is given by formula 1.2.1.

3.4. <u>Theorem</u>: The space Der $\Omega(M;E)$ of (graded) derivations of the $\Omega(M)$-module $\Omega(M;E)$ is a graded Lie algebra with the graded commutator (1.1) as bracket. We have $\overline{[D_1,D_2]} = [\bar{D}_1,\bar{D}_2]$.

<u>Proof</u>: Compute.

3.5. For P in $\Omega^{p}(M;L(E,E))$ and Q in $\Omega^{q}(M;L(E,E))$ define the bracket $[\![P, Q]\!]^{\wedge}$ in $\Omega^{p+q}(M;L(E,E))$ by the formula

$$[\![P, Q]\!]^{\wedge}(X_1,\dots,X_{p+q}) = \frac{1}{p!\,q!}\sum_{\sigma} \operatorname{sign}\sigma \; P(X_{\sigma 1},\dots,X_{\sigma p})\circ Q(X_{\sigma(p+1)},\dots).$$

This clearly defines a graded Lie algebra $(\Omega(M;L(E,E)),[\![\ ,\ ]\!]^{\wedge})$.

<u>Proposition</u>: For L_i in $\Omega^{k_i+1}(M;TM)$ and P_i in $\Omega^{k_i}(M;L(E,E))$ we have

$$[i(L_1) + \mu(P_1),\ i(L_2) + \mu(P_2)] =$$
$$= i([L_1, L_2]^{\wedge}) + \mu([\![P_1, P_2]\!]^{\wedge} + i(L_1)P_2 - (-1)^{k_1k_2}i(L_2)P_1).$$

So the graded Lie subalgebra of algebraic derivations of Der $\Omega(M;E)$ is the semidirect product of those of the form $\mu(P)$ (which form the ideal) and those of the form $i(L)$.

<u>Proof</u>: The graded commutator on the $\bar{D}$-level (in Der $\Omega(M)$) gives the i-part. Restriction of the graded commutator to $\Omega^{0}(M;E) = \Gamma(E)$ destroys the i-part and gives directly the content of μ. qed.

3.6. Now let ∇ be a covariant derivative on the vector bundle E, and denote again by ∇ the exteriour covariant derivative $\Omega^{p}(M;E)\to\Omega^{p+1}(M;E)$, given by

$$(\nabla\,\Phi)(X_0,\dots,X_p) = \sum_{0}^{p}(-1)^i(\nabla_{X_i}\Phi)(X_0,\dots,\hat{X}_i,\dots,X_p) +$$
$$+\sum_{i<j}(-1)^{i+j}\,\Phi([X_i,X_j],X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_p).$$

It is a graded derivation of degree 1 and $\nabla(\omega \wedge \Phi) = d\omega \wedge \Phi + (-1)^p \omega \wedge \nabla\Phi$ for a p-form ω. Also $[\nabla,\nabla] = 2\nabla\nabla = 2\mu(R)$, where R in $\Omega^2(M;L(E,E))$ is the curvature form of ∇. The Bianchi identity is $[\nabla, \mu(R)] = \nabla\,\mu(R) - \mu(R)\,\nabla = \nabla\nabla\nabla - \nabla\nabla\nabla = 0$. A convenient reference for this is [5], vol. II. Note that the exteriour covariant derivatives on E are exactly the elements D in $\mathrm{Der}_1\,\Omega(M;E)$ with $\bar{D} = d$, and two of them differ by $\mu(A)$ for A in $\Omega^1(M;L(E,E))$.

3.7. Let us fix a covariant exteriour derivative for the moment. Then for K in $\Omega^k(M;TM)$ we consider the derivation $\Theta_\nabla(K) := [i(K), \nabla]$ of degree k on $\Omega(M;E)$, which we call the <u>covariant Lie derivation along</u> K. Clearly $\overline{\Theta_\nabla(K)} = \Theta(K)$ on $\Omega(M)$. If we change the covariant derivative we get $\Theta_\nabla(K) - \Theta_{\nabla+A}(K) = [i(K), \mu(A)] = \mu(i(K)A)$. If $K = X$, a vector field, then $\Theta_\nabla(X)$ is mentioned in an exercise in [5] (vol. II, p. 352).

3.8. <u>Theorem</u>: If ∇ is a (fixed) covariant exteriour derivative on E, then any derivation D in $\mathrm{Der}_k\,\Omega(M;E)$ can be written in the form $D = \Theta_\nabla(K) + i(L) + \mu(P)$ for unique K in $\Omega^k(M;TM)$, L in $\Omega^{k+1}(M;TM)$ and P in $\Omega^k(M;L(E,E))$. D is algebraic if and only if $K = 0$.

<u>Proof</u>: By 1.3 we have $\bar{D} = \Theta(K) + i(L)$ for unique K and L as specified. The derivation $D - \Theta_\nabla(K) - i(L)$ then has $\bar{D}$-part 0 so it is of the form $\mu(P)$ by 3.2. qed.

3.9. <u>Lemma</u>: For K in $\Omega^k(M;TM)$ and L in $\Omega(M;TM)$ we have

$$[i(L), \Theta_\nabla(K)] = \Theta_\nabla(i(L)K) + (-1)^k i([L,K]).$$

If R in $\Omega^2(M;L(E,E))$ is is the curvature form of ∇, then

$$[\Theta_\nabla(K), \nabla] = \mu(i(K)R).$$

<u>Proof</u>: Let s in $\Omega^0(M;E)$ be a section. Then $[i(L), \Theta_\nabla(K)]\, s = i(L)\, \Theta_\nabla(K)\, s - 0$ $= i(L)\,[i(K), \nabla]\, s = i(L)\, i(K)\, \nabla s - 0 = i(L)(\nabla s \circ K) = \nabla s \circ (i(L)K)$ $= i(i(L)K)\, \nabla s = [i(i(L)K), \nabla]\, s = \Theta_\nabla(i(L)K)\, s$. So $[i(L), \Theta_\nabla(K)] - \Theta_\nabla(i(L)K)$

vanishes on $\Omega^0(M;E)$, thus it has no μ-part, is algebraic, thus of the form $i(L')$ for some L'. But this $i(L')$ is also the algebraic part of $\overline{[i(L), \Theta_\nabla(K)]} = [i(L), \Theta(K)]$ in Der $\Omega(M)$, which is $(-1)^k i([L, K])$ by 1.6. For the second formula we have $[\Theta_\nabla(K), \nabla] = [[i(K), \nabla], \nabla] =$ $= [i(K),[\nabla,\nabla]] - (-1)^{k-1}[\nabla, [i(K),\nabla]] = [i(K), 2\mu(R)] - [\Theta_\nabla(K),\nabla]$, by 3.6 and the graded Jacobi identity. So $[\Theta_\nabla(K),\nabla] = [i(K), \mu(R)] = \mu(i(K)R)$ by 3.5.

3.10 <u>Lemma</u>: For K_i in $\Omega^{k_i}(M;TM)$ and a fixed covariant derivative ∇ on (E,p,M) we have $[\Theta_\nabla(K_1), \Theta_\nabla(K_2)] = \Theta_\nabla([K_1, K_2]) + \mu(\lambda(K_1,K_2)R)$, where R in $\Omega^2(M;L(E,E))$ is the curvature of ∇ and where

$$\begin{aligned}\lambda(K_1,K_2) &= -(-1)^{k_2}(i(K_1)\, i(K_2) - i(i(K_1)K_2))\\ &= -\frac{(-1)^{k_2}}{2}\{i(K_1)\, i(K_2) + (-1)^{(k_1-1)(k_2-1)}\, i(K_2)\, i(K_1) -\\ &\qquad - i(\, i(K_1)K_2 + (-1)^{(k_1-1)(k_2-1)}\, i(K_2)K_1\,)\}\,.\end{aligned}$$

<u>Proof</u>: $[\Theta_\nabla(K_1), \Theta_\nabla(K_2)] = [\Theta_\nabla(K_1), [i(K_2), \nabla]]=$

$= [[\Theta_\nabla(K_1), i(K_2)], \nabla] + (-1)^{k_1(k_2-1)}\, [i(K_2), [\Theta_\nabla(K_1), \nabla]]$ by Jacobi

$= [i([K_1, K_2]) - (-1)^{k_1(k_2-1)}\, \Theta_\nabla(i(K_2)K_1), \nabla]$ by 3.9

$+ (-1)^{k_1(k_2-1)}[\, i(K_2), \mu(i(K_1)R)]$ by 3.9

$= \Theta_\nabla([K_1, K_2]) + \mu(\lambda(K_1,K_2)R)$ by using 3.9 and 3.5 and exchanging the role of K_1 and and K_2 in λ. Clearly λ is graded antisymmetric, and the second formula for λ is the arithmetic mean of two expressions. qed.

<u>Remark</u>: Note that in the second formula for λ the graded anticommutator appears. Here graded Jordan algebras enter.

3.11. The covariant derivative ∇ on (E,p,M) induces a covariant derivative $\nabla^{L(E,E)}$ on the vector bundle $(L(E,E),p,M)$ by the formula $\mu(\nabla_X^{L(E,E)}P) = [\nabla_X, \mu(P)]$ on $\Gamma(E)$ for P in $\Gamma(L(E,E))$ and X in $\mathfrak{X}(M)$. This is the usual extension of ∇_X to the tensor bundle in such a way, that it is a derivation with respect to tensor products and commutes with traces.

Lemma: 1. $\nabla^{L(E,E)}\, id_E = 0$.

2. $[\nabla, \mu(P)] = \mu(\nabla^{L(E,E)}P)$ for P in $\Omega(M;L(E,E))$.

3. $\nabla^{L(E,E)}(P_1 \theta\, P_2) = (\nabla^{L(E,E)}P_1)\theta\, P_2 + (-1)^{k_1} P_1 \theta\, (\nabla^{L(E,E)}P_2)$

for P_i in $\Omega^{k_i}(M;L(E,E))$, where $P_1 \theta\, P_2 = \frac{1}{k!k!}$ Alt $P_1 \circ P_2$.

Proof: Read carefully [5], vol. II, page 321 and 326. Or plug in the definitions and check 2 directly (easy). 1 and 3 follow from 2. qed.

3.12. Lemma: For K in $\Omega(M;TM)$ and P in $\Omega(M;L(E,E))$ we have

$$[\Theta_\nabla(K), \mu(P)] = \mu(\Theta_{\nabla L(E,E)}(K)\, P).$$

Proof: Use $\Theta_\nabla(K) = [i(K), \nabla]$, the graded Jacobi identity, 3.11 and 3.5. qed.

3.13. Theorem: Let K_i be in $\Omega^{k_i}(M;TM)$, L_i in $\Omega^{k_i+1}(M;TM)$ and P_i in $\Omega^{k_i}(M;L(E,E))$.

Then we have $[\, \Theta_\nabla(K_1) + i(L_1) + \mu(P_1), \Theta_\nabla(K_2) + i(L_2) + \mu(P_2)\,] =$

$$\begin{aligned}
&= \Theta_\nabla([K_1, K_2] + i(L_1)K_2 - (-1)^{k_1k_2} i(L_2)K_1) \\
&\quad + i([L_1, L_2]^\wedge + [K_1, L_2] - (-1)^{k_1k_2}[K_2, L_1]) \\
&\quad + \mu([\![P_1, P_2]\!]^\wedge + \Theta_{\nabla L(E,E)}(K_1)P_2 - (-1)^{k_1k_2}\Theta_{\nabla L(E,E)}(K_2)P_1 + \\
&\qquad\qquad + i(L_1)P_2 - (-1)^{k_1k_2} i(L_2)P_1 + \\
&\qquad\qquad + \lambda(K_1,K_2)R\ ),
\end{aligned}$$

where R is the curvature of ∇, and $\lambda(K_1,K_2)$ is from 3.10.

This follows directly from the lemmas above. Note in $\mu(\quad)$ the representations $\Theta_{\nabla L(E,E)}$ and i. Both of them are graded derivations of $(\Omega(M;L(E,E)), [\![\ ,\ ]\!]^\wedge)$, so they describe semidirect products. The mysterious summand deforming the graded Lie algebra structure is $\lambda(K_1,K_2)R$. The ingredients in the Θ_∇ - part and the i - part are again a derivatively knitted pair of representations, see 1.11.

3.14. The space Der $\Omega(M; E)$ of derivations of the graded $\Omega(M)$ - module $\Omega(M; E)$ is itself a graded $\Omega(M)$ - module with the action $(\omega \wedge D)\Phi = \omega \wedge D\Phi$ and $\overline{(\omega \wedge D)} = \omega \wedge \bar{D}$.

Proposition: 1. For D_i in $\mathrm{Der}_{k_i}\ \Omega(M;E)$ and ω in $\Omega^q(M)$ we have:

$$[\omega \wedge D_1, D_2] = \omega \wedge [D_1, D_2] - (-1)^{(q+k_1)k_2}\ \bar{D}_2\omega \wedge D_1$$

2. $\omega \wedge i(K) = i(\omega \wedge K)$.

3. $\omega \wedge \Theta_\nabla(K) = \Theta_\nabla(\omega \wedge K) - (-1)^{q+k}\ i(d\omega \wedge K)$.

4. $\omega \wedge \mu(P) = \mu(\omega \wedge P)$.

3.15. Proposition: For K in $\Omega^k(M;TM)$, Φ in $\Omega^q(M;E)$ and vector fields X_i we have: $(\Theta_\nabla(K)\Phi)(X_1,\ldots,X_{k+q}) =$

$$= \frac{1}{k!\ q!} \Sigma \operatorname{sign}\sigma\ \ \nabla_{K(X_{\sigma 1},\ldots,X_{\sigma k})}\ \Phi(X_{\sigma(k+1)},\ldots,X_{\sigma(k+q)})$$
$$+ \frac{-1}{k!\ (q-1)!} \Sigma \operatorname{sign}\sigma\ \ \Phi([K(X_{\sigma 1},\ldots,X_{\sigma k}),\ X_{\sigma(k+1)}],X_{\sigma(k+2)},\ldots)$$
$$+ \frac{(-1)^{k-1}}{(k-1)!\ (q-1)!\ 2!}\ \Sigma \operatorname{sign}\sigma\ \ \Phi(K([X_{\sigma 1},\ X_{\sigma 2}],X_{\sigma 3},\ldots,X_{\sigma(k+1)}),X_{\sigma(k+2)},\ldots)$$

Proof: This can either be proved by combinatorics from the formula in 1.2.1 (difficult) or by using 3.14.3 and starting with $k = 0$. qed.

3.16. Theorem: Let ∇ be a symmetric covariant derivative on TM. Then for K_i in $\Omega^{k_i}(M;TM)$ the Frölicher Nijenhuis bracket satisfies

$$[K_1, K_2] = \Theta_\nabla(K_1)K_2 - (-1)^{k_1 k_2}\Theta_\nabla(K_2)K_1.$$

Proof: If $k_1 = k_2 = 0$ then $K_1 = X$, $K_2 = Y$ are just vector fields and $\Theta_\nabla(X)Y = [i(X),\nabla]Y = i(X)\nabla Y - 0 = \nabla_X Y$, so the formula just says that ∇ is symmetric. Now we use induction on $k_1 + k_2$ and 1.10.2. Let X be a vector field.

$$i(X)[K_1, K_2] = [i(X)K_1, K_2] + (-1)^{k_1}[K_1, i(X)K_2] -$$
$$-((-1)^{k_1} i([K_1, X])K_2 - (-1)^{k_1 k_2 - k_2} i([K_2,X])K_1) \qquad \text{by 1.10.2.}$$
$$= \Theta_\nabla(i(X)K_1)K_2 - (-1)^{(k_1-1)k_2}\Theta_\nabla(K_2)\ i(X)K_1 + (-1)^{k_1}\{\Theta_\nabla(K_1)i(X)K_2 -$$
$$- (-1)^{k_1(k_2-1)}\ \Theta_\nabla(i(X)K_2)K_1\} -(-1)^{k_1} i([K_1,X])K_2 + (-1)^{k_1 k_2 - k_2} i([K_2,X])K_1,$$

by the induction hypothesis. Now use lemma 3.9 to obtain $i(X)(\Theta_\nabla(K_1)K_2 - (-1)^{k_1 k_2}\Theta_\nabla(K_2)K_1)$. This holds for all vector fields, so the result follows. qed.

3.17. Proof of proposition 1.9: In 3.16 use 3.15 and once more the symmetry of ∇. qed.

4. Derivations from the algebra of differential forms into the graded module of vector bundle valued forms.

4.1. Let (E,p,M) be a vector bundle over a manifold M. Then a linear mapping $D\colon \Omega(M) \to \Omega(M;E)$ will be called a graded derivation of degree k if $D(\Omega^q(M)) \subset \Omega^{q+k}(M;E)$ and $D(\omega \wedge \psi) = D\omega \wedge \psi + (-1)^{kq}\omega \wedge D\psi$ in $\Omega(M;E)$ for a form ω in $\Omega^q(M)$, and an arbitratry form ψ. Here we also consider $\Omega(M;E)$ as a graded right $\Omega(M)$-module with multiplication $\Phi \wedge \omega$ = $= (-1)^{pq}\omega \wedge \Phi$ for ω in $\Omega^q(M)$ and Φ in $\Omega^p(M;E)$.
A derivation D is called algebraic if $D|\Omega^0(M) = 0$. We will denote by $Der_k(\Omega(M), \Omega(M;E))$ the space of all derivations of degree k from $\Omega(M)$ into $\Omega(M;E)$.

4.2. Let $D_1 = (D_1, \bar{D}_1)$ in $Der_{k_1}\Omega(M;E)$ be a derivation of the $\Omega(M)$-module $\Omega(M;E)$ as studied in section 3. Let D_2 be in $Der_{k_2}(\Omega(M), \Omega(M;E))$. Then we define $[D_1, D_2] := D_1 \circ D_2 - (-1)^{k_1 k_2} D_2 \circ \bar{D}_1$.

Proposition: Then $[D_1, D_2]$ is a graded derivation of degree k_1+k_2 from $\Omega(M)$ into $\Omega(M;E)$. Thus we get a sort of graded commutator

$[\ ,\ ] : Der\Omega(M;E) \times Der(\Omega(M),\Omega(M;E)) \to Der(\Omega(M),\Omega(M;E))$.

For D_1, D_2 in $Der\Omega(M;E)$ of degree k_1, k_2, respectively, and for D_3 in $Der(\Omega(M),\Omega(M;E))$ we have the "graded Jacobi identity":

$[D_1, [D_2, D_3]] = [[D_1, D_2], D_3] + (-1)^{k_1 k_2}[D_2, [D_1, D_3]]$.

The proof is by computation. The graded Jacobi identity is equivalent to: $D \to [D, \ ]$ is a homomorphism of graded Lie algebras $\mathrm{Der}\Omega(M;E) \to$ $\to \mathrm{End}(\mathrm{Der}(\Omega(M),\Omega(M;E)))$.

4.3. Any algebraic derivation is of tensorial character and is uniquely determined by $D|\Omega^1(M)$: $\Omega^1(M) \to \Omega^{k+1}(M;E)$, which is given by the action of a vector bundle homomorphism $T^*M \to \Lambda^{k+1}T^*M \times E$, which we view as an element Q of $\Omega^{k+1}(M; L(T^*M,E))$. We write $D =: \rho(Q)$ to express this dependence.

Lemma: 1. For ω_i in $\Omega^1(M)$ and Q in $\Omega^{k+1}(M;L(T^*M,E))$ we have

$$\rho(Q)(\omega_1 \wedge \dots \wedge \omega_p) = \Sigma\, (-1)^{(i-1)k}\, \omega_1 \wedge \dots \wedge (\rho(Q)\omega_i) \wedge \dots \wedge \omega_p.$$

2. If $Q = L \otimes s$ for L in $\Omega^{k+1}(M;TM)$ and s in $\Gamma(E)$, then

$$\rho(L \otimes s)\psi = i(L)\psi \otimes s \text{ for all } \psi \text{ in } \Omega(M).$$

3. For Q in $\Omega^{k+1}(M;L(T^*M,E))$, ω in $\Omega^q(M)$, and vector fields X_i,

$$(\rho(Q)\omega)(X_1,\dots,X_{q+k}) = \frac{1}{(k+1)!\,(q-1)!}\,\Sigma \operatorname{sign}\sigma\; Q(X_{\sigma 1},\dots,X_{\sigma(k+1)})(\omega(\ ,X_{\sigma(k+2)},\dots)).$$

Proof: 1 follows by induction from the derivation property. Both sides in 2 define derivations $\Omega(M) \to \Omega(M;E)$ and they coincide on $\Omega^1(M)$ and vanish on $\Omega^0(M)$, so they are equal. 3 follows from 2 and 1.2.1. qed.

4.4. For L in $\Omega^{k+1}(M;TM)$ and Q in $\Omega^{q+1}(M;L(T^*M,E))$ the derivation $[i(L), \rho(Q)]$ is again algebraic, so it is of the form $\rho([L,Q]^\wedge)$ for some unique element $[L, Q]^\wedge$ in $\Omega^{k+q+1}(M;L(T^*M,E))$. Note the defining equation:

$$\rho([L, Q]^\wedge) := [i(L), \rho(Q)].$$

Lemma: We have $[L, Q]^\wedge = i(L)Q - (-1)^{kq}\,\rho(Q)L$, where $i(L)Q$ is clear and

$$\rho(Q)(\psi \otimes X) = \rho(Q)\psi \otimes X \ \text{ in } \Omega^{q+k+1}(M;E \otimes TM) = \Omega^{q+k+1}(M;L(T^*M,E)).$$

Proof: It suffices to check the formula for $L = \omega \otimes X$, evaluated on a 1-form. This is easy. qed.

4.5. Corollary: With the action $[\ ,\ ]\hat{}$ of $(\Omega(M;TM),[\ ,\ ]\hat{})$ the space $\Omega^{*-1}(M;L(T^*M,E))$ is a graded Lie module also (compare 4.2).

4.6. Lemma: For P in $\Omega^p(M;L(E,E))$ and Q in $\Omega^q(M;L(T^*M,E))$ we have
$[\mu(P), \rho(Q)] = \mu(P)\ \rho(Q) - 0 = \rho(\mu(P)Q)$, where $\mu(P) = (\mu(P), 0)$ is in $\mathrm{Der}_q\Omega(M;E)$, see 3.2. $\mu(P)Q$ is given by

$$(\mu(P)Q)(X_1,\ldots,X_{p+q}) = \frac{1}{p!\ q!}\ \Sigma\ \mathrm{sign}\ \sigma\ P(X_{\sigma 1},\ldots,X_{\sigma p}) \circ Q(X_{\sigma(p+1)},\ldots).$$

Proof: Evaluate at a 1-form. qed.

4.7. Now let ∇ be any covariant derivative on (E,p,M), so (∇, d) is in $\mathrm{Der}_1\Omega(M;E)$, and for any Q in $\Omega^q(M;L(T^*M,E))$ we may consider the derivation $\Theta^\nabla(Q) := (-1)^q\ [\nabla, \rho(Q)] = \rho(Q)\ d - (-1)^{q-1}\nabla\rho(Q)$ in $\mathrm{Der}_q(\Omega(M),\Omega(M;L(T^*M,E)))$. We will call $\Theta^\nabla(Q)$ the E-valued covariant exteriour Lie derivative along Q of differential forms.

Theorem: Let D be any derivation in $\mathrm{Der}_k(\Omega(M), \Omega(M;E))$. Then there are unique elements Q in $\Omega^k(M; L(T^*M,E))$ and R in $\Omega^{k+1}(M; L(T^*M,E))$ such that $D = \Theta^\nabla(Q) + \rho(R)$.

If $\nabla + \mu(A)$ is other covariant derivative (for A in $\Omega^1(M;L(E,E))$), then $\Theta^{\nabla+\mu(A)}(Q) = \Theta^\nabla(Q) + (-1)^k\ \rho(\mu(A)Q)$.

Proof: Let X_i be vector fields on M , let s be a section of E^*. Then $f \to \langle s, (Df)(X_1,\ldots,X_k)\rangle$ is a derivation $C^\infty(M) \to C^\infty(M)$; so it is given by the action of a uniquely determined vector field $Y(s,X_1,\ldots,X_k)$, which is clearly $C^\infty(M)$-linear in s and the X_i's, and is skew in the X_i's, so we may write $Y(s,X_1,\ldots,X_k) = \langle s, Q(X_1,\ldots,X_k)\rangle$ for some unique Q in $\Omega^k(M; TM\otimes E) = \Omega^k(M; L(T^*M,E))$. Now $Df = \rho(Q)df$ by construction, so $D - \Theta^\nabla(Q)$ is algebraic and coincides with $\rho(R)$ for some uniquely determined R in $\Omega^{k+1}(M; L(T^*M,E))$ by 4.3. The last assertion is clear by 4.6. qed.

4.8. Lemma: 1. For P in $\Omega^p(M;L(E,E))$ and Q in $\Omega^q(M;L(T^*M,E))$ we have

$$[\mu(P),\ \theta^\nabla(Q)] = \theta^\nabla(\mu(P)Q) - (-1)^{p+q}\rho(\mu(\nabla^{L(E,E)}P)Q).$$

2. $[\nabla,\ \theta^\nabla(Q)] = (-1)^q \quad \rho(\mu(R)Q)$, where R is the curvature.

Proof: Plug in the definitions and use 4.6 and 3.11. For 2 use the definitions, 3.6 and again 4.6. qed.

4.9. The space $\mathrm{Der}(\Omega(M),\Omega(M;E))$ also bears a graded $\Omega(M)$-module structure given by $(\omega\wedge D)\phi = \omega\wedge D\phi$.

Lemma: 1. $\rho(\omega\wedge Q) = \omega\wedge\rho(Q)$.

2. For $D_1 = (D_1, \bar{D}_1)$ in $\mathrm{Der}_{k_1}\Omega(M;E)$, D_2 in $\mathrm{Der}_{k_2}(\Omega(M),\ \Omega(M;E))$ and ω in $\Omega^p(M)$ we have

$$[D_1,\ \omega\wedge D_2] = \bar{D}_1\omega\wedge D_2 + (-1)^{p\,k_1}\omega\wedge[D_1,\ D_2].$$

3. For ω in $\Omega^p(M)$ and Q in $\Omega^q(M;\ L(T^*M,E))$ we have

$$\theta^\nabla(\omega\wedge Q) = \omega\wedge\theta^\nabla(Q) + (-1)^{p+q}d\omega\wedge\rho(Q).$$

Proof: Compute.

4.10. Now we want to decompose the bracket $[i(L),\ \theta^\nabla(Q)]$ for L in $\Omega^p(M;TM)$ and Q in $\Omega^q(M;\ L(T^*M,E))$. Let f be in $C^\infty(M) = \Omega^0(M)$. Then $[i(L),\ \theta^\nabla(Q)]f = i(L)\ \rho(Q)\ df = i(L)\ Q(df) = (i(L)Q)\ df = \rho(i(L)Q)\ df =$ $= \theta^\nabla(i(L)Q)f$. Therefore $[i(L),\ \theta^\nabla(Q)] - \theta^\nabla(i(L)Q)$ is algebraic, and we put (according to formula 3.9 for choosing the sign):

$$[i(L),\ \theta^\nabla(Q)] - \theta^\nabla(i(L)Q) =: (-1)^q\ \rho([L,\ Q]^\nabla).$$

Then $[\ ,\ ]^\nabla$ is a differential concomitant $\Omega^p(M;TM)\times\Omega^q(M;L(T^*M,E)) \to$ $\to \Omega^{p+q}(M;L(T^*M,E))$.

Lemma: In the situation above we also have

$$[\theta_\nabla(L),\ \rho(Q)] = \rho([L,\ Q]^\nabla) - (-1)^{(q-1)p}\theta^\nabla(\rho(Q)L).$$

Proof: This is also a routine computation.

4.11. Lemma: For K in $\Omega^k(M;TM)$ and Q in $\Omega^q(M;L(T^*M,E))$ we have

$$[\Theta_\nabla(K),\ \Theta^\nabla(Q)] = \Theta^\nabla([K,\ Q]^\nabla) +$$
$$+ (-1)^q\ \rho\big(i(K)\ \mu(R)\ Q - \mu(R)\ (\ i(K)\ Q - (-1)^{(k-1)(q-1)}\ \rho(Q)\ K)\big).$$

Here R is the curvature of ∇.

Proof: This is a routine computation, using $\Theta_\nabla(K) = [i(K),\nabla]$ and the graded Jacobi identity.

4.12. Analysis of the concomitant $[\ ,\ ]^\nabla\colon \Omega^k(M;TM)\times\Omega^q(M;L(T^*M,E)) \to$ $\to \Omega^{k+q}(M;L(T^*M,E))$.

Proposition: 1. Let ∇ and $\nabla + \mu(A)$, A in $\Omega^1(M;L(E,E))$, be two covariant derivatives on (E,p,M). Then we have:

$$[K,\ Q]^\nabla - [K,\ Q]^{\nabla+\mu(A)} = \mu(i(K)A)Q - (-1)^{q(k-1)}\rho(\mu(A)Q)K.$$

2. For ω in $\Omega^p(M)$ we have

$$[K,\ \omega\wedge Q]^\nabla = (-1)^{pk}\omega\wedge[K,\ Q]^\nabla + \Theta(K)\omega\wedge Q - (-1)^{(p+q)(k-1)}d\omega\wedge\rho(Q)K.$$

3. $$[\omega\wedge K,\ Q]^\nabla = \omega\wedge[K,\ Q]^\nabla + (-1)^{p+k}d\omega\wedge i(K)Q - $$
$$- (-1)^{(p+k)q}\Theta^\nabla(Q)\omega\wedge K.$$

4. For K_i in $\Omega^{k_i}(M;TM)$ and Φ in $\Omega^p(M;E)$ we have:

$$[K_1,\ \Phi\wedge K_2]^\nabla = \Theta_\nabla(K_1)\Phi\wedge K_2 - (-1)^{(p+k_2)(k_1-1)}\nabla\Phi\wedge i(K_2)K_1 +$$
$$+ (-1)^{pk_1}\Phi\wedge[K_1,\ K_2],$$

where $[K_1,\ K_2]$ is the Frölicher-Nijenhuis bracket.

5. $$[K_1,\ [K_2,\ Q]^\nabla]^\nabla - [[K_1,\ K_2],\ Q]^\nabla - (-1)^{k_1k_2}[K_2,\ [K_1,\ Q]^\nabla]^\nabla =$$
$$= \mu(\lambda(K_1,K_2)R)Q + (-1)^{q(k_1-1)+k_1k_2}\rho(\mu(i(K_2)R)Q)K_1 -$$
$$- (-1)^{q(k_2-1)}\ \rho(\mu(i(K_1)R)Q)K_2 - (-1)^{q(k_1+k_2)+k_1}\big(\rho(\rho(\mu(R)Q)K_1)K_2 +$$
$$+ (-1)^{(k_1-1)(k_2-1)}\ \rho(\rho(\mu(R)Q)K_2)K_1\big)\ .$$

Hints for the proof: We use the map $\rho(Q)\colon \Omega(M;TM) \to \Omega(M;L(T^*M,E))$ given by $\rho(Q)(\omega\otimes X) = \rho(Q)\omega\otimes X$. It satisfies $\rho(Q)(\psi\wedge K) = \rho(Q)\psi\wedge K + (-1)^{p(q-1)}\psi\wedge\rho(Q)K$. With this 2 up to 4 are routine computations using results of sections 1 and 4. 5 is shown for $Q = \Phi\wedge K_3$ and is rather difficult.

5. Looking for natural concomitants of vector valued differential forms.

5.1. Let us consider, for each n-dimensional manifold M, a $\mathbb{R}$-bilinear operator B_M: $\Omega^p(M;TM) \times \Omega^q(M;TM) \to \Omega^r(M;TM)$ such that for each local diffeomorphism $f: M \to N$ we have $f^* B_N(K,L) = B_M(f^*K, f^*L)$.

Then clearly each B_M is a local bilinear operator, and by the extension of Peetre's theorem to bilinear operators (see [1]) B_M is a bilinear differential operator. Let us now fix a manifold M. If $(x^1,\ldots,x^n)$ is a local coordinate system on M, then on its domain of definitions we may write $K = K^i_\alpha\, d^\alpha \otimes \partial_i$, where $\partial_i = \frac{\partial}{\partial x^i}$, $d^\alpha = dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_k}$, where $\alpha = (\alpha_1 < \alpha_2 < \ldots < \alpha_k)$ is a form index, for K in $\Omega^p(M;TM)$, and likewise $L = L^j_\beta\, d^\beta \otimes \partial_j$ for L in $\Omega^q(M;TM)$. Here we use the Einstein sum convention. Then $B(K,L) = B^{\alpha\beta RSk}_{ij\gamma}\, \partial^R(K^i_\alpha)\, \partial^S(L^j_\beta)\, d^\gamma \otimes \partial_k$, where R, S are multi-indices for iterated partial derivatives.

5.2. First of all we see from 5.1, that we may differentiate through B: Given any vector field X on M, we may consider it's local flow Fl^X_t and we get $\Theta(X)B(K,L) = \frac{d}{dt}|_0 (Fl^X_t)^*\, B(K,L) = \frac{d}{dt}|_0\, B((Fl^X_t)^*K, (Fl^X_t)^*L) =$
$= B(\Theta(X)K,L) + B(K,\Theta(X)L)$, or, in view of 2.6:

$$[X, B(K,L)] = B([X,K],L) + B(K,[X,L]).$$

5.3. One may take the local expression for B of 5.1 and express formula 5.2 in it. This gives horrible linear equations for the coefficients of B; for by choosing $X = \partial_j$ one easily sees, that all derivatives of $B^{\alpha\beta RSk}_{ij\gamma}$ vanish, so they are constants in any coordinate system.

I will not consider these equations in this paper.

5.4. Using a local coordinate system on M we see that we have to determine B in $\mathbb{R}^n$ at 0.

Lemma: Let I be the vector field $I = \sum x^i \partial_i$ on $\mathbb{R}^n$.

1. If ψ is a constant p-form on $\mathbb{R}^n$, then $\Theta(I)\psi = p.\psi$.
2. For K in $\Omega^p(\mathbb{R}^n, T\mathbb{R}^n)$ we have $[I,K](0) = (p-1)K(0)$.
3. If X is a vector field on $\mathbb{R}^n$ which is homogeneous of degree k, so $X(tx) = t^k X(x)$, then $[I,X] = (k-1)X$.
4. For any vector field Y we have $[I,Y](0) = -Y(0)$.
5. If ψ is a constant p-form and X is a vector field which is homogeneous of degree k, then $[I,\psi\otimes X] = (p+k-1)\psi\otimes X$.

Proof: We show 3: $Fl_t^I(x) = e^t x$. $((Fl_t^I)^* X)(x) = T(Fl_t^I)^{-1} \circ X \circ Fl_t^X (x) =$ $= e^{-t} X(e^t x) = e^{(k-1)t} X(x)$. Thus $[I,X] = \Theta(I)X = \frac{d}{dt}|_0 e^{(k-1)t}X = (k-1)X$.
5. $[I,\psi\otimes X] = \Theta(I)(\psi\otimes X) = \Theta(I)\psi\otimes X + \psi\otimes\Theta(I)X$, so this follows from 1 and 3. Similarly 2 follows from 1 and 4. The rest is obvious. qed.

5.5. Let ϕ be a constant p-form, let ψ be a constant q-form, and let X and Y be vector fields, homogeneous of degree k and m respectively. Then by 5.4:

$(r-1)\, B(\phi\otimes X,\psi\otimes Y)(0) = [I, B(\phi\otimes X,\psi\otimes Y)](0) =$

$= B([I,\phi\otimes X],\psi\otimes Y)(0) + B(\phi\otimes X, [I,\psi\otimes Y])(0) =$

$= (p + q + k + m - 2)\, B(\phi\otimes X,\psi\otimes Y)(0)$.

So if $p+q+k+m-1 \neq r$, then $B(\phi\otimes X,\psi\otimes Y)(0) = 0$.

Since 0 is arbitrary, we may conclude (where C is contraction, $C(\psi\otimes X) = i(X)\psi$).

Corollary: If $B: \Omega^p(M;TM) \times \Omega^q(M;TM) \to \Omega^r(M;TM)$ is a natural concomitant, then:

1. If $p+q-1 = r$, then B is algebraic. Examples of such concomitants are: $i(K)L$, $i(L)K$, $C(K)\wedge L$, $C(L)\wedge K$, $C(K)\wedge C(L)\wedge Id_{TM}$.
2. If $p+q = r$, then B is a homogeneous bilinear differential operator of total order 1. Thus on $\mathbb{R}^n$ we have $B(K,L) = A_1(DK,L) + A_2(K,DL)$, where A_i are algebraic and DK is the derivative of K. Examples are $[K,L]$, $dC(K)\wedge L$, $dC(L)\wedge K$, $dC(K)\wedge C(L)\wedge Id_{TM}$, $dC(L)\wedge C(K)\wedge Id_{TM}$. $dC(i(K)L)\wedge Id_{TM}$, $dC(i(L)K)\wedge Id_{TM}$.

3. If $p+q+1 = r$, then B is homogeneuos of total order 2. So on $\mathbb{R}^n$ $B(K,L) = A_1(D^2K,L) + A_2(DK,DL) + A_3(K,D^2L)$, where the A_i are algebraic. Examples are: $dC(K) \wedge dC(L) \wedge Id_{TM}$.
4. If $p+q-1 < r$, then $B = 0$.
5. If $p+q-1 = r+m$, then B is a bilinear differential operator, homogeneous of order m.

5.6. Remark: Of course it would be very interesting to determine the vector space of all natural concomitants for each choice of r. One way would be to solve the linear equations mentioned in 5.3. For me this was too difficult.

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