

**Singular Cotangent Bundle  
Reduction  
&  
Spin Calogero-Moser Systems**

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Dissertation zur Erlangung des Akademischen Grades  
**Doktor der Naturwissenschaften.**

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Wien, am 7. Februar 2005.

**Für Anne.**  
Ik hou van jou!



## Preface

This thesis is concerned with symplectic reduction of a cotangent bundle  $T^*Q$  with respect to a Hamiltonian action by a compact Lie group  $K$  that comes as the cotangent lifted action from the configuration manifold  $Q$ . Moreover, we assume that  $Q$  is Riemannian and  $K$  acts on  $Q$  by isometries. The cotangent bundle  $T^*Q$  is equipped with its canonical exact symplectic form, and we have a standard momentum map  $\mu : T^*Q \rightarrow \mathfrak{k}^*$ . Consider a coadjoint orbit  $\mathcal{O}$  that lies in the image of  $\mu$ . The goal is now to understand the symplectically reduced space

$$\mu^{-1}(\mathcal{O})/K =: T^*Q//_{\mathcal{O}}K.$$

Several difficulties arise at this point. First of all the action by  $K$  on the base  $Q$  is *not* assumed to be free. So we will get only a stratified symplectic space. Its strata will be of the form

$$(\mu^{-1}(\mathcal{O}) \cap (T^*Q)_{(L)})/K =: (T^*Q//_{\mathcal{O}}K)_{(L)}$$

where  $(L)$  is an element of the isotropy lattice of the  $K$ -action on  $T^*Q$ . This follows from the theory of singular symplectic reduction as developed in Sjamaar and Lerman [45], Bates and Lerman [6], and Ortega and Ratiu [33]. See also Theorem 1.H.1.

One of the aspects of cotangent bundle reduction is to relate the reduced space  $(T^*Q//_{\mathcal{O}}K)_{(L)}$  to the cotangent bundle of the reduced configuration space, i.e. to  $T^*(Q/K)$ . However, in this generality  $Q/K$  will not be a smooth manifold, and, worse, the mapping  $(T^*Q//_{\mathcal{O}}K)_{(L)} \rightarrow T^*(Q/K)$  (which one constructs canonically – see Section 2.D) does not have locally constant fiber type. To remedy this mess we have to assume that the base manifold is of single isotropy type, that is  $Q = Q_{(H)}$  for a subgroup  $H$  of  $K$ . Assuming this we get a first result that says that

$$\mathcal{O}//_{\mathcal{O}}H \hookrightarrow T^*Q//_{\mathcal{O}}K \longrightarrow T^*(Q/K)$$

is a singular symplectic fiber bundle, and this is Theorem 2.A.4. This result is obtained by applying the Palais Slice Theorem to the action on the base space  $Q$ , and then using the Singular Commuting Reduction Theorem of Section 1.H. This is an inroad that was also taken by Schmah [43] to get a local description of  $T^*Q//_{\mathcal{O}}K$ .

However, one can also give a global symplectic description of the reduced space, and this is done in Section 2.D. This follows an approach that is generally called gauged cotangent bundle reduction or Weinstein

construction ([50]) or also Sternberg construction. In the case that the action by  $K$  on the configuration space is free this global description was first given by Marsden and Perlmutter [25]. Their result says that the symplectic quotient  $T^*Q//_{\mathcal{O}}K$  can be realized as the fibered product

$$T^*(Q/K) \times_{Q/K} (Q \times_K \mathcal{O})$$

and they compute the reduced symplectic structure in terms of data intrinsic to this realization – [25, Theorem 4.3].

In the presence of a single non-trivial isotropy on the configuration space one gets a non-trivial isotropy lattice on  $T^*Q$  and thus has to use stratified symplectic reduction. Therefore, there is quite a difference between the regular case (where  $K$  acts freely on  $Q$ ) and the single isotropy type case (where  $Q = Q_{(H)}$ .) The result for the single isotropy type case is the following: Each symplectic stratum  $(T^*Q//_{\mathcal{O}}K)_{(L)}$  of the reduced space can be globally realized as

$$(\mathcal{W}//_{\mathcal{O}}K)_{(L)} = T^*(Q/K) \times_{Q/K} (\bigsqcup_{q \in Q} \mathcal{O} \cap \text{Ann } \mathfrak{k}_q)_{(L)}/K$$

where

$$\mathcal{W} := (Q \times_{Q/K} T^*(Q/K)) \times_Q \bigsqcup_{q \in Q} \text{Ann } \mathfrak{k}_q \cong T^*Q$$

as symplectic manifolds with a Hamiltonian  $K$ -action. Moreover, we compute the reduced symplectic structure in terms intrinsic to this realization. This is the content of Theorem 2.D.4.

In Section 2.E we are concerned with understanding the Poisson reduced space  $(T^*Q)/K \cong \mathcal{W}/K$  by means of the Weinstein construction in the single isotropy type case, i.e. where  $Q = Q_{(H)}$ . Indeed, we find in Theorem 2.E.3 that the Poisson reduced space  $(T^*Q)/K$  may be realized as the singular fibered product

$$T^*(Q/K) \times_{Q/K} (\bigsqcup_{q \in Q} \text{Ann } \mathfrak{k}_q)/K,$$

and we compute the reduced Poisson structure in terms intrinsic to this realization. The smooth strata of this singularly reduced space are of the form

$$T^*(Q/K) \times_{Q/K} (\bigsqcup_{q \in Q} \text{Ann } \mathfrak{k}_q)_{(L)}/K,$$

where  $(L)$  is an element of the isotropy lattice of the  $K$ -action on  $T^*Q$ . In particular, it follows that the reduced Poisson space is canonically a singular fiber bundle over the cotangent bundle of the reduced configuration space, that is

$$\text{Ann } \mathfrak{h}/H \hookrightarrow (T^*Q)/K \longrightarrow T^*(Q/K)$$

We refer to Section 2.B for the notion of a singular fiber bundle. The smooth strata  $\mathcal{W}_L/K$  thus fiber over  $T^*(Q/K)$  according to

$$(\text{Ann } \mathfrak{h})_{(L_0)H}/H \hookrightarrow (T^*Q)_{(L)}/K \longrightarrow T^*(Q/K)$$

where  $L_0$  is a subgroup of  $H$  which is conjugate to  $L$  within  $K$ , and  $(L_0)^H$  denotes the isotropy class of  $L_0$  with respect to the  $\text{Ad}^*(H)$ -action on  $\text{Ann } \mathfrak{h}$ . This generalizes the work of Zaalani [53] and Perlmutter and Ratiu [37] who considered free actions on the base  $Q$ .

It is rather surprising that the subject of cotangent bundle reduction, albeit so important to Hamiltonian mechanics, still is very untouched. Even in the case of a free action on the base the results are rather new, and there is not much to be found about singular cotangent bundle reduction in the literature. One of the first to study this subject is Schmah [43]. The other important paper on singular cotangent bundle reduction is the one by Perlmutter, Rodriguez-Olmos and Sousa-Diaz [38]. By restricting to do reduction at fully isotropic values of the momentum map  $\mu : T^*Q \rightarrow \mathfrak{k}^*$  they are able to drop all assumptions on the isotropy lattice of the  $K$ -action on  $Q$ , and give a very complete description of the reduced symplectic space. There is also a very recent paper by Perlmutter, Rodriguez-Olmos, and Sousa-Diaz [39] where the authors provide a Witt-Artin decomposition of the cotangent bundle  $T^*Q$  in the presence of a cotangent lifted action by  $K$ . This may make it possible to generalize the results on  $T^*Q//_{\mathcal{O}}K$  and  $(T^*Q)/K$  to the fully singular case.

As an application of these cotangent bundle reduction techniques we consider Calogero-Moser systems with spin in Section 4. In fact, it was an idea of Alekseevsky, Kriegl, Losik, Michor [2] to consider polar representations of compact Lie groups  $G$  on a Euclidean vector space  $V$  to obtain new versions of Calogero-Moser systems. We make these ideas precise by using the singular cotangent bundle reduction machinery. Thus let  $\Sigma$  be a section for the  $G$ -action in  $V$ , let  $C$  be a Weyl chamber in this section, and put  $M := Z_G(\Sigma)$ . Under a strong but not impossible condition on a chosen coadjoint orbit in  $\mathfrak{g}^*$  we get

$$T^*V//_{\mathcal{O}}G = T^*C_r \times \mathcal{O}//_0M$$

from the general theory, where  $C_r$  denotes the sub-manifold of regular elements in  $C$ . This is the effective phase space of the Spin Calogero-Moser system. The corresponding Calogero-Moser function is obtained as a reduced Hamiltonian from the free Hamiltonian on  $T^*V$ . The resulting formula is

$$H_{\text{CM}}(q, p, [Z]) = \frac{1}{2} \sum_{i=1}^l p_i^2 + \frac{1}{2} \sum_{\lambda \in R} \frac{\sum_{i=1}^{k_\lambda} z_\lambda^i z_\lambda^i}{\lambda(q)^2}.$$

This is made precise with the necessary notation in Section 4.

Finally, we study integrability properties of the thus obtained reduced Hamiltonian system. Here we consider two ways to exhibit integrability of the Calogero-Moser system. The first is to use a result from Zung [54, Theorem 2.3] to show that the system is integrable in a generalized Liouville sense. The second is to make stronger use of the

gauged reduction picture and to show that the Calogero-Moser Hamiltonian is actually integrable via a symplectically complete bifoliation. This approach has the advantage of providing a deeper insight into the dynamical behavior of the system.

Having said this, the setup of the thesis is as follows. In Chapter 1 we introduce important concepts from singular geometry which will be heavily used throughout the thesis. In Chapter 2 we come to the core subject and develop the results referred to and stated above. Chapter 3 follows Nehorošev [30], Zung [54, 55], Fasso and Ratiu [16], and Fasso [15] in introducing generalized notions of integrability as well as the appropriate concept of generalized action-angle variables. In Chapter 4 the ideas from the previous development are employed in the study of spin Calogero-Moser systems obtained from polar representations of compact Lie groups.

**Thanks.** I am grateful to Peter Michor for introducing me to symplectic geometry and proposing the subject of Calogero-Moser systems associated to polar representations of compact Lie groups. I am also thankful to Stefan Haller and Armin Rainer for helpful remarks and comments.

Vienna, February 2005



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## CHAPTER 1

### Singular geometry

Let  $K$  be a compact Lie group that acts by Hamiltonian transformations on a symplectic manifold  $(M, \omega)$  such that there is an equivariant momentum map  $J : M \rightarrow \mathfrak{k}^*$  where  $\mathfrak{k}$  is the Lie algebra of  $K$ . In such a situation one can pass to the symplectic quotient of  $(M, \omega)$  by  $K$ , that is one considers the symplectically reduced space

$$J^{-1}(\mathcal{O})/K =: M//_{\mathcal{O}}K$$

where  $\mathcal{O}$  is a coadjoint orbit in the image of  $J$ . If the  $K$  action on  $M$  is free then it follows that  $M//_{\mathcal{O}}K$  is a smooth symplectic manifold, and is also called the Marsden-Weinstein-Meyer reduced space. In general, however, the reduced space will fail to be a manifold. It will rather be a topological space that is decomposed in a technical way into pieces which can be shown to be smooth symplectic manifolds. This way to look at  $M//_{\mathcal{O}}K$  renders it a singular symplectic space, and it is obtained from  $(M, \omega)$  via a process called singular symplectic reduction. The terms singular space, singular symplectic space, and singular symplectic reduction are explained in this chapter. The reduction process in this generality is due to Sjamaar and Lerman [45].

#### 1.A. Compact transformation groups

Let  $K$  be a compact Lie group that acts by isometries on a Riemannian manifold  $M$ , i.e.  $M$  is a Riemannian  $K$ -space. The action will be written as  $l : K \times M \rightarrow M$ ,  $(k, x) \mapsto l(k, x) = l_k(x) = l^x(k) = k.x$ . Sometimes the action will be lifted to the tangent space  $TM$ . That is, we will consider  $h.(x, v) := (h.x, h.v) := Tl_h.(x, v) = (l_h(x), T_x l_h.v)$  where  $(x, v) \in TM$ . As the action is a transformation by a diffeomorphism it may also be lifted to the cotangent bundle. This is the cotangent lifted action which is defined by  $h.(x, p) := (h.x, h.p) := T^*l_h.(x, p) = (h.x, T_{h.x}^* l_{h^{-1}.p})$  where  $(x, p) \in T^*M$ .

The FUNDAMENTAL VECTOR FIELD is going to repeatedly play an important role. It is defined by

$$\zeta_X(x) := \left. \frac{\partial}{\partial t} \right|_0 l(\exp(+tX), x) = T_e l^x(X)$$

where  $X \in \mathfrak{k}$ . The fundamental vector field mapping  $\mathfrak{k} \rightarrow \mathfrak{X}(M)$ ,  $X \mapsto \zeta_X$  is clearly linear. By definition the flow of  $\zeta_X$  is given by  $l_{\exp(tX)}$ . Moreover, it intertwines Lie bracket with the negative of the

bracket of vector fields. Indeed, notice first, that

$$\begin{aligned} T_x l_k \cdot \zeta_X(x) &= \frac{\partial}{\partial t} |_0 l_k(l(\exp(tX), x)) \\ &= \frac{\partial}{\partial t} |_0 l(\text{conj}_k(\exp(tX)), k.x) \\ &= T_e l^{k.x} \cdot T_e \text{conj}_k \cdot (X) \\ &= \zeta_{\text{Ad}_{(k)X}}(k.x) \end{aligned}$$

that is  $\zeta_X$  and  $\zeta_{\text{Ad}_{(k)X}}$  are  $l_k$ -related. Now we use this to compute the bracket to be

$$\begin{aligned} [\zeta_X, \zeta_Y](x) &= \frac{\partial}{\partial s} |_0 (\text{Fl}_s^{\zeta_X})^* \zeta_Y \\ &= \frac{\partial}{\partial s} |_0 T_{\exp(sX).x} l_{\exp(-sX)} \cdot \zeta_Y(\exp(sX).x) \\ &= \frac{\partial}{\partial s} |_0 \zeta_{\text{Ad}_{(\exp(-sX))}Y}(x) \\ &= -\zeta_{[X,Y]}(x) \end{aligned}$$

where  $X, Y \in \mathfrak{k}$  and  $x \in M$ . Had we chosen the sign negatively in the fundamental vector field mapping then it were Lie algebra homomorphism. However, this choice of sign is the standard one in conjunction with Hamiltonian group actions and momentum maps.

We endow the set of orbits  $M/K$  with the quotient topology and call it the orbit space. Let  $H$  be a subgroup of  $K$ . A point  $x \in M$  is said to be of ISOTROPY TYPE  $H$  if its isotropy group  $K_x = \{k \in K : k.x = x\}$  is conjugate to  $H$  within  $K$ . If  $H'$  is conjugate to  $H$  within  $K$  we shall also write  $H' \sim H$ . The family of subgroups of  $K$  conjugate to  $H$  within  $K$  is denoted by  $(H)$  and called the CONJUGACY CLASS of  $H$ . All conjugacy classes of possible subgroups of  $K$  that show up as isotropy groups of points in  $M$  taken together constitute the ISOTROPY LATTICE

$$\mathcal{IL}_K(M) := \{(L) : L = K_x \text{ for some } x \in M\}$$

of the action under consideration. If it is clear which action is being looked at we simply write  $\mathcal{IL}(M)$ . The set of points that have isotropy  $H$  is denoted by

$$M_{(H)} := \{x \in M : K_x \text{ is conjugate to } H\},$$

and is called the ISOTROPY TYPE SUB-MANIFOLD of  $M$  of type  $(H)$ . It will be shown below that this terminology is justified, i.e.  $M_{(H)}$  indeed is a sub-manifold. Furthermore, there is

$$M_H := \{x \in M : K_x = H\}$$

which is called the set of points that have SYMMETRY  $H$ , and the set of points that are fixed by  $H$ ,

$$\text{Fix}(H) := M^H := \{x \in M : H \subseteq K_x\}$$

There is a natural partial ordering on the isotropy lattice as follows:

$$(H) \leq (L) : \iff \text{there is } k \in K \text{ such that } H \subseteq kLk^{-1}$$

where  $(H)$  and  $(L)$  are in the isotropy lattice of  $M$ . This relation is anti-symmetric because  $K$  was assumed compact. An element  $x \in M$  is called **REGULAR** if it has an open neighborhood such that  $(K_x) \leq (K_y)$  for all  $y$  in this neighborhood, and the set of regular points is denoted by  $M_{\text{reg}}$ . If a point is not regular it is said to be **SINGULAR**.

**DEFINITION 1.A.1 (Slices).** A subset  $S \subseteq M$  is called a **SLICE** at  $x \in M$  if there are an open  $K$ -invariant neighborhood  $U$  of  $K.x$  in  $M$  and a smooth  $K$ -equivariant retraction  $r : U \rightarrow K.x$  such that  $S = r^{-1}(x)$ .

**PROPOSITION 1.A.2.** *Assume  $S$  is a slice at  $x \in M$  for the  $K$ -action, and  $U$  an open  $K$ -invariant neighborhood of  $K.x$  as in the definition. Then the following are true.*

- (i) *The slice  $S$  is a manifold and  $K_x$  acts on  $S$ .*
- (ii)  *$k.S \cap S \neq \emptyset$  if and only if  $k \in K_x$ .*
- (iii)  *$K.S = U$ .*
- (iv)  *$K_s \subseteq K_x$  for all  $s \in S$*
- (v) *If  $x$  is regular then there is an open neighborhood  $V$  of  $x$  in  $S$  such that  $V \subseteq M_{\text{reg}}$ .*
- (vi) *If  $s_1, s_2 \in S$  have the same isotropy type as with regard to the  $K_x$ -action on  $S$  then their isotropy types with regard to the  $K$ -action coincide. The converse is, however, false.*
- (vii) *The topological spaces  $S/K_x$  and  $(K.S)/K$  are homeomorphic. Moreover, this is a typical open neighborhood of  $K.x$  in the orbit space  $M/K$ .*

**PROOF.** (i): Since  $r$  is a retraction,  $S$  is a sub-manifold of  $U$ , and hence also of  $M$  as  $U$  is open in  $M$ . By equivariance of  $r$  it follows that  $K_x$  acts on  $S$ .

(ii): Let  $s \in S$  and  $k \in K$  such that  $k.s = s' \in S$ . Then  $k.x = k.r(s) = r(k.s) = r(s') = x$  by equivariance, and the other direction is clear.

(iii):  $K.S = K.r^{-1}(x) = r^{-1}(K.x) = U$ .

(iv): This follows from property (ii).

(v): Is clear from the definition and the above properties.

(vi): The positive part of the assertion is obvious. A counterexample is given by [27, Remark 4.14].

(vii): We consider the map  $S/K_x \rightarrow U/K$ ,  $K_x.s \mapsto K.s$ . Clearly this map is surjective. It is also injective: assume  $K.s = K.s_1$ ; then there is a  $k \in K$  such that  $k.s = s_1$  whence  $k \in K_x$  by (ii). Now since the quotient spaces are endowed with the final topology with respect to the respective projections, the vertical arrows in the diagram

$$\begin{array}{ccc} S & \xrightarrow{i} & U \\ \downarrow & & \downarrow \\ S/K_x & \xrightarrow{\iota} & U/K \end{array}$$

are final by tautology. Thus the induced map  $\iota$  is continuous. To see that it is open, i.e. maps open subsets to open ones, take an open  $K_x$ -invariant neighborhood  $U_1$  of  $x$  in  $S$ . Then  $K.U_1$  is open in  $U$ , since the action  $K \times S \rightarrow U$  is a submersion, thus an open mapping. Hence also  $\iota(U_1/K_x) = (K.U_1)/K$  is open in  $U/K$ . Therefore also  $\iota^{-1}$  is continuous.  $\square$

**THEOREM 1.A.3 (Tube Theorem).** *Let  $S$  be a slice for the  $K$ -action at  $x \in M$ ,  $H = K_x$ , and  $U = K.S$ . Then there is a  $K$ -equivariant diffeomorphism*

$$\begin{aligned} f : K \times_H S &\longrightarrow U \\ [(k, s)] &\longmapsto k.s. \end{aligned}$$

*In particular the section  $K \times_H \{x\}$  in the associated bundle is mapped to the orbit  $K.x$ . Here the action by  $H$  on  $K \times S$  is given by  $h.(k, s) = (kh^{-1}, h.s)$ . The  $K$  action on  $K \times_H S$  is induced by left multiplication on the first factor of  $K \times S$ .*

The neighborhood  $U$  is called a **TUBE** around the orbit  $K.x$ .

**PROOF.** Consider the diagram

$$\begin{array}{ccc} K \times S & \xrightarrow{l} & K.S = U \\ \rho \downarrow & \nearrow f & \\ K \times_H S & & \end{array}$$

Since  $H$  acts freely on  $K$  it also acts freely on  $K \times S$ . Thus  $K \times_H S$  is a smooth manifold, which carries the final smooth structure with respect to the projection  $\rho$ . Therefore,  $f$  is smooth. It is clearly equivariant. The map is surjective since  $l$  is. Because  $k.s = k_1 k.s_1$  if and only if  $s = k_1^{-1}.s_1$  and this implies that  $k_1^{-1}k \in H$  it follows that  $f$  is also injective. Finally,  $l$  and  $\rho$  both are submersions. Thus this is also true for  $f$ . So  $f$  is a bijective submersion, hence a diffeomorphism.  $\square$

Since  $M$  is a Riemannian  $K$ -space we can define the normal bundle to an orbit  $K.x$  as

$$\text{Nor}(K.x) := T(K.x)^\perp = \bigsqcup_{y \in K.x} T_y(K.x)^\perp.$$

Furthermore, let  $\text{Nor}(K.x)_\varepsilon := \{X \in \text{Nor}(K.x) : |X| < \varepsilon\}$ , and we will make use of the geodesic exponential mapping  $TM \rightarrow M \times M$ ,  $\xi \mapsto (\tau(\xi), \exp_{\tau(\xi)} \xi)$  of Riemannian geometry where  $\tau$  is the foot point projection.

**THEOREM 1.A.4 (Slices).** *There is a number  $r > 0$  such that the exponential mapping  $\exp|_{\text{Nor}(K.x)_r} : \text{Nor}(K.x)_r \rightarrow U \subseteq M$ , where  $U$*

is the image, is a diffeomorphism onto  $U$ . Moreover,  $U$  is an open neighborhood of  $K.x$ , and

$$S := \exp_x(\text{Nor}_x(K.x)_r)$$

is a slice at  $x$  such that  $U = K.S$ .

A similar theorem also holds for the more general case of a proper action on a smooth manifold, and this was proved by Palais [34], see also Michor [27, Theorems 5.6 & 5.7].

PROOF. Clearly there is a  $r > 0$  such that

$$\exp | \text{Nor}(K.x)_r : \text{Nor}(K.x)_r \rightarrow \exp(\text{Nor}(K.x)_r) =: U \subseteq M$$

is a diffeomorphism. Moreover, since the metric is invariant under the group action it follows that  $\text{Nor}(K.x)_r \subseteq T_x M$  is  $K_x$ -invariant.

Since  $l_k(\exp_y(Y)) = \exp_{k.y}(T_y l_k.Y)$  for all  $y \in M$ ,  $Y \in T_y M$ , and  $k \in K$  it follows that  $U = K.S$ . Finally the retraction  $U \rightarrow K.x$  is given by  $y = \exp_{k.x}(Y) \mapsto k.x$ , and this is well-defined.  $\square$

Locally, a Riemannian action is an orthogonal action on a Euclidean vector space, in the following sense. If  $x \in M$  and  $H = K_x$  then the representation

$$\begin{aligned} H &\longrightarrow O(\text{Nor}_x(K.x)), \\ h &\longmapsto T_x l_h \end{aligned}$$

is called the SLICE REPRESENTATION of  $K$  at  $x$ . It is immediate from Proposition 1.A.2 that a point  $x$  is regular if and only if the slice representation at  $x$  is trivial.

THEOREM 1.A.5. *Let  $(H)$  be in the isotropy lattice of the Riemannian  $K$ -action on  $M$ . Then the following are true.*

- (i) *The subset  $\text{Fix}(H) = M^H = \{x \in M : H \subseteq K_x\}$  is a totally geodesic sub-manifold of  $M$ .*
- (ii)  *$M_H$  is an open dense sub-manifold of  $M^H$ , and if  $M = M_{(H)}$  then  $M^H = M_H$ . Moreover,  $M_H/N(H)$  is a smooth manifold, where  $N(H)$  is the normalizer of  $H$  in  $K$ .*
- (iii) *Both  $M_{(H)}$  and  $M_{(H)}/K$  are smooth manifolds, albeit possibly with countably many connected components that may differ in dimension.*
- (iv) *The orbit projection  $M_{(H)} \rightarrow M_{(H)}/K$  is a smooth fiber bundle with typical fiber  $K/H$ .*
- (v) *The orbit projection  $M_H \rightarrow M_H/N(H)$  is a smooth fiber bundle with typical fiber  $N(H)/H$ . Moreover,  $M_{(H)}/K$  and  $M_H/N(H)$  are diffeomorphic.*

PROOF. (i). Let  $x \in M^H$ , and  $U \subseteq M$  a chart centered at  $x$  such that  $\exp_x^{-1} : U \rightarrow V$  is a diffeomorphism where  $V \subseteq T_x M$

is an appropriate open neighborhood of 0. Then  $V \cap (T_x M)^H := \{v \in V : h.v = v\}$  is a chart of  $M^H$  which is centered at  $V$ , and, moreover,  $T_x(M^H) = (T_x M)^H$ ; indeed, let  $v \in V \cap (T_x M)^H$  then  $h.\exp_x(v) = \exp_x(h.v) = \exp_x(v) \in M^H$ . The last equality follows since  $H$  acts by isometries, and  $\frac{\partial}{\partial t}|_0 h.\exp_x(tv) = \frac{\partial}{\partial t}|_0 \exp_x(th.v)$ . This also shows that  $M^H$  is totally geodesic, i.e. any geodesic in  $M$  that starts in  $M^H$  will stay in  $M^H$  for all times.

(ii). Let  $x \in M_H \subseteq M^H$ . Consider an element  $k \in K$  such that  $k.M^H \subseteq M^H$  and some arbitrary  $h \in H$ . Then we have  $x = kk^{-1}.x = khk^{-1}.x$ , that is  $khk^{-1} \in K_x = H$ . On the other hand,  $N(H)$  obviously acts on  $M^H$ . Therefore,  $N(H)$  is the largest subgroup of  $K$  that acts on  $M^H$ . Moreover,  $H$  acts trivially and so we actually have an action of  $N(H)/H$  on  $M^H$ . Now it is clear that the set of regular points with respect to this action is just  $M_H$ . Thus this is an open dense sub-manifold of  $M^H$  by Theorem 1.A.6.

It is clear that  $M_{(H)} = M$  implies  $M_H = M^H$ .

Now  $M_H/N(H)$  is a smooth manifold: indeed, let  $x \in M_H$  and consider a slice  $S$  of this action at  $x$  with tubular  $N(H)$ -invariant neighborhood  $U$ . Then the tube theorem says that  $U$  is diffeomorphic to  $N(H)/H \times S$ . Thus a typical neighborhood of  $N(H).x$  in  $M_H/N(H)$  is of the form  $S$ .

(iii). Let  $x \in M_{(H)}$  with  $H = K_x$ . Let  $S$  be a slice at  $x$  and  $U$  a tube around  $K.x$ . We claim that  $U \cap M_{(H)} = K.S_H$ :

$\subseteq$ : Let  $y \in U \cap M_{(H)}$ . There is  $k \in K$  such that  $k.y \in S$ . But then  $H \sim K_{k.y} \subseteq H$  by Proposition 1.A.2 whence  $K_{k.y} = H$ .

$\supseteq$ : Is clear.

Thus we have the following

$$\begin{array}{ccc} K \times S_H & \xrightarrow{l} & K.S_H \\ \downarrow & & \cong \uparrow \\ K \times_H S_H & \xlongequal{\quad} & K/H \times S_H \end{array}$$

Note that  $K \times S_H \subseteq K \times S$  is a sub-manifold by the previous point. Therefore,  $M_{(H)}$  is a sub-manifold with typical neighborhood of the form  $K/H \times S_H$ .

(iv). This also shows that locally the orbit projection  $M_{(H)} \rightarrow M_{(H)}/K$  is of the form  $K/H \times S_H \rightarrow S_H$ , i.e. a smooth fiber bundle with typical fiber  $K.x \cong K/H$ .

(v). Finally, continuing the notation, the same arguments show that the projection  $M_H \rightarrow M_H/N(H)$  locally is of the form  $N(H)/H \times S_H \rightarrow S_H$ . Moreover, it is easy to see that  $M_{(H)}/K$  and  $M_H/N(H)$  are homeomorphic, and locally both spaces are modeled on  $S_H$ . Thus they are diffeomorphic.  $\square$



For completeness sake we also record the following theorem in which we return to a more general setting.

**THEOREM 1.A.6.** *Let  $M$  be a proper smooth  $G$ -manifold which need not be connected. Then the following are true.*

- (i) *The set of regular points  $M_{\text{reg}}$  is open and dense in  $M$ .*
- (ii) *Around any point in  $M$  there is an open  $G$ -invariant neighborhood which is only met by finitely many types of orbits.*
- (iii) *The set  $M_{\text{sing}}/G$  of all singular orbits does not locally disconnect the orbit space  $M/G$ .*

**PROOF.** See Michor [27, Section 6] or Palais and Terng [35].  $\square$

### 1.B. Whitney stratified spaces

In this subsection we introduce the Whitney conditions which will be necessary in the definition of Whitney stratified spaces – see Definition 1.C.5. We follow the approach of Mather [26].

**DEFINITION 1.B.1.** Let  $M$  be a manifold and  $X, Y$  sub-manifolds such that  $X \cap Y = \emptyset$ . The pair  $(X, Y)$  is said to satisfy **CONDITION (a) AT A POINT  $y \in Y$**  if the following holds. Consider an arbitrary sequence of points  $(x_i)_i$  in  $X$  tending to  $y$  such that  $T_{x_i}X$  converges to some  $r$ -plane  $\tau \subseteq T_yM$  in the Grassmanian bundle of  $r$ -planes in  $TM$ . Then it is true that  $T_yY \subseteq \tau$ . The pair  $(X, Y)$  is said to satisfy **condition (a)** if it does so at every point  $y \in Y$ .

**EXAMPLE 1.B.2.** Let  $M = \mathbb{C}^3 = \{(x, y, z)\}$ , and consider the complex analytic sub-manifolds  $X := \{(x, y, z) : zx^2 - y^2 = 0\} \setminus \{(0, 0, z)\}$  and  $Y := \{(0, 0, z)\}$ . The pair  $(X, Y)$  satisfies **condition (a)** at all points of  $Y$  except at the origin:

However,  $Y$  can be further decomposed: consider  $Y_0 = Y \setminus \{0\}$  and  $Z = \{0\}$ . Now the pairs  $(X, Y_0)$ ,  $(X, Z)$ , and  $(Y_0, Z)$  obviously do satisfy **condition (a)**.  $\square$

**DEFINITION 1.B.3 (Whitney condition (b) in  $\mathbb{R}^n$ ).** Let  $X, Y$  be disjoint sub-manifolds of  $\mathbb{R}^n$  with  $\dim X = r$ . The pair  $(X, Y)$  is said to satisfy **CONDITION (b) AT  $y \in Y$**  if the following is true. Consider sequences  $(x_i)_i, (y_i)_i$  in  $X, Y$ , respectively, such that  $x_i \rightarrow y, y_i \rightarrow y$ . Assume that  $T_{x_i}X$  converges to some  $r$ -plane  $\tau \subseteq T_y\mathbb{R}^n = \mathbb{R}^n$ , and that the lines spanned by the vectors  $y_i - x_i$  converge – in  $\mathbb{R}P^{n-1}$  – to some line  $l \subseteq \mathbb{R}^n = T_y\mathbb{R}^n$ . Then  $l \subseteq \tau$ . The pair  $(X, Y)$  satisfies **CONDITION (b)** if it does so at every  $y \in Y$ .

Obviously **condition (b)** behaves well under diffeomorphisms in the following sense: for  $i = 1, 2$  consider pairs  $(X_i, Y_i)$  in  $\mathbb{R}^n$ , points  $y_i \in Y_i$ , open neighborhoods  $U_i \subseteq \mathbb{R}^n$  of  $y_i$ , and a diffeomorphism  $\phi : U_1 \rightarrow U_2$  sending  $y_1$  to  $y_2$  and satisfying  $\phi(U_1 \cap X_1) = U_2 \cap X_2$  as well as

$\phi(U_1 \cap Y_1) = U_2 \cap Y_2$ . Thus it makes sense to formulate this condition for manifolds.

**DEFINITION 1.B.4** (Whitney condition (b)). Let  $M$  be a manifold and  $X, Y$  disjoint sub-manifolds. Now  $(X, Y)$  is said to satisfy **CONDITION (b)** if the following holds for all  $y \in Y$ . Let  $(U, \phi)$  be a chart around  $y$ . Then the pair  $(\phi(X \cap U), \phi(Y \cap U))$  satisfies condition (b) at  $\phi(y)$ .

By the above this definition is independent of the chosen chart in the formulation.

**EXAMPLE 1.B.5.** Consider  $M = \mathbb{C}^3 = \{(x, y, z)\}$  with  $Y$  the  $z$ -axis, and  $X = \{(x, y, z) : y^2 + x^3 - z^2x^2 = 0\} \setminus Y$ . Then the pair  $(X, Y)$  satisfies condition (a). It satisfies condition (b) at all points in  $Y$  except at  $y = 0$ .  $\square$

**PROPOSITION 1.B.6.** *Let  $M$  be a manifold and  $(X, Y)$  a pair of disjoint sub-manifolds of  $M$ .*

- (i) *If  $(X, Y)$  satisfies condition (b) at  $y \in Y$  then it also satisfies condition (a) at  $y$ .*
- (ii) *If  $(X, Y)$  satisfies condition (b) at  $y \in Y \cap \overline{X}$  then  $\dim X > \dim Y$ .*

Notice that the assumption  $\overline{X} \cap Y \neq \emptyset$  is necessary for the second statement in the proposition.

**PROOF.** Since both assertions are of local nature it suffices to consider the case  $M = \mathbb{R}^n$ .

First assertion: Let  $(x_i)_i$  be a sequence in  $X$  such that  $x_i \rightarrow y \in Y$ . Suppose  $T_{x_i}X \rightarrow \tau \subseteq T_y\mathbb{R}^n = \mathbb{R}^n$ . By contradiction we assume that  $T_yY \subseteq \mathbb{R}^n$  is not contained in  $\tau$ . Thus there is a line  $l \subseteq T_yY$  which intersects  $\tau$  at the origin only. Now we choose a sequence  $(y_i)_i$  in  $Y$  so that the difference  $y_i - x_i$  spans a line converging to the line  $l$  which lies in  $T_yY$ . This, however, contradicts condition (b).

Second assertion: Let  $(x_i)_i$  be a sequence in  $X$  such that  $x_i \rightarrow y \in Y$ . By compactness of the Grassmanian we can (passing to a subsequence if necessary) assume that  $T_{x_i}X$  converges to some plane  $\tau$ . From the above we know that (b) implies (a) and hence  $T_yY \subseteq \tau$ . If  $x_i$  is close enough to  $Y$ , i.e., for  $i$  large enough, we can find  $y_i \in Y$  minimizing the distance to  $x_i$  and so that  $y_i \rightarrow y$ . It follows that  $y_i - x_i$  is orthogonal to  $T_yY$ . Let  $l_i$  denote the line spanned by  $y_i - x_i$ . Passing to a subsequence if necessary, the  $l_i$  converge to a line  $l$  in  $\mathbb{R}P^{n-1}$  and  $l$  is still orthogonal to  $T_yY$ . By condition (b) we have  $l \subseteq \tau$ , and clearly  $\dim X = \dim \tau \geq \dim Y + \dim l > \dim Y$ .  $\square$

### 1.C. Singular spaces and smooth structures

Let  $X$  be a para-compact and second countable topological Hausdorff space, and let  $(I, \leq)$  be a partially ordered set.

DEFINITION 1.C.1 (Decomposed space). An  $I$ -DECOMPOSITION of  $X$  is a locally finite partition of  $X$  into smooth manifolds  $S_i$ ,  $i \in I$  which are disjoint (but may consist of finitely many connected components with differing dimension), and satisfy:

- (i) Each  $S_i$  is locally closed in  $X$ ;
- (ii)  $X = \bigcup_{i \in I} S_i$ ;
- (iii)  $S_j \cap \overline{S_i} \neq \emptyset \iff S_j \subseteq \overline{S_i} \iff j \leq i$ .

The third condition is called **CONDITION OF THE FRONTIER**. The manifolds  $S_i$  are called **STRATA** or **PIECES**. In the case that  $j < i$  one often writes  $S_j < S_i$  and calls  $S_j$  **INCIDENT** to  $S_i$  or says  $S_j$  is a **BOUNDARY PIECE** of  $S_i$ .

We define the dimension of a manifold consisting of finitely many connected components to be the maximum of the dimensions of the manifold's components.

The **DIMENSION** of the decomposed space  $X$  is defined as

$$\dim X := \sup_{i \in I} \dim S_i$$

and we will only be concerned with spaces where this supremum is attained.

The **DEPTH OF THE STRATUM**  $S_i$  of the decomposed space  $X$  is defined as

$$\text{depth } S_i := \sup \{ l \in \mathbb{N} : \text{there are strata } S_{i_0} = S_i, S_{i_1}, \dots, S_{i_l} \\ \text{such that } S_{i_0} < \dots < S_{i_l} \},$$

Notice that  $\text{depth } S_i$  is always finite; indeed, else there would be an infinite family  $(S_j)_{j \in J}$  with  $S_j > S_i$  thus making any neighborhood of any point in  $S_i$  meet all of the  $S_j$  which contradicts local finiteness of the decomposition. The **DEPTH** of  $X$  is

$$\text{depth } X := \sup \{ \text{depth } S_i : i \in I \}.$$

Thus, if  $X$  consists of one just stratum then  $\text{depth } X = 0$ . From the frontier condition we have that  $\text{depth } S_i \leq \dim X - \dim S_i$ , and also  $\text{depth } X \leq \dim X$ .

A simple example for a decomposed space is a manifold with boundary with big stratum the interior and small stratum the boundary. Also manifolds with corners are decomposed spaces in the obvious way. Likewise the cone  $CM := (M \times [0, \infty)) / (M \times \{0\})$  over a manifold  $M$  is a decomposed space, the partition being that into cusp and cylinder  $M \times (0, \infty)$ .

The following definition of singular charts and smooth structures on singular spaces is due to Pflaum [40, Section 2].

DEFINITION 1.C.2 (Singular charts). Let  $X = \bigcup_{i \in I} S_i$  be a decomposed space. A SINGULAR CHART  $(U, \psi)$  with patch  $U$  an open subset of  $X$  is to satisfy the following.

- (i)  $\psi(U)$  is locally closed in  $\mathbb{R}^n$ ;
- (ii)  $\psi : U \rightarrow \psi(U)$  is a homeomorphism;
- (iii) For every stratum  $S_i$  that meets  $U$  the restriction  $\psi|_{S_i \cap U} : S_i \cap U \rightarrow \psi(S_i \cap U)$  is a diffeomorphism onto a smooth submanifold of  $\mathbb{R}^n$ .

Two singular charts  $\psi : U \rightarrow \mathbb{R}^n$  and  $\phi : V \rightarrow \mathbb{R}^m$  are called COMPATIBLE AT  $x \in U \cap V$  if there is an open neighborhood  $W$  of  $x$  in  $U \cap V$ , a number  $N \geq \max\{n, m\}$ , and a diffeomorphism  $f : W_1 \rightarrow W_2$  between open subsets of  $\mathbb{R}^N$  such that:

$$\begin{array}{ccccc}
 & & W & & \\
 & & \downarrow \psi & \searrow \phi & \\
 & & \psi(W) & \xrightarrow{f|_{\psi(W)}} & \phi(W) \\
 & & \downarrow & & \downarrow \\
 \mathbb{R}^N & \longleftarrow & W_1 & \xrightarrow{f} & W_2 \hookrightarrow \mathbb{R}^N
 \end{array}$$

It follows that  $f|_{\psi(W)} : \psi(W) \rightarrow \phi(W)$  is a homeomorphism. Further, for all strata  $S$  that meet  $W$  the restriction  $f|_{\psi(W \cap S)} : \psi(W \cap S) \rightarrow \phi(W \cap S)$  is a diffeomorphism of sub-manifolds of  $\mathbb{R}^N$ . The charts  $(U, \psi)$  and  $(V, \phi)$  are called COMPATIBLE if they are so at every point of the intersection  $U \cap V$ . It is straightforward to check that compatibility of charts defines an equivalence relation.

A family of compatible singular charts on  $X$  such that the union of patches covers all of  $X$  is called a SINGULAR ATLAS. Two singular atlases are said to be compatible if all charts of the first are compatible with all charts of the second. Again it is clear that compatibility of atlases forms an equivalence relation.

Let  $\mathfrak{A}$  be a singular atlas on  $X$ . Then we can consider the family of all singular charts that belong to some atlas compatible with  $\mathfrak{A}$  to obtain a maximal atlas  $\mathfrak{A}_{\max}$ .

DEFINITION 1.C.3 (Smooth structure). Let  $X = \bigcup_{i \in I} S_i$  be a decomposed space. A maximal atlas  $\mathfrak{A}$  on  $X$  is called a SMOOTH STRUCTURE on the singular space  $X$ . A continuous function  $f : X \rightarrow \mathbb{R}$  is said to be SMOOTH if the following holds. For all charts  $\psi : U \rightarrow \mathbb{R}^n$  of the atlas  $\mathfrak{A}$  there is a smooth function  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  such that  $f|_U = F \circ \psi$ . The set of all smooth functions on  $X$  is denoted by  $C^\infty(X)$ .

A continuous map  $f : X \rightarrow Y$  between decomposed spaces with smooth structures is called SMOOTH if  $f^*C^\infty(Y) \subseteq C^\infty(X)$ . An ISOMORPHISM  $F : X \rightarrow Y$  between decomposed spaces is a homeomorphism that is

smooth in both directions and maps strata of  $X$  diffeomorphically onto strata of  $Y$ .

The smooth structure thus defined on decomposed spaces is in no way intrinsic but is a structure that is additionally defined to do analysis on decomposed spaces. Also a smooth map  $f : X \rightarrow Y$  between decomposed spaces need not at all be strata preserving.

**DEFINITION 1.C.4 (Cone space).** A decomposed space  $X = \bigcup_{i \in I} S_i$  is called a **CONE SPACE** if the following is true. Let  $x_0 \in X$  arbitrary and  $S$  the stratum passing through  $x_0$ . Then there is an open neighborhood  $U$  of  $x_0$  in  $X$ , there is a decomposed space  $L$  with global chart  $\psi : L \rightarrow S^{l-1} \subseteq \mathbb{R}^l$ , and furthermore there is an isomorphism of decomposed spaces

$$F : U \rightarrow (U \cap S) \times CL$$

such that  $F(x) = (x, c)$  for all  $x \in U \cap S$ . Here  $CL = (L \times [0, \infty)) / (L \times \{0\})$  is decomposed into the cusp  $c$  on the one hand, while the other pieces are of the form stratum of  $L$  times  $(0, \infty)$ . Thus we can take  $\Psi : CL \rightarrow \mathbb{R}^l$ ,  $[(z, t)] \mapsto t\psi(z)$  as a global chart on  $CL$  thereby defining a smooth structure on  $CL$  whence also on the product  $(U \cap S) \times CL$ . The space  $L$  is called a **LINK**, and the chart  $F$  is referred to as a **CONE CHART** or also **LINK CHART**. Of course, the link  $L$  depends on the chosen point  $x_0 \in X$ .

An example for a cone space is the quadrant  $Q := \{(x, y) \in \mathbb{R}^2 : x \geq 0 \text{ and } y \geq 0\}$ . A typical neighborhood of  $0 \in Q$  is of the form  $\{(x, y) : 0 \leq x < r \text{ and } 0 \leq y < r\}$ . The link with respect to the point 0 then is the arc  $L := \{(\cos \varphi, \sin \varphi) : 0 \leq \varphi \leq \frac{\pi}{2}\}$ . More generally manifolds with corners carry the structure of cone spaces.

**DEFINITION 1.C.5 (Stratified spaces).** Let  $X \subseteq \mathbb{R}^m$  be a subset and assume that  $X$  is a decomposed space, i.e.  $X = \bigcup_{i \in I} S_i$ , and that the strata  $S_i$  be sub-manifolds of  $\mathbb{R}^m$ . The  $I$ -decomposed space  $X$  is said to be **(WHITNEY) STRATIFIED** if all pairs  $(S_i, S_j)$  with  $i > j$  satisfy condition (b) – see Definition 1.B.4. For sake of convenience we will simply say stratified instead of Whitney stratified.

**THEOREM 1.C.6.** *Let  $X \subseteq \mathbb{R}^m$  be a subset of a Euclidean space and assume that  $X = \bigcup_{i \in I} S_i$  is decomposed. Then  $X$  is stratified if and only if  $X$  is a cone space.*

**PROOF.** It is proved in Pflaum [41] that every (Whitney) stratified space is also a cone space.

An outline of the converse direction is given in Sjamaar and Lerman [45, Section 6], and also in Goresky and MacPherson [17, Section 1.4]. This argument makes use of Mather's control theory as introduced in Mather [26] as well as Thom's First Isotopy Lemma.  $\square$

The above theorem depends crucially on the fact that the decomposed space  $X$  can be regarded as a subspace of some Euclidean space. As this assumption will always be satisfied in the present context we will take the words cone space and stratified space to be synonymous. In fact, Sjamaar and Lerman [45] take cone space to be the definition of stratified space.

**EXAMPLE 1.C.7.** As an example consider a compact Lie group  $K$  acting by isometries on a smooth Riemannian manifold  $M$ . We are concerned with the orbit projection  $\pi : M \rightarrow M/K$  and endow the orbit space with the final topology with respect to the projection map. For notation and basics on compact transformation groups see Section 1.A. Fix a point  $x_0 \in M$  with isotropy group  $K_{x_0} = H$ . The slice representation is then the action by  $H$  on  $\text{Nor}_{x_0}(K.x_0)$ . By the Tube Theorem there is a  $K$ -invariant open neighborhood  $U$  of the orbit  $K.x_0$  such that  $K \times_H V \cong U$  as smooth  $K$ -spaces where  $V$  is an  $H$ -invariant open neighborhood of 0 in  $\text{Nor}_{x_0}(K.x_0)$  – see Section 1.A.

Now let  $p = (p_1, \dots, p_k)$  be a Hilbert basis for the algebra  $\text{Poly}(V)^H$  of  $H$ -invariant polynomials on  $V$ . That is,  $p_1, \dots, p_k$  is a finite system of generators for  $\text{Poly}(V)^H$ . The Theorem of Schwarz [44, Theorem 1] now says that  $p^* : C^\infty(\mathbb{R}^k) \rightarrow C^\infty(V)^H$  is surjective. Moreover, the induced mapping  $q : V/H \rightarrow \mathbb{R}^k$  is continuous, injective, and proper. See also Michor [27].

As in Section 1.A consider the isotropy type sub-manifolds  $M_{(H)}$ . These give a  $K$ -invariant decomposition of  $M$  as  $M = \bigcup_{(H)} M_{(H)}$  where  $(H)$  runs through the isotropy lattice of the  $K$ -action on  $M$ . We thus get a decomposition of the orbit space

$$M/K = \bigcup_{(H)} M_{(H)}/K$$

where again  $(H)$  runs through the isotropy lattice of the  $K$ -action on  $M$ . By the results of Section 1.A this decomposition clearly renders  $M/K$  a decomposed space.

Now a theorem of Pflaum [40, Theorem 5.9] says that the induced mapping  $\psi : U/K \rightarrow \mathbb{R}^k$  as defined in the diagram

$$\begin{array}{ccccc}
 U & \xrightarrow{\cong} & K \times_H V & & \\
 \downarrow & & \downarrow & & \\
 U/K & \xrightarrow[\phi]{\cong} & V/H & \longleftarrow & V \\
 & \searrow \psi & \downarrow q & \swarrow p & \\
 & & \mathbb{R}^k & & 
 \end{array}$$

is a typical singular chart around the point  $K.x_0$  in the orbit space. Furthermore, the smooth functions with respect to this smooth structure are  $C^\infty(M/K) = C^\infty(M)^K$ , i.e. none other than the  $K$ -invariant smooth functions on  $M$ : indeed, by Schwarz' Theorem we have

$$\psi^*C^\infty(\mathbb{R}^k) = \phi^*q^*C^\infty(\mathbb{R}^k) = \phi^*C^\infty(V)^H = C^\infty(U)^K$$

whence  $C^\infty(U)^K = C^\infty(U/K)$ . Finally, the decomposition of  $M/K$  by orbit types turns the orbit space into a stratified space with smooth structure.  $\square$

LEMMA 1.C.8. *The quotient smooth structure  $C^\infty(M/K) = C^\infty(M)^K$  separates points in  $M/K$ .*

PROOF. Let  $\mathcal{O}_1$  and  $\mathcal{O}_2$  be arbitrary orbits of the  $K$ -action. Consider  $X = \mathcal{O}_1 \cup \mathcal{O}_2$ , and define  $f : X \rightarrow \mathbb{R}$  to be 1 on  $\mathcal{O}_1$  and 2 on  $\mathcal{O}_2$ . Then  $f$  may be extended to a smooth function on  $M$  by the Tietze-Urysohn Lemma. Thus it may be extended  $K$ -invariantly by averaging since  $K$  was assumed compact.  $\square$

### 1.D. Properties of Poisson morphisms

In this subsection we collect some properties on Poisson maps which are tacitly used throughout the text. An in-depth study of Poisson manifolds can be found in Vaisman [47] which has been used as a basic reference for the following.

Let  $(M_1, P_1)$  and  $(M_2, P_2)$  be smooth manifolds with Poisson tensors  $P_1$  and  $P_2$  respectively. A smooth map  $\varphi : (M_1, P_1) \rightarrow (M_2, P_2)$  is said to be a POISSON MORPHISM if

$$\{\varphi^*f, \varphi^*g\}_1 := \langle P_1, d\varphi^*f \wedge d\varphi^*g \rangle = \langle P_2, df \wedge dg \rangle \circ \varphi =: \{f, g\}_2 \circ \varphi$$

for all  $f, g \in C^\infty(M_2)$ .

PROPOSITION 1.D.1. *Let  $(M_1, P_1)$  and  $(M_2, P_2)$  be Poisson manifolds, and  $\varphi : M_1 \rightarrow M_2$  a smooth map. Then the following are equivalent*

- (i)  $\varphi : (M_1, P_1) \rightarrow (M_2, P_2)$  is a Poisson morphism.
- (ii) For all  $f \in C^\infty(M_2)$  the Hamiltonian vector fields  $\nabla_{\varphi^*f}^{P_1} := \check{P}_1(d\varphi^*f)$  and  $\nabla_f^{P_2}$  are  $\varphi$ -related, that is  $T\varphi.\nabla_{\varphi^*f}^{P_1} = \nabla_f^{P_2} \circ \varphi$ .
- (iii) The diagram

$$\begin{array}{ccc} \Lambda^2TM_1 & \xrightarrow{\Lambda^2T\varphi} & \Lambda^2TM_2 \\ P_1 \uparrow & & \uparrow P_2 \\ M_1 & \xrightarrow{\varphi} & M_2 \end{array}$$

is commutative.

PROOF. These assertions are well-known and straightforward to prove.  $\square$

PROPOSITION 1.D.2. *Let  $(M_1, P_1)$ ,  $(M_2, P_2)$ , and  $(M_3, P_3)$  be Poisson manifolds, and consider smooth maps  $\varphi : M_1 \rightarrow M_2$  and  $\psi : M_2 \rightarrow M_3$ . Then the following are true.*

- (i) *If  $\varphi$  and  $\psi$  are Poisson then so is the composition  $\psi \circ \varphi$ .*
- (ii) *If the first map  $\varphi$  is a surjective Poisson morphism and the composition  $\psi \circ \varphi$  is Poisson then this is also true for the second arrow  $\psi$ .*
- (iii) *If the second map  $\psi$  is an injective Poisson morphism and the composition  $\psi \circ \varphi$  is Poisson then this is also true for the first arrow  $\varphi$ .*

PROOF. This is straight forward by considering the appropriate commutative diagram.  $\square$

Consider now a Poisson manifold  $(M_1, P_1)$  and a smooth surjective map  $\varphi : M_1 \rightarrow M_2$ . If  $M_2$  carries Poisson structure  $P_2$  such that  $\varphi$  is a Poisson morphism then it is clear that  $P_2$  is uniquely determined by this property. In this case  $P_2$  is said to be COINDUCED from  $P_1$  via  $\varphi$ .

PROPOSITION 1.D.3. *Consider a Poisson manifold  $(M_1, P_1)$  and a smooth surjective map  $\varphi : M_1 \rightarrow M_2$ . There exists a coinduced Poisson structure  $P_2$  on  $M_2$  if and only if the expression  $\{f \circ \varphi, g \circ \varphi\}_1$  is constant along the fibers of  $\varphi$  for all  $f, g \in C^\infty(M_2)$ .*

PROOF. If a coinduced structure  $P_2$  exists then  $\{f \circ \varphi, g \circ \varphi\}_1 = \{f, g\}_2 \circ \varphi$  is clearly constant along the fibers of  $\varphi$ . Conversely, if  $\{f \circ \varphi, g \circ \varphi\}_1$  is constant along the fibers of  $\varphi$  we can use it to define a Poisson bracket on  $M_2$  in the obvious way.  $\square$

COROLLARY 1.D.4. *Consider a Poisson manifold  $(M_1, P_1)$  and a surjective submersion  $\varphi : M_1 \rightarrow M_2$  with connected fibers. Assume that  $\ker T\varphi$  is spanned by local Hamiltonian vector fields, that is*

$$\ker T_x\varphi = \text{span}\{\nabla_{f_1}^{P_1}(x), \dots, \nabla_{f_r}^{P_1}(x)\}$$

where  $f_1, \dots, f_r$  are smooth local functions on  $M_1$  defined around  $x$ . Then there exists a coinduced structure  $P_2$  on  $M_2$  such that  $\varphi$  is a Poisson morphism.

PROOF. By the proposition above we need to show that the expression  $\{f \circ \varphi, g \circ \varphi\}_1$  is constant along the fibers of  $\varphi$  for all  $f, g \in C^\infty(M_2)$ . As the fibers of  $\varphi$  are connected and their tangent spaces are spanned by local Hamiltonian vector fields it suffices to show that  $\nabla_h^{P_1}\{f \circ \varphi, g \circ \varphi\}_1 = 0$ . Indeed,

$$\begin{aligned} \nabla_h^{P_1}\{f \circ \varphi, g \circ \varphi\}_1 &= \{h, \{\varphi^*f, \varphi^*g\}_1\}_1 \\ &= \{\{h, \varphi^*f\}_1, \varphi^*g\}_1 + \{\varphi^*f, \{h, \varphi^*g\}_1\}_1 \\ &= \{\nabla_h^{P_1}\varphi^*f, \varphi^*g\}_1 + \{\varphi^*f, \nabla_h^{P_1}\varphi^*g\}_1 \end{aligned}$$



$$= 0$$

by the Jacobi identity, and since  $\nabla_h^{P_1}(f \circ \varphi) = 0$  and  $\nabla_h^{P_1}(g \circ \varphi) = 0$ .  $\square$

### 1.E. The Witt-Artin Decomposition

Let  $K$  be a compact Lie group acting from the left on a symplectic manifold  $(M, \omega)$  by symplectomorphisms. The Witt-Artin Theorem is a tool that gives a decomposition of the tangent space to a point  $x \in M$  that will provide us with a subspace  $V \subseteq T_x M$  that we can interpret as a symplectic normal space to  $K.x$  at  $x$ . We then use this symplectic normal space in Section 1.F to obtain a symplectic analog of the Riemannian slice theorem from Section 1.A.

**Lagrangian subspaces.** Before we come to Theorem 1.E.2 we need some preparatory material on Lagrangian subspaces of symplectic vector spaces.

REMARK 1. Let  $(E, \omega)$  be a finite dimensional symplectic vector space. A complex structure  $J$  on  $E$  is a linear transformation  $J : E \rightarrow E$  such that  $J^2 = -\text{id}_E$ . If a complex structure on  $E$  satisfies

$$\begin{aligned} \omega(x, Jx) &> 0 \text{ for all } x \neq 0, \text{ and} \\ \omega(x, y) &= \omega(Jx, Jy) \text{ for all } x, y \iff J^*\omega = \omega \end{aligned}$$

then it is said to be  $\omega$ -compatible. A pair  $(\omega, J)$  is compatible if and only if

$$g_c(x, y) = \omega(x, Jy)$$

defines a scalar product on  $E$ : indeed, if  $(\omega, J)$  are compatible then  $g_c$  is positive by assumption, and

$$g_c(x, y) = \omega(x, Jy) = \omega(Jx, JJy) = \omega(y, Jx) = g_c(y, x).$$

Conversely, if  $g_c$  defines an inner product then

$$\omega(x, y) = g_c(x, -Jy) = -g_c(Jy, x) = \omega(Jx, Jy),$$

and  $\omega(x, Jx) = g_c(x, y) > 0$  for all  $x \neq 0$ .

Moreover, on a symplectic vector space with inner product  $(E, g, \omega)$  there also is an automorphism  $J := \check{g}^{-1} \circ \check{\omega}$ ,  $g(Jx, y) = \omega(x, y)$ , and the following are equivalent:  $J^*\omega = \omega \iff J^2 = -\text{id}_E \iff J^*g = g$ . Indeed,

$$\begin{aligned} \omega(Jx, Jy) = \omega(x, y) &\iff g(J^2x, Jy) = \omega(Jx, Jy) \\ &= \omega(y, -x) \\ &= g(Jy, -x) \\ &= g(-x, Jy) \\ &\iff J^2 = -\text{id} \\ &\iff J^{-1} = -J \end{aligned}$$

$$\begin{aligned}
\iff g(Jx, Jy) &= \omega(x, Jy) \\
&= \omega(-Jy, x) \\
&= g(y, x).
\end{aligned}$$

If one of these equivalent conditions is fulfilled then it follows that  $J$  is an  $\omega$ -compatible complex structure because  $\omega(x, Jx) = g(x, x) > 0$  for all  $x \neq 0$ , and  $\omega(x, y) = \omega(Jx, Jy)$  for all  $x, y$ .  $\square$

REMARK 2. As a canonical example consider  $\mathbb{R}^{2n}$  with its standard dual basis  $(e^1, \dots, e^{2n})$ , then the canonical structures

$$J_0 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \quad \& \quad g_0 = \sum_{i=1}^{2n} e^i \otimes e^i \quad \& \quad \omega_0 = \sum_{i=1}^n e^i \wedge e^{i+n}$$

are compatible:

$$\begin{aligned}
\omega_0\left(\begin{pmatrix} x^1 \\ x^2 \end{pmatrix}, J_0\begin{pmatrix} y^1 \\ y^2 \end{pmatrix}\right) &= (e^1 \wedge e^2)\left(\begin{pmatrix} x^1 \\ x^2 \end{pmatrix}, \begin{pmatrix} -y^2 \\ y^1 \end{pmatrix}\right) \\
&= x^1 y^1 + y^2 x^2 = g_0\left(\begin{pmatrix} x^1 \\ x^2 \end{pmatrix}, \begin{pmatrix} y^1 \\ y^2 \end{pmatrix}\right),
\end{aligned}$$

and the general case follows by using more indices.  $\square$

REMARK 3. Let  $W$  be a finite dimensional vector space with dual  $W^*$ . Then  $(W \times W^*, \bar{\omega})$  is a symplectic vector space with symplectic form  $\bar{\omega}((x, x^*), (y, y^*)) := \langle y^*, x \rangle - \langle x^*, y \rangle$ , where  $\langle -, - \rangle$  denotes the duality pairing. If  $(x_1, \dots, x_n)$  is a basis of  $W = W^{**}$ ,  $(x_1^*, \dots, x_n^*)$  its dual basis then, for any  $\{a, b, c, d\} \subseteq \{1, \dots, n\}$ ,

$$\bar{\omega}((x_a, x_b^*), (x_c, x_d^*)) = \langle x_d^*, x_a \rangle - \langle x_b^*, x_c \rangle = \left(\sum_{k=1}^n l_k^* \wedge l_k\right)((x_a, x_b^*), (x_c, x_d^*))$$

implies  $\bar{\omega} = \sum_{k=1}^n l_k^* \wedge l_k$ , where  $l_k := 0_W \oplus x_k$ , and  $l_k^* := x_k^* \oplus 0_{W^*}$ . In particular the transversal subspaces  $W \times \{0\}$  and  $\{0\} \times W^*$  are Lagrangian.  $\square$

REMARK 4. Have  $(E, \omega, J, g)$  carry compatible structures as above, and define the  $\omega$ -orthogonal of a linear subspace  $L \subseteq E$  to be

$$L^\omega := \{x \in E : \omega(L, x) = \tilde{\omega}(L)(x) = \langle \tilde{\omega}(L), x \rangle = \{0\}\}.$$

The subspace  $L$  is called LAGRANGIAN if  $L = L^\omega$ , and since  $\dim L + \dim L^\omega = \dim E$  it follows that  $\dim L = \frac{1}{2} \dim E$ . Observe furthermore that

$$\begin{aligned}
JL &= J(L^\omega) = \{Jx : \omega(x, L) = \omega(Jx, JL) = \{0\}\} = (JL)^\omega, \text{ and} \\
JL &= L^\perp := \{x \in E : g(L, x) = \{0\}\}
\end{aligned}$$

Indeed,  $x \in L$  if and only if  $g(Jx, y) = \omega(x, y) = 0$  for all  $y \in L$  which happens if and only if  $Jx \in L^\perp$ . Thus  $L$  and  $JL$  are transversal Lagrangian subspaces.

Moreover, with the two-form from above the mapping

$$\begin{aligned} (E, \omega) &= (L \oplus JL, \omega) \xrightarrow{\psi} (L \oplus L^*, \bar{\omega}) \\ x \oplus Jy &\longmapsto x \oplus \tilde{\omega}(-Jy) \end{aligned}$$

becomes a symplectomorphism:

$$\begin{aligned} \omega(x_1 \oplus Jy_1, x_2 \oplus Jy_2) &= \omega(x_1, Jy_2) + \omega(Jy_1, x_2) \\ (\psi^*\bar{\omega})(x_1 \oplus Jy_1, x_2 \oplus Jy_2) &= \langle \tilde{\omega}(-Jy_2), x_1 \rangle - \langle \tilde{\omega}(-Jy_1), x_2 \rangle \\ &= \omega(x_1, Jy_2) + \omega(Jy_1, x_2) \end{aligned}$$

for all  $x_i, y_i \in L$ .  $\square$

**REMARK 5.** Assume now that  $(E, \omega, g)$  is a symplectic vector space with inner product that is acted upon by a compact Lie group  $H$  such that  $\omega$  and  $g$  are invariant under the action. Then  $J := \check{g}^{-1} \circ \tilde{\omega} : E \rightarrow E$  is  $H$ -equivariant since  $g(J(h.x), y) = \omega(h.x, y) = \omega(x, h^{-1}.y) = g(J(x), h^{-1}.y) = g(h.J(x), y)$ . Assume further that  $L$  is a  $H$ -invariant Lagrangian subspace of  $E$ . Then by the latter remark  $JL$  is a Lagrangian complement to  $L$ , and by equivariance of  $J$  it is clear that  $JL$  is  $H$ -invariant as well.  $\square$

**LEMMA 1.E.1.** *Let  $(E, \omega)$  be a symplectic vector space and  $E_1$  and  $E_2$  linear subspaces of  $E$ . Then*

$$(E_1^\omega)^\omega = E_1 \quad \text{and} \quad (E_1 \cap E_2)^\omega = E_1^\omega + E_2^\omega$$

where  $E_1^\omega$  is the  $\omega$ -orthogonal to  $E_1$  in  $E$ .

**PROOF.** The first relation is clear. Also it is obvious that we have  $(E_1 \cap E_2)^\omega \supseteq E_1^\omega + E_2^\omega$ . Thus it only remains to show the converse inclusion  $\subseteq$ . This, since  $E_1 \subseteq E_2$  if and only if  $E_1^\omega \supseteq E_2^\omega$ , is equivalent to  $E_1 \cap E_2 \supseteq (E_1^\omega + E_2^\omega)^\omega$  which again is straightforward.  $\square$

**The Witt-Artin decomposition.** The following theorem can also be found in Ortega and Ratiu [33, Theorem 7.1.1]. However, we simplify matters by considering only compact groups  $K$ .

**THEOREM 1.E.2 (Witt-Artin decomposition).** *Let  $K$  be a compact Lie group acting on the symplectic manifold  $(M, \omega)$  from the left by symplectomorphisms, and denote this action by  $l : K \times M \rightarrow M$ . Then the tangent space at any point  $x \in M$  may be decomposed as follows.*

$$T_x M = \mathfrak{l}.x \oplus \mathfrak{q}.x \oplus V \oplus W$$

where definitions and properties of the objects involved are as follows.

- (i)  $\mathfrak{l} := \{X \in \mathfrak{k} : \zeta_X(x) \in (\mathfrak{k}.x)^\omega\} \subseteq \mathfrak{k}$  is a Lie subalgebra.
- (ii) By the compactness of  $K$  we can fix a  $K$  invariant inner product on  $\mathfrak{k}$  and decompose orthogonally

$$\mathfrak{l} = \mathfrak{k}_x \oplus \mathfrak{m} \quad \text{and} \quad \mathfrak{k} = \mathfrak{k}_x \oplus \mathfrak{m} \oplus \mathfrak{q}.$$

The isomorphism  $\mathfrak{k} \cong \mathfrak{k}^*$  given by the inner product induces dual splittings

$$\mathfrak{l}^* = \mathfrak{k}_x^* \oplus \mathfrak{m}^* \quad \text{and} \quad \mathfrak{k}^* = \mathfrak{k}_x^* \oplus \mathfrak{m}^* \oplus \mathfrak{q}^*.$$

- (iii)  $\mathfrak{q}.x \subseteq T_x M$  is a symplectic subspace.
- (iv) By the compactness of  $K$  we can choose a  $K$ -invariant inner product on  $T_x M$ . Now, we define  $V$  as

$$V := (\mathfrak{k}.x \cap (\mathfrak{k}.x)^\omega)^\perp \cap (\mathfrak{k}.x)^\omega \cong (\mathfrak{k}.x)^\omega / (\mathfrak{k}.x \cap (\mathfrak{k}.x)^\omega)$$

where the orthogonal  $\perp$  is taken with respect to the chosen inner product. Then  $V \subseteq T_x M$  is  $K_x$ -invariant symplectic subspace. Moreover,  $V \cap \mathfrak{q}.x = \{0\}$ .

- (v)  $\mathfrak{l}.x \subseteq (V \oplus \mathfrak{q}.x)^\omega$  is a Lagrangian subspace and is  $K_x$ -invariant.
- (vi)  $W$  is a Lagrangian  $K_x$ -invariant complement to  $\mathfrak{l}.x$  in  $(V \oplus \mathfrak{q}.x)^\omega$ .
- (vii) The map  $f : W \rightarrow \mathfrak{m}^*$  defined by

$$\langle f(w), Y \rangle = \omega_x(\zeta_Y(x), w)$$

for  $Y \in \mathfrak{m}$  is a  $K_x$ -equivariant isomorphism.

PROOF. (i). Clearly,  $\mathfrak{l} \subseteq \mathfrak{k}$  is a linear subspace. Let  $X, Y \in \mathfrak{l}$  and  $Z \in \mathfrak{k}$ . Then

$$\begin{aligned} \omega(\zeta_{[X,Y]}(x), \zeta_Z(x)) &= (i_{\zeta_Z} i_{[\zeta_X, \zeta_Y]} \omega)(x) \\ &= (i_{\zeta_Z} \mathcal{L}_{\zeta_X} i_{\zeta_Y} - i_{\zeta_Z} i_{\zeta_Y} \mathcal{L}_{\zeta_X})_x \omega \\ &= ((\mathcal{L}_{\zeta_X} i_{\zeta_Z} - i_{[\zeta_X, \zeta_Z]}) i_{\zeta_Y})_x \omega - 0 \\ &= (\mathcal{L}_{\zeta_X} (\omega(\zeta_Y, \zeta_Z)))_x - 0 \\ &= (i_{\zeta_X} di_{\zeta_Z} i_{\zeta_Y})_x \omega \\ &= (i_{\zeta_X} (\mathcal{L}_{\zeta_Z} - i_{\zeta_Z} d) i_{\zeta_Y})_x \omega \\ &= (i_{\zeta_X} \mathcal{L}_{\zeta_Z} i_{\zeta_Y})_x \omega - 0 \\ &= (i_{\zeta_X} i_{[\zeta_Z, \zeta_Y]})_x \omega + (i_{\zeta_X} i_{\zeta_Y} \mathcal{L}_{\zeta_Z})_x \omega \\ &= 0 + 0 \end{aligned}$$

where we heavily used the Cartan formulas  $\mathcal{L}_\xi = i_\xi d + di_\xi$  and  $\mathcal{L}_\xi i_\eta - i_\eta \mathcal{L}_\xi = i_{[\xi, \eta]}$  for vector fields  $\xi, \eta \in \mathfrak{X}(M)$ , as well as  $di_{\zeta_Z} \omega = \mathcal{L}_{\zeta_Z} \omega = 0$  for arbitrary  $Z \in \mathfrak{k}$ . Thus  $\mathfrak{l} \subseteq \mathfrak{k}$  is a Lie subalgebra.

- (ii). This is clear by compactness of  $K$ .
- (iii). To prove that  $\mathfrak{q}.x \subseteq T_x M$  is a symplectic subspace we must show that  $\mathfrak{q}.x \cap (\mathfrak{q}.x)^\omega = \{0\}$ . Let  $X \in \mathfrak{q}$  and assume that  $\omega_x(\zeta_X(x), \zeta_Y(x)) = 0$  for all  $Y \in \mathfrak{q}$ . Then choosing  $Z \in \mathfrak{l}$  arbitrary we see that

$$\begin{aligned} \omega_x(\zeta_X(x), \zeta_{Z+Y}(x)) &= \omega_x(\zeta_X(x), \zeta_Z(x)) + \omega_x(\zeta_X(x), \zeta_Y(x)) \\ &= 0 + 0 \end{aligned}$$

whence  $X \in \mathfrak{q} \cap \mathfrak{l} = \{0\}$ .

(iv). To prove the assertion about  $V$  we use the above lemma to see that

$$\begin{aligned}
V \cap V^\omega &= (\mathfrak{l}.x)^\perp \cap (\mathfrak{k}.x)^\omega \cap ((\mathfrak{l}.x)^\perp \cap (\mathfrak{k}.x)^\omega)^\omega \\
&= (\mathfrak{l}.x)^\perp \cap (\mathfrak{k}.x)^\omega \cap (((\mathfrak{l}.x)^\perp)^\omega + \mathfrak{k}.x) \\
&\subseteq (\mathfrak{l}.x)^\perp \cap (\mathfrak{k}.x)^\omega \cap (\mathfrak{k}.x) \\
&= (\mathfrak{l}.x)^\perp \cap (\mathfrak{l}.x) \\
&= \{0\}.
\end{aligned}$$

Clearly,  $V$  is  $K_x$ -invariant where the action is given by  $K_x \times V \rightarrow V$ ,  $(h, v) \mapsto T_x l_h \cdot v$ . Finally, assume  $\zeta_X(x) \in V$  with  $X \in \mathfrak{q}$ . Then

$$\omega_x(\zeta_X(x), \zeta_Y(x)) = 0 \text{ for all } Y \in \mathfrak{k}$$

whence it follows that  $X \in \mathfrak{l}$ . Thus, by assumption,  $X \in \mathfrak{l} \cap \mathfrak{q} = \{0\}$ .

(v). We firstly show that  $\mathfrak{l}$  is  $K_x$ -invariant. Let  $X \in \mathfrak{l}$  and  $h \in K_x$  arbitrary. Since

$$\zeta_{\text{Ad}(h).X}(x) = \zeta_{\text{Ad}(h).X}(h.x) = T_x l_h \cdot \zeta_X(x)$$

we see that

$$\begin{aligned}
\omega_x(\zeta_{\text{Ad}(h).X}(x), \zeta_{\text{Ad}(hh^{-1}).Y}(x)) &= \omega_{h.x}(T_x l_h \cdot \zeta_X(x), T_x l_h \cdot \zeta_{\text{Ad}(h^{-1}).Y}(x)) \\
&= (l_h^* \omega)_x(\zeta_X(x), \zeta_{\text{Ad}(h^{-1}).Y}(x)) \\
&= \omega_x(\zeta_X(x), \zeta_{\text{Ad}(h^{-1}).Y}(x)) \\
&= 0
\end{aligned}$$

for all  $Y \in \mathfrak{k}$ . Now we show that  $\mathfrak{l}.x$  is a Lagrangian subspace of  $(V \oplus \mathfrak{q}.x)^\omega$ , and to do so we have to prove that  $(\mathfrak{l}.x)^\omega \cap (V \oplus \mathfrak{q}.x)^\omega = \mathfrak{l}.x$ . Notice here that  $V$  and  $\mathfrak{q}.x$  are symplectic by the above. The inclusion  $\mathfrak{l}.x \subseteq (\mathfrak{l}.x)^\omega$  is clear by the definition of  $\mathfrak{l}$ . Assume now that

$$\xi \in (\mathfrak{l}.x)^\omega \cap (V \oplus \mathfrak{q}.x)^\omega \subseteq (\mathfrak{l}.x)^\omega \cap (\mathfrak{q}.x)^\omega = (\mathfrak{k}.x)^\omega.$$

But then also

$$\xi \in (\mathfrak{k}.x)^\omega \cap V^\omega \subseteq (\mathfrak{k}.x)^\omega \cap V^\perp = (\mathfrak{k}.x)^\omega \cap \mathfrak{k}.x = \mathfrak{l}.x$$

by definition of  $V$ .

(vi). Now we have an induced inner product and symplectic form on  $(V \oplus \mathfrak{q}.x)^\omega$ , and  $\mathfrak{l}.x$  is a  $K_x$ -invariant subspace of  $(V \oplus \mathfrak{q}.x)^\omega$  as was just shown. Thus we can apply Remark 5 from above to find a Lagrangian  $K_x$ -invariant complement  $W$  to  $\mathfrak{l}.x$  in  $(V \oplus \mathfrak{q}.x)^\omega$ .

(vii). Since  $W$  is a Lagrangian complement to  $\mathfrak{l}.x$  in  $(V \oplus \mathfrak{q}.x)^\omega$  it is clear that  $\dim W = \dim \mathfrak{l}.x = \dim \mathfrak{m}$ . Now, the map  $f : W \rightarrow \mathfrak{m}^*$  defined by

$$\langle f(w), Y \rangle = \omega_x(\zeta_Y(x), w)$$

for  $Y \in \mathfrak{m}$  is linear since  $\omega_x$  is bilinear. It is also injective, for assume that  $f(w) = 0$ . Then  $w \in (\mathfrak{m}.x)^\omega = (\mathfrak{l}.x)^\omega$ , and thus  $w \in (\mathfrak{l}.x)^\omega \cap W = \mathfrak{l}.x \cap W = \{0\}$ . Finally,  $f$  is  $K_x$ -equivariant since

$$\begin{aligned} \langle f(h.w), Y \rangle &= \omega_x(\zeta_Y(x), T_x l_h.w) \\ &= \omega_x(T_x l_h.\zeta_{\text{Ad}(h^{-1}), Y}(x), T_x l_h.w) \\ &= \omega_x(\zeta_{\text{Ad}(h^{-1}), Y}(x), w) \\ &= \langle f(w), \text{Ad}(h^{-1}).Y \rangle \\ &= \langle \text{Ad}^*(h).f(w), Y \rangle \end{aligned}$$

by  $H$ -invariance of  $\omega_x$ .  $\square$

### 1.F. The Hamiltonian slice theorem

Let  $K$  be a compact Lie group acting from the left on a symplectic manifold  $(M, \omega)$  by symplectomorphisms. Assume this action is Hamiltonian with equivariant momentum map  $J : M \rightarrow \mathfrak{k}^*$ . Presuming  $K$  to be compact has the advantage that coadjoint orbits in  $\mathfrak{k}^*$  are submanifolds. Ortega and Ratiu [32] prove an equally strong symplectic slice theorem under the relaxed condition of having  $K$  act properly on  $(M, \omega)$  by symplectomorphisms. We will follow mainly the exposition of Ortega and Ratiu [33]. However, we will simplify matters due to the stronger conditions on the  $K$ -action.

We continue the notation from the previous section. In particular we have the Witt-Artin decomposition

$$T_x M = \mathfrak{l}.x \oplus \mathfrak{q}.x \oplus V \oplus W$$

from Theorem 1.E.2. Moreover, we put  $H := K_x$  with Lie algebra  $\mathfrak{h} = \mathfrak{k}_x$ , and assume that  $\mathfrak{l} = \mathfrak{k}_\alpha$  with Lie group  $L = K_\alpha$  and  $\alpha = J(x)$ . Note that always  $H \subseteq L$  by equivariance of the momentum map.

**LEMMA 1.F.1.** *With these assumptions  $K \times \mathfrak{l}^*$  is a symplectic manifold with symplectic form*

$$\Omega_{(g, \lambda)}^1((X_1, \lambda_1), (X_2, \lambda_2)) = \langle \lambda_2, Y_1 \rangle - \langle \lambda_1, Y_2 \rangle + \langle \lambda, [Y_1, Y_2] \rangle + \langle \alpha, [Z_1, Z_2] \rangle$$

where  $(g, \lambda) \in K \times \mathfrak{l}^*$ ,  $X_i \in \mathfrak{k}$ ,  $\lambda_i \in \mathfrak{l}^*$ , and  $X_i = Y_i + Z_i \in \mathfrak{l} \oplus \mathfrak{q}$  is the respective decomposition into vertical and horizontal part of the projection  $K \twoheadrightarrow K/L$ .

Moreover, the form  $\Omega^1$  is  $K$ -invariant with respect to the left  $K$ -action on  $K \times \mathfrak{l}^*$  given by  $(k, g, \lambda) \mapsto (kg, \lambda)$ .

Notice that  $\langle \lambda, [Z_1, Z_2] \rangle = \Omega^+(\alpha)(\zeta_{X_1}(\alpha), \zeta_{X_2}(\alpha)) = \omega_x(\zeta_{X_1}(x), \zeta_{X_2}(x))$  where  $\Omega^+$  is the positive Kostant-Kirillov-Souriau symplectic form on the coadjoint orbit  $\text{Ad}^*(K).\alpha$ .

PROOF. Let  $(g, \lambda) \in K \times \mathfrak{l}^*$ ,  $X_i \in \mathfrak{k}$ ,  $\lambda_i \in \mathfrak{l}^*$ , and  $X_i = Y_i + Z_i \in \mathfrak{l} \oplus \mathfrak{q}$  be the respective decomposition into vertical and horizontal part of the orbit projection  $\pi : K \rightarrow K/L$ . Now consider

$$\pi \circ \text{pr}_1 : K \times \mathfrak{l}^* \longrightarrow K/L = \text{Ad}^*(K).\alpha$$

where we symplectically identify  $K/L$  with the coadjoint orbit passing through  $\alpha$ .

Further, we consider the 1-form  $\theta_1$  on  $K \times \mathfrak{l}^*$  given by  $\theta_1(g, \lambda)(X_1, \lambda_1) = \langle \lambda, Y_1 \rangle$ . Now, since  $\mathfrak{l}$  is closed under Lie brackets, we obtain

$$\Omega^1 = -d\theta_1 + (\pi \circ \text{pr}_1)^*\Omega^+.$$

Therefore,  $\Omega^1$  is closed. It is also non-degenerate since it is so on every tangent space

$$T_{(g,\lambda)}(K \times \mathfrak{l}^*) = \mathfrak{k} \times \mathfrak{l}^* = (\mathfrak{l} \oplus \mathfrak{q}) \times \mathfrak{l}^* \cong (\mathfrak{l} \times \mathfrak{l}^*) \oplus T_\alpha(\text{Ad}^*(K).\alpha)$$

as follows from the construction. As tangent and cotangent lifted action of left multiplication by  $K$  on itself both are trivial, the form  $\Omega^1$  obviously is  $K$ -invariant.  $\square$

A symplectic subspace  $V \subseteq T_x M$  constructed as in Theorem 1.E.2 above will be called SYMPLECTIC NORMAL SPACE of the  $K$ -action on  $M$  at  $x$ . Clearly, the isotropy subgroup  $K_x = H$  acts linearly on  $(V, \omega_x)$  by symplectomorphisms. Therefore, there is an equivariant momentum map  $J_V : V \rightarrow \mathfrak{h}^*$  which is given by  $\langle J_V(v), X \rangle = \frac{1}{2}\omega_x(\zeta_X(v), v)$  for  $X \in \mathfrak{h}$ .

Let  $\Omega = \Omega^1 + \omega_x$  denote the product symplectic form on  $K \times \mathfrak{l}^* \times V$  where  $\Omega^1$  is the symplectic form constructed in the lemma above.

We consider the left action  $H \times K \rightarrow K$ ,  $(h, g) \mapsto gh^{-1}$ , and the corresponding cotangent lifted action  $H \times K \times \mathfrak{k}^* \rightarrow K \times \mathfrak{k}^*$ ,  $(h, g, \alpha) \mapsto (gh^{-1}, \text{Ad}^*(h^{-1}).\alpha)$ . This action is Hamiltonian with momentum map  $J_{T^*K} : K \times \mathfrak{k}^* \rightarrow \mathfrak{h}^*$ ,  $(g, \lambda) \mapsto -\lambda|_{\mathfrak{h}}$ . See Section 2.A for a proof of these formulas.

Take these considerations as a motivation for the following. With notation as in the above lemma consider the  $H$ -action on  $K \times \mathfrak{l}^*$  given by  $h.(k, \lambda) = (kh^{-1}, \text{Ad}^*(h).\lambda)$ . This action preserves the symplectic form  $\Omega^1$ :

$$\begin{aligned} \Omega_{h.(g,\lambda)}^1((\text{Ad}(h).X_1, \text{Ad}^*(h).\lambda_1), (\text{Ad}(h).X_2, \text{Ad}^*(h).\lambda_2)) \\ &= \langle \lambda_2, Y_1 \rangle - \langle \lambda_1, Y_2 \rangle + \langle \lambda, [Y_1, Y_2] \rangle + \langle \alpha, [Z_1, Z_2] \rangle \\ &= \Omega_{(g,\lambda)}^1((X_1, \lambda_1), (X_2, \lambda_2)) \end{aligned}$$

since  $\text{Ad}^*(h).\alpha = \alpha$  as  $h \in H \subseteq L = K_\alpha$  by equivariance of the momentum mapping. Furthermore, the action is Hamiltonian with global  $H$ -equivariant momentum map  $J_1 : K \times \mathfrak{l}^* \rightarrow \mathfrak{h}^*$ ,  $(k, \lambda) \mapsto$

$-\lambda|\mathfrak{h}$ . To see this we have to show that  $d\langle J_1, H_0\rangle(k, \lambda)(X_2, \lambda_2) = \Omega_{(k, \lambda)}^1(\zeta_{H_0}(k, \lambda), (X_2, \lambda_2))$ . Indeed,

$$d\langle J_1, H_0\rangle(k, \lambda)(X_2, \lambda_2) = \frac{\partial}{\partial t}|_0 \langle J_1, H_0\rangle(k \exp(tX_2), \lambda + t\lambda_2) = -\langle \lambda_2, H_0\rangle$$

and also

$$\begin{aligned} \Omega_{(k, \lambda)}^1(\zeta_{H_0}(k, \lambda), (X_2, \lambda_2)) &= \Omega_{(k, \lambda)}^1((k \exp(-tH_0), \text{Ad}^*(\exp(tH_0)).\lambda), (X_2, \lambda_2)) \\ &= \langle \lambda_2, -H_0\rangle - \langle \text{ad}^*(H_0).\lambda, Y_2\rangle + \langle \lambda, [-H_0, Y_2]\rangle + \langle \alpha, [0, Z_2]\rangle \\ &= -\langle \lambda_2, H_0\rangle \end{aligned}$$

as claimed.

Now we can consider the Hamiltonian diagonal action by  $H$  on  $K \times \mathfrak{l}^* \times V$  with equivariant momentum map  $J_1 + J_V : K \times \mathfrak{l}^* \times V \rightarrow \mathfrak{h}^*$ ,  $(k, \lambda, v) \mapsto -\lambda|\mathfrak{h} + J_V(v)$ .

**PROPOSITION 1.F.2** (Constructing the symplectic tube). *Let  $K$  be a compact Lie group which acts on  $(M, \omega)$  by symplectomorphisms such that there is an equivariant momentum map  $J : M \rightarrow \mathfrak{k}^*$ , and let  $V$  be a symplectic normal space at  $x \in M$ . Define further  $\alpha = J(x)$ ,  $H = K_x$ , and  $L = K_\alpha$  with Lie algebras  $\mathfrak{h}$  and  $\mathfrak{l}$  respectively. Then the reduced space*

$$Y := (K \times \mathfrak{l}^* \times V) //_0 H := (J_1 + J_V)^{-1}(0) / H = K \times_H (\mathfrak{m}^* \times V)$$

where  $\mathfrak{m} = \mathfrak{l} \cap \mathfrak{h}^\perp$  is as in Theorem 1.E.2, is a smooth symplectic manifold with induced symplectic structure  $\omega_Y$ .

Let  $\pi : K \times \mathfrak{m}^* \times V \rightarrow Y$  denote the projection, and consider  $(k, \mu, v) \in K \times \mathfrak{m}^* \times V$ , and  $(X_i, \mu_i, v_i) \in \mathfrak{k} \times \mu^* \times V$  where  $i = 1, 2$ . The reduced symplectic form  $\omega_Y$  is uniquely characterized by the formula

$$\begin{aligned} \omega_Y(\pi(k, \mu, v))(T\pi.(X_1, \mu_1, v_1), T\pi.(X_2, \mu_2, v_2)) &= \langle T_v J_V.v_2 + \mu_2, Y_1\rangle - \langle T_v J_V.v_1 + \mu_1, Y_2\rangle \\ &\quad + \langle J_V(v) + \mu, [Y_1, Y_2]\rangle + \langle \alpha, [Z_1, Z_2]\rangle + \omega_x(v_1, v_2) \end{aligned}$$

where  $X_i = Y_i + Z_i$  is the decomposition into vertical and horizontal part with respect to the orbit projection  $K \twoheadrightarrow K/L$ .

**PROOF.** Since  $H$  acts freely on  $K \times \mathfrak{l}^* \times V$  we can use regular symplectic reduction. First of all we note that  $Y$  clearly is non-empty. Consider a point  $(k, \lambda, v)$  in the zero fiber of the momentum map  $J_1 + J_V$ . According to the direct sum decomposition  $\mathfrak{l} = \mathfrak{h} \oplus \mathfrak{m}$  we may decompose  $\lambda$  as  $\lambda = \lambda_0 + \lambda_1$ . By assumption we then have  $\lambda_0 = J_V(v)$  while we have no restriction on the  $\mathfrak{m}$ -component. Therefore,

$$\begin{aligned} K \times \mathfrak{m}^* \times V &\cong (J_1 + J_V)^{-1}(0), \\ (k, \mu, v) &\mapsto (k, J_V(v) + \mu, v) \end{aligned}$$



which has an obvious smooth inverse and thus exhibits the zero set  $(J_1 + J_V)^{-1}(0)$  as a manifold. Moreover, this diffeomorphism is  $H$ -equivariant and thus drops to a diffeomorphism  $(K \times \mathfrak{l}^* \times V) //_0 H := (J_1 + J_V)^{-1}(0) / H \cong K \times_H (\mathfrak{m}^* \times V)$ . Finally, note that

$$\begin{aligned} \Omega^1(k, J_V(v) + \mu, v) & ((Y_1, T_v J_V \cdot v_1 + \mu_1), (Y_2, T_v J_V \cdot v_2 + \mu_2)) \\ &= \langle T_v J_V \cdot v_2 + \mu_2, Y_1 \rangle - \langle T_v J_V \cdot v_1 + \mu_1, Y_2 \rangle \\ &\quad + \langle J_V(v) + \mu, [Y_1, Y_2] \rangle + \langle \alpha, [Z_1, Z_2] \rangle \end{aligned}$$

whence the characterizing property follows.  $\square$

Note that the left  $K$ -action  $l : K \times K \times \mathfrak{l}^* \times V \rightarrow \mathfrak{k}^*$  given by  $l_k(g, \lambda, v) = (kg, \lambda, v)$  commutes with the  $H$ -action, and respects the symplectic form  $\Omega^1 + \omega_V$ .

REMARK 1. Using the compactness of the group  $K$  there is a quick way to see that the action  $l : K \times K \times \mathfrak{l}^* \times V \rightarrow K \times \mathfrak{l}^* \times V$  possesses a momentum map  $J_l : K \times \mathfrak{l}^* \times V \rightarrow \mathfrak{k}^*$ . Indeed, consider the diagram

$$\begin{array}{ccccc} C^\infty(K \times \mathfrak{l}^* \times V) & \xrightarrow{(\tilde{\Omega}^1)^{-1}} & \mathfrak{X}(K \times \mathfrak{l}^* \times V, \Omega^1) & \xrightarrow{\gamma} & H^1(K \times \mathfrak{l}^* \times V) \\ \uparrow & & \swarrow \text{---} j_l \text{---} & & \uparrow \zeta^l \\ H^0(K \times \mathfrak{l}^* \times V) & & & & \mathfrak{k} \end{array}$$

where  $j_l(X)(k, \lambda, v) = \langle J_l(k, \lambda, v), X \rangle$  and  $\gamma(\xi) = [i_\xi \Omega^1]$ . It is well known and straight forward to show that the top row of this diagram is an exact sequence of Lie algebra homomorphisms going from Poisson to Lie to trivial bracket. Since  $\mathfrak{k}$  is the Lie algebra of a compact Lie group it is reductive and we may decompose it as  $\mathfrak{k} = Z(\mathfrak{k}) \oplus [\mathfrak{k}, \mathfrak{k}]$  with respect to the fixed  $\text{Ad}(K)$ -invariant inner product on  $\mathfrak{k}$ . Here  $Z(\mathfrak{k})$  is the center of  $\mathfrak{k}$  and  $[\mathfrak{k}, \mathfrak{k}]$  is the semisimple subalgebra consisting of all linear combinations of commutators of elements in  $\mathfrak{k}$ ; see Knapp [21, Corollary 4.25] for a proof of this fact.

We want to show that  $\gamma \circ \zeta^l$  vanishes. Since both  $\gamma$  and  $\zeta^l$  are linear it suffices to show this for elements  $Z \in Z(\mathfrak{k})$  and  $[X, Y] \in [\mathfrak{k}, \mathfrak{k}]$  separately. Indeed, notice that  $\zeta_Z^l(g, \lambda, v) = (\text{Ad}(g^{-1}).Z, 0, 0) = (Z, 0, 0)$  whence  $i_{\zeta_Z^l} \Omega^1 = df$  is exact where  $f : K \times \mathfrak{l}^* \times V \rightarrow \mathbb{R}$ ,  $(g, \lambda, v) \mapsto \langle \lambda, Z \rangle$ . On the other hand  $\gamma(\zeta_{[X, Y]}^l) = \gamma(-[\zeta_X^l, \zeta_Y^l]) = 0$  since  $\gamma$  is a Lie algebra homomorphism into the zero bracket.  $\square$

REMARK 2. We claim that the momentum map in question is equivariant and is given by

$$J_l : (k, \lambda, v) \mapsto \text{Ad}^*(k).(\lambda + \alpha).$$

To see this we need to show that  $d\langle J_l, X_1 \rangle = i_{\zeta_{X_1}^l} \Omega^1$ . Notice firstly that  $\langle J_l(k, \lambda, v), X_1 \rangle = (i_{\zeta_{X_1}^l} \theta^1)(k, \lambda) + \langle \text{Ad}^*(k).\alpha, X_1 \rangle$  where  $\theta^1 \in \Omega^1(K \times \mathfrak{l}^*)$

is as in the proof of Lemma 1.F.1. Now,

$$\begin{aligned}
\Omega_{(k,\lambda,v)}^1(\zeta_{X_1}^l(k, \lambda, v), (X_2, \lambda_2, v_2)) &= \\
&= -d\theta_{(k,\lambda,v)}^1(\zeta_{X_1}^l(k, \lambda, v), (X_2, \lambda_2, v_2)) \\
&\quad + \langle \alpha, [\text{Ad}(g^{-1})X_1, X_2] \rangle + \omega_x(0, v_2) \\
&= (di_{\zeta_{X_1}^l} \theta^1)(k, \lambda)(X_2, \lambda_2) \\
&\quad - (\mathcal{L}_{\zeta_{X_1}^l} \theta^1)(k, \lambda)(X_2, \lambda_2) + \langle \alpha, [\text{Ad}(g^{-1})X_1, X_2] \rangle.
\end{aligned}$$

Because the flow up to time  $t$  of the fundamental vector field is given by  $\text{Fl}_t^{\zeta_{X_1}^l}(k, \lambda, v) = (k \exp(tX_1), \lambda, v)$  and we identify the Lie algebra  $\mathfrak{k}$  with the algebra of all left invariant vector fields on  $K$  it follows that  $T_{(k,\lambda,v)}\text{Fl}_t^{\zeta_{X_1}^l} = \text{id}$  whence

$$(\mathcal{L}_{\zeta_{X_1}^l} \theta^1)(k, \lambda)(X_2, \lambda_2) = \frac{\partial}{\partial t} |_0 \langle \lambda, Y_2 \rangle = 0.$$

Since

$$\begin{aligned}
d\langle J_l, X_1 \rangle(X_2, \lambda_2, v_2) &= \langle dJ_l(k, \lambda, v).(X_2, \lambda_2, v_2), X_1 \rangle \\
&= (di_{\zeta_{X_1}^l} \theta^1)(k, \lambda).(X_2, \lambda_2) \\
&\quad + \frac{\partial}{\partial t} |_0 \langle \text{Ad}^*(k \exp(tX_2)).\alpha, X_1 \rangle
\end{aligned}$$

and  $\frac{\partial}{\partial t} |_0 \langle \text{Ad}^*(k \exp(tX_2)).\alpha, X_1 \rangle = \langle \alpha, [\text{Ad}(g^{-1})X_1, X_2] \rangle$  the desired formula  $d\langle J_l, X_1 \rangle = i_{\zeta_{X_1}^l} \Omega^1$  follows.  $\square$

REMARK 3. Now  $H$  and  $K$  both act on  $K \times \mathfrak{l}^* \times V$  in a Hamiltonian fashion with momentum map  $J_1 + J_V : K \times \mathfrak{l}^* \times V \rightarrow \mathfrak{h}^*$ ,  $(k, \lambda, v) \mapsto -\lambda | \mathfrak{h} + J_V(v)$  and  $J_l : K \times \mathfrak{l}^* \times V \rightarrow \mathfrak{k}^*$ ,  $(k, \lambda, v) \mapsto \text{Ad}^*(k).(\lambda + \alpha)$  respectively. These momentum maps have the property that they are invariant with respect to the action corresponding to the other momentum map. In particular  $(J_1 + J_V)^{-1}(0)$  is invariant under the  $K$  action, and by invariance of  $J_l$  under the  $H$  action it follows that  $J_l$  drops to a momentum map  $J_Y$  on

$$Y := (K \times \mathfrak{l}^* \times V) //_0 H := (J_1 + J_V)^{-1}(0) / H = K \times_H (\mathfrak{m}^* \times V)$$

with respect to the induced  $K$  action. Thus

$$\begin{aligned}
J_Y : Y = K \times_H (\mathfrak{m}^* \times V) &\longrightarrow \mathfrak{k}^*, \\
[(k, \mu, v)] &\longmapsto \text{Ad}^*(k).(J_V(v) + \mu + \alpha)
\end{aligned}$$

where we used the same notation as above. This form of a momentum map is also called Guillemin-Marle normal form due to its relevance coming from Theorem 1.F.3 below.  $\square$

REMARK 4. For a  $r > 0$  we define  $\mathfrak{k}_r := \{X \in \mathfrak{k} : |X| < r\}$  where we compute the absolute value with respect to the same  $K$ -invariant inner product on  $\mathfrak{k}$  as in Theorem 1.E.2. Also we put  $\mathfrak{m}_r := \mathfrak{m} \cap \mathfrak{k}_r$ . Further, we choose a fixed  $K$ -invariant Riemannian metric on  $M$ , and

define  $(T_x M)_r := \{\xi \in T_x M : |\xi| < r\}$  and  $V_r = V \cap (T_x M)_r$ . A typical neighborhood of the zero section in  $Y$  is given as

$$Y_r = K \times_H (\mathfrak{m}_r \times V_r)$$

where  $r$  is chosen so that  $\mathfrak{m}_r$  and  $V_r$  are  $H$ -invariant. Moreover, the induced  $K$ -action restricts to a Hamiltonian action on  $Y_r$  with momentum map  $J_{Y_r} := J_Y|_{Y_r}$ .  $\square$

**THEOREM 1.F.3** (Hamiltonian slice theorem). *Let  $K$  be a compact Lie group which acts on  $(M, \omega)$  by symplectomorphisms such that there is an equivariant momentum map  $J : M \rightarrow \mathfrak{k}^*$ , and let  $Y$  be the symplectic tube around  $x \in J^{-1}(\alpha) \subseteq M$  as constructed in Proposition 1.F.2. Then there is an open  $K$ -invariant neighborhood  $U$  of  $K.x$  in  $M$  and an open  $K$ -invariant neighborhood  $Y_r$  of the zero section in  $Y$ , and a  $K$ -equivariant symplectomorphism*

$$\Psi : U \longrightarrow Y_r$$

such that  $\Psi(x) = [(e, 0, 0)]$ . Moreover,  $J_{Y_r} \circ \Psi$  is an equivariant momentum map for the induced action on  $(U, \omega|_U)$ , and if  $K/H$  is connected it follows that  $J_{Y_r} \circ \Psi = J|_U$ .

**PROOF.** Apart from the constructions in this section the proof of this theorem is an application of the Witt-Artin decomposition, the Palais slice theorem, and the relative Darboux theorem.

Let  $r > 0$  be small enough such that  $S = \exp_x(\text{Nor}_x(K.x)_r)$  is a slice for the Riemannian  $K$ -action at  $x$  as in Theorem 1.A.4 with respect to the Riemannian metric on  $M$  from Remark 4. Thus there is an open  $K$ -invariant neighborhood of  $K.x$  in  $M$ , and  $K$ -equivariant diffeomorphism

$$\begin{aligned} \phi : U &\longrightarrow K \times_H \text{Nor}_x(K.x)_r, \\ k.y &\longmapsto [(k, \exp_x^{-1}(y))] = [(k, \xi)] \end{aligned}$$

where  $H = K_x$ . In the notation of Theorem 1.E.2 we have that  $\text{Nor}(K.x) = V \oplus W$ , and we define  $V_r \oplus W_r := \text{Nor}(K.x)_r$ . Denote the orthogonal projections onto  $V$  and  $W$  by  $\text{pr}_V : \text{Nor}(K.x) \rightarrow V$  and  $\text{pr}_W : \text{Nor}(K.x) \rightarrow W$ , respectively. Now recall the  $H$ -equivariant isomorphism  $f : W \rightarrow \mathfrak{m}^*$  which is given by  $\langle f(w), Y \rangle = \omega_x(\zeta_Y(x), w)$  from that same theorem. Restrict this isomorphism to an isomorphism  $f_r := f|_{W_r} : W_r \rightarrow f(W_r) =: \mathfrak{m}_r^* \subseteq \mathfrak{m}^*$ , and consider

$$\begin{aligned} \Phi : U &\longrightarrow K \times_H (\mathfrak{m}_r^* \times V_r) = Y_r, \\ k. \exp_x(\xi) &\longmapsto [(k, f_r(\text{pr}_W(\xi)), \text{pr}_V(\xi))] \end{aligned}$$

which clearly is smooth and has a smooth and well-defined inverse given by  $[(k, \mu, v)] \mapsto k. \exp_x(v + f_r^{-1}(\mu))$ .

Thus there are now two symplectic forms on  $U$ . Namely  $\omega|_U$  and  $\Phi^*\omega_Y|_{Y_r}$ , and the  $K$ -action is Hamiltonian with respect to both these

forms. Since  $K.x$  is a closed sub-manifold of  $U$  we are in a situation to apply the relative Darboux theorem of Section 1.I. We need only to show that  $\omega|U$  and  $\Phi^*\omega_Y|Y_r$  coincide along  $K.x$ . However, both forms are  $K$ -invariant, and therefore it actually suffices to show that  $\omega_x = (\Phi^*\omega_Y)_x$ . Indeed, consider  $\xi_1, \xi_2 \in T_x M$ . According to the Witt-Artin decomposition theorem we may write these vectors as

$$\xi_i = \zeta_{Y_i}(x) + \zeta_{Z_i}(x) + v_i + w_i \in \mathfrak{l}.x \oplus \mathfrak{q}.x \oplus V \oplus W$$

and we put also  $X_i = Y_i + Z_i$ . In this notation we have that

$$\begin{aligned} \omega_x(\xi_1, \xi_2) &= \omega_x(\zeta_{Y_1}(x) + \zeta_{Z_1}(x) + v_1 + w_1, \zeta_{Y_2}(x) + \zeta_{Z_2}(x) + v_2 + w_2) \\ &= \omega_x(\zeta_{Y_1}(x), w_2) + \omega_x(\zeta_{Z_1}(x), \zeta_{Z_2}(x)) + \omega_x(v_1, v_2) \\ &\quad + \omega_x(w_1, \zeta_{Z_2}(x)) \end{aligned}$$

since all other terms cancel by the properties of the decomposition. On the other hand, we see that

$$\begin{aligned} (\Phi^*\omega_Y)_x(\xi_1, \xi_2) &= \\ &= \omega_Y[(e, 0, 0)](T_x \Phi.(\zeta_{X_1}(x) + v_1 + w_1), T_x \Phi.(\zeta_{X_2}(x) + v_2 + w_2)) \\ &= \omega_Y[(e, 0, 0)](T_x \pi.(X_1, f_r(w_1), v_1), T_x \pi.(X_2, f_r(w_2), v_2)) \\ &= \Omega^1(e, 0, 0)((X_1, T_0 J_V.v_1 + f_r(w_1), v_1), (X_2, T_0 J_V.v_2 + f_r(w_2), v_2)) \\ &= \langle T_0 J_V.v_2 + f_r(w_2), Y_1 \rangle - \langle T_0 J_V.v_1 + f_r(w_1), Y_2 \rangle \\ &\quad + \langle J_V(0) + 0, [Y_1, Y_2] \rangle + \langle \alpha, [Z_1, Z_2] \rangle + \omega_x(v_1, v_2) \\ &= \omega_x(\zeta_{Y_1}(x), w_2) + \omega_x(\zeta_{Z_1}(x), \zeta_{Z_2}(x)) + \omega_x(v_1, v_2) \\ &\quad + \omega_x(w_1, \zeta_{Z_2}(x)) \\ &= \omega_x(\xi_1, \xi_2) \end{aligned}$$

where  $\pi : K \times \mathfrak{m}^* \times V \rightarrow K \times_H (\mathfrak{m}^* \times V)$  is the orbit projection, and we used the characterizing formula for  $\omega_Y$  from Proposition 1.F.2. Note also that  $T_0 J_V.v_1 = 0$  because  $\langle T_0 J_V.v_1, H_1 \rangle = \omega_x(\zeta_{H_1}(0), v_1) = 0$  for all  $H_1 \in \mathfrak{h}$  since 0 is fixed by the  $H$ -action, and  $J_V(0) = 0$ . Now we are in a position to apply the relative Darboux theorem. By eventually shrinking  $U$  to a smaller  $K$ -invariant neighborhood the relative Darboux theorem provides us with a  $K$ -equivariant symplectomorphism  $h : (U, \omega|U) \rightarrow (U, \Phi^*\omega_Y)$ . So we can consider the composite  $\Psi := \Phi \circ h : (U, \omega|U) \rightarrow (Y_r, \omega_Y|Y_r)$ , and this is the desired  $K$ -equivariant symplectomorphism as asserted.

Finally, it is easy to see from the diagram in the remark above that the two momentum maps  $J|U : U \rightarrow \mathfrak{k}^*$  and  $J_{Y_r} \circ \Phi : U \rightarrow \mathfrak{k}^*$  differ by an element in  $H^0(U) = H^0(K \times_H (\mathfrak{m}_r^* \times V_r))$ . However, the bundle projection  $Y_r = K \times_H (\mathfrak{m}_r^* \times V_r) \rightarrow K/H \cong K.x$  is a smooth deformation retract. Therefore,  $H^0(U) = H^0(K/H) = \mathbb{R}$  if  $K/H$  is connected, and the two momentum maps differ only by a constant which we may choose to be zero.  $\square$

### 1.G. Singular Poisson reduction

Let  $K$  be a Lie group acting properly on a smooth manifold  $M$ . We equip the orbit space  $M/K$  with the quotient topology with respect to the canonical projection  $\pi : M \rightarrow M/K$ . The set of smooth functions on  $M/K$  is defined by the requirement that  $\pi$  is a smooth map, i.e.,

$$C^\infty(M/K) := \{f \in C^0(M/K) : f \circ \pi \in C^\infty(M)\}.$$

**THEOREM 1.G.1** (Singular Poisson reduction). *Let  $(M, \{\cdot, \cdot\})$  be a Poisson manifold,  $K$  a Lie group, and let  $l : K \times M \rightarrow M$  be a smooth proper Poisson action, i.e.,  $l_k^*\{f, g\} = \{l_k^*f, l_k^*g\}$  for  $f, g \in C^\infty(M)$  and  $k \in K$ . Then we have:*

- (i) *The pair  $(C^\infty(M/K), \{\cdot, \cdot\}^{M/K})$  is a Poisson algebra, where the Poisson bracket  $\{\cdot, \cdot\}^{M/K}$  is characterized by  $\{f, g\}^{M/K} \circ \pi = \{f \circ \pi, g \circ \pi\}$ , for any  $f, g \in C^\infty(M/K)$ , and  $\pi : M \rightarrow M/K$  denotes the canonical smooth projection.*
- (ii) *Let  $h \in C^\infty(M)^K$  be a  $K$ -invariant function on  $M$ . The flow  $\text{Fl}_t$  of the Hamiltonian vector field  $\nabla_h$  commutes with the  $K$ -action, so it induces a flow  $\text{Fl}_t^{M/K}$  on  $M/K$  which is Poisson and is characterized by the identity  $\pi \circ \text{Fl}_t = \text{Fl}_t^{M/K} \circ \pi$ .*
- (iii) *The flow  $\text{Fl}_t^{M/K}$  is the unique Hamiltonian flow defined by the function  $H \in C^\infty(M/K)$  which is given by  $H \circ \pi = h$ .*

**PROOF.** This can be found in Ortega and Ratiu [31]. □

If, in particular,  $K$  is a compact Lie group acting by isometries on a smooth Riemannian manifold  $M$ , then by the Tube Theorem [35] and Schwarz' Theorem [44] we may identify  $C^\infty(M/K)$  with  $C^\infty(M)^K$ . See Example 1.C.7.

**DEFINITION 1.G.2** (Poisson stratified space). Let  $X$  be a stratified space with smooth structure in the sense of Section 1.C. Then  $X$  is said to be a **SINGULAR POISSON SPACE** if there is a Poisson bracket

$$\{\cdot, \cdot\} : C^\infty(X) \times C^\infty(X) \longrightarrow C^\infty(X)$$

on the algebra of smooth functions determined by the smooth structure such that the inclusion of each stratum  $S \hookrightarrow X$  is a Poisson morphism. In particular, the strata  $S$  are Poisson manifolds in the usual sense.

An alternative definition of a singular Poisson space in terms of a Poisson bivector on the stratified space is given in Pflaum [40].

**PROPOSITION 1.G.3** (Reduced Poisson structure). *Let  $(M, \{\cdot, \cdot\})$  be a Poisson manifold,  $K$  a compact Lie group, and let  $K$  act on  $M$  by Poisson morphisms. Then  $(C^\infty(M/K), \{\cdot, \cdot\}^{M/K})$  is a singular Poisson space.*

PROOF. By Example 1.C.7 the algebra  $C^\infty(M/K)$  is indeed determined by a smooth structure on  $M/K$ . Thus it only remains to check that the inclusion of each stratum  $M_{(H)}/K \hookrightarrow M/K$  is a Poisson morphism. This is, however, obvious.  $\square$

### 1.H. Singular symplectic reduction

The machinery of singular symplectic reduction is due to Sjamaar and Lerman [45] who prove that the singular symplectic quotient is a Whitney stratified space that has symplectic manifolds as its strata. This result which is the Singular Reduction Theorem was then generalized to the case of proper actions by Bates and Lerman [6], Ortega and Ratiu [33], and also others.

Let  $(M, \omega)$  be a connected symplectic manifold, and  $K$  a compact connected Lie group that acts on  $(M, \omega)$  in a Hamiltonian fashion such that there is an equivariant momentum map  $J : M \rightarrow \mathfrak{k}^*$ .

**THEOREM 1.H.1** (Singular symplectic reduction). *Let  $(H)$  be in the isotropy lattice of the  $K$ -action on  $M$ , and suppose that  $J^{-1}(\mathcal{O}) \cap M_{(H)} \neq \emptyset$  for a coadjoint orbit  $\mathcal{O} \subseteq \mathfrak{k}^*$ . Then the following are true.*

- *The subset  $J^{-1}(\mathcal{O}) \cap M_{(H)}$  is an initial sub-manifold of  $M$ .*
- *The topological quotient  $J^{-1}(\mathcal{O}) \cap M_{(H)}/K$  has a unique smooth structure such that the projection map*

$$J^{-1}(\mathcal{O}) \cap M_{(H)} \xrightarrow{\pi} J^{-1}(\mathcal{O}) \cap M_{(H)}/K$$

*is a smooth surjective submersion.*

- *Let  $\iota : J^{-1}(\mathcal{O}) \cap M_{(H)} \hookrightarrow M$  denote the inclusion mapping. Then  $J^{-1}(\mathcal{O}) \cap M_{(H)}/K$  carries a symplectic structure  $\omega_0$  which is uniquely characterized by the formula*

$$\pi^* \omega_0 = \iota^* \omega - (J|_{(J^{-1}(\mathcal{O}) \cap M_{(H)})})^* \Omega^\mathcal{O}$$

*where  $\Omega^\mathcal{O}$  is the canonical (positive Kirillov-Kostant-Souriau) symplectic form on  $\mathcal{O}$ .*

- *Consider a  $K$ -invariant function  $H \in C^\infty(M)^K$ . Then the flow to the Hamiltonian vector field  $\nabla_H^\omega$  leaves the connected components of  $J^{-1}(\mathcal{O}) \cap M_{(H)}$  invariant. Moreover,  $H$  factors to a smooth function  $h$  on the quotient  $J^{-1}(\mathcal{O}) \cap M_{(H)}/K$ . Finally,  $\nabla_H^\omega$  and the Hamiltonian vector field to  $h$  are related via the canonical projection  $\pi$ , whence the flow of the former projects to the flow of the latter.*
- *The collection of all strata of the form  $J^{-1}(\mathcal{O}) \cap M_{(H)}/K$  constitutes a Whitney stratification of the topological space  $J^{-1}(\mathcal{O})/K$ .*

PROOF. This theorem is the content of Ortega and Ratiu [33, Section 8] where it is proved in the more relaxed setting of a proper action

by a possibly non-connected group and a not necessarily equivariant momentum map. See also Bates and Lerman [6, Corollary 14].

As an application of the Hamiltonian slice theorem we will show how to obtain smooth charts for the strata  $J^{-1}(\mathcal{O}) \cap M_{(H)}$  and then also on  $(J^{-1}(\mathcal{O}) \cap M_{(H)})/K$ . Let  $\mathcal{O} = \text{Ad}^*(K).\alpha$  and  $x \in J^{-1}(\mathcal{O}) \cap M_{(H)}$  such that  $K_x = H$  and  $J(x) = \alpha$ . By Theorem 1.F.3 we can find an open  $K$ -invariant neighborhood  $U$  of  $K.x$  in  $M$ , and a  $K$ -equivariant symplectomorphism  $\Psi : U \rightarrow Y_r$  where  $Y_r$  is the local model

$$Y_r = K \times_H (\mathfrak{m}_r^* \times V_r)$$

obtained through the Witt-Artin decomposition of  $T_x M$  as in Section 1.F. Now we know that  $J|_U : U \rightarrow \mathfrak{k}^*$  factors over  $\Psi$  to

$$\begin{aligned} J_{Y_r} : Y_r = K \times_H (\mathfrak{m}_r^* \times V_r) &\longrightarrow \mathfrak{k}^*, \\ [(k, \mu, v)] &\longmapsto \text{Ad}^*(k).(J_V(v) + \mu + \alpha) \end{aligned}$$

which we regard as the local normal form of  $J$ . By equivariance of the symplectomorphism  $\Psi$  it follows that  $\Psi$  restricts to a homeomorphism from  $J^{-1}(\mathcal{O}) \cap M_{(H)}$  to  $J_{Y_r}^{-1}(\mathcal{O}) \cap Y_{(H)} \cap Y_r$ , and thus it suffices to show that the latter is a smooth subspace of  $Y_r$ . Indeed, notice that  $[(k, \mu, v)] \in J_{Y_r}^{-1}(\mathcal{O})$  if and only if  $\mu = 0$  and  $J_V(v) = 0$ . But since  $\zeta_Z(v) = 0$  for all  $v \in V_{(H)} = V_H$  and  $Z \in \mathfrak{h}$ , and, moreover,  $\langle J_V(v), Z \rangle = \omega_x(\zeta_Z(v), v)$  it follows that  $V_H \subseteq J_V^{-1}(0)$ . Therefore, we conclude

$$J_{Y_r}^{-1}(\mathcal{O}) \cap Y_{(H)} \cap Y_r = K \times_H (V_r)_H = K/H \times (V_r)_H$$

which clearly is a smooth submanifold of  $Y_r$ , and this is our local model for  $J^{-1}(\mathcal{O}) \cap M_{(H)}$ .

By equivariance of  $\Psi$  we further note that we thus obtain a chart

$$(V_r)_H = (J_{Y_r}^{-1}(\mathcal{O}) \cap (Y_r)_{(H)})/K \cong (U \cap J^{-1}(\mathcal{O}) \cap M_{(H)})/K$$

which is the desired local symplectic model of a typical stratum.  $\square$

As a matter of convention we write shorthand  $M//_{\mathcal{O}}K := J^{-1}(\mathcal{O})/K$  for the reduced space of  $M$  with respect to the Hamiltonian action by  $K$ . If  $\mathcal{O}$  is the coadjoint orbit passing through  $\alpha$  then we shall also abbreviate  $J^{-1}(\alpha)/K_{\alpha} = M//_{\alpha}K = M//_{\mathcal{O}}K$ .

**DEFINITION 1.H.2** (Singular symplectic space). Let  $X$  be a singular Poisson space in the sense of Definition 1.G.2. Then  $X$  is called a **SINGULAR SYMPLECTIC SPACE** if each stratum  $S$  is a symplectic manifold such that the inclusion

$$S \hookrightarrow X$$

is a Poisson morphism with respect to the Poisson structure on  $S$  which is determined by the symplectic structure.

This definition is essentially due to Sjamaar and Lerman [45].

**PROPOSITION 1.H.3.** *Let  $(M, \omega)$  be a connected symplectic manifold, and  $K$  a compact connected Lie group that acts on  $(M, \omega)$  in a Hamiltonian fashion such that there is an equivariant momentum map  $J : M \rightarrow \mathfrak{k}^*$ . Let  $\mathcal{O}$  be a coadjoint orbit in the image of  $J$ . Then  $M//_{\mathcal{O}}K$  is a singular symplectic space.*

**PROOF.** Using the Symplectic Slice Theorem of Section 1.F one can show as in Example 1.C.7 that there is a smooth structure on  $M//_{\mathcal{O}}K$  such that

$$C^\infty(M//_{\mathcal{O}}K) \cong \{f \in C^0(J^{-1}(\mathcal{O})) : \text{there is } F \in C^\infty(M) : F|_{J^{-1}(\mathcal{O})} = f\}^K.$$

The rest is now just a rephrasing of Theorem 1.H.1.  $\square$

**THEOREM 1.H.4** (Singular commuting reduction). *Let  $G$  and  $K$  be compact Lie groups that act by symplectomorphisms on  $(M, \omega)$  with momentum maps  $J_G$  and  $J_K$  respectively. Assume that the actions commute, that  $J_G$  is  $K$ -invariant, and that  $J_K$  is  $G$ -invariant. Let  $\alpha \in \mathfrak{g}^*$  be in the image of  $J_G$  and  $\beta \in \mathfrak{k}^*$  in the image of  $J_K$ .*

*Then the  $G$  action drops to a Poisson action on  $M//_{\beta}K$  and  $J_G$  factors to a momentum map  $j_G$  for the induced action. Likewise, the  $K$  action drops to a Poisson action on  $M//_{\alpha}G$  and  $J_K$  factors to a momentum map  $j_K$  for the induced action. Furthermore, we have*

$$(M//_{\alpha}G)//_{\beta}K \cong M//_{(\alpha, \beta)}(G \times K) \cong (M//_{\beta}K)//_{\alpha}G$$

*as symplectically stratified spaces.*

**PROOF.** This is proved and discussed (in greater generality) in [23, Section 10.4].  $\square$

### 1.I. Appendix: Relative Darboux Theorem

In this section we follow the development in Michor [28]. The presentation differs, however, in so far as we add a  $K$ -action to the situation. Let  $t \mapsto f_t$  be a curve of diffeomorphisms on a manifold  $M$  which are defined locally for all  $t \in \mathbb{R}$  such that  $f_0 = \text{id}_M$ . We define the time dependent vector fields on  $M$

$$\xi_t(x) := (T_x f_t)^{-1} \frac{\partial}{\partial t} f_t(x)$$

and

$$\eta_t(x) := \left(\frac{\partial}{\partial t} f_t\right)(f_t^{-1}(x)).$$

We have by definition that  $Tf_t \cdot \xi_t = \eta_t \circ f_t$  so that  $\xi_t$  and  $\eta_t$  are  $f_t$ -related.

**PROPOSITION 1.I.1** (Time dependent vector fields). *Let  $\omega \in \Omega^k(M)$ . Then the following are true for the above defined time dependent vector fields.*



- (i)  $i_{\xi_t} f_t^* \omega = f_t^* i_{\eta_t} \omega.$
- (ii)  $\frac{\partial}{\partial t} f_t^* \omega = f_t^* \mathcal{L}_{\eta_t} \omega = \mathcal{L}_{\xi_t} f_t^* \omega.$

PROOF. Let  $x \in M$  and  $X_1, \dots, X_k \in \mathfrak{X}(M)$ . Then

$$\begin{aligned} (i_{\xi_t} f_t^* \omega)_x(X_1, \dots, X_k) &= \omega_{f_t(x)}(T_x f_t \cdot \xi_t(x), T_x f_t \cdot X_1, \dots, T_x f_t \cdot X_k) \\ &= (f_t^* i_{\eta_t} \omega)_x(X_1, \dots, X_k) \end{aligned}$$

since  $\xi_t$  and  $\eta_t$  are  $f_t$ -related.

To prove the second part we have to use the evolution operator  $\Phi^\eta : \mathbb{R} \times \mathbb{R} \times M \rightarrow M$  for the time dependent vector field  $\eta_t$  which has the defining properties

$$\frac{\partial}{\partial t} \Phi_{t,s}^\eta(x) = \eta_t(\Phi_{t,s}^\eta(x)), \text{ and } \Phi_{s,s}^\eta(x) = x.$$

If we define  $\bar{\eta} \in \mathfrak{X}(\mathbb{R} \times M)$  by  $\bar{\eta}(t, x) = (\partial_t, \eta_t(x))$  then we have

$$(t, \Phi_{t,s}^\eta(x)) = \text{Fl}_{t-s}^{\bar{\eta}}(s, x)$$

and thus also

$$\Phi_{t,s}^\eta = \Phi_{t,r}^\eta \circ \Phi_{r,s}^\eta.$$

Moreover, by definition of  $\eta_t$  we have  $\frac{\partial}{\partial t} f_t = \eta_t \circ f_t$  and also  $f_0 = \text{id}_M$ . Therefore, it follows that  $f_t(x) = \Phi_{t,0}^\eta(x)$  which we may rewrite as  $(t, f_t(x)) = \text{Fl}_t^{\bar{\eta}}(0, x)$  or also as

$$f_t = \text{pr}_2 \circ \text{Fl}_t^{\bar{\eta}} \circ \text{ins}_0$$

where  $\text{ins}_0$  means insertion of 0 in the first variable. We thus compute

$$\begin{aligned} \frac{\partial}{\partial t} f_t^* \omega &= \frac{\partial}{\partial t} (\text{pr}_2 \circ \text{Fl}_t^{\bar{\eta}} \circ \text{ins}_0)^* \omega \\ &= \text{ins}_0^* \frac{\partial}{\partial t} (\text{Fl}_t^{\bar{\eta}})^* \text{pr}_2^* \omega \\ &= \text{ins}_0^* \frac{\partial}{\partial s} \Big|_0 (\text{Fl}_{t+s}^{\bar{\eta}})^* \text{pr}_2^* \omega \\ &= \text{ins}_0^* (\text{Fl}_t^{\bar{\eta}})^* \mathcal{L}_{\bar{\eta}} (\text{pr}_2^* \omega) \end{aligned}$$

where we used that  $\text{ins}_0^*$  is linear on  $\Omega^k(\mathbb{R} \times M)$ . Now notice that

$$(\mathcal{L}_{\bar{\eta}} \text{pr}_2^* \omega)(t, x) = (d i_{\bar{\eta}} \text{pr}_2^* \omega)(t, x) - (i_{\bar{\eta}} d \text{pr}_2^* \omega)(t, x) = (\mathcal{L}_{\eta_t} \omega)(x)$$

where we used that pull back commutes with  $d$ . Thus we obtain

$$\begin{aligned} \left( \frac{\partial}{\partial t} f_t^* \omega \right)(x) &= (\text{ins}_0^* (\text{Fl}_t^{\bar{\eta}})^* \mathcal{L}_{\bar{\eta}} (\text{pr}_2^* \omega))(x) \\ &= (\mathcal{L}_{\bar{\eta}} \text{pr}_2^* \omega)(t, f_t(x)) \circ \Lambda^k(\text{id}_{\mathbb{R}} \times T_x f_t) \\ &= (\mathcal{L}_{\eta_t} \omega)(x) \circ \Lambda^k T_x f_t \\ &= (f_t^* \mathcal{L}_{\eta_t} \omega)(x) \end{aligned}$$

which is the first part of the second assertion. To see the second part we use the first assertion and get

$$f_t^* \mathcal{L}_{\eta_t} \omega = f_t^* (d i_{\eta_t} - i_{\eta_t} d) \omega = (d i_{\xi_t} - i_{\xi_t} d) f_t^* \omega = \mathcal{L}_{\xi_t} f_t^* \omega$$

as required.  $\square$

**PROPOSITION 1.I.2 (Relative Poincare Lemma).** *Let  $\iota : N \hookrightarrow M$  be a closed submanifold and  $\omega \in \Omega^{k+1}$  such that  $\iota^*\omega = 0$ . Then there exists an open neighborhood  $U$  of  $N$  in  $M$  and a form  $\theta \in \Omega^k(U)$  such that  $d\theta = \omega|_U$  and  $\theta|_N = 0$ . Moreover, if  $\omega$  vanishes when evaluated on  $TM|_N$  then so do the first derivatives of  $\theta$  along  $N$ .*

*Assume furthermore that  $M$  is acted upon by a compact Lie group  $K$ , and that  $\omega$  and  $N$  are  $K$ -invariant. Then also  $U$  and  $\theta$  may be chosen  $K$ -invariant.*

**PROOF.** By restricting to a ( $K$ -invariant) tubular neighborhood of  $N$  we may assume that  $p : M = E \rightarrow N$  is fiber bundle with zero section  $\iota : N \rightarrow E$ . We define fiber wise multiplication  $\mu : \mathbb{R} \times E \rightarrow E$  by  $\mu(t, x) = \mu_t(x) = tx$ . Clearly  $\mu_1 = \text{id}_E$  and  $\mu_0 = \iota \circ p$ . Consider the vertical vector field  $\xi \in \mathfrak{X}(E)$  given by  $\xi(x) = x$ . Also we consider the curve of diffeomorphisms  $f_t := \text{Fl}_{\log t}^\xi = \mu_t$ . As above we define the time dependent vector fields corresponding to this curve by

$$\xi_t = (T_x f_t)^{-1} \frac{\partial}{\partial t} f_t(x) = \frac{1}{t} (\text{Fl}_{\log t}^\xi)^* \xi \text{ and } \eta_t(x) = \left( \frac{\partial}{\partial t} \mu_t \right) (\mu_t^{-1}(x)) = \frac{1}{t} \xi(x).$$

Using the proposition on time dependent vector fields we get

$$\frac{\partial}{\partial t} \mu_t^* \omega = \mu_t^* \mathcal{L}_{\eta_t} \omega = d\mu_t^* i_{\eta_t} \omega$$

Notice that

$$(\mu_t^* i_{\eta_t} \omega)_x (X_1, \dots, X_k) = \omega_{tx} (\eta_t(tx), tX_1, \dots, tX_k)$$

defines a smooth curve  $\mu_t^* i_{\eta_t} \omega$  in  $\Omega^k(E)$  since  $\eta_t(tx) = \xi(x) = x$ . Moreover, if  $\omega$  is  $K$ -invariant then so is  $\mu_t^* i_{\eta_t} \omega$  by construction. Now since  $\mu_1^* \omega = \omega$  and  $\mu_0^* \omega = p^* \circ \iota^* \omega = 0$  we conclude that

$$\begin{aligned} \omega &= \mu_1^* \omega - \mu_0^* \omega \\ &= \int_0^1 \frac{\partial}{\partial t} \mu_t^* \omega dt \\ &= d \left( \int_0^1 \mu_t^* i_{\eta_t} \omega dt \right) =: d\theta. \end{aligned}$$

Thus  $d\theta = \omega$  on  $E$ , and  $\iota^* \theta = 0$  since  $\iota^* (\mu_t^* i_{\eta_t} \omega) = 0$ . Moreover, if  $\omega$  vanishes on  $TE|_N$  then so do the first derivatives of  $\theta$  along  $N$ , and the asserted  $K$ -invariance is also obvious.  $\square$

**THEOREM 1.I.3 (Relative Darboux Theorem).** *Let  $N$  be a closed submanifold of  $M$ , and let  $\omega_0$  and  $\omega_1$  be symplectic forms on  $M$  that coincide along  $N$ . Then there are open neighborhoods  $U$  and  $V$  of  $N$  in  $M$ , and a diffeomorphism  $f : U \rightarrow V$  which satisfies  $f|_N = \text{id}_N$ ,  $Tf|(TM|_N) = \text{id}_{TM|_N}$ , and  $f^* \omega_1 = \omega_0$ .*

*If furthermore  $M$  is acted upon by a compact Lie group  $K$  such that both  $\omega_0$  and  $\omega_1$  are  $K$ -invariant, and  $N$  is a  $K$ -invariant closed submanifold then  $U$  and  $V$  may be chosen  $K$ -invariant, and  $f : (U, \omega_0|_U) \rightarrow (V, \omega_1|_V)$  is a  $K$ -equivariant symplectomorphism.*

PROOF. Suppose we are in the situation of a given  $K$ -action as formulated in the second part of the theorem. Let  $\omega_t := \omega_0 + t(\omega_1 - \omega_0)$  which clearly is a closed form on  $M$ . Since non-degeneracy is an open condition we find an open neighborhood  $U$  of  $N$  in  $M$  such that  $\omega_t|_U$  is symplectic. Moreover, we choose  $U$  to be  $K$ -invariant. By passing to a smaller neighborhood if necessary we find by the Relative Poincare Lemma a  $K$ -invariant one-form  $\theta \in \Omega^1(U)$  such that  $(\omega_1 - \omega_0)|_U = d\theta$ . Moreover,  $\theta|_N = 0$  and also all first derivatives along  $N$  of  $\theta$  vanish.

Now consider the time dependent vector field  $\eta_t \in \mathfrak{X}(U)$  defined by the equation  $i_{\eta_t}\omega_t|_U = -\theta$ . Correspondingly we have the curve of local diffeomorphisms defined by  $f_t = \text{pr}_2 \circ \text{Fl}_t^{\bar{\eta}} \circ \text{ins}_0$  where  $\bar{\eta}(t, x) = (\partial_t, \eta_t(x))$  defines an ordinary vector field as in the proof of the proposition on time dependent vector fields above. Notice that by  $K$ -invariance of  $\omega_t$  and  $\theta$  we may conclude that  $Tl_k.\eta_t = \eta_t \circ l_k$  where  $l_k : M \rightarrow M$  is the  $K$ -action on  $M$ . That is  $\eta_t$  is  $l_k$ -related to itself, whence the corresponding flow commutes with the action, thus implying that  $f_t$  is  $K$ -equivariant.

To see that  $f_t$  leads to desired symplectomorphism we notice that

$$\begin{aligned} \frac{\partial}{\partial t} f_t^* \omega_t &= \frac{\partial}{\partial s} \Big|_0 f_{t+s}^* \omega_t + \frac{\partial}{\partial s} \Big|_0 f_t^* \omega_{t+s} \\ &= f_t^* \mathcal{L}_{\eta_t} \omega_t + f_t^* (\omega_1 - \omega_0) \\ &= f_t^* (di_{\eta_t} \omega_t - (\omega_1 - \omega_0)) \\ &= f_t^* (d\theta - (\omega_1 - \omega_0)) = 0. \end{aligned}$$

Therefore,  $f_t^* \omega_t$  is constant in  $t$ , and we see that  $f_1$  is the desired  $K$ -equivariant symplectomorphism since  $\omega_0 = f_0^* \omega_0 = f_1^* \omega_1$ .  $\square$



## CHAPTER 2

### Singular cotangent bundle reduction

This chapter is concerned with symplectic and Poisson reduction of a cotangent bundle  $T^*Q$  with respect to a Hamiltonian action by a compact Lie group  $K$  that comes as the cotangent lifted action from the configuration manifold  $Q$ . Moreover, we assume that  $Q$  is Riemannian and  $K$  acts on  $Q$  by isometries. The cotangent bundle  $T^*Q$  is equipped with its canonical exact symplectic form, and we have a standard momentum map  $\mu : T^*Q \rightarrow \mathfrak{k}^*$ . Consider a coadjoint orbit  $\mathcal{O}$  that lies in the image of  $\mu$ . The goal is now to understand the symplectically reduced space

$$\mu^{-1}(\mathcal{O})/K =: T^*Q//_{\mathcal{O}}K.$$

Several difficulties arise at this point. First of all the action by  $K$  on the base  $Q$  is *not* assumed to be free. So we will get only a stratified symplectic space. Its strata will be of the form

$$(\mu^{-1}(\mathcal{O}) \cap (T^*Q)_{(L)})/K =: (T^*Q//_{\mathcal{O}}K)_{(L)}$$

where  $(L)$  is an element of the isotropy lattice of the  $K$ -action on  $T^*Q$ . This follows from the theory of singular symplectic reduction as developed in Sjamaar and Lerman [45], Bates and Lerman [6], and Ortega and Ratiu [33]. See also Theorem 1.H.1.

One of the aspects of cotangent bundle reduction is to relate the reduced space  $(T^*Q//_{\mathcal{O}}K)_{(L)}$  to the cotangent bundle of the reduced configuration space, i.e. to  $T^*(Q/K)$ . However, in this generality  $Q/K$  will not be a smooth manifold, and, worse, the mapping  $(T^*Q//_{\mathcal{O}}K)_{(L)} \rightarrow T^*(Q/K)$  (which one constructs canonically – see Section 2.D) does not have locally constant fiber type. To remedy this mess we have to assume that the base manifold is of single isotropy type, that is  $Q = Q_{(H)}$  for a subgroup  $H$  of  $K$ . Assuming this we get a first result that says that

$$\mathcal{O}//_{\mathcal{O}}H \hookrightarrow T^*Q//_{\mathcal{O}}K \longrightarrow T^*(Q/K)$$

is a symplectic fiber bundle, and this is Theorem 2.A.4. This result is obtained by applying the Palais Slice Theorem to the action on the base space  $Q$ , and then using the Singular Commuting Reduction Theorem of Section 1.H. This is an inroad that was also taken by Schmah [43] to get a local description of  $T^*Q//_{\mathcal{O}}K$ .

However, one can also give a global symplectic description of the reduced space, and this is done in Section 2.D. This follows an approach

that is generally called gauged cotangent bundle reduction or Weinstein construction ([50]) or also Sternberg construction. In the case that the action by  $K$  on the configuration space is free this global description was first given by Marsden and Perlmutter [25]. Their result says that the symplectic quotient  $T^*Q//_{\mathcal{O}}K$  can be realized as the fibered product

$$T^*(Q/K) \times_{Q/K} (Q \times_K \mathcal{O})$$

and they compute the reduced symplectic structure in terms of data intrinsic to this realization – [25, Theorem 4.3].

In the presence of a single non-trivial isotropy on the configuration space one gets a non-trivial isotropy lattice on  $T^*Q$  and thus has to use stratified symplectic reduction. The result is then the following: Each symplectic stratum  $(T^*Q//_{\mathcal{O}}K)_{(L)}$  of the reduced space can be globally realized as

$$(\mathcal{W}//_{\mathcal{O}}K)_{(L)} = T^*(Q/K) \times_{Q/K} (\bigsqcup_{q \in Q} \mathcal{O} \cap \text{Ann } \mathfrak{k}_q)_{(L)}/K$$

where

$$\mathcal{W} := (Q \times_{Q/K} T^*(Q/K)) \times_Q \bigsqcup_{q \in Q} \text{Ann } \mathfrak{k}_q \cong T^*Q$$

as symplectic manifolds with a Hamiltonian  $K$ -action. Moreover, we compute the reduced symplectic structure in terms intrinsic to this realization. This is the content of Theorem 2.D.4.

It is rather surprising that the subject of cotangent bundle reduction, albeit so important to Hamiltonian mechanics, still is very untouched. Even in the case of a free action on the base the results are rather new, and there is not much to be found about singular cotangent bundle reduction in the literature. One of the first to study this subject is Schmah [43]. The other important paper on singular cotangent bundle reduction is the one by Perlmutter, Rodriguez-Olmos and Sousa-Diaz [38]. By restricting to do reduction at fully isotropic values of the momentum map  $\mu : T^*Q \rightarrow \mathfrak{k}^*$  they are able to drop all assumptions on the isotropy lattice of the  $K$ -action on  $Q$ , and give a very complete description of the reduced symplectic space.

In order to understand the Poisson reduced space  $T^*Q = \mathcal{W}/K$  via the Weinstein construction we follow a similar program. That is, we compute the coinduced Poisson bracket on

$$\mathcal{W}/K = T^*(Q/K) \times_{Q/K} (\bigsqcup_{q \in Q} \text{Ann } \mathfrak{k}_q)/K$$

in terms intrinsic to this realization. The resulting formula involves the canonical symplectic form on  $T^*(Q/K)$  and vertical differentiation and curvature form with respect to a certain connection on the bundle

$$\text{Ann}/H \hookrightarrow \mathcal{W}/K \longrightarrow T^*(Q/K)$$

which is a bundle with smooth base and singular fiber. This is the content of Theorem 2.E.9.

In the case that  $K$  acts freely on  $Q$  the Poisson bracket on the reduced Poisson manifold  $T^*Q/K$  is determined in Zaalani [53] and in Perlmutter and Ratiu [37].

## 2.A. Bundle picture

Let us introduce the basic assumptions of the thesis. That is  $Q$  is a Riemannian manifold,  $K$  is a compact connected Lie group which acts on  $Q$  by isometries. The  $K$  action then induces a Hamiltonian action on the cotangent bundle  $T^*Q$  by cotangent lifts. This means that the lifted action respects the canonical symplectic form  $\Omega = -d\theta$  on  $T^*Q$  where  $\theta$  is the Liouville form on  $T^*Q$ , and, moreover, there is a momentum map  $\mu : T^*Q \rightarrow \mathfrak{k}^*$  given by  $\langle \mu(q, p), X \rangle = \theta(\zeta_X^{T^*Q})(q, p) = \langle p, \zeta_X(q) \rangle$  where  $(q, p) \in T^*Q$ ,  $X \in \mathfrak{k}$ ,  $\zeta_X$  is the fundamental vector field associated to the  $K$ -action on  $Q$ , and  $\zeta_X^{T^*Q} \in \mathfrak{X}(T^*Q)$  is the fundamental vector field associated to the cotangent lifted action.

In this section we want to apply the slice theorem of Section 1.A to the action of  $K$  on  $Q$  to get a local model of the singularly symplectic reduced space  $T^*Q//_{\mathcal{O}}K = \mu^{-1}(\mathcal{O})/K$  where  $\mathcal{O}$  is a coadjoint orbit in the image of  $\mu$ .

Thus we consider a tube  $U$  in  $Q$  around an orbit  $K.q$  with  $K_q = H$ . And we denote the slice at  $q$  by  $S$  such that

$$U \cong K \times_H S.$$

In particular it follows that  $U/K \cong S/H$ .

Assume for a moment that the action by  $K$  on  $U$  is free, that is  $U \cong K \times S$ . Let  $\mu : T^*U \rightarrow \mathfrak{k}^*$  be the canonical momentum mapping,  $\lambda \in \mathfrak{k}^*$  a regular value in the image of  $\mu$ , and  $\mathcal{O}$  the coadjoint orbit passing through  $\lambda$ . Then we have

$$\begin{aligned} (T^*U)//_{\mathcal{O}}K &= (T^*U)//_{\lambda}K = (T^*K \times T^*S)//_{\lambda}K \\ &= (T^*K)//_{\lambda}K \times T^*S \\ &= \mathcal{O} \times T^*(U/K) \end{aligned}$$

as symplectic spaces; since  $T^*K//_{\lambda}K = \mathcal{O}$ . The aim of this section is to drop the freeness assumption. To do so we will take the same approach as Schmah [43] and use singular commuting reduction.

Now we return to the case where  $U = K \times_H S$  as introduced above. On  $K \times S$  we will be concerned with two commuting actions. These are

$$\begin{aligned} \lambda : K \times K \times S &\longrightarrow K \times S, & \lambda_g(k, s) &= (gk, s) \\ \tau : H \times K \times S &\longrightarrow K \times S, & \tau_h(k, s) &= (kh^{-1}, h.s). \end{aligned}$$

These actions obviously commute. The latter, i.e.  $\tau$  is called the twisted action by  $H$  on  $K \times S$ . We can cotangent lift  $\lambda$  and  $\tau$  to give Hamiltonian transformations on  $T^*(K \times S)$  with momentum mappings

$J^\lambda$  and  $J^\tau$ , respectively. By left translation we trivialize  $T^*(K \times S) = (K \times \mathfrak{k}^*) \times T^*S$ .

To facilitate the notation we will denote the cotangent lifted action of  $\lambda, \tau$  again by  $\lambda, \tau$  respectively.

LEMMA 2.A.1. *Let  $(k, \eta; s, p) \in K \times \mathfrak{k}^* \times T^*S$ . Then we have the following formulas*

$$\begin{aligned} J^\lambda(k, \eta; s, p) &= \text{Ad}(k^{-1})^*.\eta =: \text{Ad}^*(k).\eta \in \mathfrak{k}^*, \\ J^\tau(k, \eta; s, p) &= -\eta|\mathfrak{h} + \mu(s, p) \in \mathfrak{h}^* \end{aligned}$$

where  $\mu$  is the canonical momentum map on  $T^*S$ . Moreover, the actions  $\lambda$  and  $\tau$  commute, and  $J^\lambda$  is  $H$ -invariant and  $J^\tau$  is  $K$ -invariant.

Since the canonical momentum map on  $T^*S$  is the same as that on  $T^*U$  restricted to  $T^*S$  the use of the symbol  $\mu$  for both these maps is unambiguous.

The signs in the formulas for the momentum mappings depend on the choice of sign in the definition of the fundamental vector field map as defined in Section 1.A.

PROOF. We denote the left action by  $K$  on itself by  $L$ , the right action by  $R$ , and the conjugate action by  $\text{conj}$ . In this notation we then have  $\text{Ad}(k).X = T_e \text{conj}_k.X$ , and  $\text{conj}_k = L_k \circ R^{k^{-1}} = R^{k^{-1}} \circ L_k$ . Notice the choice of sign in the definition of the fundamental vector field associated to left actions in Section 1.A. For right actions we need to choose the opposite sign. It is straightforward to verify that the cotangent lifted actions of  $L$  and  $R$  on  $T^*K = K \times \mathfrak{k}^*$  are given by

$$\begin{aligned} T^*L_g(k, \eta) &= (gk, -\eta) = (gk, \eta \circ \zeta^R(g)) \\ T^*R^g(k, \eta) &= (kg, \text{Ad}(g^{-1})^*.\eta) = (kg, \eta \circ \zeta^L(g)) \end{aligned}$$

where  $\zeta^L$  and  $\zeta^R$  denote the fundamental vector field mappings associated to  $L$  and  $R$  respectively.

Thus  $\langle J^\lambda(k, \eta; s, p), X \rangle = \langle \eta, \zeta_X^L(k) \rangle = \langle \text{Ad}^*(k).\eta, X \rangle$  for all  $X \in \mathfrak{k}$  which shows the first claim. Also, it follows that  $\langle J^\tau(k, \eta; s, p), Z \rangle = \langle -\eta, Z \rangle + \langle p, \zeta_X(s) \rangle$  for all  $Z \in \mathfrak{h}$ . The invariance of  $J^\lambda$  and  $J^\tau$  is immediate from the formulas of the trivialized cotangent lifted actions.  $\square$

COROLLARY 2.A.2. *Let  $\alpha \in \mathfrak{k}^*$  and  $\beta \in \mathfrak{h}^*$  such that  $\alpha, \beta$  is in the image of  $J^\lambda, J^\tau$  respectively. Then the following are true.*

- (i) *The action  $\lambda$  descends to a Hamiltonian action on the Marsden-Weinstein reduced space  $T^*(K \times S) //_\beta H$ . Moreover,  $J^\lambda$  factors to a momentum map  $j_\lambda : T^*(K \times S) //_\beta H \rightarrow \mathfrak{k}^*$  for this action.*
- (ii) *The action  $\tau$  descends to a Hamiltonian action on the Marsden-Weinstein reduced space  $T^*(K \times S) //_\alpha K$ . Moreover,  $J^\tau$  factors to a momentum map  $j_\tau : T^*(K \times S) //_\alpha K \rightarrow \mathfrak{h}^*$  for this action.*



(iii) *The product action  $K \times H \times T^*(K \times S) \rightarrow T^*(K \times S)$ ,  $(k, h, u) \mapsto \lambda_k \cdot \tau_h \cdot u$  is Hamiltonian with momentum map  $(J^\lambda, J^\tau)$ . Moreover,*

$$\begin{aligned} (T^*(K \times S) //_\alpha K) //_\beta H &= T^*(K \times S) //_{(\alpha, \beta)} (K \times H) \\ &= (T^*(K \times S) //_\beta H) //_\alpha K \end{aligned}$$

*as singular symplectic spaces.*

PROOF. Since the actions by  $\lambda$  and  $\tau$  are free the first two assertions can be deduced from the regular commuting reduction theorem with the necessary conditions being verified in the above lemma. Clearly, the product action by  $K \times H$  is well-defined and Hamiltonian with asserted momentum map. However, the product action will not be free in general. Thus the last point is a consequence of the singular commuting reduction theorem of Section 1.H.  $\square$

We will only be interested in the case where  $\beta = 0$ . There are more than one Hamiltonian cotangent lifted actions on  $T^*K$ . However, when it comes to reduction we will be only concerned with the lifted action  $\lambda$ . Thus the expression  $T^*K //_\alpha K$  unambiguously stands for  $(J^\lambda)^{-1}(\alpha) / K_\alpha$ .

PROPOSITION 2.A.3. *Clearly, 0 is in the image of  $J^\tau$ . Therefore,*

$$\begin{aligned} T^*U //_\alpha K &\cong T^*(K \times_H S) //_\alpha K \\ &= T^*(K \times S) //_0 H //_\alpha K \\ &= (T^*K //_\alpha K \times T^*S) //_0 H \\ &= (\mathcal{O} \times T^*S) //_0 H \end{aligned}$$

*as stratified symplectic spaces, and where  $\mathcal{O} = \text{Ad}^*(K) \cdot \alpha$ .*

PROOF. Since the isomorphism  $T^*U \xrightarrow{\cong} T^*(K \times_H S)$  comes from an equivariant diffeomorphism  $U \xrightarrow{\cong} K \times_H S$  on the base it is an equivariant symplectomorphism that intertwines the respective momentum maps. Now the regular reduction theorem for cotangent bundles at zero momentum says that  $T^*(K \times_H S)$  and  $T^*(K \times S) //_0 H$  are symplectomorphic. Further it is well-known that  $T^*K //_\alpha K = \mathcal{O}$ . The rest is a direct consequence of Theorem 1.H.4 on singular commuting reduction.  $\square$

We continue to assume that  $K$  acts on the Riemannian manifold  $Q$  by isometries. But now we also make the rather strong assumption that

$$Q = Q_{(H)}$$

i.e. all isotropy subgroups of points  $q \in Q$  are conjugate within  $K$  to  $H$ .

**THEOREM 2.A.4** (Bundle picture). *Let  $Q = Q_H$  and let  $\mathcal{O} \subseteq \mathfrak{k}^*$  be a coadjoint orbit in the image of the momentum map  $\mu : T^*Q \rightarrow \mathfrak{k}^*$ . Then we have a singular symplectic fiber bundle*

$$\mathcal{O} //_0 H \hookrightarrow T^*Q //_{\mathcal{O}} K \longrightarrow T^*(Q/K)$$

*in the sense of Section 2.B with stratified typical fiber  $\mathcal{O} //_0 H$  and smooth base  $T^*(Q/K)$ .*

This theorem is to say that the singularities of the reduced phase space are confined to the fiber direction which also will be referred to as the spin direction.

**PROOF.** This follows from the above in the following way. Consider a tube  $U$  of the  $K$ -action on  $Q$ . Then the slice theorem tells us that there is a slice  $S$  such that there is a  $K$ -equivariant diffeomorphism

$$U \cong K \times_H S = K/H \times S$$

since all points of  $Q$  by assumption are regular whence the slice representation is trivial. We can lift this diffeomorphism to a symplectomorphism of cotangent bundles to get

$$T^*U //_{\mathcal{O}} K \cong \mathcal{O} //_0 H \times T^*S$$

as in Proposition 2.A.3 above. Since  $T^*S$  is a typical neighborhood in  $T^*(Q/K)$  the result follows.  $\square$

## 2.B. Interlude on singular fiber bundles

**Singular bundles.** By Theorem 2.A.4 we are led to the following generalization of the concept of smooth bundles.

**DEFINITION 2.B.1** (Singular fiber bundles). . Let  $F$  and  $P$  be stratified spaces (Definition 1.C.5) with smooth structure (Definition 1.C.3) and  $M$  be a smooth manifold. We say that the topological fiber bundle

$$F \hookrightarrow P \xrightarrow{\pi} M$$

is a **SINGULAR FIBER BUNDLE** if for each trivializing patch  $U \subseteq M$  the homeomorphism

$$P|U \cong U \times F$$

is an isomorphism of stratified spaces.

There are two reasons for defining singular fiber bundles in this way. Firstly, it is the kind of structure encountered in Theorem 2.A.4, and secondly by Mather's control theory [26] these bundles possess many features similar to ordinary smooth fiber bundles.

Note that if  $M$  is a Riemannian manifold which is acted upon by a compact Lie group  $K$  through isometries then the orbit projection mapping  $M \rightarrow M/K$  is, in general, not a singular fiber bundle according to this

definition. Indeed, the fiber type of  $M \rightarrow M/K$  need not be locally constant.

LEMMA 2.B.2. *Let  $\pi : P \rightarrow M$  be a singular fiber bundle with typical fiber  $F$ . Let  $S$  be a stratum of  $P$ . Then  $\pi|_S : S \rightarrow M$  is a smooth fiber bundle.*

PROOF. Indeed, locally the stratum  $S$  is diffeomorphic to  $U \times S_F$  where  $S_F$  is a stratum of  $F$  and  $U$  is a trivializing neighborhood in  $M$ .  $\square$

DEFINITION 2.B.3 (Singular symplectic fiber bundles). Let  $F$  and  $P$  be stratified symplectic spaces (Definition 1.H.2) with smooth structure and  $M$  be a smooth symplectic manifold. We say that the singular fiber bundle

$$F \hookrightarrow P \xrightarrow{\pi} M$$

is a SINGULAR SYMPLECTIC FIBER BUNDLE if for each trivializing patch  $U \subseteq M$  the homeomorphism

$$P|U \cong U \times F$$

is an isomorphism of stratified symplectic spaces with respect to the inherited symplectic structures. It follows, in particular, that  $\pi$  is a Poisson morphism.

**Control data.** The theory of control data is due to Mather [26], and we follow in our presentation of the subject that of [26]. Let  $N$  be a smooth manifold, and  $X \subseteq N$  a stratified subset endowed with the relative topology with strata  $S_i$  where  $i \in I$  as in Section 1.C.

A TUBULAR NEIGHBORHOOD of a stratum  $S_i$  in  $X$  is a closed neighborhood of  $S_i$  in  $N$  which is diffeomorphic to an inner product bundle  $\pi_i : E_i \rightarrow S_i$ . Via the inner product we can measure the vertical distance of a point in  $E_i$  to  $S_i$  and call this the tubular neighborhood function  $\rho_i : E_i \rightarrow \mathbb{R}$ . Clearly,  $\rho_i(x) = 0$  if and only if  $x \in S_i$ . We can also think of the tubular neighborhood as being retracted onto  $S_i$  via the projection  $\pi_i$ . CONTROL DATA associated to the stratification  $\{S_i : i \in I\}$  of  $X$  is a system of tubular neighborhoods  $\pi_i : E_i \rightarrow S_i$  satisfying the following commutation relations:

$$\begin{aligned} (\pi_j \circ \pi_i)(x) &= \pi_j(x), \\ (\rho_j \circ \pi_i)(x) &= \rho_j(x) \end{aligned}$$

whenever  $j \leq i$  and both sides are defined.

PROPOSITION 2.B.4. *There exist control data to the stratification  $\{S_i : i \in I\}$  of  $X$ . If  $M$  is another manifold, and  $f : N \rightarrow M$  a smooth mapping such that  $f|_{S_i} : S_i \rightarrow M$  is a submersion for all  $i \in I$  then the control data may be chosen so that  $f \circ \pi_{S_i} = f$  for all  $i \in I$ .*

PROOF. See Mather [26, Proposition 7.1].  $\square$

If  $f : N \rightarrow M$  is as in Proposition 2.B.4 then  $f$  is said to be a CONTROLLED SUBMERSION from  $X$  to  $M$ .

By a STRATIFIED VECTOR FIELD  $\eta$  on  $X$  we mean a collection  $\{\eta_i : i \in I\}$  where each  $\eta_i$  is a smooth vector field on  $S_i$ . Assume we are given a system of control data associated to the stratification of  $X$ , and identify the tubular neighborhoods of the strata with the corresponding inner-product bundles. Then the stratified vector field  $\eta$  on  $X$  is said to be a CONTROLLED VECTOR FIELD if the following conditions are met. For any stratum  $S_j$  there is an open neighborhood  $B_j$  of  $S_j$  in the tubular neighborhood  $E_j$  such that for any stratum  $S_i$  with  $i > j$  the conditions

$$\begin{aligned} \mathcal{L}_{\eta_i}(\rho_j|_{B_j \cap S_i}) &= 0, \\ T_x(\pi_j|_{B_j \cap S_i}) \cdot \eta_i(x) &= \eta_j(\pi_j(x)) \end{aligned}$$

are satisfied for all  $x \in B_j \cap S_i$ .

Let  $J$  be an open neighborhood of  $\{0\} \times X$  in  $\mathbb{R} \times X$ , and assume that  $\alpha : J \rightarrow X$  is a local one-parameter group which is smooth in the sense of Definition 1.C.3. We say that  $\alpha$  GENERATES the stratified vector field  $\eta$  if  $J$  is maximal such that each stratum  $S_i$  is invariant under  $\alpha$  and

$$\frac{\partial}{\partial t} \Big|_0 \alpha(t, x) = \eta_i(x)$$

for all  $x \in S_i$  and all  $i \in I$ .

PROPOSITION 2.B.5. *Assume  $\eta$  is a controlled vector field on  $X$ . Then there is a unique smooth one-parameter group which generates  $\eta$ .*

PROOF. See Mather [26, Proposition 10.1]. □

PROPOSITION 2.B.6. *Assume  $f : N \rightarrow M$  is a smooth map such that  $f|_X : X \rightarrow M$  is a controlled surjective submersion. Then the following are true.*

- *Let  $\xi$  be a smooth vector field on  $M$ . Then there is a controlled vector field  $\eta$  on  $X$  such that  $\eta_i$  and  $\xi$  are  $f|_{S_i}$ -related for all  $i \in I$ .*
- *Suppose further that  $f|_X \rightarrow M$  is a proper map. Then  $f : X \rightarrow M$  is a singular fiber bundle.*

PROOF. See Mather [26, Proposition 9.1] for the first statement. Concerning the second assertion, [26, Proposition 11.1] states that under these assumptions the mapping  $f : X \rightarrow M$  is locally topologically trivial, and it follows from [26, Corollary 10.3] that the trivializing homeomorphisms are, in fact, isomorphisms of stratified spaces. Thus  $f : X \rightarrow M$  is a singular fiber bundle in the sense of Definition 2.B.1. □

**Pullback bundles.** Let  $M$  and  $Y$  be smooth manifolds, and let  $\tau : Y \rightarrow M$  be a smooth mapping. Consider a singular fiber bundle  $\pi : X \rightarrow M$  with typical fiber  $F$  as in Definition 2.B.1. We consider further the topological pullback bundle of  $X$  and  $Y$  over  $M$  with the following notation.

$$\begin{array}{ccccc} X \times_M Y & \equiv & \tau^* X & \xrightarrow{\pi^* \tau} & X \\ & & \tau^* \pi \downarrow & & \downarrow \pi \\ & & Y & \xrightarrow{\tau} & M \end{array}$$

Now we can endow  $X \times_M Y$  with the product stratification given by strata of the form  $S \times_M Y$  which is the smooth fibered product of a stratum  $S$  of  $X$  with  $Y$  over  $M$ . Note that  $S \times_M Y$  is a well defined pull back bundle by Lemma 2.B.2. Moreover,  $X \times_M Y$  inherits a smooth structure in the sense of Section 1.C from the canonical topological inclusion  $X \times_M Y \hookrightarrow X \times Y$ . The singular space with smooth structure thus obtained is called the **FIBERED PRODUCT** of  $X$  and  $Y$  over  $M$ . Since  $\pi : X \rightarrow M$  is a singular fiber bundle it follows that

$$\tau^* \pi : X \times_M Y \rightarrow Y$$

is a singular fiber bundle as well with the same typical fiber  $F$ . Moreover, this construction satisfies the following universal property. Let  $Z$  be a singular space with smooth structure and  $f_1 : Z \rightarrow X$ ,  $f_2 : Z \rightarrow Y$  be smooth mappings satisfying  $\pi \circ f_1 = \tau \circ f_2$ . Then there is a unique smooth map  $f = (f_1, f_2)$  such that the following commutes.

$$\begin{array}{ccccc} & & Z & & \\ & f_1 \swarrow & \downarrow f & \searrow f_2 & \\ X & \xleftarrow{\pi^* \tau} & X \times_M Y & \xrightarrow{\tau^* \pi} & Y \\ & \searrow \pi & & \swarrow \tau & \\ & & M & & \end{array}$$

Therefore, in this sense, pull backs exist in the category of singular fiber bundles. There is, in fact, a similar notion of pull backs in the work of Davis [12].

## 2.C. Weinstein construction

For this section we continue with the basic assumptions of the paper. That is  $Q$  is a Riemannian manifold,  $K$  is a compact connected Lie group which acts on  $Q$  by isometries. Moreover,  $Q$  is supposed to be of single isotropy type, i.e.  $Q = Q_{(H)}$  where  $H$  is an isotropy subgroup of  $K$ . The  $K$  action then induces a Hamiltonian action on the cotangent bundle  $T^*Q$  by cotangent lifts. This means that the lifted action respects the canonical symplectic form  $\Omega = -d\theta$

on  $T^*Q$  and there is a momentum map  $\mu : T^*Q \rightarrow \mathfrak{k}^*$  given by  $\langle \mu(q, p), X \rangle = \theta(\zeta_X^{T^*Q})(q, p) = \langle p, \zeta_X(q) \rangle$  where  $(q, p) \in T^*Q$ ,  $X \in \mathfrak{k}$ ,  $\zeta_X$  is the fundamental vector field associated to the  $K$ -action on  $Q$ , and  $\zeta_X^{T^*Q} \in \mathfrak{X}(T^*Q)$  is the fundamental vector field associated to the cotangent lifted action.

Since the  $K$  action on  $Q$  has only a single isotropy type the orbit space  $Q/K$  is a smooth manifold, and the projection  $\pi : Q \rightarrow Q/K$  is a surjective Riemannian submersion with compact fibers. However, the lifted action by  $K$  on  $T^*Q$  is already much more complicated, and the quotient space  $(T^*Q)/K$  is only a stratified space in general. Its strata are of the form  $(T^*Q)_{(L)}/K$  where  $(L)$  is in the isotropy lattice of  $T^*Q$ . The vertical sub-bundle of  $TQ$  with respect to  $\pi : Q \rightarrow Q/K$  is  $\text{Ver} := \ker T\pi$ . Via the  $K$ -invariant Riemannian metric we obtain the horizontal sub-bundle as  $\text{Hor} := \text{Ver}^\perp$ . We define the dual horizontal sub-bundle of  $T^*Q$  as the sub-bundle  $\text{Hor}^*$  consisting of those co-vectors that vanish on all vertical vectors. Likewise, we define the dual vertical sub-bundle of  $T^*Q$  as the sub-bundle  $\text{Ver}^*$  consisting of those co-vectors that vanish on all horizontal vectors.

Fix an  $\text{Ad}(K)$ -invariant inner product on the Lie algebra  $\mathfrak{k}$  of the compact Lie group  $K$ . For  $X, Y \in \mathfrak{k}$  and  $q \in Q$  we define

$$\mathbb{I}_q(X, Y) := \langle \zeta_X(q), \zeta_Y(q) \rangle$$

and call this the INERTIA TENSOR. This gives a non-degenerate pairing on  $\mathfrak{k}_q^\perp \times \mathfrak{k}_q^\perp$ , whence it gives an identification  $\check{\mathbb{I}}_q : \mathfrak{k}_q^\perp \rightarrow (\mathfrak{k}_q^\perp)^* = \text{Ann } \mathfrak{k}_q$ . We use this isomorphism to define a one-form on  $Q$  with values in the bundle  $\bigsqcup_{q \in Q} \mathfrak{k}_q^\perp$  by the following:

$$\begin{array}{ccc} T_q^*Q & \xrightarrow{\mu_q} & \text{Ann } \mathfrak{k}_q \\ \simeq \uparrow & & \downarrow (\check{\mathbb{I}}_q)^{-1} \\ T_qQ & \xrightarrow{A_q} & \mathfrak{k}_q^\perp \end{array}$$

The form  $A$  shall be called the MECHANICAL CONNECTION on  $Q \rightarrow Q/K$ . The inertia tensor and mechanical connection first appeared in Smale [46] in the context of an Abelian group action. See also Marsden, Montgomery, and Ratiu [24, Section 2]. Blaom [8] generalized these concepts to the case of a free action of the group on the configuration space. In the present situation, however, we are concerned with a non-trivial single isotropy type on the base manifold  $Q$ , and we will see in the next paragraphs that the thus defined form  $A$  indeed is a connection form on  $Q \rightarrow Q/K$  – albeit in a generalized sense.

The mechanical connection has the following properties. It follows from its definition that  $TQ \rightarrow \bigsqcup_{q \in Q} \mathfrak{k}_q^\perp$ ,  $(q, v) \mapsto (q, A_q(v))$  is equivariant,  $\ker A_q = T_q(K \cdot q)^\perp$ , and  $A_q(\zeta_X(q)) = X$  for all  $X \in \mathfrak{k}_q^\perp$ .

This means that  $A \in \Omega^1(Q; \mathfrak{k})$  given by  $A : TQ \rightarrow \mathfrak{k}_q^\perp \hookrightarrow \mathfrak{k}$ ,  $(q, v) \mapsto A_q(v)$  is a principal connection form on the  $K$ -manifold  $Q$  in the sense of Alekseevsky and Michor [3, Section 3.1]. According to [3, Section 4.6] the curvature form associated to  $A$  is defined by

$$\text{Curv}^A := dA + \frac{1}{2}[A, A]^\wedge$$

where

$$[\varphi, \psi]^\wedge := \frac{1}{k!l!} \sum_{\sigma} \text{sign } \sigma [\varphi(v_{\sigma 1}, \dots, v_{\sigma l}), \psi(v_{\sigma(l+1)}, \dots, v_{\sigma(l+k)})]$$

is the graded Lie bracket on  $\Omega(Q; \mathfrak{k}) := \bigoplus_{k=0}^{\infty} \Gamma(\Lambda^k T^*Q \otimes \mathfrak{k})$ , and  $\varphi \in \Omega^l(Q; \mathfrak{k})$  and  $\psi \in \Omega^k(Q; \mathfrak{k})$ .

We define a point-wise dual  $A_q^* : \text{Ann } \mathfrak{k}_q \rightarrow \text{Ver}_q^* \subseteq T_q^*Q$  by the formula  $A_q^*(\lambda)(v) = \lambda(A_q(v))$  where  $\lambda \in \text{Ann } \mathfrak{k}_q$  and  $v \in T_qQ$ . Notice that

$$A_q^*(\mu_q(p)) = p$$

for all  $p \in \text{Ver}_q^*$  and

$$\mu_q(A_q^*(\lambda)) = \lambda$$

for all  $\lambda \in \text{Ann } \mathfrak{k}_q$ .

Using the horizontal lift mapping which identifies  $\text{Hor} \cong (Q \times_{Q/K} T(Q/K))$  on the one hand and the mechanical connection  $A$  on the other hand we obtain an isomorphism

$$TQ = \text{Hor} \oplus \text{Ver} \longrightarrow (Q \times_{Q/K} T(Q/K)) \times_Q \bigsqcup_{q \in Q} \mathfrak{k}_q^\perp$$

of bundles over  $Q$ . Via the Riemannian structure there is a dual version to this isomorphism, and to save on typing we will abbreviate

$$\mathcal{W} := (Q \times_{Q/K} T^*(Q/K)) \times_Q \bigsqcup_{q \in Q} \text{Ann } \mathfrak{k}_q \cong \text{Hor}^* \oplus \text{Ver}^*.$$

To set up some notation for the upcoming proposition, and clarify the picture consider the following stacking of pull-back diagrams.

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{\rho^* \tilde{\tau} = \tilde{\tilde{\tau}}} & \bigsqcup_q \text{Ann } \mathfrak{k}_q \\ \tilde{\tau}^* \rho = \tilde{\rho} \downarrow & & \downarrow \rho \\ Q \times_{Q/K} T^*(Q/K) & \xrightarrow{\pi^* \tau = \tilde{\tau}} & Q \\ \tau^* \pi = \tilde{\pi} \downarrow & & \downarrow \pi \\ T^*(Q/K) & \xrightarrow{\tau} & Q/K \end{array}$$

The upper stars in this diagram are, of course, not pull-back stars. It is in fact the transition functions that are being pulled-back, whence the name.

**PROPOSITION 2.C.1** (Symplectic structure on  $\mathcal{W}$ ). *There is a dual isomorphism*

$$\psi = \psi(A) : (Q \times_{Q/K} T^*(Q/K)) \times_Q \bigsqcup_{q \in Q} \text{Ann } \mathfrak{k}_q = \mathcal{W} \longrightarrow T^*Q,$$

$$(q, \eta, \lambda) \longmapsto (q, \eta + A(q)^* \lambda)$$

where we identify elements in  $\{q\} \times T_{[q]}^*(Q/K)$  with elements in  $\text{Hor}_q^*$  via the dual of the inverse of the horizontal lift.

This isomorphism can be used to induce a symplectic form on the connection dependent realization of  $T^*Q$ , namely  $\sigma = \psi^* \Omega$  where  $\Omega = -d\theta$  is the canonical form on  $T^*Q$ . Moreover, there is an explicit formula for  $\sigma$  in terms of the chosen connection:

$$\sigma = (\tilde{\pi} \circ \tilde{\rho})^* \Omega^{Q/K} - d\tilde{\tau}^* B$$

where  $\Omega^{Q/K}$  is the canonical symplectic form on  $T^*(Q/K)$ , and furthermore  $B \in \Omega^1(\bigsqcup_q \text{Ann } \mathfrak{k}_q)$  is given by

$$B_{(q,\lambda)}(v_1, \lambda_1) = \langle \lambda, A_q(v_1) \rangle.$$

The explicit formula now is

$$\begin{aligned} (dB)_{(q,\lambda)}((v_1, \lambda_1), (v_2, \lambda_2)) \\ = \langle \lambda, \text{Curv}_q^A(v_1, v_2) \rangle + \langle \lambda, [Z_1, Z_2] \rangle - \langle \lambda_2, Z_1 \rangle + \langle \lambda_1, Z_2 \rangle \end{aligned}$$

where  $(q, \lambda) \in \bigsqcup_q \text{Ann } \mathfrak{k}_q$ ,  $(v_i, \lambda_i) \in T_{(q,\lambda)}(\bigsqcup_q \text{Ann } \mathfrak{k}_q)$  for  $i = 1, 2$ , and

$$v_i = \zeta_{Z_i}(q) \oplus v_i^{\text{hor}} \in \text{Ver}_q \oplus \text{Hor}_q$$

is the decomposition into vertical and horizontal part with  $Z_i \in \mathfrak{k}$ .

Furthermore, there clearly is an induced action by  $K$  on  $\mathcal{W}$ . This action is Hamiltonian with momentum mapping

$$\mu_A = \mu \circ \psi : \mathcal{W} \longrightarrow \mathfrak{k}^*, (q, \eta, \lambda) \longmapsto \lambda,$$

where  $\mu$  is the momentum map  $T^*Q \rightarrow \mathfrak{k}^*$ , and  $\psi$  is equivariant.

PROOF. Clearly, the isomorphism  $\psi$  does induce a symplectic form  $\sigma = \psi^* \Omega$  on  $\mathcal{W}$ , and it only remains to verify the asserted formula. Let

$$w = (q; [q], \eta; q, \lambda) = (q, \eta, \lambda) \in \mathcal{W},$$

and  $\xi_i \in \mathfrak{X}(\mathcal{W})$  for  $i = 1, 2$ . We use the notation

$$\xi_i(w) = (v_i(q), \eta_i([q], \eta), \lambda_i(q, \lambda)).$$

That is,  $v_i \in \mathfrak{X}(Q)$ ,  $\eta_i \in \mathfrak{X}(T^*(Q/K))$ , and  $\lambda_i \in \mathfrak{X}(\bigsqcup_q \text{Ann } \mathfrak{k}_q)$ . By definition of pulling back of forms we have

$$\sigma_w(\xi_1, \xi_2) = \Omega_{(q,\eta+A^*(\lambda))}(T_w \psi \cdot \xi_1(w), T_w \psi \cdot \xi_2(w)).$$

Denoting the horizontal lift of  $(\text{Fl}_t^{v_i}(q), \text{Fl}_t^{\eta_i}([q], \eta))$  simply by  $\text{Fl}_t^{\eta_i}(\eta)$ , and considering  $\lambda_i(q, \lambda)$  as an element of  $\text{Ann } \mathfrak{k}_q$  (effectively forgetting the  $Q$ -component which is just  $v_i(q)$ ) we compute

$$\begin{aligned} T_w \psi \cdot \xi_i(w) &= \frac{\partial}{\partial t} \Big|_0 \psi(\text{Fl}_t^{\xi_i}(w)) \\ &= \frac{\partial}{\partial t} \Big|_0 (\text{Fl}_t^{v_i}(q), \text{Fl}_t^{\eta_i}(\eta) + A(\text{Fl}_t^{v_i}(q))^*(\text{Fl}_t^{\lambda_i}(q, \lambda))) \\ &= (v_i(q), \eta_i(\eta) + A_q^*(\lambda_i(q, \lambda)) + (\mathcal{L}_{v_i} A + A \circ \text{ad}(v_i))_q^*(\lambda)) \end{aligned}$$



where the last equality is true since:

$$\begin{aligned} \frac{\partial}{\partial t} |_0 \langle A(\mathrm{Fl}_t^{v_i}(q))^*(\mathrm{Fl}_t^{\lambda_i}(q, \lambda)), X \rangle &= \frac{\partial}{\partial t} |_0 \langle \mathrm{Fl}_t^{\lambda_i}(q, \lambda), A(X)(\mathrm{Fl}_t^{v_i}(q)) \rangle \\ &= \langle \lambda_i(q, \lambda), A_q(X) \rangle + \langle \lambda, \mathcal{L}_{v_i}(A(X))_q \rangle \\ &= \langle A_q^*(\lambda_i(q, \lambda)) \\ &\quad + (\mathcal{L}_{v_i}A + A \circ \mathrm{ad}(v_i))_q^*(\lambda), X \rangle \end{aligned}$$

for  $X \in TQ$  and we used the formal notation  $\mathrm{ad}(v_1)_q(v_2) = [v_1, v_2](q)$ . Therefore,

$$\begin{aligned} \sigma_w(\xi_1, \xi_2) &= \Omega_{(q, \eta + A_q^*(\lambda))}((v_1(q), \eta_1(\eta) + A_q^*(\lambda_1(q, \lambda)) \\ &\quad + (\mathcal{L}_{v_1}A + A \circ \mathrm{ad}(v_1))_q^*(\lambda)), \\ &\quad (v_2(q), \eta_2(\eta) + A_q^*(\lambda_2(q, \lambda)) \\ &\quad + (\mathcal{L}_{v_2}A + A \circ \mathrm{ad}(v_2))_q^*(\lambda))) \\ &= \langle \eta_2(\eta) + A_q^*(\lambda_2(q, \lambda)) + (\mathcal{L}_{v_2}A + A \circ \mathrm{ad}(v_2))_q^*(\lambda), v_1(q) \rangle \\ &\quad - \langle \eta_1(\eta) + A_q^*(\lambda_1(q, \lambda)) + (\mathcal{L}_{v_1}A + A \circ \mathrm{ad}(v_1))_q^*(\lambda), v_2(q) \rangle \\ &\quad + \langle \eta + A(q)^*(\lambda), [v_1, v_2](q) \rangle \\ &= \langle \eta_2(\eta), v_1^{\mathrm{hor}}(q) \rangle - \langle \eta_1(\eta), v_2^{\mathrm{hor}}(q) \rangle + \langle \eta, [v_1^{\mathrm{hor}}, v_2^{\mathrm{hor}}](q) \rangle \\ &\quad + \langle \lambda, \mathcal{L}_{v_2}(A(v_1))_q \rangle - \langle \lambda, \mathcal{L}_{v_1}(A(v_2))_q \rangle \\ &\quad + \langle \lambda, A(q)[v_1, v_2](q) \rangle \\ &\quad + \langle \lambda_2(\lambda), A(q).v_1(q) \rangle - \langle \lambda_1(\lambda), A(q).v_2(q) \rangle \\ &= \Omega_{([q], \eta)}^{Q/K}((v_1^{\mathrm{hor}}, \eta_1), (v_2^{\mathrm{hor}}, \eta_2)) \\ &\quad - \langle \lambda, \mathrm{Curv}_q^A(v_1, v_2) \rangle - \langle \lambda, [Z_1, Z_2] \rangle \\ &\quad + \langle \lambda_2(\lambda), Z_1 \rangle - \langle \lambda_1(\lambda), Z_2 \rangle \end{aligned}$$

where  $v_i(q) = v_i(q)^{\mathrm{hor}} \oplus \zeta_{Z_i}(q) \in \mathrm{Hor}_q \oplus \mathrm{Ver}_q$ . Note also that  $[\zeta_{Z_1}, \zeta_{Z_2}] = -\zeta_{[Z_1, Z_2]}$  which is due to the sign chosen in the definition of the fundamental vector field in Section 1.A. The curvature two-form is given by

$$\mathrm{Curv}_q^A(v_1, v_2) = dA_q(v_1, v_2) + A(q)[v_1, v_2](q) \in \mathfrak{k}.$$

For  $B \in \Omega^1(\bigsqcup_q \mathrm{Ann} \mathfrak{k}_q)$  given by  $B(q, \lambda)(v_1, \lambda_1) = \langle \lambda, A(q).v_1(q) \rangle$  we have

$$\begin{aligned} dB(q, \lambda)((v_1, \lambda_1), (v_2, \lambda_2)) &= \mathcal{L}_{(v_1, \lambda_1)}(B(v_2, \lambda_2))_{(q, \lambda)} \\ &\quad - \mathcal{L}_{(v_2, \lambda_2)}(B(v_1, \lambda_1))_{(q, \lambda)} \\ &\quad - B(q, \lambda)([(v_1, \lambda_1), (v_2, \lambda_2)]) \\ &= \frac{\partial}{\partial t} |_0 B(\mathrm{Fl}_t^{(v_1, \lambda_1)}(q, \lambda))(v_2, \lambda_2) \\ &\quad - \frac{\partial}{\partial t} |_0 B(\mathrm{Fl}_t^{(v_2, \lambda_2)}(q, \lambda))(v_1, \lambda_1) \\ &\quad - \langle \lambda, A(q)[v_1, v_2](q) \rangle \end{aligned}$$

$$\begin{aligned}
&= \frac{\partial}{\partial t} \Big|_0 \langle \mathrm{Fl}_t^{\lambda_1}(\lambda), A(\mathrm{Fl}_t^{v_1}(q))(v_2) \rangle \\
&\quad - \frac{\partial}{\partial t} \Big|_0 \langle \mathrm{Fl}_t^{\lambda_2}(\lambda), A(\mathrm{Fl}_t^{v_2}(q))(v_1) \rangle \\
&\quad + \langle \lambda, [Z_1, Z_2] \rangle \\
&= \langle \lambda_1(\lambda), Z_2 \rangle - \langle \lambda_2(\lambda), Z_1 \rangle \\
&\quad + \langle \lambda, \mathcal{L}_{v_1}(A(v_2))_q \rangle \\
&\quad - \langle \lambda, (\mathcal{L}_{v_2}A)_q(v_1(q)) \rangle \\
&\quad + \langle \lambda, [Z_1, Z_2] \rangle.
\end{aligned}$$

Putting this together we find

$$\begin{aligned}
\sigma_w(\xi_1, \xi_2) &= \Omega_{([q], \eta)}^{Q/K}((v_1^{\mathrm{hor}}, \eta_1), (v_2^{\mathrm{hor}}, \eta_2)) - (\langle \lambda_1(\lambda), Z_2 \rangle - \langle \lambda_2(\lambda), Z_1 \rangle \\
&\quad + \langle \lambda, \mathcal{L}_{v_1}(A(v_2))_q \rangle - \langle \lambda, \mathcal{L}_{v_2}(A(v_1))_q \rangle + \langle \lambda, [Z_1, Z_2] \rangle) \\
&= ((\tilde{\pi} \circ \tilde{\rho})^* \Omega^{Q/K} - d\tilde{\tau}^* B)_w(\xi_1, \xi_2)
\end{aligned}$$

which is the desired formula. Finally, the statement about the  $K$  action on  $\mathcal{W}$  is obvious since  $\psi$  is equivariantly symplectomorphic by construction.  $\square$

## 2.D. Gauged cotangent bundle reduction

**PROPOSITION 2.D.1** (Poisson structure on Weinstein space). *There are stratified isomorphisms of singular bundles over  $Q/K$ :*

$$\begin{aligned}
\alpha = \alpha(A) : \bigsqcup_{(L)} (TQ)_{(L)}/K &\longrightarrow T(Q/K) \times_{Q/K} \bigsqcup_{(L)} (\bigsqcup_{q \in Q} \mathfrak{k}_q^\perp)_{(L)}/K, \\
[(q, v)] &\longmapsto (T\pi(q, v), [(q, A_q v)])
\end{aligned}$$

where  $(L)$  runs through the isotropy lattice of  $TQ$ . The dual isomorphism is given by

$$\begin{aligned}
\beta = (\alpha^{-1})^* : (T^*Q)/K &\longrightarrow T^*(Q/K) \times_{Q/K} (\bigsqcup_{q \in Q} \mathrm{Ann} \mathfrak{k}_q)/K =: W, \\
[(q, p)] &\longmapsto (C^*(q, p), [(q, \mu(q, p))])
\end{aligned}$$

where the stratification was suppressed. Here

$$C^* : T^*Q \rightarrow \mathrm{Hor}^* \rightarrow T^*(Q/K)$$

is constructed as the point wise dual to the horizontal lift mapping  $C : T(Q/K) \times_{Q/K} Q \rightarrow \mathrm{Hor} \subseteq TQ$ ,  $([q], v; q) \rightarrow C_q(v)$ .

Moreover,  $\beta$  is an isomorphism of Poisson spaces as follows. There is a natural isomorphism

$$\begin{aligned}
\mathcal{W}/K &\xrightarrow{\cong} W, \\
[(q; [q], \eta; q, \lambda)] &\longmapsto ([q], \eta; [(q, \lambda)])
\end{aligned}$$

thus inducing a quotient Poisson bracket on  $C^\infty(W) \cong C^\infty(\mathcal{W})^K$  as the quotient Poisson bracket.

In the case that  $K$  acts on  $Q$  freely the first assertion of the above proposition can also be found in Cendra, Holm, Marsden, Ratiu [10]. Following Ortega and Ratiu [33, Section 6.6.12] the above constructed interpretation  $W$  of  $(T^*Q)/K$  is called WEINSTEIN SPACE referring to Weinstein [50] where this universal construction first appeared.

PROOF. As already noted above,  $(TQ)/K$  is a stratified space. Since the base  $Q$  is stratified as consisting only of a single stratum, the equivariant foot point projection map  $\tau : TQ \rightarrow Q$  is trivially a stratified map. Using the Slice Theorem on the base  $Q$  it is easy to see that both  $(TQ)/K \rightarrow Q/K$  and the projection  $(\bigsqcup_{q \in Q} \mathfrak{k}_q^\perp)/K \rightarrow Q/K$  are singular bundle maps in the sense of Definition 2.B.1. Hereby  $(\bigsqcup_{q \in Q} \mathfrak{k}_q^\perp)/K$  is stratified into isotropy types, According to Davis [12] or also Section 2.B pullbacks are well defined in the category of stratified spaces and thus it makes sense to define  $T(Q/K) \times_{Q/K} (\bigsqcup_{q \in Q} \mathfrak{k}_q^\perp)/K$  as a stratified space with smooth structure.

The map  $\alpha$  is well defined: indeed, for  $(q, v) \in TQ$  and  $k \in K$  we have

$$\begin{aligned} T\pi(k.q, k.v) &= (\pi(k.q), T_{k.q}\pi(T_q l_k(v))) \\ &= (\pi(q), T_q(\pi \circ l_k)(v)) \\ &= T_q\pi(v), \end{aligned}$$

and  $[(k.q, A(k.q, k.v))] = [(q, A(q, v))]$  by equivariance of  $A$ . It is clearly continuous as a composition of continuous maps. Moreover, since  $C^\infty((TQ)/K) = C^\infty(TQ)^K$  by Example 1.C.7 it follows that  $\alpha$  is a smooth map of singular spaces.

We claim that  $\alpha$  maps strata onto strata, and moreover we have the formula

$$\alpha((TQ)_{(L)}/K) = T(Q/K) \times_{Q/K} (\bigsqcup_{q \in Q} \mathfrak{k}_q^\perp)_{(L)}/K.$$

Indeed, consider  $(q, v) \in (TQ)_{(L)}$ , that is  $H \cap K_v = L' \sim L$  where  $H = K_q$ . The notation  $L' \sim L$  means that  $L'$  is conjugate to  $L$  within  $K$ . Now we can decompose  $v$  as  $v = v_0 \oplus \zeta_X(q) \in \text{Hor}_q \oplus T_q(K.q)$  for some appropriate  $X \in \mathfrak{k}$ . Since  $Q$  consists only of a single isotropy type we have  $T_qQ = T_qQ_H + T_q(K.q)$  – which is not a direct sum decomposition. As usual,  $Q_H = \{q \in Q : K_q = H\}$ . This shows that  $v_0 \in T_qQ_H$ , and hence  $H \subseteq K_{v_0}$ . By equivariance of  $A$  it follows that

$$K_q \cap K_v = H \cap K_{v_0} \cap K_{\zeta_X(q)} = H \cap K_{\zeta_X(q)} = H \cap K_{A(q,v)}$$

which is independent of the horizontal component. Hence the claim. The restriction of  $\alpha$  to any stratum clearly is smooth as a composition of smooth maps.

Since  $A_q(\zeta_X(q)) = X$  for  $X \in \mathfrak{k}_q^\perp$  we can write down an inverse as

$$\alpha^{-1} : ([q], v; [(q, X)]) \rightarrow [(q, C_q(v) + \zeta_X(q))]$$

and again it is an easy matter to notice that this map is well defined, continuous, and smooth on each stratum. Again, it follows from the definition of the smooth structures on the respective spaces that  $\alpha^{-1}$  is smooth.

It makes sense to define the dual  $\beta$  of the inverse map  $\alpha^{-1}$  in a point wise manner, and it only remains to compute this map.

$$\begin{aligned} \langle \beta[(q, p)], ([q], v; [(q, X)]) \rangle &= \langle [(q, p)], [(q, C_q(v) + \zeta_X(q))] \rangle \\ &= \langle p, C_q(v) \rangle + \langle p, \zeta_X(q) \rangle \\ &= \langle C^*(q, p), v \rangle + \langle \mu(q, p), X \rangle \\ &= \langle (C^*(q, p), [(q, \mu(q, p)]), ([q], v; [(q, X)]) \rangle \end{aligned}$$

where we used the  $K$ -invariance of the dual pairing over  $Q$ .

Finally,  $\beta$  is an isomorphism of singular Poisson spaces: note first that the identifying map  $\mathcal{W}/K \rightarrow W$ ,  $[(q; [q], \eta; q, \lambda)] \mapsto ([q], \eta; [(q, \lambda)])$  is well-defined because  $K_q$  acts trivially on  $\text{Hor}_q^* = T_{[q]}^*(Q/K) \ni \eta$  which in turn is due to the fact that all points of  $Q$  are regular. Moreover, by the universal property for singular pull back bundles from Section 2.B it is obvious that this map  $\mathcal{W}/K \rightarrow W$  is smooth and has a smooth inverse. The quotient Poisson bracket is well-defined since  $C^\infty(\mathcal{W})^K \subseteq C^\infty(\mathcal{W})$  is a Poisson sub-algebra. The statement now follows because the diagram

$$\begin{array}{ccc} T^*Q & \xrightarrow{\psi^{-1}} & \mathcal{W} \\ \downarrow & & \searrow \\ (T^*Q)/K & \xrightarrow{\beta} & W = \mathcal{W}/K \end{array}$$

is commutative, and composition of top and down-right arrow is Poisson and the left vertical arrow is surjective.  $\square$

LEMMA 2.D.2. *Let  $\mathcal{O} \subseteq \mathfrak{k}^*$  be a coadjoint orbit,  $\mu : T^*Q \rightarrow \mathfrak{k}^*$  the canonical momentum mapping, and  $\mu_q := \mu|_{(T_q^*Q)}$ . Then either  $\mu_q^{-1}(\mathcal{O}) = \emptyset$  for all  $q \in Q$  or  $\mu_q^{-1}(\mathcal{O}) \neq \emptyset$  for all  $q \in Q$ . In the latter case we have*

$$\mu_q^{-1}(\mathcal{O}) = \text{Ann}_q(T_q(K \cdot q)) \times \{A_q^*(\lambda) : \lambda \in \text{Ann } \mathfrak{k}_q \cap \mathcal{O}\}$$

which is an equality of topological spaces and where  $A_q^*$  is the adjoint of  $A_q : T_qQ \rightarrow \mathfrak{k}_q^\perp$ .

PROOF. Via the  $K$ -invariant inner product on  $\mathfrak{k}$  we identify  $\mathfrak{k}^*$  with  $\mathfrak{k}$ . Thus  $\mathcal{O}$  is an  $\text{Ad}(K)$ -orbit. Moreover, we identify  $T^*Q$  and  $TQ$  via the  $K$ -invariant metric on  $Q$ . The statement in the lemma is now equivalent to the following assertion. Given  $q_1, q_2 \in Q$ , then  $A_{q_1}^{-1}(\mathcal{O}) \neq \emptyset$  if and only if  $A_{q_2}^{-1}(\mathcal{O}) \neq \emptyset$ . If  $q_1$  and  $q_2$  are in the same  $K$ -orbit then this is obvious. Therefore we can assume without loss of generality

that  $K_{q_1} = K_{q_2}$ : all isotropy subgroups are conjugate to each other, and  $q_2$  can be moved around in its orbit.

Let  $X = A_{q_1}(v_1) \in \mathcal{O} \cap \mathfrak{k}_{q_1}^\perp$  with  $v_1 = \zeta_X(q_1) \in \text{Ver}_{q_1}$ . Then also  $A_{q_2}(\zeta_X(q_2)) = X \in \mathcal{O} \cap \mathfrak{k}_{q_2}^\perp$  since  $\mathfrak{k}_{q_1}^\perp = \mathfrak{k}_{q_2}^\perp$ .  $\square$

The action of Proposition 2.C.1 by  $K$  on  $\mathcal{W}$  is Hamiltonian with momentum map  $\mu_A = \mu \circ \psi : \mathcal{W} \rightarrow \mathfrak{k}^*$ .

LEMMA 2.D.3. *Let  $\mathcal{O}$  be a coadjoint orbit in the image of the momentum map  $\mu_A : \mathcal{W} \rightarrow \mathfrak{k}^*$ . Further, let  $(L)$  be in the isotropy lattice of the  $K$ -action on  $\mathcal{W}$  such that  $\mu_A^{-1}(\mathcal{O}) \cap \mathcal{W}_{(L)} \neq \emptyset$ . Then*

$$\mathcal{W}_{(L)} = (Q \times_{Q/K} T^*(Q/K)) \times_Q (\bigsqcup_{q \in Q} \text{Ann } \mathfrak{k}_q)_{(L)}$$

and

$$\mathcal{W}_{(L)} \cap \mu_A^{-1}(\mathcal{O}) = (Q \times_{Q/K} T^*(Q/K)) \times_Q (\bigsqcup_{q \in Q} \text{Ann } \mathfrak{k}_q \cap \mathcal{O})_{(L)}$$

are smooth manifolds. Moreover,

$$\mathcal{O}_{(L_0)H} \cap \text{Ann } \mathfrak{h} \longrightarrow (\bigsqcup_{q \in Q} \mathcal{O} \cap \text{Ann } \mathfrak{k}_q)_{(L)} \longrightarrow Q$$

is a smooth fiber bundle where  $L_0$  is a subgroup of  $H$  such that  $L_0$  is conjugate to  $L$  within  $K$ . The space  $\mathcal{O}_{(L_0)H}$  denotes the isotropy sub-manifold of type  $L_0$  of  $\mathcal{O}$  with regard to the  $\text{Ad}^*(H)$ -action on  $\mathcal{O}$ .

Notice that  $\mathcal{O} \cap \text{Ann } \mathfrak{h}$  is not smooth, in general.

PROOF. The statement about  $\mathcal{W}_{(L)}$  is clear. Thus also the description of  $\mathcal{W}_{(L)} \cap \mu_A^{-1}(\mathcal{O})$  follows from the previous lemma together with Theorem 1.H.1.

Now to the second assertion. Let  $q_0 \in Q$  with  $K_{q_0} = H$ . Then

$$(q_0, \lambda) \in (\bigsqcup_{q \in Q} \mathcal{O} \cap \text{Ann } \mathfrak{k}_q)_{(L)}$$

if and only if

$$\lambda \in \mathcal{O} \cap \text{Ann } \mathfrak{h} \text{ and } H \cap K_\lambda = H_\lambda = L_0 \sim L \text{ within } K$$

which is true if and only if

$$\lambda \in (\mathcal{O} \cap \text{Ann } \mathfrak{h})_{(L_0)H} = \mathcal{O}_{(L_0)H} \cap \text{Ann } \mathfrak{h}$$

where  $L_0$  is a subgroup of  $H$  conjugate to  $L$  within  $K$ .

Consider the  $\text{Ad}^*(H)$  action on  $\mathcal{O}$ . This action is a Hamiltonian one with momentum map given by  $\rho : \mathcal{O} \rightarrow \mathfrak{h}^*$ ,  $\lambda \mapsto \lambda|_{\mathfrak{h}}$ , i.e. by restriction. Thus  $\mathcal{O} \cap \text{Ann } \mathfrak{h} = \rho^{-1}(0)$  which however is not smooth but only a stratified space in general. Typical smooth strata of this space are of the form  $\mathcal{O}_{(L_0)H} \cap \text{Ann } \mathfrak{h}$  with  $L_0$  a subgroup of  $H$ .

To see smooth local triviality we proceed as follows. Let again  $q_0 \in Q$  with  $K_{q_0} = H$ , and let  $S$  be a slice at  $q_0$  and  $U$  a tube around  $K.q_0$ .

That is,  $K/H \times S \cong U$ ,  $(kH, s) \mapsto k.s$ . Then we consider the smooth trivializing map

$$\begin{aligned} S \times K \times_H (\mathcal{O} \cap \text{Ann } \mathfrak{h})_{(L_0)H} &\longrightarrow (\bigsqcup_{q \in Q} \mathcal{O} \cap \text{Ann } \mathfrak{k}_q)_{(L)}|U, \\ (s, [(k, \lambda)]) &\longmapsto (k.s, \text{Ad}^*(k).\lambda) \end{aligned}$$

which is well defined since  $\text{Ad}^*(k).\lambda \in (\text{Ann } \mathfrak{k}_{k.s} \cap \mathcal{O})_{(kL_0k^{-1})K_{k.s}}$ , and the uncertainty coming from the diagonal  $H$ -action just cancels out. Clearly this map is smooth with the obvious smooth inverse

$$(q, \lambda) = (k.s, \text{Ad}^*(k).\lambda_0) \longmapsto (s, [(k, \lambda_0)]).$$

In particular this construction constructs smooth bundle charts of the total space  $(\bigsqcup_{q \in Q} \mathcal{O} \cap \text{Ann } \mathfrak{k}_q)_{(L)}$ .  $\square$

The singular reduction diagram of Ortega and Ratiu [33, Theorem 8.4.4] adjoined to the universal reduction procedure of Arms, Cushman, and Gotay [4], see also [33, Section 10.3.2] applied to the Weinstein space has the following form.

$$\begin{array}{ccccc} \mu_A^{-1}(\mathcal{O}) & \longleftarrow & \mu_A^{-1}(\lambda) & \hookrightarrow & \mathcal{W} \\ \downarrow & & \downarrow & & \downarrow \\ \mu_A^{-1}(\mathcal{O})/K & \xleftarrow{\simeq} & \mu_A^{-1}(\lambda)/K & \hookrightarrow & \mathcal{W}/K = W \end{array}$$

where  $\lambda \in \mu_A(\mathcal{W})$  and  $\mathcal{O}$  is the coadjoint orbit passing through  $\lambda$ . Therefore it is a sensible generalization of the smooth case to interpret the reduced  $\mu_A^{-1}(\mathcal{O})/K = \mathcal{W} //_{\mathcal{O}} K$  as a typical stratified symplectic leaf of the stratified Poisson space  $W$ . The following thus generalizes the result of Marsden and Perlmutter [25, Theorem 4.3] to the case of a non-free but single isotropy type action of  $K$  on  $Q$ .

What now follows is notation for the upcoming theorem. Let  $\mathcal{O}$  be a coadjoint orbit in the image of the momentum map  $\mu_A : \mathcal{W} \rightarrow \mathfrak{k}^*$ , and let  $(L)$  be in the isotropy lattice of the  $K$ -action on  $\mathcal{W}$  such that  $\mathcal{W}_{(L)}^{\mathcal{O}} := \mu_A^{-1}(\mathcal{O}) \cap \mathcal{W}_{(L)} \neq \emptyset$ . Then we have

$$\iota_{(L)}^{\mathcal{O}} : \mathcal{W}_{(L)}^{\mathcal{O}} \hookrightarrow \mathcal{W},$$

the canonical embedding, and the orbit projection mapping

$$\pi_{(L)}^{\mathcal{O}} : \mathcal{W}_{(L)}^{\mathcal{O}} \rightarrow \mathcal{W}_{(L)}^{\mathcal{O}}/K =: (\mathcal{W} //_{\mathcal{O}} K)_{(L)}.$$

Consider furthermore

$$\rho_{(L)}^{\mathcal{O}} : (\bigsqcup_{q \in Q} \mathcal{O} \cap \text{Ann } \mathfrak{k}_q)_{(L)} \rightarrow (\bigsqcup_{q \in Q} \mathcal{O} \cap \text{Ann } \mathfrak{k}_q)_{(L)}/K$$

and

$$\phi_{(L)}^{\mathcal{O}} : (\bigsqcup_{q \in Q} \mathcal{O} \cap \text{Ann } \mathfrak{k}_q)_{(L)} \longrightarrow \mathcal{O}, \quad (q, \lambda) \longmapsto \lambda$$

as well as the embedding

$$j_{(L)}^{\mathcal{O}} : (\bigsqcup_{q \in Q} \mathcal{O} \cap \text{Ann } \mathfrak{k}_q)_{(L)} \hookrightarrow \bigsqcup_{q \in Q} \text{Ann } \mathfrak{k}_q.$$

Finally, we denote the Kirillov-Kostant-Souriau symplectic form on  $\mathcal{O}$  by  $\Omega^\mathcal{O}$ , that is  $\Omega^\mathcal{O}(\lambda)(\text{ad}^*(X).\lambda, \text{ad}^*(Y).\lambda) = \langle \lambda, [X, Y] \rangle$ . Remember from Proposition 2.C.1 that the symplectic structure on  $\mathcal{W}$  is denoted by  $\sigma$ .

**THEOREM 2.D.4** (Gauged symplectic reduction). *Let  $Q = Q_{(H)}$ , let  $\mathcal{O}$  be a coadjoint orbit in the image of the momentum map  $\mu_A : \mathcal{W} \rightarrow \mathfrak{k}^*$ , and let  $(L)$  be in the isotropy lattice of the  $K$ -action on  $\mathcal{W}$  such that  $\mathcal{W}_{(L)}^\mathcal{O} := \mu_A^{-1}(\mathcal{O}) \cap \mathcal{W}_{(L)} \neq \emptyset$ . Then the following are true.*

(i) *The smooth manifolds  $(\mathcal{W} //_{\mathcal{O}} K)_{(L)}$  and*

$$(\mathcal{O} //_0 H)_{(L_0)^H} =: (\mathcal{O} \cap \text{Ann } \mathfrak{h})_{(L_0)^H} / H$$

*are typical symplectic strata of the stratified symplectic spaces  $\mathcal{W} //_{\mathcal{O}} K$  and  $\mathcal{O} //_0 H$  respectively. Here  $L_0$  is an isotropy subgroup of the induced  $H$ -action on  $\mathcal{O}$  and  $(L_0)^H$  denotes its isotropy class in  $H$ .*

(ii) *The symplectic stratum  $(\mathcal{W} //_{\mathcal{O}} K)_{(L)}$  can be globally described as*

$$(\mathcal{W} //_{\mathcal{O}} K)_{(L)} = T^*(Q/K) \times_{Q/K} (\bigsqcup_{q \in Q} \mathcal{O} \cap \text{Ann } \mathfrak{k}_q)_{(L)} / K$$

*whence it is the total space of the smooth symplectic fiber bundle*

$$(\mathcal{O} //_0 H)_{(L_0)^H} \hookrightarrow (\mathcal{W} //_{\mathcal{O}} K)_{(L)} \longrightarrow T^*(Q/K)$$

*Hereby  $L_0$  is an isotropy subgroup of the induced  $H$ -action on  $\mathcal{O}$  which is conjugate in  $K$  to  $L$ , and  $(L_0)^H$  denotes its isotropy class in  $H$ .*

(iii) *The symplectic structure  $\sigma_{(L)}^\mathcal{O}$  on  $(\mathcal{W} //_{\mathcal{O}} K)_{(L)}$  is uniquely determined and given by the formula*

$$(\pi_{(L)}^\mathcal{O})^* \sigma_{(L)}^\mathcal{O} = (\iota_{(L)}^\mathcal{O})^* \sigma - (\mu_A|_{\mathcal{W}_{(L)}^\mathcal{O}})^* \Omega^\mathcal{O}.$$

*More precisely,*

$$\sigma_{(L)}^\mathcal{O} = \Omega^{Q/K} - \beta_{(L)}^\mathcal{O}$$

*where  $\beta_{(L)}^\mathcal{O} \in \Omega^2((\bigsqcup_{q \in Q} \mathcal{O} \cap \text{Ann } \mathfrak{k}_q)_{(L)} / K)$  is defined by*

$$(\rho_{(L)}^\mathcal{O})^* \beta_{(L)}^\mathcal{O} = (j_{(L)}^\mathcal{O})^* dB + (\phi_{(L)}^\mathcal{O})^* \Omega^\mathcal{O}.$$

*Finally  $B$  is the form that was introduced in Proposition 2.C.1.*

*Thus for  $(q, \lambda) \in (\bigsqcup_{q \in Q} \mathcal{O} \cap \text{Ann } \mathfrak{k}_q)_{(L)}$  and*

$$(v_i, \text{ad}^*(X_i).\lambda) \in T_{(q, \lambda)}(\bigsqcup_{q \in Q} \mathcal{O} \cap \text{Ann } \mathfrak{k}_q)_{(L)}$$

*where  $i = 1, 2$  we have the explicit formulas*

$$B_{(q, \lambda)}(v_i, \text{ad}^*(X_i).\lambda) = \langle \lambda, A_q(v_i) \rangle$$

*and also*

$$\begin{aligned} dB_{(q, \lambda)}((v_1, \text{ad}^*(X_1).\lambda), (v_2, \text{ad}^*(X_2).\lambda)) \\ = \langle \lambda, \text{Curv}_q^A(v_1, v_2) \rangle + \langle \lambda, [X_2, Z_1] \rangle - \langle \lambda, [X_1, Z_2] \rangle + \langle \lambda, [Z_1, Z_2] \rangle \end{aligned}$$

where  $v_i = \zeta_{Z_i}(q) \oplus v_i^{\text{hor}} \in \text{Ver}_q \oplus \text{Hor}_q$  is the decomposition into vertical and horizontal parts with  $Z_i \in \mathfrak{k}$ .

(iv) The stratified symplectic space can be globally described as

$$\mathcal{W} //_{\mathcal{O}} K = T^*(Q/K) \times_{Q/K} \bigsqcup_{q \in Q} \mathcal{O} \cap \text{Ann } \mathfrak{k}_q / K$$

whence it is canonically the total space of

$$\mathcal{O} //_0 H \hookrightarrow \mathcal{W} //_{\mathcal{O}} K \longrightarrow T^*(Q/K)$$

which is a singular symplectic fiber bundle with singularities confined to the fiber direction in the sense of Definition 2.B.1.

PROOF. Assertion (i). This is well-known as a general principle of stratified symplectic reduction – see Ortega and Ratiu [33, Section 8.4] or Section 1.H.

Assertion (ii). We know from above that all spaces involved in the diagram really are smooth. As in the proof of Lemma 2.D.3 let  $q_0 \in Q$  with  $K_{q_0} = H$ ,  $S$  a slice at  $q_0$ , and  $U \cong K/H \times S$  a tube around the orbit  $K.q_0$ . Then we get the local description

$$\begin{aligned} (\mathcal{W} //_{\mathcal{O}} K)_{(L)}|U &= T^*S \times_S (\bigsqcup_{q \in U} \mathcal{O} \cap \text{Ann } \mathfrak{k}_q)_{(L)} / K \\ &\cong T^*S \times_S S \times (\mathcal{O} \cap \text{Ann } \mathfrak{h})_{(L_0)H} / H \\ &= T^*S \times (\mathcal{O} //_0 H)_{(L_0)H} \end{aligned}$$

as claimed.

The bundle is symplectic: This follows from Theorem 2.A.4.

Assertion (iii). The defining property of the reduced symplectic form  $\sigma_{(L)}^{\mathcal{O}}$ , namely,

$$(\pi_{(L)}^{\mathcal{O}})^* \sigma_{(L)}^{\mathcal{O}} = (\iota_{(L)}^{\mathcal{O}})^* \sigma - (\mu_A|_{\mathcal{W}_{(L)}^{\mathcal{O}}})^* \Omega^{\mathcal{O}}$$

is a well-established fact, see e.g. Bates and Lerman [6, Proposition 11]. Thus it is clear from Proposition 2.C.1 that

$$\sigma_{(L)}^{\mathcal{O}} = \Omega^{Q/K} - \beta_{(L)}^{\mathcal{O}}$$

– if  $\beta_{(L)}^{\mathcal{O}}$  is a well defined two-form on  $(\bigsqcup_{q \in Q} \mathcal{O} \cap \text{Ann } \mathfrak{k}_q)_{(L)} / K$  such that

$$(\rho_{(L)}^{\mathcal{O}})^* \beta_{(L)}^{\mathcal{O}} = (j_{(L)}^{\mathcal{O}})^* dB + (\phi_{(L)}^{\mathcal{O}})^* \Omega^{\mathcal{O}}.$$

To see this notice firstly that

$$\tilde{\beta} := (j_{(L)}^{\mathcal{O}})^* dB + (\phi_{(L)}^{\mathcal{O}})^* \Omega^{\mathcal{O}} \in \Omega^2((\bigsqcup_{q \in Q} \mathcal{O} \cap \text{Ann } \mathfrak{k}_q)_{(L)})$$

is  $K$ -invariant. Furthermore, we claim that  $\tilde{\beta}$  is horizontal, i.e. vanishes upon insertion of a vertical vector field. Indeed, let

$$(q, \lambda) \in (\bigsqcup_{q \in Q} \mathcal{O} \cap \text{Ann } \mathfrak{k}_q)_{(L)},$$

and  $Z_i \in \mathfrak{k}$  for  $i = 1, 2$ , and  $Y \in \mathfrak{k}$  such that

$$\text{ad}^*(Z_1).\lambda, \text{ad}^*(Y).\lambda \in T_{\lambda} \mathcal{O}_{(L_0)K_q} \cap \text{Ann } \mathfrak{k}_q,$$



and consider  $v_2(q) = v_2^{\text{hor}} \oplus \zeta_{Z_2}(q) \in \text{Hor}_q \oplus \text{Ver}_q$  as in the proof of Proposition 2.C.1. Then we have

$$\begin{aligned}
& \tilde{\beta}_{(q,\lambda)}((\zeta_{Z_1}(q), \text{ad}^*(Z_1).\lambda), (v_2(q), \text{ad}^*(Y).\lambda)) \\
&= 0 + \langle \lambda, [Z_1, Z_2] \rangle - \langle \text{ad}^*(Y).\lambda, Z_1 \rangle + \langle \text{ad}^*(Z_1).\lambda, Z_2 \rangle \\
&\quad + \langle \lambda, [Z_1, Y] \rangle \\
&= \langle \lambda, [Z_1, Z_2] \rangle + \langle \lambda, [Y, Z_1] \rangle - \langle \lambda, [Z_1, Z_2] \rangle \\
&\quad + \langle \lambda, [Z_1, Y] \rangle \\
&= 0.
\end{aligned}$$

That is  $\tilde{\beta}$  is a basic form and thus descends to a form  $\beta_{(L)}^{\mathcal{O}}$ .

Assertion (iv) is a pasting together of the results in (ii).  $\square$

**COROLLARY 2.D.5.** *Let  $\mathcal{O}$  be a coadjoint orbit in the image of the momentum map  $\mu_A : \mathcal{W} \rightarrow \mathfrak{k}^*$ , and let  $(L)$  be in the isotropy lattice of the  $K$ -action on  $\mathcal{W}$  such that  $\mathcal{W}_{(L)}^{\mathcal{O}} := \mu_A^{-1}(\mathcal{O}) \cap \mathcal{W}_{(L)} \neq \emptyset$ . Assume further that there is a global slice  $S$  such that  $Q \cong K/H \times S$ . Then we have the global description*

$$\mathcal{W} //_{\mathcal{O}} K = T^*S \times \mathcal{O} //_0 H.$$

Moreover, the reduced symplectic form  $\sigma_{(L)}^{\mathcal{O}}$  on a symplectic stratum  $(\mathcal{W} //_{\mathcal{O}} K)_{(L)} = T^*S \times (\mathcal{O} //_0 H)_{(L_0)H}$  is given by the formula

$$\sigma_{(L)}^{\mathcal{O}} = \Omega^{Q/K} - \Omega_{(L_0)H}^{\mathcal{O}}$$

where  $\Omega_{(L_0)H}^{\mathcal{O}}$  is the canonically reduced symplectic form on  $(\mathcal{O} //_0 H)_{(L_0)H}$ , and  $L_0$  is a subgroup of  $H$  which is conjugate to  $L$  within  $K$ .

**PROOF.** This is an immediate consequence of Theorems 2.A.4 and 2.D.4.  $\square$

## 2.E. Gauged Poisson reduction

Let us introduce the abbreviation  $E := \bigsqcup_{q \in Q} \text{Ann } \mathfrak{k}_q$ .

**LEMMA 2.E.1.** *The natural projection  $\rho : E \rightarrow Q$  is a smooth fiber bundle with typical fiber  $\text{Ann } \mathfrak{h}$ .*

**PROOF.** We trivialize at an arbitrary point  $q_0 \in Q$ . We may assume  $K_{q_0} = H$ . Let  $U \subseteq Q$  be a tube around  $K.q_0$  such that  $U \cong S \times K/H$  as  $K$ -spaces, where  $S$  is a slice at  $q_0$ . Then it is possible to trivialize as

$$\begin{array}{ccc}
E|_U & \xrightarrow{\cong} & S \times (K \times \text{Ann } \mathfrak{h})/H & \xlongequal{\quad} & S \times K \times_H \text{Ann } \mathfrak{h} \\
\downarrow & & \downarrow & & \\
U & \xrightarrow{\cong} & S \times K/H & & 
\end{array}$$

with trivializing map given by

$$\begin{aligned} S \times K \times_H \text{Ann } \mathfrak{h} &\longrightarrow E|_U \\ (s, [(k, \lambda)]) &\longmapsto (k.s, \text{Ad}^*(k)(\lambda)) \end{aligned}$$

where  $\text{Ad}^*(k)(\lambda) := \text{Ad}(k^{-1})^*(\lambda)$ . This map is well-defined and smooth with inverse given by

$$(q, \lambda) = (k.s, \text{Ad}^*(k)(\lambda_0)) \longmapsto (s, [(k, \lambda_0)])$$

which is well-defined and smooth as well. Notice furthermore that the trivializing map is  $K$ -equivariant with respect to the  $K$ -action on  $S \times K \times_H \text{Ann } \mathfrak{h}$  given by  $g.(s, [(k, \lambda)]) = (s, [(gk, \lambda)])$ . Indeed, this follows immediately from the proof of Lemma 2.A.1.  $\square$

LEMMA 2.E.2. *Let  $E := \bigsqcup_{q \in Q} \text{Ann } \mathfrak{k}_q$ .*

- (i) *Let  $U \subseteq Q$  be a trivializing neighborhood for  $\rho : E \rightarrow Q$  as in the proof of Lemma 2.E.1. Then  $E|_U$  is  $K$ -invariant, and if  $(L)$  is an element of the isotropy lattice for the  $K$ -action on  $E$  then the corresponding stratum is trivialized as*

$$(E|_U)_{(L)} \cong S \times K \times_H (\text{Ann } \mathfrak{h})_{(L_0)^H} / H$$

*where  $L_0 \subseteq H$  is an isotropy subgroup for the  $H$ -action conjugate to  $L$  in  $K$ , and  $(L_0)^H$  is the conjugacy class of  $L_0$  in  $H$ . Moreover, the strata of  $\text{Ann } \mathfrak{h}/H$  are of the form  $(\text{Ann } \mathfrak{h})_{(L_0)^H} / H$ .*

- (ii) *The induced mapping  $\rho_0 : E/K \rightarrow Q/K$  is a singular fiber bundle with typical fiber  $\text{Ann } \mathfrak{h}/H$  in the sense of Definition 2.B.1.*

PROOF. The proof of this Lemma works like that of Lemma 2.D.3. Indeed, let  $q_0 \in Q$  with  $K_{q_0} = H$ . Then

$$(q_0, \lambda) \in (\bigsqcup_{q \in Q} \text{Ann } \mathfrak{k}_q)_{(L)} = E_{(L)}$$

if and only if

$$\lambda \in \text{Ann } \mathfrak{h} \text{ and } H \cap K_\lambda = H_\lambda = L_0 \sim L \text{ within } K$$

which is true if and only if

$$\lambda \in (\text{Ann } \mathfrak{h})_{(L_0)^H}$$

where  $L_0$  is a subgroup of  $H$  conjugate to  $L$  within  $K$ . Notice that it follows from the Slice Theorem for Riemannian actions that  $(\text{Ann } \mathfrak{h})_{(L_0)^H}$  is a smooth manifold. Therefore, also the second assertion follows.  $\square$

THEOREM 2.E.3. *There is a stratified isomorphism of stratified bundles over  $Q/K$*

$$\begin{aligned} \psi_0 = \psi_0(A) : T^*(Q/K) \times_{Q/K} (\bigsqcup_{q \in Q} \text{Ann } \mathfrak{k}_q) / K =: W &\longrightarrow (T^*Q)/K, \\ (C^*(q, p), [(q, \mu(q, p))]) &\longmapsto [(q, p)] \end{aligned}$$

*where the stratification was suppressed. Here*

$$C^* : T^*Q \rightarrow \text{Hor}^* \rightarrow T^*(Q/K)$$

is constructed as the point wise dual to the horizontal lift mapping  $C : T(Q/K) \times_{Q/K} Q \rightarrow \text{Hor} \subseteq TQ$ ,  $([q], v; q) \rightarrow C_q(v)$  associated to the connection  $A \in \Omega^1(Q; \mathfrak{k})$ .

If  $(L)$  is an isotropy class of the  $K$ -action on  $T^*Q$ , then  $\psi_0^{-1}$  maps the isotropy stratum  $(T^*Q)_{(L)}/K$  onto

$$T^*(Q/K) \times_{Q/K} (\bigsqcup_{q \in Q} \text{Ann } \mathfrak{k}_q)_{(L)}/K =: W_{(L)}.$$

Moreover, the natural projection

$$\tilde{\rho}_0^{(L)} : W_{(L)} \rightarrow T^*(Q/K)$$

is a smooth Poisson fiber bundle with typical fiber of the form  $(\text{Ann } \mathfrak{h})_{(L_0)H}/H$  in the sense that  $\tilde{\rho}_0^{(L)}$  is a Poisson morphism. Here  $L_0 \subseteq H$  is an isotropy subgroup for the  $H$ -action conjugate to  $L$  in  $K$ , and  $(L_0)^H$  is the conjugacy class of  $L_0$  in  $H$ .

Therefore,  $\tilde{\rho}_0 : W \rightarrow T^*(Q/K)$  is a singular Poisson fiber bundle in the sense of Section 2.B.

In the case that  $K$  acts on  $Q$  freely the first assertion of the above theorem can also be found in Cendra, Holm, Marsden, Ratiu [10]. Following Ortega and Ratiu [33, Section 6.6.12] the above constructed interpretation  $W$  of  $(T^*Q)/K$  is called WEINSTEIN SPACE referring to Weinstein [50] where this universal construction first appeared.

PROOF. It only remains to show that  $\tilde{\rho}_0 : W \rightarrow T^*(Q/K)$  is a Poisson morphism. Indeed, all the other assertions follow from Proposition 2.D.1 and Lemma 2.E.2 together with Section 2.B. To see that  $\tilde{\rho}_0 : W \rightarrow T^*(Q/K)$  is a Poisson morphism notice firstly that

$$\text{Hor}^*(\pi : Q \rightarrow Q/K) = \mu^{-1}(0)$$

where  $\mu : T^*Q \rightarrow \mathfrak{k}^*$  is the canonical momentum map. Since the inclusion mapping  $\mu^{-1}(0) \hookrightarrow T^*Q$  is an injective Poisson mapping it follows that also the horizontal projection  $T^*Q \rightarrow \text{Hor}^*(\pi) = \mu^{-1}(0)$  is a Poisson morphism. Therefore, the thus constructed projection mapping

$$T^*Q \longrightarrow \mu^{-1}(0) \twoheadrightarrow \mu^{-1}(0)/K \cong T^*(Q/K)$$

is a morphism of Poisson manifolds. Hereby, it is important to remark that the isomorphism  $\mu^{-1}(0)/K \cong T^*(Q/K)$  is a symplectomorphism from the Marsden-Weinstein reduced structure on  $\mu^{-1}(0)/K$  into the canonical cotangent bundle symplectic form on  $T^*(Q/K)$ . By the characterization of the Poisson structure on  $W$  from Proposition 2.D.1 this implies that  $\tilde{\rho}_0 : W \rightarrow T^*(Q/K)$  indeed is a Poisson morphism, and the same is true for the smooth bundle projection map  $\tilde{\rho}_0^{(L)} : W_{(L)} \rightarrow T^*(Q/K)$ .  $\square$

Next we shall construct a connection on  $\rho : E \rightarrow Q$  which will provide a connection on  $\tilde{\rho} : \mathcal{W} \rightarrow Q \times_{Q/K} T^*(Q/K)$ . Recall the mechanical connection  $A \in \Omega^1(Q; \mathfrak{k})$  from Section 2.C. Consider the embedding

$$\begin{aligned} \iota : E &\longrightarrow T^*Q \\ (q, \lambda) &\longmapsto (q, A_q^*(\lambda)). \end{aligned}$$

On  $\varrho : T^*Q \rightarrow Q$  we choose the canonical (with respect to the metric) linear connection  $\Phi(\varrho) : TT^*Q \rightarrow V(\varrho)$ . Consider the following diagram

$$\begin{array}{ccc} TE & \dashrightarrow & \mathfrak{k}^* \\ T\iota \downarrow & & \uparrow d\mu|_{V(\varrho)} \\ TT^*Q & \xrightarrow{\Phi(\varrho)} & V(\varrho) \end{array}$$

which induces a connection on  $\rho : E \rightarrow Q$ . Via the pullback construction this also induces a connection on  $\tilde{\rho} : \mathcal{W} \rightarrow Q \times_{Q/K} T^*(Q/K)$ . We denote this connection by  $\tilde{A} : T\mathcal{W} \rightarrow V(\tilde{\rho})$ .

The connection  $\tilde{A}$  and the momentum map  $\mu_A := \mu \circ \psi : \mathcal{W} \rightarrow \mathfrak{k}^*$  are related by

$$\tilde{A}(q, \eta, \lambda)(\xi) = d\mu_A(q, \eta, \lambda)(\xi),$$

where  $\xi \in T_{(q, \eta, \lambda)}\mathcal{W}$ , and  $(q, \eta, \lambda)$  is short-hand for  $(q; [q], \eta; q, \lambda) \in \mathcal{W}$ .

We will use the connection  $\tilde{A}$  to decompose an arbitrary vector  $\xi \in T_{(q, \eta, \lambda)}\mathcal{W}$  as

$$\xi = (v(q); \eta'([q], \eta); v_1(q), \nu(q, \lambda)),$$

where  $\nu(q, \lambda) = \tilde{A}(q, \eta, \lambda)(\xi)$  is independent of  $\eta$ . Notice also that  $v_1(q) = v(q)$  by the pullback property. Further we can decompose  $v(q) \in T_qQ$  according to

$$v(q) = v(q)^{\text{hor}(\pi)} + \zeta_Z(q) \in H_q(\pi) \oplus V_q(\pi)$$

with respect to the connection  $A$  on  $\pi : Q \rightarrow Q/K$ . The same can be done with  $\eta'([q], \eta) \in T_{(q, \eta, \lambda)}(T^*(Q/K))$  as

$$\eta'([q], \eta) = \eta'([q], \eta)^{\text{hor}(\tau)} + \eta'([q], \eta)^{\text{ver}(\tau)} \in H_{([q], \eta)}(\tau) \oplus V_{([q], \eta)}(\tau)$$

with respect to the canonical connection on  $\tau : T^*(Q/K) \rightarrow Q/K$  which comes from the induced metric on  $Q/K$ . Notice that we have  $\eta'([q], \eta)^{\text{hor}(\tau)} = v(q)^{\text{hor}(\pi)}$  by the pullback property.

**DEFINITION 2.E.4** (Vertical differentiation on). Consider the bundle

$$\tilde{\rho} : \mathcal{W} \longrightarrow Q \times_{Q/K} T^*(Q/K)$$

Let  $\nu \in V_{(q, \eta, \lambda)}(\tilde{\rho})$  and  $F \in C^\infty(\mathcal{W})$ . We define

$$d_v F(q, \eta, \lambda)(\nu) := \left. \frac{\partial}{\partial t} \right|_0 F(q, \eta, \lambda + t\nu)$$

to be the VERTICAL DERIVATIVE of  $F$  at  $(q, \eta, \lambda)$ .

DEFINITION 2.E.5 (Horizontal differentiation on). Consider the bundle

$$\tilde{\rho} : \mathcal{W} \longrightarrow Q \times_{Q/K} T^*(Q/K).$$

The HORIZONTAL DERIVATIVE of  $F \in C^\infty(\mathcal{W})$  is defined as

$$d_{\tilde{A}}F := dF \circ \chi$$

where

$$\chi := 1 - \tilde{A} : T\mathcal{W} \rightarrow H(\tilde{\rho})$$

denotes the horizontal projection with respect to  $\tilde{\rho}$ .

LEMMA 2.E.6. Let  $F \in C^\infty(\mathcal{W})^K$ , and decompose the Hamiltonian vector field of  $F$  at  $(q, \eta, \lambda) \in \mathcal{W}$  as

$$\nabla_F^\sigma(q, \eta, \lambda) = (v(q); \eta'([q], \eta); v(q), \nu(q, \lambda))$$

with  $\nu(q, \lambda) = \tilde{A}(q, \eta, \lambda)(\nabla_F^\sigma(q, \eta, \lambda))$  according to above. Here  $\sigma$  is the symplectic structure on  $\mathcal{W}$  from Proposition 2.C.1. Then,  $\nu(q, \lambda) = 0$ .

PROOF. This is a consequence of Noether's Theorem. We have

$$\nu(q, \lambda) = \tilde{A}(q, \eta, \lambda)(\nabla_F^\sigma(q, \eta, \lambda)) = d\mu_A(q, \eta, \lambda)(\nabla_F^\sigma(q, \eta, \lambda)) = 0,$$

since  $\mu_A$  is constant along flow lines of Hamiltonian vector fields of invariant functions.  $\square$

LEMMA 2.E.7. Let  $F \in C^\infty(\mathcal{W})^K$ , and decompose the Hamiltonian vector field of  $F$  at  $(q, \eta, \lambda) \in \mathcal{W}$  as

$$\nabla_F^\sigma(q, \eta, \lambda) = (v(q); \eta'([q], \eta); v(q), 0).$$

Then,

$$\eta'([q], \eta) = (\Omega_{([q], \eta)}^{Q/K})^{-1} (d_{\tilde{A}}F(q, \eta, \lambda)),$$

where we consider  $d_{\tilde{A}}F(q, \eta, \lambda)$  as an element of  $T_{([q], \eta)}^*(T^*(Q/K))$ . Recall that  $\Omega^{Q/K}$  denotes the canonical symplectic form on  $T^*(Q/K)$ .

PROOF. Let us assume that  $F \in C^\infty(\mathcal{W})^K$  factors through  $\tilde{\pi} \circ \tilde{\rho}$ :

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{F} & \mathbb{R} \\ \tilde{\pi} \circ \tilde{\rho} \downarrow & \nearrow f & \\ T^*(Q/K) & & \end{array}$$

Then,

$$d_{\tilde{A}}F(q, \eta, \lambda) = df((\tilde{\pi} \circ \tilde{\rho})(q, \eta, \lambda)) = df([q], \eta).$$

By Theorem 2.E.3 the projection  $\tilde{\pi} \circ \tilde{\rho}$  is Poisson as a composition of Poisson maps: Indeed we have

$$\begin{array}{ccc} \mathcal{W} & \twoheadrightarrow & \mathcal{W}/K \\ \tilde{\pi} \circ \tilde{\rho} \downarrow & & \parallel \\ T^*(Q/K) & \xleftarrow{\tilde{\rho}_0} & W \end{array}$$

where  $\tilde{\rho}_0$  is as in Theorem 2.E.3. Therefore, we find

$$\begin{aligned}\eta'([q], \eta) &= T_{(q, \eta, \lambda)}(\tilde{\pi} \circ \tilde{\rho}) \cdot \nabla_F^\sigma(q, \eta, \lambda) \\ &= \nabla_f^{Q/K}((\tilde{\pi} \circ \tilde{\rho})(q, \eta, \lambda)) \\ &= (\Omega_{([q], \eta)}^{Q/K})^{-1}(df([q], \eta)) \\ &= (\Omega_{([q], \eta)}^{Q/K})^{-1}(d_{\tilde{A}}F(q, \eta, \lambda)).\end{aligned}$$

This yields the statement for general  $F \in C^\infty(\mathcal{W})^K$ .  $\square$

LEMMA 2.E.8. *Let  $F \in C^\infty(\mathcal{W})^K$ , and decompose the Hamiltonian vector field of  $F$  at  $(q, \eta, \lambda) \in \mathcal{W}$  as*

$$\nabla_F^\sigma(q, \eta, \lambda) = (v(q); \eta'([q], \eta); v(q), 0).$$

*Via the connection  $A \in \Omega^1(Q; \mathfrak{k})$  we can write*

$$v(q) = v(q)^{\text{hor}(\pi)} + \zeta_Z(q) \in H_q(\pi) \oplus V_q(\pi).$$

*Then,  $Z = d_v F(q, \eta, \lambda)$ .*

PROOF. It suffices to consider  $F \in C^\infty(\bigsqcup_{q \in Q} \text{Ann } \mathfrak{k}_q)^K$ . We work in tube coordinates around  $q \in Q$ . Thus let  $S$  be a slice through  $q$  for the  $K$ -action such that  $U \cong S \times K/H$ , where  $U$  is a tube around  $K \cdot q$  and  $H = K_q$ . Then we have

$$\bigsqcup_{q \in Q} \text{Ann } \mathfrak{k}_q|_U \cong S \times K \times_H \text{Ann } \mathfrak{h},$$

by Lemma 2.E.1. Since we already know the part of the Hamiltonian vector field of  $F$  that is tangent to  $S$ , we may further reduce the problem to considering a function  $F \in C^\infty(K \times_H \text{Ann } \mathfrak{h})^K = C^\infty(\text{Ann } \mathfrak{h}/H)$ . Now

$$K \times_H \text{Ann } \mathfrak{h} = T^*K //_0 T^*R(H),$$

where  $T^*R(H)$  is the cotangent lifted action of the right multiplication of  $H$  on  $K$ . Thus, there exists a function  $F_1 \in C^\infty(T^*K) = C^\infty(K \times \mathfrak{k}^*)$  that is  $T^*L(K)$ -invariant ( $L$  denotes the left multiplication on  $K$ ) such that the following diagram commutes

$$\begin{array}{ccc} K \times \text{Ann } \mathfrak{h} & \hookrightarrow & K \times \mathfrak{k}^* \\ \downarrow & & \downarrow F_1 \\ K \times_H \text{Ann } \mathfrak{h} & \xrightarrow{F} & \mathbb{R} \end{array}$$

We can choose local cotangent bundle coordinates  $a_i, b_i$  where  $i = 1, \dots, m$  on  $T^*K$  such that  $b_1, \dots, b_l$  are coordinates on  $\mathfrak{h}^*$ ,  $b_{l+1}, \dots, b_m$  are coordinates on  $\text{Ann } \mathfrak{h}$ , and such that

$$\frac{\partial}{\partial a_1}, \dots, \frac{\partial}{\partial a_l} \text{ are a basis of } \mathfrak{h}, \text{ and } \frac{\partial}{\partial a_{l+1}}, \dots, \frac{\partial}{\partial a_m} \text{ are a basis of } \mathfrak{h}^\perp.$$

Then for the canonical Poisson bracket on  $T^*K$  we obtain

$$\{F_1, \cdot\}^{T^*K} = \sum_{i=1}^m \left( \frac{\partial F_1}{\partial b_i} \frac{\partial}{\partial a_i} - \frac{\partial F_1}{\partial a_i} \frac{\partial}{\partial b_i} \right) = \sum_{i=l+1}^m \frac{\partial F_1}{\partial b_i} \frac{\partial}{\partial a_i} \in \mathfrak{h}^\perp$$

which is the vertical derivative of  $F_1$  identified with an element of  $\mathfrak{h}^\perp$ . Since the projection  $K \times \text{Ann } \mathfrak{h} \rightarrow K \times_H \text{Ann } \mathfrak{h}$  is Poisson, Hamiltonian vector fields project to Hamiltonian vector fields, and therefore the Hamiltonian vector field of  $F$  on  $K \times_H \text{Ann } \mathfrak{h}$  is  $d_v F$ , and this is tangent to the  $K$ -factor. Thus  $Z = d_v F(q, \eta, \lambda) \in \mathfrak{k}_q^\perp$ .  $\square$

**THEOREM 2.E.9** (Poisson structure on Weinstein space). *The natural identification*

$$\begin{aligned} \mathcal{W}/K &\xrightarrow{=} W, \\ [(q; [q], \eta; q, \lambda)] &\longmapsto ([q], \eta; [(q, \lambda)]) \end{aligned}$$

*gives an induced Poisson bracket on  $C^\infty(W) = C^\infty(\mathcal{W})^K$  which makes the stratified isomorphism*

$$\psi_0 = \psi_0(A) : T^*(Q/K) \times_{Q/K} (\bigsqcup_{q \in Q} \text{Ann } \mathfrak{k}_q)/K = W \longrightarrow (T^*Q)/K$$

*from Theorem 2.E.3 into an isomorphism of Poisson spaces.*

*Let  $[(q, \eta, \lambda)] \in \mathcal{W}/K = W$ , and  $f_1, f_2 \in C^\infty(W)$ . Assume that  $F_1, F_2 \in C^\infty(\mathcal{W})^K$  are lifts of  $f_1, f_2$  to  $\mathcal{W}$ . Then the induced Poisson bracket on  $W$  is given by*

$$\begin{aligned} \{f_1, f_2\}^W [(q, \eta, \lambda)] &= \\ &= \Omega_{([q], \eta)}^{Q/K} \left( (\Omega_{([q], \eta)}^{Q/K})^{\check{-1}} (d_{\bar{A}} F_1(q, \eta, \lambda)), (\Omega_{([q], \eta)}^{Q/K})^{\check{-1}} (d_{\bar{A}} F_2(q, \eta, \lambda)) \right) \\ &\quad - \langle \lambda, \text{Curv}_q^A \left( (\Omega_{([q], \eta)}^{Q/K})^{\check{-1}} (d_{\bar{A}} F_1(q, \eta, \lambda)), (\Omega_{([q], \eta)}^{Q/K})^{\check{-1}} (d_{\bar{A}} F_2(q, \eta, \lambda)) \right) \rangle \\ &\quad - \langle \lambda, [d_v F_1(q, \eta, \lambda), d_v F_2(q, \eta, \lambda)] \rangle, \end{aligned}$$

*where  $\Omega^{Q/K}$  is the canonical symplectic form on  $T^*(Q/K)$ , the horizontal components  $d_{\bar{A}} F_i(q, \eta, \lambda)$  are considered as elements of  $T_{([q], \eta)}^*(T^*(Q/K))$ , and  $\text{Curv}^A$  is the curvature form associated to the mechanical connection  $A$  on  $Q \rightarrow Q/K$  from Section 2.C.*

In the case that  $K$  acts freely on  $Q$  the Poisson bracket on the reduced Poisson manifold  $T^*Q/K$  is determined in Zaalani [53] and in Perlmutter and Ratiu [37]. In the first paper the realization of  $T^*Q/K$  as Weinstein space is used, the latter deals with its realization as Sternberg and Weinstein space.

**PROOF.** The identifying map  $\mathcal{W}/K \rightarrow W$ ,  $[(q; [q], \eta; q, \lambda)] \mapsto ([q], \eta; [(q, \lambda)])$  is well-defined because  $K_q$  acts trivially on  $\text{Hor}_q^* = T_{[q]}^*(Q/K) \ni \eta$  which in turn is due to the fact that all points of  $Q$  are regular. The quotient Poisson bracket is well-defined since

$C^\infty(\mathcal{W})^K \subseteq C^\infty(\mathcal{W})$  is a Poisson sub-algebra. The first statement in the theorem now follows because the diagram

$$\begin{array}{ccc} T^*Q & \xrightarrow{\psi^{-1}} & \mathcal{W} \\ \downarrow & & \searrow \\ (T^*Q)/K & \xrightarrow{\psi_0^{-1}} & W \equiv \mathcal{W}/K \end{array}$$

is commutative, and composition of top and down-right arrow is Poisson and the left vertical arrow is surjective.

Let  $f_1, f_2 \in C^\infty(W)$  and let  $F_1, F_2 \in C^\infty(\mathcal{W})^K$  be its unique lifts to  $\mathcal{W}$ . In order to establish the formula for the reduced Poisson bracket we decompose the Hamiltonian vector fields of  $F_1$  and  $F_2$  as above

$$\nabla_{F_i}^\sigma(q, \eta, \lambda) = (v_i(q); \eta'_i([q], \eta); v_i(q), \nu_i(q, \lambda)) \quad (i = 1, 2).$$

With the intrinsic symplectic form  $\sigma$  on  $\mathcal{W}$  from Proposition 2.C.1 we have

$$\{f_1, f_2\}^W[(q, \eta, \lambda)] = \{F_1, F_2\}^{\mathcal{W}}(q, \eta, \lambda) = \sigma(\nabla_{F_1}^\sigma, \nabla_{F_2}^\sigma)(q, \eta, \lambda)$$

which turns to the desired formula by the identity  $\eta'([q], \eta)^{\text{hor}(\tau)} = v(q)^{\text{hor}(\pi)}$ , and Lemmas 2.E.6, 2.E.7, and 2.E.8.  $\square$



## CHAPTER 3

### Non-commutative integrability

The idea of non-commutative integrability under the name of degenerate integrability is due to Nehorošev [30] who also introduced the appropriate concept of action-angle variables. This presentation follows mostly the approach of Zung [54, 55], that of Fasso and Ratiu [16], and Fasso [15]. See also Mishchenko and Fomenko [29].

#### 3.A. Generalized Liouville integrability

Let  $M$  be a smooth manifold and  $f_1, \dots, f_r \in C^\infty(M)$ . The family  $f_1, \dots, f_r$  is said to be FUNCTIONALLY INDEPENDENT if there is an open and dense subset  $U \subseteq M$  such that  $df_1(x), \dots, df_r(x)$  are linearly independent for all  $x \in U$ . The FUNCTIONAL DIMENSION of a family  $\mathcal{F}$  of smooth functions on  $M$  is the maximal number of elements in  $\mathcal{F}$  which are functionally independent.

The following definition is less general than that given in the above cited references but better suited for the applications in this paper.

DEFINITION 3.A.1. Let  $(M, \{ \cdot, \cdot \})$  be a Poisson manifold, and consider a Hamiltonian function  $H : M \rightarrow \mathbb{R}$ . We denote the Poisson sub-algebra of all first integrals of  $H$  by  $\mathcal{F}_H$ , that is

$$\mathcal{F}_H := \{F \in C^\infty(M) : \{F, H\} = 0\}.$$

The Hamiltonian system is called GENERALIZED LIOUVILLE INTEGRABLE if there is a finite dimensional Poisson vector space  $W$  and a generalized momentum map  $\Phi : M \rightarrow W$  which is a Poisson morphism with respect to the Poisson structure on  $W$  such that the following are satisfied.

- $\Phi^* : C^\infty(W) \rightarrow \mathcal{F}_H$  is an isomorphism of Lie-Poisson algebras.
- $\dim M = \text{ddim } C^\infty(W) + \text{ddim } Z(C^\infty(W))$  where  $Z(C^\infty(W))$  denotes the commutative sub-algebra of Casimir functions on  $W$ , and  $\text{ddim } C^\infty(W) = \dim W$  is the functional dimension of  $C^\infty(W)$ .

For the following assume that  $(M, \omega, H)$  is a generalized Liouville integrable system on a symplectic manifold with  $\dim M = 2n$ . Let  $l = \text{ddim } Z(C^\infty(W))$  and  $m = \text{ddim } C^\infty(W)$ . Assume we are given a smooth vector valued function

$$u = (u_1, \dots, u_l, u_{l+1}, \dots, u_m) : W \rightarrow \mathbb{R}^m = \mathbb{R}^{2n-l}$$

such that  $u_1, \dots, u_l$  generate the center of  $C^\infty(W)$  and  $u_1, \dots, u_m$  generate  $C^\infty(W)$  as Poisson algebras. Assume further for simplicity that  $u$  and  $\Phi$  both are submersions. If we define

$$\Phi_i := u_i \circ \Phi$$

we get the following assertions.

- $\{\Phi_i, \Phi_j\} = 0$  for all  $i \leq l$  and  $j$  arbitrary.
- $\{\Phi_j, H\} = 0$  for all  $j$ .
- the set  $\{d\Phi_1(x), \dots, d\Phi_l(x), d\Phi_{l+1}(x), \dots, d\Phi_{2n-l}(x)\}$  is linearly independent for all  $x \in M$ .

If  $l = n$  this is one of the usual definitions of complete integrability.

REMARK 1. Consider the level set  $\Phi^{-1}(\Phi(x))$  and the connected component  $L$  containing  $x$  thereof. By definition all points in  $M$  are regular with respect to  $\Phi$ . Thus  $L$  is a closed sub-manifold of dimension  $l$ .  $\square$

REMARK 2. Consider the  $l$ -dimensional integrable distribution spanned by the  $\nabla_{\Phi_i}^\omega$  where  $i \leq l$ . Since

$$\frac{\partial}{\partial t} \Big|_0 (\Phi_j \circ \text{Fl}_t^{\nabla_{\Phi_i}^\omega})(x) = \{\Phi_j, \Phi_i\} = 0,$$

the leaf passing through  $x$  is just the connected component  $L \subseteq \Phi^{-1}(\Phi(x))$  containing  $x$ . Obviously,  $L$  is an isotropic submanifold of  $M$ , and for the annihilator with respect to  $\omega$  we have

$$(T_x L)^\omega := \omega\text{-Ann}(T_x L) = \text{Span}\{\nabla_{\Phi_j}^\omega(x) : 1 \leq j \leq 2n - l\}.$$

Moreover, Hamiltonian flow lines of  $H$  are parallel to this foliation by isotropic sub-manifolds. Therefore,  $\nabla_H^\omega(x) \in \text{Span}\{\nabla_{\Phi_i}^\omega(x) : 1 \leq i \leq l\}$  and applying  $\tilde{\omega}^{-1}$  yields

$$dH(x) = \sum_{i=1}^l \frac{\partial H}{\partial \Phi_i}(x) d\Phi_i(x).$$

$\square$

REMARK 3. As above let  $L$  be the leaf through  $x$  corresponding to the distribution spanned by  $\nabla_{\Phi_i}^\omega$  where  $i \leq l$ . Since  $L$  is at the same time the connected component of  $\Phi^{-1}(\Phi(x))$  containing  $x$ , the equation

$$dH \cdot \nabla_{\Phi_i}^\omega = \{H, \Phi_i\} = 0$$

where  $i \leq l$  implies that  $H$  is constant on the fibers of the submersion  $\Phi : M \rightarrow W$ . Thus there is a smooth mapping  $h : W \rightarrow \mathbb{R}$  such that  $\Phi^*h = H$ . Moreover, as  $\Phi$  is a surjective Poisson morphism it follows that  $h$  lies in the center of the Poisson algebra  $C^\infty(W)$ .  $\square$

REMARK 4. Assume  $(\varphi, I, q, p)$  are GENERALIZED ACTION-ANGLE VARIABLES in the sense of Nehorošev [30] on an open subset  $U \subseteq M$ . That is

$$\omega|_U = \sum_{j=1}^l dI_j \wedge d\varphi_j + \sum_{j=1}^{n-l} dp_j \wedge dq_j,$$

and thus  $\nabla_{I_j}^\omega = \tilde{\omega}^{-1}(dI_j) = \partial_{\varphi_j}$ , for example. Assume furthermore the commutation relations

$$\{I_j, H\} = 0 \quad \text{and} \quad \{q_i, H\} = \{p_i, H\} = 0.$$

In these coordinates the flow equations to the Hamiltonian  $H$  then assume the canonical form

$$\begin{aligned} \frac{\partial}{\partial t} |_0 (\varphi_i \circ \text{Fl}_t^{\nabla_{I_j}^\omega})(x) &= d\varphi_i(x) \cdot \tilde{\omega}^{-1} \left( \sum_{j=1}^l \frac{\partial H}{\partial I_j}(x) dI_j(x) \right) = \frac{\partial H}{\partial I_i}(x), \\ \frac{\partial}{\partial t} |_0 (I_i \circ \text{Fl}_t^{\nabla_{I_j}^\omega})(x) &= 0, \\ \frac{\partial}{\partial t} |_0 (q_j \circ \text{Fl}_t^{\nabla_{I_j}^\omega})(x) &= 0, \\ \frac{\partial}{\partial t} |_0 (p_j \circ \text{Fl}_t^{\nabla_{I_j}^\omega})(x) &= 0. \end{aligned}$$

Thus we have  $H = H(I)$  in these coordinates. Generalizing in accordance to the harmonic oscillator, the numbers

$$\nu_i(x) = \frac{\partial H}{\partial I_i}(x)$$

are called frequencies of the system. They are said to be independent if they are linearly independent over the rationals.  $\square$

The following is a generalized Liouville-Arnold theorem, and is the main theorem of Nehorošev [30].

**THEOREM 3.A.2.** *Let  $(M, \omega)$  be a symplectic manifold with  $\dim M = 2n$ . Assume the Hamiltonian system  $(M, \omega, H)$  is generalized Liouville integrable with integrals  $\Phi_1, \dots, \Phi_l, \Phi_{l+1}, \dots, \Phi_{2n-l}$  defined by the momentum map  $\Phi : M \rightarrow W$  as above and where  $l \leq n$ . Assume further that the Hamiltonian vector fields corresponding to the integrals  $\Phi_1, \dots, \Phi_l$  are complete. Then a connected component  $L$  of a non-empty level surface  $\Phi^{-1}(\Phi(x))$ , where  $x$  is a regular point of  $\Phi$ , is an isotropic submanifold of dimension  $l$ . On an open neighborhood of  $L$  there exist generalized action-angle variables  $(\varphi, I, q, p)$ . Moreover  $I_j = I_j(\Phi_1, \dots, \Phi_l)$ , and  $q_j = q_j(\Phi)$  and  $p_j = p_j(\Phi)$ .*

*The Hamiltonian flow lines are affine in these coordinates, and are given by the set of equations in Remark 4 above. If  $L$  is compact it is diffeomorphic to a  $l$ -torus,<sup>1</sup> otherwise it is diffeomorphic to the product of a torus by a vector space.*

**PROOF.** See Nehorošev [30, Theorem 1].  $\square$

Say we are given a generalized Liouville integrable system  $(M, \omega, H)$  as above which we now additionally assume to be invariant under the Hamiltonian action by a compact Lie group  $G$ . That is  $G$  acts on  $M$  by symplectomorphisms, there is a standard momentum map  $J : M \rightarrow \mathfrak{g}^*$ , and  $H \in C^\infty(M)^G$ . We know that we can do singular Poisson reduction or singular symplectic reduction with this system to obtain a reduced

<sup>1</sup>In this case one speaks of conditionally periodic or quasi periodic motion.

Hamiltonian system. However, what happens to the integrability of the system? Curiously, this question seems to not have been formally addressed until Zung [54, 55].

**THEOREM 3.A.3.** *Assume the Hamiltonian system  $(M, \omega, H)$  is invariant under a Hamiltonian action of a compact Lie group  $G$ . If  $(M, \omega, H)$  is generalized Liouville integrable then the reduced system is integrable as well:*

- *The singularly Poisson reduced system is generalized Liouville integrable.*
- *The singularly symplectic reduced system is generalized Liouville integrable.*

**PROOF.** This theorem is proved by Zung [54, Theorem 2.3]. For material on singular reduction we refer to Sections 1.G and 1.H.  $\square$

It is crucial in the formulation of the above theorem that  $\dim M = \text{ddim } \mathcal{F}_H + \text{ddim } Z(\mathcal{F}_H)$ , and  $\mathcal{F}_H$  is the set of *all* first integrals of  $H$ . Thus generalized Liouville integrability is in the context of reduction better suited than classical integrability.

### 3.B. Symplectically complete foliations

The aim of the following sections is to investigate the geometric meaning of non-commutative integrability. In doing so we will follow the presentation of Fasso and Ratiu [16] and that of Fasso [15].

Let  $(M, \omega)$  be a symplectic manifold with dimension  $2n = \dim M$ . Let  $\mathcal{A}$  be a distribution on  $M$  with coisotropic subspaces. That is, for every  $x \in M$  the space  $\mathcal{A}_x$  is a coisotropic subvector space of  $T_x M$ , and  $\mathcal{A} = \bigsqcup_{x \in M} \mathcal{A}_x$ . The POLAR DISTRIBUTION  $\mathcal{A}^\omega$  to  $\mathcal{A}$  is defined as the symplectic orthogonal distribution by which we mean the distribution given by

$$\bigsqcup_{x \in M} \mathcal{A}_x^\omega =: \mathcal{A}^\omega.$$

By coisotropy we have that  $\mathcal{A}_x^\omega \subseteq \mathcal{A}_x$  and thus also  $\mathcal{A}^\omega \subseteq \mathcal{A}$ . Therefore, leaves of  $\mathcal{A}$  are, if they exist, unions of leaves of  $\mathcal{A}^\omega$ , if these exist.

**PROPOSITION 3.B.1.** *Let  $\alpha : M \rightarrow A$  be a surjective submersion such that  $\{\alpha^* f, \alpha^* g\} = 0$  for all  $f, g \in C^\infty(A)$ . Then*

$$\mathcal{A} := \bigsqcup_{x \in M} T_x \alpha^{-1}(\alpha(x))$$

*defines a smooth coisotropic distribution which is integrable. Moreover, the polar distribution  $\mathcal{B} := \mathcal{A}^\omega$  of  $\mathcal{A}$  is integrable. Its leaves are isotropic initial submanifolds of  $M$ .*

**PROOF.** Since  $\alpha : M \rightarrow A$  is a surjective submersion the fibers  $\alpha^{-1}(\alpha(x))$  are closed submanifolds of  $M$ . Thus the distribution  $\mathcal{A} := \bigsqcup_{x \in M} T_x \alpha^{-1}(\alpha(x))$  is trivially integrable with leaves the connected components of the fibers of  $\alpha$ .

Since  $\{\alpha^*f, \alpha^*g\} = 0$  for all  $f, g \in C^\infty(A)$  we can consider  $\alpha$  as a Poisson morphism into the trivial bracket on  $A$ . Let  $X$  be a local vector field on  $M$  defined on an open neighborhood of  $x$  such that  $\omega_x(X_x, Y_x) = 0$  for all  $Y \in \mathfrak{X}(M)$  which are tangent to the fiber  $\alpha^{-1}(\alpha(x))$ . Then we can find a smooth function defined on an open neighborhood  $U$  of  $x$  in  $M$  satisfying  $\tilde{\omega}|_U(X) = df$ . Therefore,  $df$  is horizontal with respect to  $\alpha$  since  $df(Y) = \omega(X, Y) = 0$  for all vertical vector fields  $Y$ . Thus  $f = \alpha^*f_0$  where  $f_0$  is a local smooth function on  $A$ . Because  $\alpha$  is a Poisson morphism it follows that the Hamiltonian vector field  $X$  corresponding to  $f$  and the Hamiltonian vector field on  $A$  corresponding to  $f_0$  are  $\alpha$ -related. However, since the Poisson structure on  $A$  is trivial it follows that the latter is zero. That is,  $T\alpha.X = 0$  whence  $X$  is tangent to the fibers of  $\alpha$ . Thus we have shown that  $X_x \in \mathcal{A}_x$  which proves  $\mathcal{A}$  a coisotropic distribution.

By a general feature of the  $\omega$ -annihilator we have that

$$\dim \mathcal{B}_x = \dim \mathcal{A}_x^\omega = \dim M - \dim \mathcal{A}_x,$$

and the latter number is independent of  $x$ , whence  $\mathcal{B}$  is a distribution of constant rank. Let  $x \in M$  arbitrary and  $f_1, \dots, f_l$  be smooth functions on  $A$  such that  $df_1(\alpha(x)), \dots, df_l(\alpha(x))$  are linearly independent. Then it follows from the same argument as above that

$$\mathcal{B}_x = \text{span}\{\nabla_{\alpha^*f_i}^\omega(x) : i = 1, \dots, l\}$$

where  $\nabla^\omega$  is the symplectic gradient. Therefore,  $\mathcal{B}$  is a smooth distribution, and it is also involutive since

$$[\nabla_{\alpha^*f_i}^\omega, \nabla_{\alpha^*f_j}^\omega] = \nabla_{\{\alpha^*f_i, \alpha^*f_j\}}^\omega = 0$$

since  $\{\alpha^*f_i, \alpha^*f_j\} = \alpha^*\{f_i, f_j\} = 0$ . Thus integrability follows from the Frobenius theorem.  $\square$

**DEFINITION 3.B.2** (Polar foliation). Let  $\mathcal{F}$  be a foliation on the symplectic manifold  $(M, \omega)$ . The foliation  $\mathcal{F}$  is called **SYMPLECTICALLY COMPLETE** if it has a polar foliation  $\mathcal{F}^\omega$ . Hereby, a foliation  $\mathcal{F}^\omega$  is said to be **POLAR** to  $\mathcal{F}$  if for all  $x \in M$  the tangent space at  $x$  to the leaf of  $\mathcal{F}^\omega$  is the symplectic orthogonal of the tangent space at  $x$  to the leaf of  $\mathcal{F}$  that passes through  $x$ ; if the polar foliation exists it is clearly unique.

The notion of symplectic completeness is due to Dazord and Delzant [13].

**EXAMPLE 3.B.3.** A foliation into Lagrangian submanifolds is symplectically complete as it is its own polar.

**EXAMPLE 3.B.4.** Let  $h \in C^\infty(M)$ . Then the flow lines of the Hamiltonian vector field  $\nabla_h^\omega$  constitute a symplectically complete foliation. Its polar is given by the foliation into connected level sets of  $h$ . Indeed, if a vector field  $\xi$  is tangent to the level sets of  $h$  then  $\omega(\nabla_h^\omega, \xi) = dh.\xi = 0$ .

EXAMPLE 3.B.5. Generalizing the last example, let a compact connected Lie group  $K$  act on  $(M, \omega)$  in a Hamiltonian fashion with momentum map  $J : M \rightarrow \mathfrak{k}^*$ . Since

$$\langle dJ(x) \cdot \xi(x), X \rangle = (i_\xi \langle dJ, X \rangle)(x) = (i_\xi i_{\zeta_X} \omega)(x) = \langle \check{\omega}_x(\zeta_X(x)), \xi \rangle$$

for all  $x \in M$ ,  $\xi \in \mathfrak{X}(M)$ , and  $X \in \mathfrak{k}$  it follows that the transpose to  $dJ(x) : T_x M \rightarrow \mathfrak{k}^*$  is given by

$$(dJ(x))^t = \check{\omega}_x \circ \zeta(x) : \mathfrak{k} \longrightarrow T_x^* M.$$

This implies the equation

$$\text{rank } T_x J = \dim(K \cdot x)$$

for all  $x \in M$  which is also known as the Bifurcation Lemma.

Assume further that all  $\alpha \in \mathfrak{k}^*$  are weakly regular with respect to  $J$  in the sense that  $J^{-1}(\alpha) \subseteq M$  is a submanifold and we have  $T_x J^{-1}(\alpha) = \ker T_x J$  for all  $x \in J^{-1}(\alpha)$ . Thus we have that

$$\dim J^{-1}(\alpha) + \dim(K \cdot x) = \dim M - \text{rank } T_x J + \dim(K \cdot x) = \dim M$$

on the one hand. On the other hand we have that

$$\omega_x(\zeta_X(x), \xi(x)) = \langle (dJ \cdot \xi)(x), X \rangle = 0$$

for all  $\xi$  tangent to the fibers of  $J$  and all  $X \in \mathfrak{k}$ . Therefore, the orbits of the Hamiltonian group action define a foliation of  $M$  which is symplectically complete. Its polar is given by the connected components of the fibers of  $J$ .

EXAMPLE 3.B.6. By Proposition 3.B.1 every foliation of  $(M, \omega)$  which is given by the connected components of the fibers of a surjective submersion  $\alpha : M \rightarrow A$  with the property that  $\{\alpha^* f, \alpha^* g\} = 0$  for all  $f, g \in C^\infty(A)$  is symplectically complete.

LEMMA 3.B.7. *Let  $\alpha : M \rightarrow A$  be a surjective submersion such that  $\{\alpha^* f, \alpha^* g\} = 0$  for all  $f, g \in C^\infty(A)$ . Define  $\mathcal{A}$  to be the foliation corresponding to the constant rank integrable distribution*

$$\bigsqcup_{x \in M} T_x \alpha^{-1}(\alpha(x)),$$

*and let  $\mathcal{B} = \mathcal{A}^\omega$  be its polar foliation, as in Proposition 3.B.1. Assume further that the leaf space  $M/\mathcal{B}$  is a smooth manifold such that the projection  $\beta : M \rightarrow M/\mathcal{B}$  is a smooth surjective submersion. Then there exists a uniquely induced Poisson structure on  $M/\mathcal{B}$  such that  $\beta : M \rightarrow M/\mathcal{B}$  is a Poisson morphism.*

PROOF. Let us introduce the notation  $B := M/\mathcal{B}$ . By assumption  $\beta : M \rightarrow B$  is a surjective submersion with connected fibers. Moreover, we know from the proof of Proposition 3.B.1 that

$$T_x \beta^{-1}(\beta(x)) = \mathcal{B}_x = \mathcal{A}_x^\omega = \text{span}\{\nabla_{\alpha^* f_i}^\omega(x) : i = 1, \dots, l\}$$

whence we can invoke Corollary 1.D.4 to conclude as claimed.  $\square$

PROPOSITION 3.B.8. *Let  $\alpha : M \rightarrow A$  be a surjective submersion such that  $\{\alpha^*f, \alpha^*g\} = 0$  for all  $f, g \in C^\infty(A)$ . Define  $\mathcal{A}$  to be the foliation corresponding to the constant rank integrable distribution*

$$\bigsqcup_{x \in M} T_x \alpha^{-1}(\alpha(x)),$$

*and let  $\mathcal{B} = \mathcal{A}^\omega$  be its polar foliation, as in Proposition 3.B.1. Assume further that the leaf space  $M/\mathcal{B}$  is a smooth manifold such that the projection  $\beta : M \rightarrow M/\mathcal{B}$  is a smooth surjective submersion which is Poisson with respect to the induced Poisson bracket on  $M/\mathcal{B}$ . Then there exists a unique surjective submersion  $\gamma : M/\mathcal{B} =: B \rightarrow A = M/\mathcal{A}$  which is a Poisson morphism such that the diagram*

$$\begin{array}{ccc} M & \xrightarrow{\beta} & B \\ & \searrow \alpha & \downarrow \gamma \\ & & A \end{array}$$

*commutes. Moreover, the symplectic leaves of  $B$  are the connected components of the fibers of  $\alpha$ .*

PROOF. Let  $x \in M$  arbitrary and denote the leaf of  $\mathcal{A}$  passing through  $x$  by  $\mathcal{A}_x$  and that of  $\mathcal{B}$  passing through  $x$  by  $\mathcal{B}_x$ . Since

$$T_x \mathcal{B}_x = T_x \mathcal{A}_x^\omega \subseteq T_x \mathcal{A}_x$$

by coisotropy of  $\mathcal{A}_x$  it follows that every leaf of  $\mathcal{B}$  is a union of leaves of  $\mathcal{A}$ . Thus  $\mathcal{A}$  induces a foliation  $\beta(\mathcal{A})$  of  $B$ , and thus a well-defined set theoretic map  $\gamma : B \rightarrow A$  such that  $\alpha = \gamma \circ \beta$ . Since  $B$  is a quotient manifold of  $M$  it carries the final topology and the final smooth structure with respect to the projection  $\beta : M \rightarrow B$ . Thus  $\gamma$  is continuous and smooth since this is true for the composition map  $\gamma \circ \beta = \alpha$ .

Moreover,  $\gamma$  is a surjective submersion since this is true for  $\alpha$ .

It remains to check the assertion about the symplectic leaves of  $B$ . Firstly, notice that  $B$  is a Poisson manifold with Poisson structure  $P^B$  coinduced from  $M$  via  $\beta$  as Lemma 3.B.7 shows. By the theorem about the symplectic foliation of a Poisson manifold we need to show that

$$\check{P}_b^B(T_b B) = T_b \gamma^{-1}(\gamma(b)) = \ker T_b \gamma$$

for all  $b \in B$ . Indeed, to see the inclusion ( $\subseteq$ ) let  $x \in \beta^{-1}(b)$  and  $f$  a smooth local function on  $B$  defined locally around  $b$ . Then

$$\begin{aligned} T_b \gamma \cdot \nabla_f^B(b) &= T_b \gamma \cdot T_x \beta \cdot \nabla_{\beta^* f}^M(x) \\ &= T_x \alpha \cdot \nabla_{\beta^* f}^M(x) \\ &= 0 \end{aligned}$$

since  $\gamma$  is a Poisson morphism and  $\alpha$  is a Poisson morphism into the trivial bracket by assumption.

To see the inclusion ( $\supseteq$ ) let  $x \in \beta^{-1}(b)$  and  $\xi_0$  a local vector field on  $B$  defined locally around  $b$ . Then we choose a local lift  $\xi$  of  $\xi_0$  defined around  $x \in M$ . The equation

$$T_x \alpha. \xi(x) = T_b \gamma. T_x \beta. \xi_0(b) = 0$$

implies that  $\xi = \nabla_{\alpha^* f}^M$  for a local smooth function  $f$  on  $A$ , and thus

$$\xi_0(x) = T_x \beta. \nabla_{\beta^* \gamma^* f}^M(x) = \nabla_{\gamma^* f}^B(b)$$

which shows the converse inclusion.  $\square$

### 3.C. Integrability via bifoliations

**DEFINITION 3.C.1 (Bifoliation).** Let  $\mathcal{F}$  be a foliation of  $(M, \omega)$  which is symplectically complete with polar  $\mathcal{F}^\omega$ . Then the pair  $(\mathcal{F}, \mathcal{F}^\omega)$  is called a **BIFOLIATION** of  $(M, \omega)$ . We say that the pair  $(\mathcal{F}, \mathcal{F}^\omega)$  is a **REGULAR BIFOLIATION** of  $(M, \omega)$  if it is a bifoliation, and the leaf spaces  $M/\mathcal{F}$  and  $M/\mathcal{F}^\omega$  are smooth (Hausdorff) manifolds such that the canonical projections  $M \rightarrow M/\mathcal{F}$  and  $M \rightarrow M/\mathcal{F}^\omega$  are surjective submersions.

**THEOREM 3.C.2 (Liouville-Arnold).** *Let  $(M, \omega)$  be a symplectic manifold with regular bifoliation  $(\mathcal{A}, \mathcal{B})$  such that the canonical projection  $\alpha : M \rightarrow M/\mathcal{A} =: A$  has coisotropic fibers. Let  $\dim M = 2n$  and  $\dim B = l$ , and assume that  $I_1^A, \dots, I_l^A$  are local coordinates on  $A$  defined on an open subset  $U_A$  in  $A$  such that the local Hamiltonian vector fields*

$$\nabla_{I_j^A}^M := \tilde{\omega}^{-1}(dI_j^A) \text{ are complete}$$

where  $I_j := \alpha^* I_j^A$  and  $j = 1, \dots, l$ . Then the following are true.

- (i) *The leaves of  $\mathcal{B}$  are isotropic submanifolds of  $M$ .*
- (ii) *If we equip  $A$  with the trivial Poisson structure  $\alpha$  is a Poisson morphism into the zero bracket. The quotient  $M/\mathcal{B} =: B$  is a smooth manifold which carries a Poisson structure coinduced by the canonical projection  $\beta : M \rightarrow B$ . Moreover, there exists surjective submersive Poisson mapping  $\gamma : B \rightarrow A$  such that  $\alpha = \gamma \circ \beta$ .*
- (iii) *For all  $b \in \gamma^{-1}(U_A)$  there is an open neighborhood  $U_B$  in  $\gamma^{-1}(U_A)$  on which there are coordinates*

$$I_1^B := \gamma^* I_1^A, \dots, I_l^B := \gamma^* I_l^A, \quad q_1^B, \dots, q_{n-l}^B, \quad p_1^B, \dots, p_{n-l}^B,$$

that satisfy the commutation relations

$$\begin{aligned} \{I_k^B, I_j^B\}^B &= 0, \\ \{I_k^B, q_j^B\}^B &= \{I_k^B, p_j^B\}^B = 0, \\ \{p_i^B, q_j^B\}^B &= \delta_{ij} \end{aligned}$$

with respect to the induced bracket on  $B$ .



(iv) Let  $b$  and  $U_B$  be fixed as in point (iii). Then there exist local coordinates

$$\begin{aligned} \varphi_1, \dots, \varphi_l, \quad I_1 &:= \alpha^* I_1^A, \dots, I_l := \alpha^* I_l^A, \\ q_1 &:= \beta^* q_1^B, \dots, q_{n-l} := \beta^* q_{n-l}^B, \\ p_1 &:= \beta^* p_1^B, \dots, p_{n-l} := \beta^* p_{n-l}^B, \end{aligned}$$

defined on an open neighborhood  $U$  of  $\beta^{-1}(b)$  such that

$$\omega|_U = \sum_{j=1}^l dI_j \wedge d\varphi_j + \sum_{j=1}^{n-l} dp_j \wedge dq_j$$

in these coordinates.

PROOF. Assertion (i) is obvious, and assertion (ii) is just a reformulation of Proposition 3.B.8.

(iii). This assertion makes use of the Darboux Theorem applied to the symplectic manifold  $\gamma^{-1}(\gamma(b))$ .

(iv). This assertion is proved as in the usual Liouville-Arnold Theorem. In particular, it follows that the coordinates  $\varphi_1, \dots, \varphi_l$  can be computed from the other data exclusively through employment of differentiation, integration, and use of the implicit function theorem.  $\square$

The coordinates

$$q_1^B, \dots, q_{n-l}^B, \quad p_1^B, \dots, p_{n-l}^B,$$

in the above theorem should be thought of as local coordinates on the fiber  $\gamma^{-1}(\gamma(b))$  which is the symplectic leaf passing through  $b$  by Proposition 3.B.8. If the fiber is compact then there is, of course, no hope to find such coordinates globally on the fiber.

DEFINITION 3.C.3 (Non-commutative integrability). A Hamiltonian system  $(M, \omega, H)$  is called NON-COMMUTATIVELY INTEGRABLE if the following requirements are met. There is a regular bifoliation  $(\mathcal{A}, \mathcal{B})$  where  $\mathcal{A}$  is a coisotropic foliation and a smooth function  $h : M/\mathcal{A} =: A \rightarrow \mathbb{R}$  such that

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & A \\ & \searrow H & \downarrow h \\ & & \mathbb{R} \end{array}$$

commutes. Moreover, the Hamiltonian vector field  $\nabla_{\alpha^* I}^M$  is complete for all  $I \in C^\infty(A)$ . In this situation, the space  $A$  is called ACTION SPACE of the system.

A few remarks are in order. If the leaves of the foliation  $\mathcal{A}$  are compact then the Hamiltonian vector field  $\nabla_{\alpha^* I}^M$  is automatically complete for all  $I \in C^\infty(A)$ . This is due to the fact that the flow of  $\nabla_{\alpha^* I}^M$  acts on the leaves, and thus is always confined by a compact submanifold of  $M$ .

If the foliation  $\mathcal{A}$  is Lagrangian such that  $\mathcal{A} = \mathcal{B}$  the system is integrable in the usual (commutative) sense.

Various synonyms are used for the term non-commutative integrability. The most common ones are degenerate integrability (Nehorošev [30]), superintegrability (Fasso [15]), and generalized Liouville integrability (Zung [54]). The latter terminology was also used in Section 3.A to distinguish the approach via first integrals from this more geometric approach.

One should also note that Fasso and Ratiu [16] consider the system  $(M, \omega, H)$  to be non-commutatively integrable if the Hamiltonian vector field  $\nabla_F^M$  is tangent to the leaves of the *isotropic* foliation  $\mathcal{B}$  which are assumed to be compact. We get rid of the compactness condition by assuming the Hamiltonian vector fields  $\nabla_{\alpha^* I}^M$  to be complete for all  $I \in C^\infty(A)$ . Further, it follows from our definition that  $\nabla_H^M$  is tangent to the leaves of  $\mathcal{B}$ . Conversely, if  $\nabla_H^M$  is assumed to be tangent to the leaves of  $\mathcal{B}$  then  $dh \cdot \nabla_{\alpha^* f}^M = \omega(\nabla_H^M, \nabla_{\alpha^* f}^M) = 0$  for all functions  $f \in C^\infty(A)$ . Since  $T_x \alpha^{-1}(\alpha(x))$  is spanned by Hamiltonian vector fields of the type  $\nabla_{\alpha^* f}^M(x)$  for all  $x \in M$  this implies that  $H$  is constant along the (connected) fibers of  $\alpha$ , and thus factors over  $\alpha$  to a mapping  $h : A \rightarrow \mathbb{R}$ .

Notice also the connection with the definition of generalized Liouville integrability from Section 3.A. Indeed, there the central object of interest was the set  $\mathcal{F}_H$  of all first integrals of the Hamiltonian  $H$ . Now, if the dimension of the action space  $A$  equals the number independent frequencies of the system, then  $\mathcal{F}_H = \beta^* C^\infty(B)$ .

The following is a corollary of the Liouville-Arnold Theorem 3.C.2.

**COROLLARY 3.C.4.** *Let  $(M, \omega, H)$  be a non-commutatively integrable system. Assume  $(\varphi, I, q, p)$  are generalized action-angle variables on an open subset  $U \subseteq M$ . That is*

$$\omega|_U = \sum_{j=1}^l dI_j \wedge d\varphi_j + \sum_{j=1}^{n-l} dp_j \wedge dq_j,$$

*In these coordinates the flow equations to the Hamiltonian  $H$  then assume the canonical form*

$$\begin{aligned} \frac{\partial}{\partial t} \Big|_0 (\varphi_i \circ \text{Fl}_t^{\nabla_H^\omega})(x) &= d\varphi_i(x) \cdot \tilde{\omega}^{-1} \left( \sum_{j=1}^l \frac{\partial H}{\partial I_j}(x) dI_j(x) \right) = \frac{\partial H}{\partial I_i}(x), \\ \frac{\partial}{\partial t} \Big|_0 (I_i \circ \text{Fl}_t^{\nabla_H^\omega})(x) &= 0, \\ \frac{\partial}{\partial t} \Big|_0 (q_j \circ \text{Fl}_t^{\nabla_H^\omega})(x) &= 0, \\ \frac{\partial}{\partial t} \Big|_0 (p_j \circ \text{Fl}_t^{\nabla_H^\omega})(x) &= 0. \end{aligned}$$

*In particular, we have  $H = H(I)$  in these coordinates.*

**PROOF.** By Theorem 3.C.2 the appropriate set  $(\varphi, I, q, p)$  of generalized action-angle variable indeed does exist. By construction via the bifoliation of these coordinates the Hamiltonian  $H$  satisfies the same

commutation relations with  $(\varphi, I, q, p)$  as in Remark 4 of Section 3.A. Thus the flow equations follow.  $\square$

### 3.D. Bifoliations with symmetry

Suppose we are given a non-commutatively integrable system  $(M, \omega, H)$  with symmetries. That is, continuing the notation from the previous section, there is a regular bifoliation  $(\mathcal{A}, \mathcal{B})$  of  $M$  where  $\mathcal{A}$  has coisotropic leaves, and  $H$  factors to a mapping  $h$  on  $A := M/\mathcal{A}$  through the projection  $\alpha : M \rightarrow A$  onto the action space. The symmetries of the system are expressed by a Hamiltonian action  $K \times M \rightarrow M$  of a compact connected Lie group with equivariant momentum map  $J : M \rightarrow \mathfrak{k}^*$ . We suppose that  $K$  also acts on the action space  $A$  such that  $\alpha$  is  $K$ -equivariant, and that  $h$  is  $K$ -invariant. Thus also  $H$  is  $K$ -invariant.

In this setting it is natural to reduce the system  $(M, \omega, H)$  in the following sense. Let  $\mathcal{O}$  be a coadjoint in the image of the momentum map  $J$  such that  $J$  is of constant rank in an open neighborhood of  $J^{-1}(\mathcal{O})$  in  $M$ . Then it is well known that the symplectic reduced space  $J^{-1}(\mathcal{O})/K = M//_{\mathcal{O}}K$  is a smooth symplectic manifold with uniquely induced symplectic structure, and that  $H$  reduces to a Hamiltonian  $H_0$  on  $M//_{\mathcal{O}}K$ . By this we mean the following. Let  $\iota : J^{-1}(\mathcal{O}) \hookrightarrow M$  and  $\pi : J^{-1}(\mathcal{O}) \rightarrow M//_{\mathcal{O}}K$  denote inclusion and projection respectively. Then the reduced Hamiltonian satisfies  $\pi^*H_0 = \iota^*H$ . This terminology is justified due to Noether's Theorem which guarantees that the flow lines of the Hamiltonian vector field of  $H$  are confined within  $J^{-1}(\mathcal{O})$  and, furthermore, project onto the flow lines of the Hamiltonian vector field of  $H_0$ .

As it is natural to reduce the system it is also natural to suspect that some features of the non-commutative integrability will be valid for the reduced system. For the case of Liouville integrable systems this has been dealt with by Zung [54, 55] as was shown in Section 3.A. However, for the case of non-commutative integrability via bifoliations this topic seems to be, so far, untouched. We try to give a partial answer to this problem in this section.

**PROPOSITION 3.D.1.** *Let  $(M, \omega, H)$  be a non-commutatively integrable Hamiltonian system which is invariant under the action by a compact connected Lie group  $K$  such that there is an equivariant momentum map  $J : M \rightarrow \mathfrak{k}^*$ . Assume that  $(M, \omega, H)$  is integrable via the regular bifoliation  $(\mathcal{A}, \mathcal{B})$  where  $\mathcal{A}$  has coisotropic leaves such that  $H$  factors over  $\alpha : M \rightarrow M/\mathcal{A} =: A$  to a mapping  $h : A \rightarrow \mathbb{R}$ . Furthermore,  $K$  acts on  $A$  such that  $\alpha : M \rightarrow A$  is equivariant, and  $h$  is  $K$ -invariant. Let  $\mathcal{O}$  be a coadjoint orbit in the image of  $J$  such that  $J$  is of constant rank on an open neighborhood of  $J^{-1}(\mathcal{O})$  in  $M$ , and let the orbit spaces  $M/K$  and  $A/K$  be smooth manifolds.*

Assume also that  $A$  is connected and that  $\alpha : M \rightarrow A$  possesses a global section  $S_\alpha : A \hookrightarrow M$  such that  $S_\alpha(A) \cap J^{-1}(\mathcal{O}) \neq \emptyset$ .

Then there exists a surjective and submersive Poisson morphism  $\alpha_0 : M//_{\mathcal{O}}K \rightarrow A/K$  into the trivial Poisson bracket on  $A/K$  such that

$$\begin{array}{ccccc} M//_{\mathcal{O}}K & \hookrightarrow & M/K & \longleftarrow & M \\ H_0 \downarrow & \searrow \alpha_0 & \downarrow & & \downarrow \alpha \\ \mathbb{R} & \xleftarrow{h} & A/K & \longleftarrow & A \end{array}$$

is commutative, and where  $H_0$  is the reduced Hamiltonian.

PROOF. Via the section  $S_\alpha$  let us consider  $A$  as a submanifold of  $M$ . Notice that  $A$  is isotropic and  $K$ -invariant in  $(M, \omega)$ . We claim that

$$dJ|_{TA} = 0$$

whence  $J$  is locally constant along  $A$ . Indeed, let  $\xi$  be a vector field that is tangent to  $A$  and  $X \in \mathfrak{k}$  arbitrary. Then

$$\langle dJ(x) \cdot \xi_x, X \rangle = \omega_x(\zeta_X(x), \xi_x) = 0$$

for all  $x \in A$  since  $A$  is  $K$ -invariant and isotropic. Therefore,  $J$  is locally constant and by connectedness also globally constant on  $A$ . By assumption there is a point  $a \in A$  such that  $J(a) \in \mathcal{O}$ , and thus it follows that  $A \subseteq J^{-1}(\mathcal{O})$ .

By the assumptions  $J^{-1}(\mathcal{O})$  is an immersed initial submanifold of  $M$ , and so  $\alpha$  restricts to a smooth surjective mapping  $\alpha|_{J^{-1}(\mathcal{O})} : J^{-1}(\mathcal{O}) \rightarrow A$ . However, since

$$T_x A \subseteq \ker T_x J \subseteq T_x J^{-1}(\mathcal{O})$$

for all  $x \in A \subseteq J^{-1}(\mathcal{O})$ . By equivariance  $\alpha$  thus induces a surjective submersion  $\alpha_0 : M//_{\mathcal{O}}K \rightarrow A/K$ , since  $M//_{\mathcal{O}}K = J^{-1}(\mathcal{O})/K$  is a smooth manifold as it follows from the assumptions. Moreover, because the inclusion  $M//_{\mathcal{O}}K \hookrightarrow M/K$  is a Poisson morphism and  $\alpha$  itself is Poisson it follows from the diagram in the assertion that also  $\alpha_0$  is a Poisson morphism into the coinduced bracket on  $A/K$  which is the trivial bracket.

Finally it is clear that  $H_0$  factors over  $\alpha_0$  to a map  $h$  since  $H$  factors over  $\pi_A \circ \alpha : M \rightarrow A \rightarrow A/K$  by assumption.  $\square$

As a corollary to this proposition we find out that the dynamics of the reduced system behave like those of a (non-commutatively) integrable system in the following way.

COROLLARY 3.D.2. *Let assumptions and notation be the same as in Proposition 3.D.1. Then the following are true.*

- (i) The projection  $\alpha_0 : M//_{\mathcal{O}}K \rightarrow A/K$  onto the reduced action space determines a coisotropic foliation  $\mathcal{A}_0$  whose leaves are the (connected) fibers of  $\alpha_0$ .
- (ii) The coisotropic foliation  $\mathcal{A}_0$  is symplectically complete, and we denote its polar foliation which is isotropic by  $\mathcal{B}_0$ .
- (iii) The Hamiltonian vector field on  $M//_{\mathcal{O}}K$  of  $\alpha_0^*f$  is complete for all  $f \in C^\infty(A/K)$ .
- (iv) The flow lines of the Hamiltonian vector field of the reduced Hamiltonian  $H_0$  are parallel to the isotropic leaves of  $\mathcal{B}_0$ .
- (v) If the foliation  $\mathcal{B}_0$  is regular then the reduced system  $(M//_{\mathcal{O}}K, \omega_0, H_0)$  is non-commutatively integrable in the sense of Definition 3.C.3

PROOF. (i) and (ii). Note that the fibers of  $\alpha_0$  are connected since this is true for the fibers of  $\alpha$ . Thus we obtain a coisotropic foliation  $\mathcal{A}_0$  which is symplectically complete as in Proposition 3.B.1.

(iii). Let  $f_0 \in C^\infty(A/K)$  which we may identify with a function  $f \in C^\infty(A)^K$ . Since  $(M, \omega, H)$  is non-commutatively integrable it follows the Hamiltonian vector field  $\nabla_{\alpha^*f}^\omega$  is complete by Definition 3.C.3. Since  $\alpha^*f \in C^\infty(M)^K$  by equivariance the vector field  $\nabla_{\alpha^*f}^\omega$  is tangent to  $J^{-1}(\mathcal{O})$  by virtue of the Noether Theorem. Now via reduction the Hamiltonian vector field of  $\alpha^*f$  on  $M$  projects to the Hamiltonian vector field of  $\alpha_0^*f_0$  on  $M//_{\mathcal{O}}K$ , and since the former is complete so is the latter.

(iv). This follows from the construction of  $\mathcal{B}_0$  (see Proposition 3.B.1) and the fact that  $H_0 = \alpha_0^*h$ .

(v). This last point is clear from the definition.  $\square$

Motivated by the above conclusion we introduce the following terminology.

DEFINITION 3.D.3 (Weak non-commutative integrability). A Hamiltonian system  $(M, \omega, H)$  is called WEAKLY NON-COMMUTATIVE INTEGRABLE if the following requirements are met. There is a bifoliation  $(\mathcal{A}, \mathcal{B})$  where  $\mathcal{A}$  is a regular coisotropic foliation and a smooth function  $h : M/\mathcal{A} =: A \rightarrow \mathbb{R}$  such that

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & A \\ & \searrow H & \downarrow h \\ & & \mathbb{R} \end{array}$$

commutes. Moreover, the Hamiltonian vector field  $\nabla_{\alpha^*I}^M$  is complete for all  $I \in C^\infty(A)$ .

Thus we can rephrase Corollary 3.D.2 by saying that, under appropriate assumptions, a non-commutatively integrable system with symmetries reduces to a weakly non-commutative integrable system.

One of the drawbacks of dropping the regularity assumption on the isotropic foliation  $\mathcal{B}$  is that there need not exist generalized action-angle coordinates on neighborhood of a full leaf of  $\mathcal{B}$ . However, we can pass to the local leaf space with respect to the foliation  $\mathcal{B}$ , and then locally carry out the same construction as in the regular case.

The motivating example for this construction is the Calogero-Moser system which is obtained via a cotangent bundle reduction from a simple (integrable) system on a cotangent bundle.

## CHAPTER 4

### Spin Calogero-Moser systems

Consider the mechanical system that consists of  $l$  particles which move on the real line and interact via an inverse square potential. That is we are concerned with the Hamiltonian system which is described by the Hamiltonian function

$$H_{\text{CM}}(q_1, \dots, q_l, p_1, \dots, p_l) = \frac{1}{2} \sum_{i=1}^l p_i^2 + \sum_{i>j} V(q_{ij})$$

where  $q_i$  and  $p_i$  denote position and momenta of the particles, respectively, and  $q_{ij} := q_i - q_j$ . The potential is given by

$$V(\xi) = \frac{1}{\xi^2},$$

and  $H_{\text{CM}}$  is called the classical, i.e. spin less, rational CALOGERO-MOSER Hamiltonian function. One can alter the potential function to consider trigonometric, hyperbolic, and elliptic Calogero-Moser systems.

In general, when one is given a Hamiltonian system with symmetries one can reduce the system to obtain a system that depends on fewer variables than the original one. The idea is that the reduced system will be easier to solve than the original one since there are fewer equations. Then one usually wants to reconstruct solutions of the original system out of solutions for the reduced system. However, it may also occur that the original system is of a very simple form while the reduced system is very complicated. In this case one can picture the reduced system as a system that has hidden symmetries. Indeed, the Calogero-Moser system is such a system that can be constructed through a reduction process from a very simple, namely a free, Hamiltonian system with symmetries on a cotangent bundle. This observation is due to Kazhdan, Kostant, Sternberg [20, Section 2] for the case of classical rational Calogero-Moser systems. For non-rational Calogero-Moser systems see, for example, Gorsky and Nekrasov [18].

The point of this chapter is to apply the cotangent bundle reduction procedure of Section 2.D to obtain the Calogero-Moser system as a *reduced* Hamiltonian system. Therefore, we consider only the case of rational potential since the other cases involve infinite dimensional Hamiltonian systems, and this is beyond the scope of the techniques from Section 2.D.

The observation that the construction of the Calogero-Moser system via Hamiltonian reduction should prove it integrable in a non-commutative way was made by Reshetikhin [42].

#### 4.A. $\mathrm{SL}(m, \mathbb{C})$ by hand

As an example consider  $G = \mathrm{SL}(m, \mathbb{C})$ . Here we work along the lines of Kazhdan, Kostant, Sternberg [20, Section 2] who considered the case  $G = \mathrm{SU}(m, \mathbb{C})$ . See also Alekseevsky, Kriegel, Losik, Michor [2, Section 5.7]. The point to this example is that we try to say as much as possible about the reduced phase space by using an *ad hoc* approach. The result may then be taken as motivation for the general theory of cotangent bundle reduction.

Let  $\mathcal{O} = \mathrm{Ad}(G)Z_0$  be an orbit passing through a semisimple element  $Z_0$ . Consider  $(a, \alpha) \in G_r \times \mathfrak{g}$  with  $\alpha - a\alpha a^{-1} = \mu(a, \alpha) = Z$ . As usual  $G_r$  denotes the set of regular elements, that is  $G_r$  consists of those matrices that have  $m$  different eigenvalues. Moreover, we let  $H$  denote the subgroup of diagonal matrices, and  $H_r := H \cap G_r$ . Via the  $\mathrm{Ad}(G)$ -action we can bring  $a$  in diagonal form with entries  $a_i \neq a_j$  for  $i \neq j$ . Since  $Z_{ij} = \alpha_{ij} - \frac{a_i}{a_j}\alpha_{ij}$  the following are coordinates on  $(\mu^{-1}(\mathcal{O}) \cap (G_r \times \mathfrak{g}))/\mathrm{Ad}(G)$ .

- $a_i$  for  $i = 1, \dots, m$ .
- $\alpha_i := \alpha_{ii}$  for  $i = 1, \dots, m$ .
- $\alpha_{ij} = (1 - \frac{a_i}{a_j})^{-1}Z_{ij}$  for  $i \neq j$ .

These coordinates give an identification

$$(\mu^{-1}(\mathcal{O}) \cap (G_r \times \mathfrak{g}))/\mathrm{Ad}(G) = (T^*H_r \times (\mathcal{O} \cap \mathfrak{h}^\perp)/\mathrm{Ad}(H))/W$$

where  $W = N(H)/H$  is the Weyl group. CLAIM: *If  $\mathcal{O}$  is an orbit which is of minimal non-zero dimension then we have that  $\mathcal{O} \cap \mathfrak{h}^\perp/\mathrm{Ad}(H) = \{\text{point}\}$ . Moreover, the reduced phase space can be described as  $(\mu^{-1}(\mathcal{O}) \cap (G_d \times \mathfrak{g}))/\mathrm{Ad}(G) \cong T^*H_r/W$ .*

Here  $G_d$  denotes the open and dense subset of all diagonalizable elements in  $\mathrm{SL}(m, \mathbb{C})$ . Indeed, let  $\mu(a, \alpha) = Z \in \mathcal{O} \cap \mathfrak{h}^\perp$  with  $a$  in diagonal form. Thus  $Z = vw^t - cI$  where  $c := \frac{1}{m}\langle v, w \rangle \neq 0$ ,  $v, w \in \mathbb{C}^m$ , and  $w^t$  is the transposed to the column vector  $w$ . Since  $Z \in \mathfrak{h}^\perp$  we infer that  $v_i w_i = c$ . Hence

$$\mathcal{O} \cap \mathfrak{h}^\perp = \{(\frac{c}{v_1}v, \dots, \frac{c}{v_m}v) - cI : v_i \in \mathbb{C} \setminus \{0\}\}.$$

Take such an  $(\frac{c}{v_1}v, \dots, \frac{c}{v_m}v) - cI =: Z_1$ . Let  $h = \prod_{i=1}^m v_i \cdot \mathrm{diag}(v_1^{-1}, \dots, v_m^{-1})$ . Then we can bring  $Z_1$  into the normal form  $\mathrm{Ad}(h)Z_1 = c(1)_{ij} - cI$  where  $(1)_{ij}$  denotes the  $m \times m$ -matrix with all entries equal to 1. Finally note that  $\alpha_{ij} - \frac{a_i}{a_j}\alpha_{ij} = \frac{c}{v_j}v_i \neq 0$  implies that  $a = \mathrm{diag}(a_1, \dots, a_m)$  is actually regular.



### 4.B. Application: Hermitian matrices

Consider  $V$  the space of complex Hermitian  $n \times n$  matrices as the configuration space to start from. This space shall be endowed with the inner product  $V \times V \rightarrow \mathbb{R}$ ,  $(a, b) \mapsto \text{Tr}(ab)$ . Moreover, we let  $G = \text{SU}(n, \mathbb{C})$  act on  $V$  by conjugation. Clearly this action leaves the trace form invariant. Via the inner product we can trivialize the cotangent bundle as  $T^*V = V \times V^* = V \times V$ , and the cotangent lifted action of  $G$  is simply given by the diagonal action. The canonical symplectic form on  $T^*V$  is given by

$$\Omega_{(a, \alpha)}((a_1, \alpha_1), (a_2, \alpha_2)) = \text{Tr}(\alpha_2 a_1) - \text{Tr}(\alpha_1 a_2).$$

The free Hamiltonian on  $T^*V = V \times V$  is given by

$$H_{\text{free}} : (a, \alpha) \mapsto \frac{1}{2} \text{Tr}(\alpha \alpha).$$

Trajectories of this Hamiltonian are given by straight lines of the form  $t \mapsto a + t\alpha$  in the configuration space  $V$ .

Let us further identify  $\mathfrak{su}(n)^* = \mathfrak{su}(n)$  via the Killing form. The momentum mapping is then given by

$$\mu : (a, \alpha) \mapsto [a, \alpha] = \text{ad}(a).\alpha.$$

Consider also an orbit  $\mathcal{O}$  together with its canonically induced symplectic structure in the image of the momentum mapping.

**Assumption:** The orbit  $\mathcal{O}$  is such that  $\mu^{-1}(\mathcal{O}) \subseteq V_r \times V$ . Here  $V_r$  denotes the set of regular elements in  $V$  with respect to the  $G$  action. This assumption is for example fulfilled if the projection from  $\mathcal{O}$  to any root space is non-trivial.

Let  $\Sigma$  denote the subspace of  $V$  consisting of diagonal matrices. Then  $\Sigma$  is a section of the  $G$ -action on  $V$ , see Section 4.F. Further, we define  $\Sigma_r := V_r \cap \Sigma$ . Within  $\Sigma$  we choose the positive Weyl chamber  $C := \{\text{diag}(q_1, \dots, q_n) : q_1 > \dots > q_n\}$  so that  $C = \Sigma/W$  where  $W = W(\Sigma) = N_G(\Sigma)/Z_G(\Sigma)$ . Thus  $C_r := \Sigma_r \cap C$  may be considered as a global slice for the  $G$ -action on  $V_r$  so that  $G/M \times C_r \cong V_r$ ,  $(gM, a) \mapsto g.a$  where  $M := Z_G(\Sigma_r) = Z_G(\Sigma)$ . That is  $M$  is the subgroup of  $\text{SU}(n)$  consisting of diagonal matrices only. Now we may apply Corollary 2.D.5 to get

$$T^*V //_{\mathcal{O}} G = T^*C_r \times \mathcal{O} //_0 M$$

as symplectically stratified spaces. The strata are of the form

$$(T^*V //_{\mathcal{O}} G)_{(L)} = T^*C_r \times (\mathcal{O} //_0 M)_{(L_0)M}$$

where  $L_0$  is a subgroup of  $M$  conjugate to  $L$  within  $G$ . Moreover, the reduced symplectic structure  $\sigma_{(L)}^{\mathcal{O}}$  on  $(T^*V //_{\mathcal{O}} G)_{(L)}$  is of product form, i.e.

$$\sigma_{(L)}^{\mathcal{O}} = \Omega^{C_r} - \Omega_{(L_0)M}^{\mathcal{O}}$$

where  $\Omega_{(L_0)M}^{\mathcal{O}}$  is the canonically reduced symplectic form on  $(\mathcal{O} //_0 M)_{(L_0)M}$ .

From the general theory we know that the Hamiltonian  $H_{\text{free}}$  reduces to a Hamiltonian  $H_{\text{CM}}^{(L)}$  on the stratum  $(T^*V //_{\mathcal{O}} G)_{(L)}$ , and that integral curves of  $H_{\text{free}}$  project to integral curves of  $H_{\text{CM}}^{(L)}$ . In particular the dynamics remain confined to the symplectic stratum. The reduced Hamiltonian is thus given by

$$H_{\text{CM}}^{(L)}(q, p, [\lambda]) = H_{\text{free}}(q, p + A_q^*(\lambda))$$

where  $[\lambda]$  is the class of  $\lambda$  in  $(\mathcal{O} //_0 M)_{(L_0)M}$  and  $A_q^* : \mathfrak{g}_q^\perp = \mathfrak{m}^\perp \rightarrow T_q(G \cdot q) = \Sigma^\perp$  is the point wise dual to the mechanical connection as introduced in Section 2.D. Assume that  $q = \text{diag}(q_1 > \dots > q_n)$  and that  $\lambda = (\lambda_{ij})_{ij} \in (\mathcal{O} \cap \mathfrak{m}^\perp)_{(L_0)M}$ . Then

$$A_q^*(\lambda)_{ij} = \frac{\lambda_{ij}}{q_i - q_j} \text{ for } i \neq j, \text{ and } A_q^*(\lambda)_{ii} = 0.$$

Therefore, for  $p = \text{diag}(p_1, \dots, p_n) \in \Sigma$  and  $q, [\lambda]$  as introduced we obtain

$$\begin{aligned} H_{\text{CM}}^{(L)}(q, p, [\lambda]) &= \frac{1}{2} \text{Tr}(p)^2 + \frac{1}{2} \text{Tr}(A_q^*(\lambda))^2 \\ &= \frac{1}{2} \sum_{i=1}^n p_i^2 + \frac{1}{2} \sum_{i \neq j} \frac{\lambda_{ij} \lambda_{ji}}{(q_i - q_j)(q_j - q_i)} \\ &= \frac{1}{2} \sum_{i=1}^n p_i^2 + \sum_{i > j} \frac{|\lambda_{ij}|^2}{(q_i - q_j)^2} \end{aligned}$$

since  $\lambda_{ji} = -\overline{\lambda_{ij}}$  and  $\text{Tr}(pA_q^*(\lambda)) = \text{Tr}(A_q^*(\lambda)p) = 0$ . This is the Hamiltonian function of the Calogero-Moser system with spin. Integrability of this system in the non-commutative sense is proved in the next section in a more general context.

#### 4.C. Calogero-Moser systems associated to polar representations

The idea of considering polar representations of compact Lie groups to obtain new versions of Spin Calogero-Moser systems is due to Alekseevsky, Kriegl, Losik, Michor [2].

As in Section 4.F let  $V$  be a real Euclidean vector space and  $G$  a connected compact Lie group that acts on  $V$  via a polar representation. Via the inner product we consider the cotangent bundle of  $V$  as a product  $T^*V = V \times V$ . The canonical symplectic form  $\Omega$  is thus given by

$$\Omega_{(a,\alpha)}((a_1, \alpha_1), (a_2, \alpha_2)) = \langle \alpha_2, a_1 \rangle - \langle \alpha_1, a_2 \rangle$$

where  $\langle \cdot, \cdot \rangle$  is the inner product on  $V$ . The free Hamiltonian on  $T^*V = V \times V$  is given by

$$H_{\text{free}} : (a, \alpha) \longmapsto \frac{1}{2} \langle \alpha, \alpha \rangle.$$

Trajectories of this Hamiltonian are given by straight lines of the form  $t \mapsto a + t\alpha$  in the configuration space  $V$ .

Of course, we want to play with the cotangent lifted action of  $G$ , and this is just the diagonal action of  $G$  on  $V \times V$ . By Section 4.F we may think of the action by  $G$  on  $V$  as a symmetric space representation and thus consider  $\mathfrak{g} \oplus V =: \mathfrak{l}$  as a real semisimple Lie algebra with Cartan decomposition into  $\mathfrak{g}$  and  $V$ , and with bracket relations  $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}$ ,  $[\mathfrak{g}, V] \subseteq V$ , and  $[V, V] \subseteq V$ . The momentum mapping corresponding to the  $G$ -action on  $T^*V = V \times V$  is now given by

$$\mu : V \times V \longrightarrow \mathfrak{g}^* = \mathfrak{g}, \quad (a, \alpha) \longmapsto [a, \alpha] = \text{ad}(a).\alpha$$

where we identify  $\mathfrak{g} = \mathfrak{g}^*$  via an  $\text{Ad}(G)$ -invariant inner product. Consider also an orbit  $\mathcal{O}$  together with its canonically induced symplectic structure in the image of the momentum mapping.

**Assumption:** The orbit  $\mathcal{O}$  is such that  $\mu^{-1}(\mathcal{O}) \subseteq V_r \times V$ . Here  $V_r$  denotes the set of regular elements in  $V$  with respect to the  $G$  action. We proceed as above, and let  $\Sigma$  denote a fixed section of the  $G$ -action on  $V$ , consider  $C$  a Weyl chamber in  $\Sigma$ , and put  $M := Z_G(\Sigma)$ . Now we may apply Corollary 2.D.5 to get

$$T^*V //_{\mathcal{O}} G = T^*C_r \times \mathcal{O} //_0 M$$

as symplectically stratified spaces. The strata are of the form

$$(T^*V //_{\mathcal{O}} G)_{(L)} = T^*C_r \times (\mathcal{O} //_0 M)_{(L_0)M}$$

where  $L_0$  is a subgroup of  $M$  conjugate to  $L$  within  $G$ . Moreover, the reduced symplectic structure  $\sigma_{(L)}^{\mathcal{O}}$  on  $(T^*V //_{\mathcal{O}} G)_{(L)}$  is of product form, i.e.

$$\sigma_{(L)}^{\mathcal{O}} = \Omega^{C_r} - \Omega_{(L_0)M}^{\mathcal{O}}$$

where  $\Omega_{(L_0)M}^{\mathcal{O}}$  is the canonically reduced symplectic form on  $(\mathcal{O} //_0 M)_{(L_0)M}$ .

From the general theory we know that the Hamiltonian  $H_{\text{free}}$  reduces to a Hamiltonian  $H_{\text{CM}}^{(L)}$  on the stratum  $(T^*V //_{\mathcal{O}} G)_{(L)}$ , and that integral curves of  $H_{\text{free}}$  project to integral curves of  $H_{\text{CM}}^{(L)}$ . In particular the dynamics remain confined to the symplectic stratum. The reduced Hamiltonian is thus given by

$$H_{\text{CM}}^{(L)}(q, p, [Z]) = H_{\text{free}}(q, p + A_q^*(\lambda))$$

where  $[Z]$  is the class of  $Z$  in  $(\mathcal{O} //_0 M)_{(L_0)M}$  and  $A_q^* : \mathfrak{g}_q^{\perp} = \mathfrak{m}^{\perp} \rightarrow T_q(G.q) = \Sigma^{\perp}$  is the point wise dual to the mechanical connection as introduced in Section 2.D. Let

$$q \in C_r, \quad p = \sum_{i=1}^l p_i B_0^i, \quad \text{and} \quad Z = \sum_{\lambda \in R} \sum_{i=1}^{k_{\lambda}} z_{\lambda}^i E_{\lambda}^i \in (\mathcal{O} \cap \mathfrak{m}^{\perp})_{(L_0)M}$$

where  $l = \dim \Sigma$  and  $k_\lambda = \frac{1}{2} \dim \mathfrak{l}_\lambda$ . Here notation is as in Section 4.F, and  $R = R(\mathfrak{l}, \Sigma) \subseteq \Sigma^*$  denotes the set of restricted roots, in particular. With these definitions the dual mapping to the mechanical connection is given by

$$A_q^*(Z) = \sum_{\lambda \in R} \sum_{i=1}^{k_\lambda} \frac{z_\lambda^i}{\lambda(q)} B_\lambda^i.$$

Note that  $\lambda(q) \neq 0$  for all  $\lambda \in R$  since  $q \in C_r$  is regular. The reduced Hamiltonian thus computes to

$$H_{\text{CM}}^{(L)}(q, p, [Z]) = \frac{1}{2} \langle p + A_q^*(Z), p + A_q^*(Z) \rangle = \frac{1}{2} \sum_{i=1}^l p_i^2 + \frac{1}{2} \sum_{\lambda \in R} \frac{\sum_{i=1}^{k_\lambda} z_\lambda^i z_\lambda^i}{\lambda(q)^2}.$$

The reduced Hamiltonian system  $(T^*V //_{\mathcal{O}} G, \sigma^{\mathcal{O}}, H_{\text{CM}})$  is thus a new version of a Calogero-Moser system with spin. It is in fact a stratified Hamiltonian system in the sense that it is a Hamiltonian system on each symplectic stratum  $(T^*V //_{\mathcal{O}} G)_{(L)}$ , and the Hamiltonian dynamics stay confined to these strata.

#### 4.D. Integrability of Calogero-Moser systems

**Liouville integrability of the Calogero-Moser system.** We now show that the thus obtained Calogero-Moser system is integrable in the generalized Liouville sense. To do so we will use Theorem 3.A.3. We start by choosing coordinates  $q_1, \dots, q_n, p_1, \dots, p_n$  on  $T^*V = V \times V$  such that the Poisson bracket of functions  $f, g \in C^\infty(V \times V)$  is given by the usual equation

$$\{f, g\} = \sum_{i=1}^n \left( \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right).$$

Moreover we assume that  $q_1, \dots, q_l, p_1, \dots, p_l$  are coordinates on  $\Sigma \times \Sigma \hookrightarrow V \times V$ . Let us now consider the map

$$\Phi : V \times V \longrightarrow \Sigma^\perp \times V$$

given by projection, and endow  $\Sigma^\perp \times V$  with the inherited Poisson structure. Clearly,  $C^\infty(\Sigma^\perp \times V)$  has a center and this is just generated by  $p_1, \dots, p_l$ . Thus we may identify  $Z(C^\infty(\Sigma^\perp \times V)) = C^\infty(\Sigma)$ . Now the set of all first integrals of  $H_{\text{free}}$ , i.e.

$$\mathcal{F}_{H_{\text{free}}} = \{F \in C^\infty(V \times V) : \{F, H\} = 0\}$$

can be identified with  $C^\infty(\Sigma^\perp \times V)$  via  $\Phi$  since  $H_{\text{free}}$  factors over the projection onto the second factor and is  $G$ -invariant, and thus can be considered as a function on  $\Sigma$ . Therefore,

$$\begin{aligned} \dim V \times V &= \dim \Sigma^\perp \times V + \dim \Sigma \\ &= \text{ddim } \mathcal{F}_{H_{\text{free}}} + \text{ddim } Z(C^\infty(\Sigma^\perp \times V)), \end{aligned}$$

and we are exactly in the situation of Theorem 3.A.3 to conclude generalized Liouville integrability of the reduced system.

**Non-commutative integrability of the system.** The aim of this section is to employ the methods of Section 3.D to better understand the geometry of the reduced Hamiltonian system on  $T^*V_r//_{\mathcal{O}}G$ . We continue to stick to the notation from the previous paragraphs. That is  $\Sigma$  is a section of the polar representation of  $G$  on  $V$ , the subset  $C$  is a Weyl chamber in this section, and  $M = Z_G(\Sigma)$  is the centralizer of  $\Sigma$  in the group. Moreover,  $W = W(\Sigma) := N_G(\Sigma)/M$  denotes the Weyl group associated to the section  $\Sigma$ .

Consider the projection of  $T^*V = V \times V$  onto the second factor

$$T^*V = V \times V \xrightarrow{\alpha := \text{pr}_2} V =: A$$

which has the property the  $H_{\text{free}}$  factors over  $\alpha$  to a mapping

$$h : A \longrightarrow \mathbb{R}, \quad a \longmapsto \frac{1}{2}\langle a, a \rangle.$$

Notice also that  $\alpha$  is equivariant with respect to the diagonal  $G$ -action on  $T^*V$  and the action on  $V$ . Thus we may interpret  $A$  as the action space of the completely integrable system  $(T^V, \Omega, H_{\text{free}})$  in the sense of Definition 3.C.3. Indeed, the system is integrable via the Lagrangian bifoliation given by the fibers of  $\alpha$ .

However, this Lagrangian bifoliation is a bifoliation with symmetries as in Section 3.D. That is, we are in the following situation.

$$\begin{array}{ccccc} T^*V & \twoheadrightarrow & (T^*V)/G & \longleftarrow & T^*V//_{\mathcal{O}}G \\ \alpha \downarrow & & \downarrow & \swarrow \alpha_0 & \downarrow H_0 \\ A & \xrightarrow{\pi_A} & A/G & \xleftarrow{h_0} & \mathbb{R} \end{array}$$

where  $H_0 = H_{\text{CM}}$  is the reduced Hamiltonian and  $h_0$  is such that  $h = \pi_A^* h_0$ . Furthermore,  $\alpha_0 : T^*V//_{\mathcal{O}}G \rightarrow A/G = \Sigma/W$  is a well-defined smooth map of stratified spaces, in the sense of Definition 1.C.3. Indeed, this is true since  $\alpha_0$  is a composition of smooth maps by Example 1.C.7 and Proposition 1.H.3.

However,  $\alpha_0$  need not be surjective nor will it be a strata-preserving map in general. This means that fibers of  $\alpha_0$  are not unions of strata of  $T^*V//_{\mathcal{O}}G$ .

Let  $p \in \Sigma_r$  be an arbitrary regular point in the section. Since  $M = Z_G(\Sigma) = Z_G(p)$  for regular  $p \in \Sigma$ , Remark 1 of Section 4.F implies that the equation

$$\mu(q, p) = \text{ad}(-p).q \in \mathcal{O} \cap \mathfrak{m}^\perp$$

always has a solution  $q$  which is given by

$$q = (\text{ad}(-p)|_{\Sigma^\perp})^{-1}(Z).$$

This shows that the image of  $\alpha_0$  contains all of  $\Sigma_r/W = C_r$ .

Employing the same methods as in Section 4.C we thus conclude that

$$\alpha_0^{-1}(C_r) \cong \Sigma \times \mathcal{O} //_0 M \times C_r \hookrightarrow (T^*V) //_{\mathcal{O}} G$$

where the isomorphism is a symplectic diffeomorphism of stratified symplectic spaces, and the inclusion is strata preserving. In particular, the fibers  $\alpha_0^{-1}([p])$  with  $p \in \Sigma_r$  are stratified into smooth strata of the form

$$\alpha_0^{-1}([p]) \cap (T^*V //_{\mathcal{O}} G)_{(L)} = \Sigma \times (\mathcal{O} //_0 M)_{(L_0)^M}$$

where  $L_0$  is a subgroup of  $M$  conjugate to  $L$  within  $G$ .

More generally, using Remarks 1 and 4 of Section 4.F we see that the fibers of  $\alpha_0$  are of the form

$$\alpha_0^{-1}([p]) = (Z_V(p) \times \mathcal{O} \cap \mathfrak{g}_p^\perp) / N_G(p) = \Sigma / W_p \times (\mathcal{O} \cap \mathfrak{g}_p^\perp) / M$$

where  $[p] \in \Sigma / W = A / K$  is an arbitrary point in the image of  $\alpha_0$ , and  $W_p$  denotes the stabilizer in  $W$  of  $p$  with respect to the effective  $W$ -action on  $\Sigma$ . This is in accordance to the description above since  $\Sigma_r$  can be characterized as the subset of  $\Sigma$  upon which  $W$  acts freely.

To understand the Hamiltonian dynamics generated on  $T^* //_{\mathcal{O}} G$  by  $H_0 = H_{\text{CM}}$  let, as above,  $p_0 = [p] \in C_r \subseteq A_0$ . Then the fiber  $\alpha_0^{-1}(p_0)$  is stratified into coisotropic strata of the form

$$\Sigma \times (\mathcal{O} //_0 M)_{(L_0)^M}$$

where  $L_0$  is a subgroup of  $M$  conjugate to  $L$  within  $G$ . The symplectic perpendicular of such a stratum obviously exists and is given by

$$\Sigma \hookrightarrow \Sigma \times (\mathcal{O} //_0 M)_{(L_0)^M}.$$

By the theory of Section 3.D we thus conclude that the Hamiltonian flow of  $H_0 = H_{\text{CM}}$  starting at a point in

$$\Sigma \times (\mathcal{O} //_0 M)_{(L_0)^M} \times \{p_0\}$$

is given by a straight line in  $\Sigma$ .

This means that we have identified the isotropic invariant submanifolds of the Hamiltonian system  $(T^*V //_{\mathcal{O}} G, H_0 = H_{\text{CM}})$ . These submanifolds have dimension  $l = \dim \Sigma$  which is in general strictly less than  $\frac{1}{2} \dim T^*V //_{\mathcal{O}} G$  whence the Hamiltonian system is a non-commutatively integrable one.

#### 4.E. Discussion of the dynamics

In the last section we discussed the Hamiltonian dynamics of the reduced system  $(T^*V //_{\mathcal{O}} G, H_0 = H_{\text{CM}})$ . However, now we want to investigate the dynamical behavior on the reduced configuration space

$$V/G = \Sigma/W = C$$

which is often also called the shape space of the system. This is interesting because the dynamics that take place on this space are those of the Calogero-Moser dynamical system.

In the previous sections we have given two isomorphic descriptions of an open dense subset of the reduced system  $(T^*V//_{\mathcal{O}}G, H_0 = H_{\text{CM}})$ . That is, we are in the following situation:

$$C_r \times (\Sigma \times \mathcal{O}//_0M) \hookrightarrow T^*V//_{\mathcal{O}}G \longleftarrow (\Sigma \times \mathcal{O}//_0M) \times C_r$$

where we have placed brackets to distinguish between reduced position and reduced momentum coordinates.

As a particular case consider the situation where  $(L_0)^M$  is an element of the isotropy lattice such that

$$(\mathcal{O}//_0M)_{(L_0)^M} = (\mathcal{O}_{(L_0)^M} \cap \text{Ann } \mathfrak{m})/M$$

is discrete. For example, this was the case in the explicit approach of Section 4.A. Since  $W = W(\Sigma)$  is a reflection group we conclude from the above that, in this case, the dynamics on the shape space is given by a line in  $\Sigma$  that is reflected at all Walls. Thus the scattering process is given by a transformation of the type

$$(x_1, \dots, x_l) \mapsto (x_l, \dots, x_1)$$

where  $l = \dim \Sigma$ .

More generally, the dynamics are more complicated, and we consider the coordinates of  $(\mathcal{O}//_0M)_{(L_0)^M}$  to be spin coordinates which keep the dynamics from hitting certain walls.

#### 4.F. Appendix: Polar representations

Let  $V$  be a real Euclidean vector space, and  $G$  be a connected compact Lie group. Further, let  $\rho : G \rightarrow \text{SO}(V, \langle -, - \rangle)$  be a POLAR REPRESENTATION of  $G$  on  $V$ . That is, there is subspace  $\Sigma \subseteq V$  (a SECTION) such that  $\Sigma$  meets all  $G$ -orbits, and does so orthogonally.

The following is due to Dadok [11] and is a consequence of his classification of polar actions.

**PROPOSITION 4.F.1.** *There exists a connected Lie group  $\tilde{G}$  together with a representation  $\tilde{\rho} : \tilde{G} \rightarrow \text{SO}(V)$  such that the following hold. There is a real reductive Lie algebra  $\mathfrak{l}$  with a Cartan decomposition  $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{p}$ . Moreover, there is a Lie algebra isomorphism  $A : \text{Lie}(\tilde{G}) = \tilde{\mathfrak{g}} \rightarrow \mathfrak{k}$  and a linear isomorphism  $B : V \rightarrow \mathfrak{p}$  such that  $B(\tilde{\rho}'(X).v) = [A(X), B(v)]$  for all  $X \in \tilde{\mathfrak{g}}$  and  $v \in V$ . Finally, the  $G$ -orbits coincide with the  $\tilde{G}$ -orbits, that is  $V/G = V/\tilde{G}$ .*

**PROOF.** See Dadok [11, Proposition 6]. □

Thus, for the purpose of this paper, it suffices to assume that the representation of  $G$  on  $V$  is a symmetric space representation whence  $\mathfrak{l} = \mathfrak{g} \oplus V$  is a Cartan decomposition, and hence  $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}$ ,  $[\mathfrak{g}, V] \subseteq V$ , and  $[V, V] \subseteq \mathfrak{g}$ . Therefore,  $G \times V \cong L$ ,  $(g, v) \mapsto g \exp(v)$  is a global Cartan decomposition of  $L$  with maximal compact subgroup  $G$  where  $\text{Lie}(L) = \mathfrak{l}$ .

An element  $v \in V$  is said to be REGULAR (with respect to the  $G$ -action) if the orbit  $\mathcal{O}(v) = \rho(G).v = G.v$  is of maximal possible dimension. The set of regular elements will be denoted by  $V_r$ . The following assertions are well known and/or obvious. See, for example, Knapp [21].

REMARK 1. Let  $v \in V$ . Then, by reason of dimension  $\text{ad}(v)|_{Z_{\mathfrak{g}}(v)^\perp} : Z_{\mathfrak{g}}(v)^\perp \rightarrow Z_V(v)^\perp$  and  $\text{ad}(v)|_{Z_V(v)^\perp} : Z_V(v)^\perp \rightarrow Z_{\mathfrak{g}}(v)^\perp$  both are linear isomorphisms.  $\square$

REMARK 2. The set  $V_r$  of regular elements is open dense in  $V$ . Moreover,  $v \in V_r$  if and only if  $Z_V(v) =: \Sigma$  is a section in  $V$ . This is the case if and only if  $\Sigma$  is maximally Abelian.  $\square$

REMARK 3. Let  $\Sigma \in V$  be a section, and put  $\mathfrak{m} := Z_{\mathfrak{g}}(\Sigma)$ . The set  $R = R(\mathfrak{l}, \Sigma) \subseteq \Sigma^*$  shall denote the set of restricted roots. This gives rise to the restricted root space decomposition

$$\mathfrak{l} = \mathfrak{m} \oplus \Sigma \oplus \bigoplus_{\lambda \in R} \mathfrak{l}_\lambda.$$

Any Cartan subalgebra  $\mathfrak{h} \subseteq \mathfrak{l}$  of  $\mathfrak{l}$  is of the form

$$\mathfrak{h} = \mathfrak{t} \oplus \Sigma$$

where  $\mathfrak{t} \subseteq \mathfrak{m}$  is a Cartan subalgebra (Lie algebra to a maximal torus) of  $\mathfrak{g}$ .

Each restricted root space  $\mathfrak{l}_\lambda$  has an orthonormal basis

$$E_\lambda^i \in \mathfrak{g}, \quad B_\lambda^i \in V$$

where  $i = 1, \dots, k_\lambda = \frac{1}{2} \dim \mathfrak{l}_\lambda$ , and which is such that  $\text{ad}(v)E_\lambda^i = \lambda(v)B_\lambda^i$  and  $\text{ad}(v)B_\lambda^i = \lambda(v)E_\lambda^i$  for all  $v \in \Sigma$ . The vectors

$$E_0^i, i = 1, \dots, \dim \mathfrak{m}, \quad B_0^j, j = 1, \dots, \dim \Sigma$$

will denote an orthonormal basis of  $\mathfrak{m}$ ,  $\Sigma$  respectively.  $\square$

REMARK 4. Consider an arbitrary point  $v \in V$ . Then the isotropy subgroup  $G_v = Z_G(v)$  acts transitively on the set of all sections containing  $v$ . Moreover, if  $\Sigma_0$  and  $\Sigma_1$  both are sections containing  $v$  then the element  $g \in G$  satisfying  $g.\Sigma_0 = \Sigma_1$  is unique up to right multiplication in the group by  $N_G(\Sigma_0)$ .  $\square$

DEFINITION 4.F.2 (Generalized Weyl group). Let  $\Sigma$  be a section in  $V$  of the  $G$ -action. Then

$$W := W(\Sigma) := N_G(\Sigma)/Z_G(\Sigma)$$



is called the GENERALIZED WEYL GROUP associated the action.

REMARK 5. The generalized Weyl group of a polar representation is a classical Weyl group. Indeed, this follows also from Dadok [11, Proposition 6]. In particular,  $W$  is a reflection group.  $\square$



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## Curriculum Vitae

- Born on October 12th 1979, in Vienna, Austria.
- 1990-1998. Attended high school at the Gymnasium auf der Schmelz, Vienna.
- 1995/96. In this academic year I took part in an AFS-program and attended a public high school in Denver, USA.
- 1998/99. Enrolled at the University of Vienna, and began to study physics.
- 1999/00. Began to study mathematics.
- The winter term 2001/02 I spent at the University of Santiago de Compostela, Spain. During this time I was supported by an Erasmus scholarship.
- I completed my diploma thesis in May 2002 under the supervision of Prof. Peter Michor. The title was *Pseudo-holomorphic curves. Transversality and a Gromov-Witten invariant of simple curves.*
- In July 2002 I concluded undergraduate studies and obtained my degree Mag. rer. nat. with distinction.
- Since October 2002 I have been working on my thesis under the supervision of Prof. Michor. This work is supported by the FWF – the Austrian science fund.
- In the wintersemester 2002/03 I was tutor for calculus at the BOKU – the University of Agriculture at Vienna.

### Conference activities: (where I held talks).

- Visitor at the Centre Bernoulli in Lausanne from September 12th to September 24th 2004 through participation in the MASIE-programme.
- The 25th Winter School of Geometry and Physics at Srni, January 15th-22nd, 2005.

### Preprints:

- *Singular cotangent bundle reduction and spin Calogero-Moser systems*, arXiv:math.SG/0411068, 2004.