REPRESENTATIONS OF THE INFINITE DIMENSIONAL HEISENBERG GROUP

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Günther Hörmann

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Für Günther Wetterhahn

“I’m riding through the desert on a horse with no name . . .”

. . . sang die zerfahrene Stimme aus dem Radio mitten in das Chaos der unzähligen Schachteln und Kisten eines typischen Wohnungswechsels. Den Text des Songs in bildliche Vorstellung übersetzend ertappte ich mich bei der Suche nach . . . — ich weiß heute noch nicht wonach. Ähnlich sah ich mich in diesen Tagen durch unvertrautes mathematisches Gelände rechnen mit Werkzeugen, die für mich noch keine klare Bedeutung hatten. Die Oase versprach ich mir von der genaueren Ausarbeitung eines konkreten Beispiels, in dem die abstrakten Namen Gestalt annehmen konnten. Mag sein, daß ich zu lange Zeit beim erfrischenden Wasser verweilt war und den Aufbruch zu gewagteren Strukturen verpaßte. Doch erzählte die Beobachtung des Einwirkens von allgemeineren Strukturen auf eine konkrete Situation mir von der Entstehung einer ersten Annäherung an neue Begriffe.

In diesem Sinne wird in vorliegender Arbeit ab Abschnitt 2 der Raum $\mathbb{R}^{(\mathbb{N})}$ unter das Vergrößerungsglas gelegt und kein Blick mehr durch das Fernglas angeboten. Dies liegt auch darin begründet, daß meine Idee zu diesem Thema entstand, als ich (vergeblich) versuchte die Konstruktionen in [G/W] vollständig zu verstehen.

Ein Großteil der Dissertation enstand aus meinem Vortrag und harten Diskussionen der noch unausgereiften Entwürfe im Seminar von Peter Michor und Andreas Kriegl. Wesentliche Hilfestellung für den dritten Teil erhielt ich von Andreas Cap, dem an dieser Stelle mein aufrichtiger Dank gilt. Für die Diskussion und den Hinweis zur Vereinfachung des Beweises von Proposition 3.2 sowie die Auseinandersetzung mit dem Nebenthema (“Riding through the desert . . .”) bedanke ich mich schließlich bei Niki.

Günther Hörmann, Wien im Juni 1993

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Notes on Notation

A special comment has to be made about usage of the symbol $\hbar$ in this thesis. We will use it to stand for any real non-zero number representing the "height" of a fixed orbit of interest. Though the usual convention in mathematical physics is the assumption of units of measurement yielding $\hbar = 1$ we will take care to make the dependence on this "constant parameter" explicit in all computations — on the other hand we will not attempt to give estimates by inserting the number $1,054589 \cdot 10^{-34}$ into the resulting expressions. We only "see" that this is a non-zero number (i. e. of absolute value greater than some epsilon).

$\mathrm{Aut}(G)$ .... smooth group automorphisms
$\mathrm{Aut}_c(G)$ .... continuous group automorphisms
$C^\infty(G,V)^K$ .... K-equivariant smooth functions $G \to V$
$\mathrm{End}(E)$ .... smooth endomorphisms
$\Gamma(B)$ .... smooth sections of the bundle B
$\mathrm{GL}(E)$ .... linear diffeomorphisms
$h(.,.)$ .... standard Hermitian form of complex sequences
$\langle .|. \rangle$ .... (complex or real) inner product
$\langle .,. \rangle$ .... action of a functional
im .... image of a mapping
Im .... imaginary part of a complex vector
$\Omega^k(M)$ .... differential forms of degree k
Re .... real part of a complex vector
S^1 .... one-dimensional sphere
$\check{\sigma}$ .... embedding $E \hookrightarrow E^*$ associated to σ, $\check{\sigma}(x) = \sigma(x,.)$
$\mathrm{Sp}(E,\sigma)$ .... continuous σ-symplectic endomorphisms on E
$S(V)$ .... symmetric algebra over the vector space V
$\mathcal{U}(V)$ .... unitary operators on the Hilbert space V
$V^{(Y)}$ .... functions $Y \to V$ (V vector space) with finite support
$\mathfrak{X}(M)$ .... smooth vector fields
$\mathcal{Z}(G)$ .... centre of the group G

Chapter I

THE HEISENBERG GROUP

1 General Algebraic Setting

Let E be a real vector space with a bilinear form $\sigma : E \times E \to \mathbb{R}$ having the following properties:

(i) σ is skew symmetric, i.e. $\forall x, y \in E : \quad \sigma(x, y) = -\sigma(y, x)$

(ii) σ is non-degenerate, i.e. $\check{\sigma} : E \to E^*, \check{\sigma}(x) = \sigma(x, .)$ is injective

σ is said to be a *weak symplectic form* and the pair (E, σ) is a *weak symplectic vector space*.

1.1. Examples:

(i) If $E = \mathbb{R}^{2n}$ then the symplectic structure is essentially unique (up to base transformations) and given by

$$\sigma(x, y) = \langle x_1 | y_2 \rangle - \langle x_2 | y_1 \rangle \ ,$$

where $x = x_1 + x_2, y = y_1 + y_2$ according to the decomposition $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$ and $\langle .|. \rangle$ denotes the standard inner product on $\mathbb{R}^n$.

(ii) Let V be a complex vector space equipped with a complex inner product $h : V \times V \to \mathbb{C}$ (conjugate linear in the first entry). Then

$$\sigma(x, y) := \operatorname{Im} h(x, y) \quad \forall x, y \in V$$

defines a weak symplectic form on the underlying real space $E := V_{\mathbb{R}}$.

Note that the situation of (i) is reproduced now by choosing V to be $\mathbb{C}^n$. The main example we will be concerned with is built up from $V = \mathbb{C}^{(\mathbb{N})} := \bigoplus_{n \in \mathbb{N}} \mathbb{C}$ with the standard Hermitian form $h(x, y) = \sum_{j \in \mathbb{N}} \bar{x}^j y^j$. Once again we can write $\sigma(x, y) = \langle x_1 | y_2 \rangle - \langle x_2 | y_1 \rangle$, where x_1, y_1 (resp. x_2, y_2) denote the real (resp. imaginary) part of x, y and $\langle .|. \rangle$ is the standard inner product on $\mathbb{R}^{(\mathbb{N})}$.

(iii) Examples of a very different kind arise from function spaces corresponding to certain symplectic field theories in physics (see [Woo, 7.9]). The symplectic forms are constructed by integrating skew symmetric combinations of the functions and their derivatives. We do not present the details here because in later sections we will concentrate on situations described in (ii).

1.2. Definition: Let (E, σ) be a weak symplectic vector space. Consider the central extension of the commutative group $(E, +)$ with $(\mathbb{R}, +)$ defined by the cocycle $c(x, y) = \frac{1}{2}\sigma(x, y)$, i.e. introduce a group multiplication on the set $E^c := E \times \mathbb{R}$ by setting

$$(x, r) \cdot (y, s) = (x + y, r + s + \frac{1}{2}\sigma(x, y)) \quad . \tag{I.1}$$

The group $(E^c, \cdot)$ is the *(simply connected) Heisenberg group over* (E, σ) and will be denoted by $H(E, \sigma)$ (or just H if σ and E are fixed).

1.3. Remark:

(i) The factor $\frac{1}{2}$ in the definition of c is due to physics, where *Heisenberg's uncertainty relation* for the observables a and b in a state ν reads

$$\Delta_\nu(a)\Delta_\nu(b) \geq \frac{1}{2}|\nu(ab - ba)| \ ,$$

with $\Delta_\nu(a) := \sqrt{\nu(a^2) - \nu(a)^2}$. So if $qp - pq = i\hbar\,\mathrm{Id}$ then we obtain the well known form of the uncertainty principle for position and momentum

$$\Delta_\nu(q)\Delta_\nu(p) \geq \frac{\hbar}{2} \ .$$

(ii) $H(E, \sigma)$ is the simply connected covering group associated to the central extension of E by S^1 and the cocycle $e^{i\sigma(x,y)/2}$ equipped with the product topology of a vector space topology on E and the usual one on S^1. This central extension is also referred to as Heisenberg group in the literature and will be denoted by $H_0(E, \sigma)$.

(iii) Note that the identity element of $H(E, \sigma)$ is $(0, 0) \in E \times \mathbb{R}$ and the inverse element of $(x, r) \in H(E, \sigma)$ is just $(-x, -r)$.

1.4. Lemma:

(i) The centre $\mathcal{Z}(H(E, \sigma))$ of the Heisenberg group is $\{0\} \times \mathbb{R}$

(ii) H is nilpotent of class 2.

Proof:

(i) Since $(y, r) \cdot (0, s) = (y, r + s) = (0, s) \cdot (y, r)$ we have $\{0\} \times \mathbb{R} \subseteq \mathcal{Z}(H)$. If $(x, r) \in \mathcal{Z}(H)$ and $(y, s) \in H$ arbitrary then the condition $(x, r) \cdot (y, s) = (y, s) \cdot (x, r)$ leads to $\sigma(x, y) = \sigma(y, x)$ and hence by skew symmetry to $\sigma(x, y) = 0$, which in turn forces $x = 0$ since σ is non-degenerate.

(ii) Let $H^1 = [H, H]$ be the subgroup generated by all commutators in H. Direct computation shows $(x, r) \cdot (y, s) \cdot (-x, -r) \cdot (-y, -s) = (0, \sigma(x, y))$ and hence $H^1 = \mathcal{Z}(H) \neq \{(0, 0)\}$. Therefore $H^2 = [H, H^1] = \{(0, 0)\}$ ■

The above Lemma indicates the algebraic basis of the attempt to apply Kirillov's orbit method for nilpotent (finite dimensional) Lie groups to construct representations of an infinite dimensional Heisenberg group.

1.5. Remark:

(i) If $W \subseteq E$ is a subspace then $W \times \mathbb{R}$ constitutes a normal subgroup of $H(E, \sigma)$.

(ii) If $F \subseteq E$ is a symplectic subspace (i.e. $\sigma\,|_{F \times F}$ is non-degenerate) then the subset $F \times \mathbb{R} \subseteq H(E, \sigma)$ defines a normal subgroup isomorphic to $H(F, \sigma\,|_{F \times F})$.

(iii) If $L \subseteq E$ is an isotropic subspace (i.e. $\sigma\,|_{L \times L} = 0$) then we have the associated commutative subgroups $L \times \{0\}$ and $L \times \mathbb{R}$.

2 Topological and Smooth Structure

In Algebraic Quantum Theory one studies representations of the *Weyl relations*

$$\text{(W)} \qquad W(x)W(y) = e^{\frac{i\hbar}{2}\sigma(x,y)}W(x+y) \tag{I.2}$$

by a family of unitary operators $(W(x))_{x \in E}$ on a Hilbert space satisfying the following *Regularity Condition*

$$\text{(R)} \qquad \forall x \in E: \quad t \mapsto W(tx) \text{ is strongly continuous} \tag{I.3}$$

where strong continuity means that $W(tx)$ converges pointwise to $W(sx)$ as $t \to s$ in $\mathbb{R}$.

2.1. Remark:

(i) The set of unitary operators on a Hilbert space equipped with the topology of pointwise convergence (or strong operator-topology) is a topological group with respect to multiplication of operators.

(ii) Property (R) guarantees the existence of the anti-self-adjoint (unbounded) generator of the 1-parameter group $(W(tx))_{t\in\mathbb{R}}$, which can be recovered by (pointwise) differentiation on a dense subspace of the Hilbert space. In a physical context the self-adjoint operators $\Phi(x) = -i\frac{d}{dt}\mid_0 W(tx)$ are called *field operators*. In the early times of Quantum Field Theory these objects together with their commutation relations served as starting point for “quantized field theories”

(iii) The relation of the Heisenberg group $H(E,\sigma)$ to the abstract C^*-algebra arising from the relations (W), the *Weyl algebra*, is sketched in the Appendix to this chapter (see p. 17).

Relation (W) defines a unitary projective representation of the group $(E,+)$ with multiplier $e^{\frac{i\hbar}{2}\sigma(x,y)}$ on a Hilbert space V. We can translate such a representation into a group homomorphism $\pi : H(E,\sigma) \to \mathcal{U}(V)$ of the central extension into the group of unitary operators by defining

$$\pi(x,r) = e^{i\hbar r}W(x) . \tag{I.4}$$

Thus π is a unitary representation of the Heisenberg group $H(E,\sigma)$ which is one dimensional when restricted to the centre $\{0\}\times\mathbb{R}$. Note that if $\rho : H \to \mathrm{GL}(W)$ is an arbitrary irreducible representation then $\rho\mid_{\mathcal{Z}(H)}$ is necessarily one dimensional, i.e. a character of $\mathbb{R}$. The converse does not hold (see the constructions in [Sla]).

On the other hand each unitary representation π of H with the property $\pi(0,r) = e^{i\hbar r}\mathrm{Id}$ defines a representation of the Weyl relations (W) by setting

$$W(x) := \pi(x,0) . \tag{I.5}$$

Let us now study condition (R) in more detail. Our aim is to equip $H(E, \sigma)$ with a topological structure having the property that continuity of a unitary representation (with respect to the strong operator-topology) is characterized exactly by condition (R). In order to do so we first translate the formulation of condition (R) into a statement about continuity along finite dimensional subspaces of E.

2.2. Lemma: Let $\pi : H(E, \sigma) \to \mathcal{U}(V)$ be a group homomorphism with $\pi(0, r) = e^{i\hbar r}\mathrm{Id} \quad \forall r \in \mathbb{R}$. Denote by $\mathcal{F}$ be the family of all finite dimensional subspaces of E. Then the following are equivalent:

(i) The operators $W(x) = \pi(x, 0)$ $(x \in E)$ satisfy condition (R)

(ii) For all $F \in \mathcal{F}$ the restricted homomorphism $\pi\,|_{F\times\mathbb{R}}$: $F \times \mathbb{R} \to \mathcal{U}(V)$ is continuous with respect to the product of the (unique) locally convex topology on F with the usual one on $\mathbb{R}$ and the strong operator topology on $\mathcal{U}(V)$.

Proof:

(i)→(ii) $F \times \mathbb{R}$ is a finite dimensional nilpotent Lie group (note that $\sigma\,|_{F\times F}$: $F \times F \to \mathbb{R}$ is smooth) with Lie algebra $F \oplus \mathbb{R}$. So $\exp : F \oplus \mathbb{R} \to F \times \mathbb{R}$ is a global diffeomorphism and choosing a base $X_1, \ldots, X_n$ of E and 1 in $\mathbb{R}$ we can write

$$\begin{aligned} \pi(x, r) &= \pi(\exp(t_1 X_1) \cdots \exp(t_n X_n) \cdot (0, r)) = \\ &= \pi(\exp(t_1 X_1)) \cdots \pi(\exp(t_n X_n)) e^{i\hbar r} . \end{aligned}$$

Now each single factor depends continuously on t_k $(k = 1, \ldots, n)$ and on r by assumption. Since exp is a diffeomorphism and by strong continuity of multiplication on the set of unitary operators the assertion follows.

(ii)→(i) set $F = \mathrm{span}\{x\}$ then (i) is just a special case of (ii) ■

The previous Lemma points out that the adequate topology on $H(E, \sigma)$ should come up from the product of the Euclidean topology ε on $\mathbb{R}$ and the inductive limit topology with respect to the embeddings of finite dimensional subspaces. Thus we equip E with its finest locally convex topology τ. To ensure that the corresponding topological space $(H(E,\sigma),\tau\times\varepsilon)$ becomes a topological group under the multiplication I.1 one has to show the continuity of $\sigma : E\times E\to\mathbb{R}$. Since on (E,τ) every linear functional is continuous the symplectic form σ is at least separately continuous. But in general a separately continuous bilinear form need not be continuous and even in the present special case of finest locally convex topologies I was not able to clarify the situation. However, this uncertainty is no dramatic restriction for two reasons.

The first is that in further course of this work we will use a more concrete example which is easily seen to have the desired continuity property. Second we note that in order to have smoothness properties in the sense of [F/K] it is necessary and sufficient to show boundedness of the bilinear form (see [K/M, Global Analysis: 3.4]). Since the bounded subsets of E are exactly the bounded subsets contained in finite dimensional subspaces ([Sch, II,Ex 7]) and a fundamental system of bounded subsets of $E\times E$ is given by products of bounded sets each bilinear form on E is bounded. Therefore $\sigma : E\times E\to\mathbb{R}$ is smooth and yields also smoothness of the group multiplication in $H(E,\sigma)$.

From now on we will consider a concrete symplectic space (E,σ) and work out some methods for constructing representations of the associated Heisenberg group.

2.3. Phase space for an arbitrary number of particles:

One can think of the (classical) phase space of an open thermodynamic system with arbitrary (variable and finite) number of particles as the space

$$E=\mathbb{R}^{2(\mathbb{N})}:=\mathbb{R}^{(\mathbb{N})}\oplus\mathbb{R}^{(\mathbb{N})} \tag{I.6}$$

where the direct sum decomposition corresponds to position and momentum part of the phase space. It is convenient to identify $\mathbb{R}^{2(\mathbb{N})}$ with $\mathbb{C}^{(\mathbb{N})}$ (as real vector space), so the momentum coordinates constitute the imaginary part.

We equip E with the canonical weak symplectic form given in 1.1(ii),i.e.

$$\sigma(x,y)=\operatorname{Im}h(x,y)\qquad\forall x,y\in E. \tag{I.7}$$

2.4. Remark:

(i) The finest locally convex topology τ on $\mathbb{C}^{(\mathbb{N})} = \bigoplus_{n\in\mathbb{N}} \mathbb{C}$ is equivalent to the locally convex direct sum topology. $(\mathbb{C}^{(\mathbb{N})}, \tau)$ is therefore a non-metrizable nuclear (LB)-space which is continuously embedded into $\ell^2_{\mathbb{C}}(\mathbb{N})$. Since $(\mathbb{C}^{(\mathbb{N})})^* = \mathbb{C}^{\mathbb{N}}$ we obtain a so-called *Gelfand triplet* if $\mathbb{C}^{\mathbb{N}}$ is given the (usual) product topology:

$$\mathbb{C}^{(\mathbb{N})} \hookrightarrow \ell^2_{\mathbb{C}}(\mathbb{N}) \hookrightarrow \mathbb{C}^{\mathbb{N}}$$

Note that $\mathbb{C}^{(\mathbb{N})}$ is a dense subspace in $\ell^2_{\mathbb{C}}(\mathbb{N})$ and $\mathbb{C}^{\mathbb{N}}$ with respect to the corresponding coarser topologies. So the out-left and out-right spaces approximate the ("physicist's") separable Hilbert space in some sense.

(ii) The only extensive attempt to classify representations of infinitely many canonical commutation relations was made in [G/W, 1954]. Though their methods are spectral- and measure-theoretic there is the structural background of the Heisenberg group $H(\mathbb{C}^{(\mathbb{N})}, \operatorname{Im} h(.,.))$ behind. So the hope of a future comparison with a new approach towards classification could be one more motivation to focus our attention mainly on this example.

(iii) There would be no gain in starting with the popular "one particle Hilbert space" $E = L^2_{\mathbb{C}}(\mathbb{R}^3)$ because the necessary change of the topology to the finest locally convex topology would bring up a totally different (and no more well known) structure on $L^2_{\mathbb{C}}(\mathbb{R}^3)$.

(iv) Let $\delta_n := (\delta_n^k)_{k\in\mathbb{N}}$ $(n \in \mathbb{N})$ then $\{\delta_1, \delta_2, \ldots, i\delta_1, i\delta_2 \ldots\}$ is a $\mathbb{R}$-Hamel base in $\mathbb{C}^{(\mathbb{N})}$. If we define $E_n = \operatorname{span}\{\delta_1, \ldots, \delta_n, i\delta_1, \ldots, i\delta_n\}$ and $\sigma_n = \sigma|_{E_n\times E_n}$ then (E_n, σ_n) is a $2n$-dimensional symplectic subspace of $(E = \mathbb{C}^{(\mathbb{N})}, \sigma)$ and the Heisenberg group H can be described as (algebraic) direct limit of the finite dimensional Heisenberg groups $H(E_n, \sigma_n)$ for $n \in \mathbb{N}$.

With the notation of 2.4 (iv) we can give Lemma 2.2 a more concrete form.

2.5. Lemma: If $\pi : H(\mathbb{C}^{(\mathbb{N})}, \sigma) \to \mathcal{U}(V)$ is a group homomorphism with $\pi(0,r) = e^{i\hbar r}\mathrm{Id}$ and $W : \mathbb{C}^{(\mathbb{N})} \to \mathcal{U}(V)$ is the corresponding projective representation (cf. equation I.5) then the following are equivalent:

(i) W satisfies condition (R)

(ii) $\forall n \in \mathbb{N} : \pi|_{H(E_n, \sigma_n)}$ is strongly continuous

Proof: Each finite dimensional subspace of $\mathbb{C}^{(\mathbb{N})}$ is contained in some E_n for n large enough. Hence the result follows from Lemma 2.2 ∎

Now we are in a position to state our desired result about characterization of continuity of irreducible unitary representations of the Heisenberg group.

2.6. Corollary: Let $\pi : H \to \mathcal{U}(V)$ be a group homomorphism with the property $\pi(0,r) = e^{i\hbar r}$ and W be the corresponding projective representation of $\mathbb{C}^{(\mathbb{N})}$. Then the following are equivalent:

(i) π is continuous with respect to the strong operator topology on $\mathcal{U}(V)$

(ii) W satisfies the regularity condition (R)

Proof:

(i)→(ii) this is obvious

(ii)→(i) we show that the inductive limit topology τ on $\mathbb{C}^{(\mathbb{N})}$ is equivalent to the final topology τ_f (in the sense of general topology) with respect to the embeddings $E_n \hookrightarrow \mathbb{C}^{(\mathbb{N})}$. Then the result follows from Lemma 2.5.

By definition we have $\tau \preceq \tau_f$. To show $\tau \succeq \tau_f$ we consider an arbitrary τ_f-neighbourhood $U(x)$ of $x \in \mathbb{C}^{(\mathbb{N})}$. Then $\forall n \in \mathbb{N} : \exists \delta_n > 0 : U \cap E_n \supseteq x_n + B^n_{\delta_n}$ where $x_n = (x^1, \ldots, x^n, 0, 0, \ldots)$ and B^n_ε denotes the $2n$-dimensional open box of width ε around $0 \in E_n$. Now the set

$$U_0 = \{(z^n)_{n\in\mathbb{N}} \in \mathbb{C}^{(\mathbb{N})} \mid \sum_n |z^n| \leq 1, \quad |z^n| < \min\delta_k, k \leq n\}$$

is a typical τ-neighbourhood of $0 \in \mathbb{C}^{(\mathbb{N})}$ (cf. [Sch, II.6]) and clearly $x + U_0 \subseteq U$ by construction. ■

2.7. Lie group structure on $H(\mathbb{C}^{(\mathbb{N})}, \sigma)$:

The topology on H is actually the locally convex direct sum topology of $\mathbb{C}^{(\mathbb{N})} \oplus \mathbb{R}$. Therefore H is homeomorphic to the nuclear (LB)-space $\mathbb{R}^{(\mathbb{N})}$ which serves as the modelling space for the manifold structure on H. It fits in this way into the concept of infinite dimensional Differential Geometry developed in [K/M, Global Analysis] upon smooth structures in convenient spaces.

As we noted in the discussion on page 7 the symplectic form σ is smooth and hence the group multiplication is smooth $H \times H \to H$. Since the inversion $(x, r)^{-1} = (-x, -r)$ is clearly smooth $H \to H$ we can thus speak of H as a (non-metrizable) nuclear (LB)-Lie group.

It is the aim of this thesis to use this point of view for transferring constructions like Kirillov's Method of Orbits (cf. [Kir, 1962]) and Kostant's Geometric Quantization (cf. [Kos, 1970]) to the infinite dimensional situation. On the way we will realize that the adequate notion of a representation turns out to be that of *smooth representation*, i.e. smooth actions of H on convenient vector spaces. Further construction of unitary representations is on the one hand, by lack of Haar measure, not as natural as in the finite dimensional case and on the other hand does destroy smoothness properties, since the derived field operators in a unitary representation of H are necessarily unbounded. Nevertheless note that we used Hilbert space structure and continuity properties from the physical context to motivate (and actually define) our topological and henceforth smooth structure on the Heisenberg group.

3 Automorphisms of the Heisenberg Group

The set of (continuous or smooth) automorphisms on the Heisenberg group $H(\mathbb{C}^{(\mathbb{N})}, \sigma)$ plays a rôle in representation theory for at least two reasons:

- if $\pi : H \to \mathrm{GL}(V)$ is a representation and α is an automorphism of H then $\pi \circ \alpha$ defines a representation which is in general not equivalent to π; in physical terms each α induces a *-automorphism of the algebra of observables which need not be a "spatial" equivalence
- inner automorphisms are leading to the adjoint and coadjoint representation on the Lie algebra and its dual vector space; stabilizer groups and orbits with respect to these actions are the basic objects in geometric quantization and orbit theory

We will give an explicit description of a parametrization of the set $\mathrm{Aut}_c(H)$ of continuous automorphisms in Proposition 3.2 showing how the group $\mathrm{Aut}_c(H)$ itself could be considered as infinite dimensional Lie group. Furthermore we will deduce that $\mathrm{Aut}_c(H) \subseteq \mathrm{Aut}(H)$, where $\mathrm{Aut}(H)$ denotes the smooth automorphisms.

3.1. Examples of automorphisms:

(i) inner automorphisms: let $(y, s) \in H$ then $\mathrm{conj}_{(y,s)} \in \mathrm{Aut}(H)$ is computed as follows:

$$(y, s) \cdot (x, r) \cdot (y, s)^{-1} = (x + y, r + s + \frac{1}{2}\sigma(y, x)) \cdot (-y, -s) =$$
$$= (x, r + \frac{1}{2}(\sigma(y, x) - \sigma(x, y))) = (x, r + \sigma(y, x))$$

and hence

$$\mathrm{conj}_{(y,s)}(x, r) = (x, r + \sigma(y, x)) \tag{I.8}$$

(ii) symplectic action: let $T \in \mathrm{Sp}(E, \sigma) = \{S \in \mathrm{GL}(E) \mid \forall x, y \in E : \sigma(Sx, Sy) = \sigma(x, y)\}$ and define $\alpha_T \in \mathrm{Aut}(H)$ by setting $\alpha_T(x, r) = (Tx, r)$. We have the relation

$$\alpha_T \circ \mathrm{conj}_{(y,s)} = \mathrm{conj}_{\alpha_T(y,s)} \circ \alpha_T$$

(iii) generalizes (i): if we take $f \in E^*, t \in \mathbb{R} \setminus \{0\}$ then $\varphi_{(f,t)}(x,r) =$
$= (\sqrt{|t|}x, f(x) + tr)$ defines an automorphism

(iv) combining (ii) and (iii) now yields an automorphism β induced from $T \in \mathrm{Sp}(E,\sigma),\ f \in E^*,\ t^2 \in \mathbb{R}^+$:

$$\beta(x,r) = (tTx, f(x) + t^2 r) \tag{I.9}$$

in other "words"

$$\beta = \alpha_T \circ \varphi_{(f,t^2)} = \varphi_{(f\circ T,t^2)} \circ \alpha_T$$

3.2. Proposition: Each continuous automorphism of $H(\mathbb{C}^{(\mathbb{N})},\sigma)$ is of the form given in 3.1 (iv) and is therefore smooth. Hence the set $\mathrm{Aut}_c(H)$ is parameterized by the set $\mathrm{Sp}(\mathbb{C}^{(\mathbb{N})},\sigma) \times \mathbb{C}^{\mathbb{N}} \times \mathbb{R}^+$.

Proof: Let $\varphi \in \mathrm{Aut}(H)$ then

$$\varphi(x,r) = \varphi((x,0)\cdot(0,r)) = \varphi(x,0)\cdot\varphi(0,r)\ .$$

Now since $\mathcal{Z}(H)$ is φ-invariant we have $\varphi(0,r) = \varphi(0,\gamma(r))$ for some $\gamma : \mathbb{R} \to \mathbb{R}$. Let $\varphi(x,0) = (\alpha(x),\beta(x))$ with $\alpha : \mathbb{C}^{(\mathbb{N})} \to \mathbb{C}^{(\mathbb{N})}$ and $\beta : \mathbb{C}^{(\mathbb{N})} \to \mathbb{R}$, so we may write

$$\varphi(x,r) = (\alpha(x), \beta(x) + \gamma(r))\ .$$

Step 1: $\gamma \in \mathrm{GL}(\mathbb{R})$

$$\varphi((0,r)\cdot(0,s)) = \varphi(0,r+s) = (0,\gamma(r+s))$$

on the other hand

$$\varphi((0,r)\cdot(0,s)) = \varphi(0,r)\cdot\varphi(0,s) = (0,\gamma(r)+\gamma(s))$$

which shows that γ has to be additive and continuous and hence homogeneous by well known arguments (first deduce $\gamma(ns) = n\gamma(s)$ for $n \in \mathbb{N}$; then $m\gamma(n\frac{s}{m}) = nm\gamma(\frac{s}{m}) = n\gamma(s)$ gives $\gamma(qs) = q\gamma(s)$ for rational q and the result follows by density of $\mathbb{Q} \subseteq \mathbb{R}$). Since φ is onto γ is given by multiplication with a non-zero constant, i.e. $\exists t \in \mathbb{R}\setminus\{0\} : \gamma(r) = tr$

To obtain information about α and β we compute as follows

$$\varphi((x,0)\cdot(y,0)) = \varphi(x+y, \frac{1}{2}\sigma(x,y)) = (\alpha(x+y), \beta(x+y) + \frac{t}{2}\sigma(x,y))$$

on the other hand this has to be equal to

$$\begin{aligned} \varphi(x,0)\cdot\varphi(y,0) &= (\alpha(x),\beta(x))\cdot(\alpha(y),\beta(y)) = \\ &= (\alpha(x)+\alpha(y), \beta(x)+\beta(y)+\frac{1}{2}\sigma(\alpha(x),\alpha(y))) \end{aligned}$$

yielding the equations

$$\begin{aligned} &(\mathrm{A}) \quad \alpha(x+y) &&= \alpha(x)+\alpha(y) \\ &(\mathrm{B}) \quad \beta(x+y) &&= \beta(x)+\beta(y)+\frac{1}{2}(\sigma(\alpha(x),\alpha(y)) - t\sigma(x,y)) \end{aligned}$$

Step 2: $\alpha \in \mathrm{GL}_{\mathbb{R}}(\mathbb{C}^{(\mathbb{N})})$

since $\mathbb{C}^{(\mathbb{N})}$ is a topological vector space the continuity of scalar multiplication makes the usual arguments work to show that an additive continuous mapping is $\mathbb{R}$-homogeneous.

Step 3: $|t|^{-\frac{1}{2}}\alpha \in \mathrm{Sp}(\mathbb{C}^{(\mathbb{N})}, \sigma)$

equation (B) and skew symmetry of σ implies

$$0 = \beta(x+y) - \beta(y+x) = \sigma(\alpha(x),\alpha(y)) - t\sigma(x,y)$$

and shows that $T := |t|^{-\frac{1}{2}}\alpha$ is symplectic

Step 4: $\beta \in E^*$

from step 3 together with equation (B) we deduce that β is additive and since it is continuous once again the linearity follows by the standard argument

To summarize we collect the results in the equation

$$\varphi(x,r) = (\alpha(x), \beta(x)+\gamma(r)) = (\sqrt{|t|}Tx, \beta(x)+tr)\ .$$

Smoothness is clear from boundedness of the involved linear operators. ■

4 The Lie Algebra and the Canonical Commutation Relations

The definition of the Lie algebra associated to a Lie group depends on the notion of tangent space, tangent bundle and vector field. Whereas in general infinite dimensional geometry the two concepts of kinematical tangent vectors (derivatives of smooth curves) and operational tangent vectors (derivations of the algebra of function germs) are no longer equivalent our modelling (and in any sense) convenient space $\mathbb{R}^{(\mathbb{N})}$ avoids such difficulties.

4.1. Tangent bundle and vector fields:

As stated in [K/M, Global Analysis: 13.6] the two sufficient properties a modelling vector space of a manifold should have to guarantee equivalence of the kinematical and operational tangent bundle are reflexivity and the bornological approximation property (i.e. $E^* \otimes E$ has to be dense in $\mathrm{End}(E)$ with respect to the bornological locally convex topology). Both of these conditions are satisfied for the space $\mathbb{R}^{(\mathbb{N})}$ which is the modelling vector space for $H(\mathbb{C}^{(\mathbb{N})}, \sigma)$ (cf. [J, 11.4.5,d and 18.2.6]).

Clearly the kinematical tangent space at an arbitrary point of H is isomorphic to $\mathbb{C}^{(\mathbb{N})} \oplus \mathbb{R}$ and the tangent bundle is given by the trivial bundle $TH = H \times (\mathbb{C}^{(\mathbb{N})} \oplus \mathbb{R})$. Smooth vector fields are therefore identified with smooth mappings $H \to \mathbb{C}^{(\mathbb{N})} \oplus \mathbb{R}$ or equivalently with bounded derivations of $C^\infty(H, \mathbb{R})$. An explicit correspondence is given by the mapping $D : C^\infty(H, \mathbb{C}^{(\mathbb{N})} \oplus \mathbb{R}) \to \mathrm{Der}_b(C^\infty(H, \mathbb{R}))$,

$$(D(f)(\varphi))(h) = \langle d\varphi_{(h)}, f(h) \rangle \qquad f \in C^\infty(H, \mathbb{C}^{(\mathbb{N})} \oplus \mathbb{R}), \varphi \in C^\infty(H, \mathbb{R}). \quad \text{(I.10)}$$

4.2. Left invariant vector fields and Lie bracket:

If $(y, s) \in H$ we denote by $\lambda_{(y,s)}$ the left translation on H by (y, s), i.e. $\lambda_{(y,s)}(x, r) = (x + y, r + s + \frac{1}{2}\sigma(y, x))$. Then the derivative $T\lambda_{(y,s)} : H \times (\mathbb{C}^{(\mathbb{N})} \oplus \mathbb{R}) \to H \times (\mathbb{C}^{(\mathbb{N})} \oplus \mathbb{R})$ is given by $((x, r), (u, a)) \mapsto (\lambda_{(y,s)}(x, r), (u, a + \frac{1}{2}\sigma(y, u)))$. We define the left invariant vector field $\xi_{(u,a)} \in C^\infty(H, \mathbb{C}^{(\mathbb{N})} \oplus \mathbb{R})$ corresponding to $(u,a) \in \mathfrak{h} := T_{(0,0)}H \cong \mathbb{C}^{(\mathbb{N})} \oplus \mathbb{R}$ by the equation

$$\xi_{(u,a)}(x,r) = T_{(0,0)}\lambda_{(x,r)} = (u, a + \frac{1}{2}\sigma(x,u)) \; . \tag{I.11}$$

It clearly satisfies $(\lambda_{(y,s)})^*\xi_{(u,a)} = (T\lambda_{(y,s)})_{-1} \circ \xi_{(u,a)} \circ \lambda_{(y,s)} = \xi_{(u,a)}$, the left invariance. Conversely each left invariant vector field is seen to be of the above form exactly as in the finite dimensional theory. Hence the Lie bracket on the vector space $\mathfrak{h} := T_{(0,0)}H$ can be introduced by this isomorphism via the commutator of the derivations corresponding to left invariant vector fields. According to equation I.10 the derivation $D(\xi_{(u,a)})$ is acting by $(D(\xi_{(u,a)})\varphi)(x,r) = \langle d\varphi_{(x,r)}, (u, a + \frac{1}{2}\sigma(x,u))\rangle$.

The Lie bracket can now be obtained by direct (simple but careful) computation of the commutator $D(\xi_{[(u,a),(v,b)]}) := D(\xi_{(u,a)}) \circ D(\xi_{(v,b)}) - D(\xi_{(v,b)}) \circ D(\xi_{(u,a)})$ or simply by using the inductive limit structure of $\mathfrak{h}$ and the finite dimensional results (note that two fixed elements $X, Y \in \mathfrak{h}$ are always contained in some finite dimensional Heisenberg algebra and so is their Lie bracket $[X,Y]$). Both ways end up with the perfect analogue to the finite dimensional situation, namely

$$[(u,a),(v,b)] = (0, \sigma(u,v)) \; . \tag{I.12}$$

4.3. Remark:

(i) The canonical commutation relations do now have a precise meaning without reference to any representation and are obtained as a special case of equation I.12. Define $Q_j := (\delta_j, 0)$ ("position operator"), $P_k := (i\delta_k, 0)$ ("momentum operator") and $\mathrm{Id} := (0,1)$ then we can write

$$[Q_j, P_k] = (0, \sigma(\delta_j, i\delta_k)) = \delta_{jk}\mathrm{Id} \tag{I.13}$$

(ii) If $c(t) = (x(t), f(t))$ is a smooth curve $c : \mathbb{R} \to H$ the differential equation $\dot{c}(t) = \xi_{(u,a)}(c(t))$ with initial condition $c(0) = 0$ corresponds to the equations

$$\begin{aligned}
\dot{x}(t) &= u, & x(0) = 0 \\
\dot{f}(t) &= a + \frac{1}{2}\sigma(u, x(t)), & f(0) = 0
\end{aligned}$$

with the unique solution $x(t) = tu$ and $f(t) = ta$. Hence the exponential map $\exp : \mathfrak{h} \to H$ is just given by $\exp(u, a) = \mathrm{Fl}^{\xi_{(u,0)}}(1, (0, 0)) = (u, a)$, which is trivially a global diffeomorphism.

5 Coadjoint Orbits

5.1. Coadjoint action:

We have $H = \mathbb{C}^{(\mathbb{N})} \times \mathbb{R}$ and $\mathfrak{h} = \mathbb{C}^{(\mathbb{N})} \oplus \mathbb{R}$ hence the dual of the Heisenberg algebra is $\mathfrak{h}^* = \mathbb{C}^{\mathbb{N}} \oplus \mathbb{R}$.

Starting with the *adjoint representation* $\mathrm{Ad} : H \to \mathrm{GL}(\mathfrak{h})$, $\mathrm{Ad}_{(x,r)}(u, a) = T_{(0,0)}\mathrm{conj}_{(x,r)} \cdot (u, a) = (u, a + \sigma(x, u))$ the *coadjoint representation* $\mathrm{Ad}^* : H \to \mathrm{GL}(\mathfrak{h}^*)$ is defined by

$$\mathrm{Ad}^*_{(x,r)} f = f \circ \mathrm{Ad}^{-1}_{(x,r)} \qquad \forall f \in \mathfrak{h}^* .$$

So if $f = (u^*, a) \in \mathfrak{h}^*$ and $(v, b) \in \mathfrak{h}$:

$$\begin{aligned}
\langle \mathrm{Ad}^*_{(x,r)}(u^*, a), (v, b)\rangle &= \langle (u^*, a), \mathrm{Ad}_{(-x,-r)}(v, b)\rangle = \langle (u^*, a), (v, b - \sigma(x, v))\rangle \\
&= \langle u^*, v\rangle + a(b - \sigma(x, v)) = \langle u^*, v\rangle + ab - \langle a\check{\sigma}(x), v\rangle \\
&= \langle u^* - a\check{\sigma}(x), v\rangle + ab = \langle (u^* - a\check{\sigma}(x), a), (v, b)\rangle
\end{aligned}$$

or in a more compact form

$$\mathrm{Ad}^*_{(x,r)}(u^*, a) = (u^* - a\check{\sigma}(x), a) . \tag{I.14}$$

5.2. Corollary:

If $(u^*, a) \in \mathfrak{h}^*$ then the orbit $\mathcal{O}_{(u^*,a)}$ of (u^*, a) with respect to the coadjoint action falls in exactly one of the following two classes:

(i) If $a = 0$ then $\mathcal{O}_{(u^*,a)} = \{(u^*, a)\}$

(ii) If $a \neq 0$ then $\mathcal{O}_{(u^*,a)} = \{(u^* + v^*, a) \mid v^* \in \operatorname{im} \check{\sigma}\} = (u^* + \operatorname{im} \check{\sigma}) \times \{a\}$

5.3. Remark: Restriction to the finite dimensional case $\mathbb{C}^n$ would reproduce the result of [Kir, 1962] since then $(\mathbb{C}^n)^* \cong \mathbb{C}^n$. In case of infinitely many degrees of freedom the hyperplanes at each "height" $(0, \hbar) \in \mathfrak{h}^*$ break up into uncountably many translates of the subset $\operatorname{im} \check{\sigma} \cong \mathbb{C}^{(\mathbb{N})}$. This should be the first hint to the existence of the immense number of non-equivalent irreducible unitary representations.

APPENDIX: The Weyl Algebra

This appendix is meant as a short note on the connection to the C^*-algebraic framework of large quantum systems based on the Weyl relations (R) on p. 4 as it can be found for example in [P],[B/R] or [T]. The constructions sketched in the following arose in a discussion with Peter Michor.

Let (E, σ) be a weak symplectic space and H_0 be the associated Heisenberg group obtained by central extension with S^1 (cf. 1.3(ii)). Our aim is to construct the *Weyl algebra* $\mathcal{W}(E, \sigma)$ as a certain subalgebra of the group C^*-algebra $C^*(H_0)$.

Equip $H_0 = E \times S^1$ with the product topology τ_0 of the discrete topology on E and the usual one on S^1. Then (H_0, τ_0) is a locally compact topological group possessing a compact neighborhood of the identity $(0, 1)$ which is invariant under all inner automorphisms, namely $\{0\} \times S^1$ is one. Hence by [D, 14.3] it is a unimodular group. Define the group *-algebra $\mathcal{A}$ to be the set $C_c(H_0, \mathbb{C})$ of continuous functions with compact support with the multiplication law

$$\begin{aligned}(F \star G)(x,w) &= \sum_{y\in E}\int_{S^1} F(y,z)G((x,w)\cdot(y,z)^{-1})\,dz = \\ &= \sum_{y\in E}\int_{S^1} F(y,z)G(x-y, w\overline{z}e^{\frac{i}{2}\sigma(y,x)})\,dz\end{aligned}$$

and the involution $F^*(x,w) = \overline{F(-x,\overline{w})}$.

Consider the subset $\mathcal{W}_0 := \mathbb{C}^{(E)} \otimes \mathrm{Id}_{S^1}$ of $\mathcal{A}$. $\mathcal{W}_0$ is a *-subalgebra of $\mathcal{A}$:

$$(f\otimes \mathrm{Id})\star(g\otimes\mathrm{Id})(x,w) = \sum_{y\in E}\int_{S^1}(f\otimes\mathrm{Id})(y,z)(g\otimes\mathrm{Id})(x-y,w\overline{z}e^{\frac{i}{2}\sigma(y,x)})\,dz =$$

$$= \sum_{y\in E}\int_{S^1} f(y)zg(x-y)w\overline{z}e^{\frac{i}{2}\sigma(y,x)}\,dz = \sum_{y\in E} f(y)g(x-y)e^{\frac{i}{2}\sigma(y,x)}\cdot\int_{S^1} w\,dz =$$

$$=: (f\star g\otimes\mathrm{Id})(x,w)$$

$$(f\otimes\mathrm{Id})^*(x,w) = \overline{f(-x)\otimes\overline{w}} = \overline{f(-x)}\otimes w =: (f^*\otimes\mathrm{Id})(x,w)$$

So $\mathcal{W}_0$ induces a *-algebra structure on the vector space $\mathbb{C}^{(E)}$. The Weyl relations can now be recovered by setting $W(x) = \delta_x \in \mathbb{C}^{(E)}$, since

$$\begin{aligned}W(x)\star W(y) = \delta_x\star\delta_y &= \sum_{v\in E}\delta_x(v)\delta_y(.-v)e^{\frac{i}{2}\sigma(v,.)} = \\ &= \delta_y(.-x)e^{\frac{i}{2}\sigma(x,.)} = \delta_{x+y}(.)e^{\frac{i}{2}\sigma(x,y)} = e^{\frac{i}{2}\sigma(x,y)}W(x+y)\,.\end{aligned}$$

Finally let $\mathcal{W}(E,\sigma)$ be the completion of $\mathcal{W}_0$ with respect to the norm inherited from $C^*(H_0)$ (which is induced by the operator norms in all non-degenerate *-representations). $\mathcal{W}(E,\sigma)$ is the unique C^*-algebra (up to *-isomorphism) generated by a set of unitary elements satisfying the Weyl relations (see [P, 2.1]) and each representation of the Weyl relations yields a *-representation of $\mathcal{W}(E,\sigma)$. Since on the other hand each regular *-representation of the Weyl algebra defines a continuous representation of the Heisenberg group (cf. I.4) we have therefore sketched the equivalence of these concepts.

Chapter II

ORBIT METHOD

The original paper [Kir, 1962] presented as a first prominent example the explicit correspondence between coadjoint orbits and the (equivalence classes of) irreducible unitary representations of the Heisenberg group.

This chapter starts with a description of adapted constructions leading to smooth representations of $H(\mathbb{C}^{(\mathbb{N})}, \sigma)$ in spaces of sections of a complex line bundle over certain quotients of the Heisenberg group. In contrast to the finite dimensional theory there is no obvious procedure to turn these into unitary representations (by lack of an invariant measure on spaces which are not locally compact). Nevertheless we are able to give explicit formulas of actions on spaces of smooth functions of infinitely many variables mirroring exactly the degrees of freedom in the underlying physical system. Furthermore we obtain the canonical commutation relations now satisfied by bounded (equivalently smooth) operators.

The input to the whole algorithm is the choice of a coadjoint orbit and an arbitrary point in it (which turns out to be not essential). The crucial point seems to be the further choice of an associated admissible subalgebra of the Lie algebra. Later sections will deal with special arrangements of the objects in question.

Complexification of the constructions will yield representations in spaces of holomorphic functions. This will enable us to recover the well known *Fock repre-*

sentations of mathematical physics and to compute its associated generalized momentum mapping (see [M, 1990]).

6 Adjusting Kirillov's Construction

In 5.2 we listed the possible orbits of the coadjoint action of $H = \mathbb{C}^{(\mathbb{N})} \times \mathbb{R}$ on $\mathfrak{h}^* = \mathbb{C}^{\mathbb{N}} \oplus \mathbb{R}$. There came up two classes:

a) each single point of the hyperplane $\mathbb{C}^{\mathbb{N}} \times \{0\}$

b) at height $\hbar$ the affine subspaces $(u^* + \operatorname{im} \check{\sigma}) \times \{\hbar\} \subseteq \mathbb{C}^{\mathbb{N}} \times \{\hbar\}$

The representations of H corresponding to orbits of class a) are the one dimensional representations $\chi^{u^*} : H \to S^1$,

$$\chi^{u^*}(x, r) = e^{i\langle u^*, x\rangle} \tag{II.1}$$

which can be considered as characters of the additive group $\mathbb{C}^{(\mathbb{N})}$. To investigate representations arising from orbits of class b) let us fix such a typical orbit, say $\mathcal{O} = (u^* + \operatorname{im}\check{\sigma}) \times \{\hbar\} \cong (u^* + \mathbb{C}^{(\mathbb{N})}) \times \{\hbar\}$, and take $(u^*, \hbar) \in \mathcal{O}$ as a representative of $\mathcal{O}$.

6.1. Admissible subalgebras of $\mathfrak{h}_{\mathbb{C}}$:

Let $\mathfrak{h}_{\mathbb{C}} = \mathfrak{h} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of the Heisenberg algebra $\mathfrak{h}$. A subalgebra $\mathfrak{n}$ of $\mathfrak{h}_{\mathbb{C}}$ is said to be *subordinated* to the functional $(u^*, \hbar) \in \mathcal{O}$ if

$$\langle (u^*, \hbar), [X, Y] \rangle = 0 \qquad \forall X, Y \in \mathfrak{n}\ . \tag{II.2}$$

This is equivalent to the assertion of $X \mapsto \langle (u^*, \hbar), X \rangle$ being a (one-dimensional) Lie algebra representation of $\mathfrak{n}$. A maximal $(u^*, \hbar)$-subordinated subalgebra $\mathfrak{n}$ is called *admissible*. If σ is extended to the complexification $E_{\mathbb{C}} = E \otimes_{\mathbb{R}} \mathbb{C}$ by $\sigma_{\mathbb{C}}(x \otimes \lambda, y \otimes \mu) := \lambda\mu\sigma(x, y)$ we can write down equation II.2 in the more concrete form (set $X = (v, a), Y = (w, b)$ and use I.12)

$$0 = \langle (u^*, \hbar), (0, \sigma_{\mathbb{C}}(v, w)) \rangle = \hbar\sigma_{\mathbb{C}}(v, w)$$

which in turn together with maximality forces $\mathfrak{n}$ to be of the form

$$\mathfrak{n} = L \oplus \mathbb{C} \ , \tag{II.3}$$

where L is a maximally isotropic (Lagrangian) subspace of $E_{\mathbb{C}}$. Now let K be the (closed) subgroup of H corresponding to the (real) subalgebra $\mathfrak{k} = \mathfrak{n} \cap \mathfrak{h}$. Since $\exp : \mathfrak{h} \to H$ is the identity map (cf. 4.3 (ii)) we can identify K as a set with $\mathfrak{k}$. Note that K is a normal subgroup of H.

6.2. A one-dimensional representation of K:

Define the Lie algebra homomorphism $\rho : \mathfrak{n} \to \mathbb{C}$ by (let $X = (v, a) \in \mathfrak{n}$)

$$\rho(X) = i\langle (u^*, \hbar), X\rangle = i\langle u^*, v\rangle + i\hbar a \ .$$

By exponentiation ρ yields a one-dimensional (unitary) representation U of K: $U(\exp X) = e^{\rho(X)} \quad \forall X \in \mathfrak{k}$ or explicitly

$$U(y, s) = e^{i\langle u^*, y\rangle + i\hbar s} \qquad \forall (y, s) \in K \subseteq H \ . \tag{II.4}$$

6.3. Induced representation in a space of sections:

Since H carries the finest locally convex topology and $K \supseteq \{0\} \times \mathbb{R}$ the quotient group H/K can be identified (as a topological space) with the topological vector space $\mathfrak{h}/\mathfrak{k}$ which carries again the finest locally convex topology (cf. [Sch, II.ex 7]). Therefore we can speak of the principal fibre bundle $(H, p, H/K, K)$ with total space H, $p : H \to H/K$ the canonical surjection, H/K the base manifold and the structure group K. The induced representation $\pi = \mathrm{ind}_K^H U$ can be realized in the space $\Gamma(H[\mathbb{C}])$ of sections of the associated vector bundle over H/K with the fibre $\mathbb{C}$ (the representation space of U). As in the finite dimensional theory (cf. [M, Lect.Notes: 15.12]) one obtains a natural isomorphism $\Gamma(H[\mathbb{C}]) \cong C^\infty(H, \mathbb{C})^K$, where $C^\infty(H, \mathbb{C})^K$ denotes the space of K-equivariant smooth functions (i.e. $\varphi : H \to \mathbb{C}$ smooth with $\varphi(hk) = U(k^{-1})\varphi(h) \quad \forall h \in H, k \in K$), given by the following diagram

($\varphi \in C^\infty(H,\mathbb{C})^K$):

$$\begin{tikzcd} H \times \mathbb{C} \arrow[r] & H[\mathbb{C}] \\ H \arrow[u, "\mathrm{Id}\times\varphi"] \arrow[r, "p"'] & H/K \arrow[u, "s_\varphi"'] \end{tikzcd}$$

So $\pi = \mathrm{ind}_K^H U$ is equivalently given by translation action on the space $C^\infty(H,\mathbb{C})^K$ (by abuse of notation we denote this again by π):

$$(\pi(h)\varphi)(g) = \varphi(h^{-1}g) = \varphi \circ \lambda_{h^{-1}}(g) \quad \forall h, g \in H\ . \tag{II.5}$$

6.4. Smoothness of the representation π:

$C^\infty(H,\mathbb{C})^K$ is a subspace of $C^\infty(H,\mathbb{C})$ which is topologized in the following way: first equip $C^\infty(\mathbb{R},\mathbb{C})$ with the topology of uniform convergence on compact sets of each derivative separately; for $c : \mathbb{R} \to H$ smooth define the linear map $c^* : C^\infty(H,\mathbb{C}) \to C^\infty(\mathbb{R},\mathbb{C})$ by $c^*(\varphi) = \varphi \circ c$; now define the locally convex topology to be the initial topology with respect to the family $(c^* : C^\infty(H,\mathbb{C}) \to C^\infty(\mathbb{R},\mathbb{C}))_{c\in C^\infty(\mathbb{R},H)}$; in this way $C^\infty(H,\mathbb{C})$ becomes a convenient vector space (see [K/M, Global Analysis,1.19 and 4.8]).

Since we have

$$C^\infty(H,\mathbb{C})^K = \bigcap_{k\in K} \{\varphi \in C^\infty(H,\mathbb{C}) \mid \varphi(hk) = U(k^{-1})\varphi(h)\}$$

and each subspace in the intersecting family is closed we conclude that $C^\infty(H,\mathbb{C})^K$ is a closed subspace. Hence it is a convenient vector space.

Proposition: Let $\pi = \mathrm{ind}_K^H U$ then the mapping $\hat{\pi} : H \times C^\infty(H,\mathbb{C})^K \to C^\infty(H,\mathbb{C})^K$, $\hat{\pi}(h,\varphi) = \pi(h)\cdot\varphi$, is smooth.

Proof:

Step 1: $\forall h \in H : \pi(h)$ is smooth $C^\infty(H,\mathbb{C})^K \to C^\infty(H,\mathbb{C})^K$

Since $\pi(h)$ is linear it is sufficient to show continuity (even boundedness would suffice). $C^\infty(H,\mathbb{C})^K$ carries an initial topology, so continuity can be checked by composition with smooth curves $c : \mathbb{R} \to H$, i.e. $\varphi \mapsto c^*(\pi(h)\varphi) = c^* \circ \lambda^*_{h^{-1}}(\varphi) = (\lambda_{h^{-1}} \circ c)^*\varphi$ should be continuous $C^\infty(H,\mathbb{C})^K \to C^\infty(\mathbb{R},H)$. But $(\lambda_{h^{-1}} \circ c)^*\varphi = \varphi \circ \lambda_{h^{-1}} \circ c$ and $t \mapsto (\lambda_{h^{-1}} \circ c)(t) = h^{-1}c(t)$ is clearly a smooth curve into H; denote this curve by d. Now $c^*(\pi(h)) = d^*$ is continuous by definition of the initial topology. Since c was arbitrary we have proved continuity and hence smoothness of $\pi(h)$.

So $\pi : H \to \mathrm{GL}(C^\infty(H,\mathbb{C})^K) \subseteq C^\infty(C^\infty(H,\mathbb{C})^K, C^\infty(H,\mathbb{C})^K)$ and smoothness of $\hat{\pi}$ can be proved by using Cartesian closedness, i.e. $\hat{\pi}$ is smooth if and only if π is smooth (cf. [K/M, Global Analysis,1,21]).

Step 2: $\forall \varphi \in C^\infty(H,\mathbb{C})^K : h \mapsto \pi(h)\varphi$ is smooth $H \to C^\infty(H,\mathbb{C})^K$ (therefore $C^\infty(H,\mathbb{C})^K$ consists entirely of *smooth vectors*)

$C^\infty(H,\mathbb{C})^K$ is a closed subspace, therefore it is enough to show smoothness of $h \mapsto \pi(h)\varphi$, $H \to C^\infty(H,\mathbb{C})$. But this is again by Cartesian closedness equivalent to showing smoothness of $(h,g) \mapsto (\pi(h)\varphi)(g)$, $H \times H \to \mathbb{C}$. Now finally, if m and inv denote the (smooth) group operations multiplication and inversion in H,

$$(\pi(h)\varphi)(g) = \varphi(h^{-1}g) = (\varphi \circ \mathrm{m})(\mathrm{inv}(h), g)$$

is clearly smooth in (h,g).

Step 3: $\pi : H \to \mathrm{GL}(C^\infty(H,\mathbb{C})^K)$ is smooth

Since π takes values in the closed subspace $\mathrm{End}(C^\infty(H,\mathbb{C})^K)$ of smooth linear mappings it suffices by [K/M, Global Analysis,4.11] to show that $ev_\varphi \circ \pi : H \to \mathrm{End}(C^\infty(H,\mathbb{C})^k) \to C^\infty(H,\mathbb{C})^K$ is smooth for all $\varphi \in C^\infty(H,\mathbb{C})^K$. But this requires exactly $h \mapsto \pi(h)\varphi$ to be smooth, i.e. φ to be a smooth vector, which was shown in step 2. $\blacksquare$

6.5. Remark: The result of 6.4 enables us to give another characterization of the subspace $C^\infty(H,\mathbb{C})^K$ in $C^\infty(H,\mathbb{C})$ by differential equations. Let $\varphi \in C^\infty(H,\mathbb{C})^K$ and U be the representation of K given in 6.2. Then for $k \in K$ we have $\varphi(hk) = U(k^{-1})\varphi(h)$ or if $k = \exp X$ with $X \in \mathfrak{k}$

$$\varphi(h \exp X) = U(\exp(-X))\varphi(h) = e^{-\rho(X)}\varphi(h)\ . \tag{II.6}$$

So by replacing X by tX and differentiating at $t = 0$ we compute

$$\frac{d}{dt}|_0\ \varphi(h \exp tX) = \frac{d}{dt}|_0\ (\varphi \circ \lambda_h)(\exp tX) = \langle d\varphi_{(h)}, T_e\lambda_h \cdot X\rangle = \xi_X(\varphi)(h)\ ,$$

where ξ_X denotes the left invariant vector field corresponding to X (cf. 2.2). On the other hand

$$\frac{d}{dt}|_0\ e^{-\rho(tX)}\varphi(h) = -\rho(X)\varphi(h)$$

yielding the differential equation

$$\xi_X(\varphi) + \rho(X)\varphi = 0\ . \tag{II.7}$$

More concrete if $X = (v, a), h = (y, s)$ this reads (by 4.2 and 6.2)

$$\langle d\varphi_{(y,s)}, (v, a + \frac{1}{2}\sigma(y, v))\rangle + i(\langle u^*, v\rangle + \hbar a)\varphi(y, s) = 0\ . \tag{II.8}$$

Now by maximality of $\mathfrak{k}$ clearly $\mathfrak{k} \supseteq \{0\} \times \mathbb{R}$ so $(0, 1) \in \mathfrak{k}$. Therefore the above equations contains as a special case

$$\langle d\varphi, (0, 1)\rangle + i\hbar\varphi = 0$$

or equivalently

$$\frac{d\varphi}{ds}(y, s) + i\hbar\varphi(y, s) = 0 \qquad \forall(y, s) \in H\ .$$

Fix $y \in E$ and let φ_y be the smooth function $\varphi_y(s) = \varphi(y, s)$. Then the last equation means

$$\varphi_y' = -i\hbar\varphi \tag{II.9}$$

and therefore $\varphi_y(s) = \varphi(y, 0)e^{-i\hbar s}$. Considering a decomposition of E into Lagrange subspaces one can obtain information about dependence on "polarized" coordinates in the same way. We will use such techniques in section 13. To summarize the above discussion we state the following

Lemma: For all $\varphi \in C^\infty(H,\mathbb{C})^K$ there exists a unique function $\psi \in C^\infty(E,\mathbb{C})$ such that

$$\varphi(y,s) = \psi(y)e^{-i\hbar s} \ . \tag{II.10}$$

ψ has the following property (reflecting the equivariance of φ):

$$\forall (v,0) \in \mathfrak{k} : \psi(y+v) = e^{-i\langle u^*,v\rangle + \frac{i\hbar}{2}\sigma(y,v)}\psi(y) \ .$$

Conversely each function $\psi \in C^\infty(E,\mathbb{C})$ having this property defines a function $\varphi \in C^\infty(H,\mathbb{C})^K$ by equation II.10.

Proof: set $\psi(y) = \varphi(y,0)$ then

$$\begin{aligned} \psi(y+v) &= \varphi(y+v,0) = \varphi((y,0)\cdot(v,-\frac{1}{2}\sigma(y,v))) = \\ &= U(-v,\frac{1}{2}\sigma(y,v))\varphi(y,0) = e^{-i\langle u^*,v\rangle + \frac{i\hbar}{2}\sigma(y,v)}\psi(y) \ . \end{aligned}$$

The converse is clear by construction. ■

If we were dealing with a finite dimensional Lie group the further procedure of constructing an irreducible unitary representation out of $(\pi, C^\infty(H,\mathbb{C})^K)$ would be routine. The choice of a translation invariant measure on H and completion of the subspace of square-integrable functions in the $\|.\|_2$-norm would guide to the desired aim. However there exists no such concept in our situation, even for this (in all senses) convenient example, and the general construction ends up with the (only ?) advantage that the resulting representation is smooth in a precise sense.

6.6. Remark:

(i) In section 8 we will describe in some detail a special situation where the physical context suggests a recipe for constructing a Hilbert space on which the above representation acts unitarily.

(ii) Even if the building of a Hilbert space fails one (or at least I) will conjecture that irreducibility of the representation still holds in the modified framework.

7 Generalized Schrödinger Representations

Now we are in a position to apply the concept developed in the previous section to special choices of admissible subalgebras $\mathfrak{n} \subseteq \mathfrak{h}_{\mathbb{C}}$. It was shown that such an $\mathfrak{n}$ has to be of the form $\mathfrak{n} = L \oplus \mathbb{C}$, where L is an appropriate Lagrangian subspace of $E_{\mathbb{C}}$ with respect to $\sigma_{\mathbb{C}}$.

Let

$$L = \mathrm{Re}\,(\mathbb{C}^{(\mathbb{N})}) \otimes_{\mathbb{R}} \mathbb{C} \subseteq E_{\mathbb{C}}\ , \tag{II.11}$$

i.e. $L = \mathrm{span}\{x \otimes \lambda \mid \bar{x} = x, \lambda \in \mathbb{C}\} \cong \mathbb{R}^{(\mathbb{N})} \otimes_{\mathbb{R}} \mathbb{C}$ (where $\bar{x}$ means complex conjugation in each component). L is isotropic since for $x, y \in \mathrm{Re}\,\mathbb{C}^{(\mathbb{N})},\ \lambda, \mu \in \mathbb{C}$

$$\sigma_{\mathbb{C}}(x \otimes \lambda, y \otimes \mu) = \lambda\mu\sigma(x + i0, y + i0) = \lambda\mu(\langle x|0\rangle - \langle 0|y\rangle) = 0$$

and on the other hand if $0 \neq iy \otimes \mu \in i\mathbb{R}^{(\mathbb{N})} \otimes \mathbb{C}$

$$\sigma_{\mathbb{C}}(y \otimes 1, iy \otimes \mu) = \mu\sigma(y, iy) = \mu\langle y|y\rangle \neq 0\ ,$$

so L is also maximal. Hence $L \oplus \mathbb{C}$ is an admissible subalgebra.

Note that $\mathfrak{n} = \mathfrak{k}_{\mathbb{C}}$ where

$$\mathfrak{k} = (\mathrm{Re}\,\mathbb{C}^{(\mathbb{N})}) \oplus \mathbb{R} \subseteq \mathfrak{h} \tag{II.12}$$

is a real subalgebra.
The corresponding subgroup is $K = \exp \mathfrak{k} = \mathrm{Re}\,\mathbb{C}^{(\mathbb{N})} \times \mathbb{R}$ with the associated representation $U : K \to S^1$, $U(l, r) = e^{i\langle u^*, l\rangle + i\hbar r}$ $(l \in L, r \in \mathbb{R})$. The base manifold of the vector bundle $H[\mathbb{C}]$ is now $H/K \cong \{0\} \times i\mathbb{R}^{(\mathbb{N})} \times \{0\} \cong \mathbb{R}^{(\mathbb{N})}$:

$$\begin{tikzcd} H \times \mathbb{C} \arrow[r, "q"] \arrow[d, "\mathrm{pr}_1"'] & H[\mathbb{C}] \arrow[d, "\bar{p}"] \\ H \arrow[r, "p"'] & H/K \cong \mathbb{R}^{(\mathbb{N})} \end{tikzcd}$$

where $q : H \times \mathbb{C} \to H[\mathbb{C}]$ sends an element (h, λ) to its corresponding K-orbit under the right action $R : H \times \mathbb{C} \times K \to H \times \mathbb{C}$,

$$R((h,\lambda),k) = (hk, U(k^{-1})\lambda) \tag{II.13}$$

which reads in more explicit terms as follows ($x = x_1 + ix_2, l \in \mathrm{Re}\,\mathbb{C}^{(\mathbb{N})}$):

$$R(((x,r),\lambda),(l,s)) = (x_1 + l + ix_2, r + s - \frac{1}{2}\langle x_2|l\rangle, e^{-i\langle u^*,l\rangle - i\hbar s}\lambda)\ . \tag{II.14}$$

By choosing an appropriate representative of the set

$$[x,r,\lambda] := \{R(((x,r),\lambda),(l,s)) \mid (l,s) \in K\}$$

one obtains a trivialization of the vector bundle $H[\mathbb{C}]$.

7.1. Lemma: The smooth mapping $q_0 : H \times \mathbb{C} \to \mathbb{R}^{(\mathbb{N})} \times \mathbb{C}$ given by $q_o((x,r),\lambda) = (x_2, e^{i\langle u^*,x_1\rangle + i\hbar r + \frac{i\hbar}{2}\langle x_1|x_2\rangle}\lambda)$ factors to a bijection $\Psi : H[\mathbb{C}] \to \mathbb{R}^{(\mathbb{N})} \times \mathbb{C}$, i.e.

$$\begin{tikzcd}
H \times \mathbb{C} \arrow[rr, "q_0"] \arrow[dr, "q"'] & & \mathbb{R}^{(\mathbb{N})} \times \mathbb{C} \\
& H[\mathbb{C}] \arrow[ur, "\Psi"'] &
\end{tikzcd}$$

Proof:

q_0 is R-invariant:

$$\begin{aligned}
&(q_0 \circ R_{(l,s)})((x,r),\lambda) = \\
&\quad = q_0((x+l, r+s-\frac{1}{2}\langle x_2|l\rangle), e^{-i\langle u^*,l\rangle - i\hbar s}\lambda) = \\
&\quad = (x_2, e^{i\langle u^*,x_1+l\rangle + \frac{i\hbar}{2}\langle x_1+l|x_2\rangle + i\hbar r + i\hbar s - \frac{i\hbar}{2}\langle x_2|l\rangle} e^{-i\langle u^*,l\rangle - i\hbar s}\lambda) = \\
&\quad = (x_2, e^{i\langle u^*,x_1\rangle + i\hbar r + \frac{i\hbar}{2}\langle x_1|x_2\rangle}\lambda) = q_0((x,r),\lambda)
\end{aligned}$$

q_0 is one-to-one: if $q_0((x,r),\lambda) = q_0((y,s),\mu)$ we have

$$(x_2, e^{i\langle u^*,x_1\rangle + \frac{i\hbar}{2}\langle x_1|x_2\rangle + i\hbar r}\lambda) = (y_2, e^{i\langle u^*,y_1\rangle + \frac{i\hbar}{2}\langle y_1|y_2\rangle + i\hbar s}\mu)$$

and hence $x_2 = y_2$ together with

$$\lambda = e{-}i\langle u^*, x_1 - y_1\rangle - \frac{i\hbar}{2}\langle x_1 - y_1|y_2\rangle - i\hbar(r-s)\mu.$$

So by putting $l = x_1 - y_1, t = r - s + \frac{1}{2}\langle l|y_2\rangle$ we obtain $R(((y,s),\mu),(l,t)) = ((x,r),\lambda)$ or equivalently $[x,r,\lambda] = [y,s,\mu]$.
$\underline{q_0 \text{ is onto}}$: take $(x,\lambda) \in \mathbb{R}^{(\mathbb{N})} \times \mathbb{C}$ arbitrary then clearly $\Psi([ix,0,\lambda]) = = q_0((ix,0),\lambda) = (x,\lambda)$ ∎

Using the trivialization Ψ of the bundle $H[\mathbb{C}]$ we are able to identify the space of sections $\Gamma(H[\mathbb{C}]) \cong C^\infty(H,\mathbb{C})^K$ with the function space $C^\infty(\mathbb{R}^{(\mathbb{N})},\mathbb{C})$ and carry over the induced representation π of H to it.

7.2. Lemma: Let $\Psi_* : C^\infty(H,\mathbb{C})^K \to C^\infty(\mathbb{R}^{(\mathbb{N})},\mathbb{C})$ be defined by the following diagram ($\varphi \in C^\infty(H,\mathbb{C})^K$):

$$\begin{tikzcd}
H \times \mathbb{C} \arrow[r, "q"] & H[\mathbb{C}] \arrow[r, "\Psi"] & \mathbb{R}^{(\mathbb{N})} \times \mathbb{C} \\
H \arrow[u, "\mathrm{Id} \times \varphi"] \arrow[r, "p"'] & H/K \arrow[u, "s_\varphi"'] \arrow[r, equal] & \mathbb{R}^{(\mathbb{N})} \arrow[u, "\mathrm{Id} \times \Psi_*(\varphi)"']
\end{tikzcd}$$

Then Ψ_* is a linear diffeomorphism and we have the explicit formulas

$$\Psi_*(\varphi) = \varphi|_{\{0\}\times i\mathbb{R}^{(\mathbb{N})}\times\{0\}} \qquad \forall \varphi \in C^\infty(H,\mathbb{C})^K$$

and

$$\Psi_*^{-1}(f)(x,r) = e^{-i\langle u^*,x_1\rangle - i\hbar r - \frac{i\hbar}{2}\langle x_1|x_2\rangle} f(x_2)\ .$$

Proof: Take $\varphi \in C^\infty(H,\mathbb{C})^K$; starting with a point $(x,r) \in H$ at the left lower corner of the diagram we get

$$((\mathrm{Id} \times \Psi_*(\varphi)) \circ p)(x,r) = (\mathrm{Id} \times \Psi_*(\varphi))(x_2) = (x_2, \Psi_*(\varphi)(x_2))\ .$$

On the other hand

$$
\begin{aligned}
(\Psi \circ q \circ (\mathrm{Id} \times \varphi))(x,r) &= (\Psi \circ q)((x,r),\varphi(x,r)) = q_0((x,r),\varphi(x,r)) = \\
&= (x_2, e^{i\langle u^*,x_1\rangle + i\hbar r + \frac{i\hbar}{2}\langle x_1|x_2\rangle}\varphi(x,r)) = (\text{now use } K\text{-equivariance of } \varphi) \\
&= (x_2, e^{i\langle u^*,x_1\rangle + i\hbar r + \frac{i\hbar}{2}\langle x_1|x_2\rangle} e^{-i\langle u^*,x_1\rangle - i\hbar r - \frac{i\hbar}{2}\langle x_1|x_2\rangle}\varphi(ix_2,0)) = \\
&= (x_2, \varphi(ix_2,0))
\end{aligned}
$$

yields the equation $\Psi_*(\varphi) = \varphi|_{i\mathbb{R}^{(\mathbb{N})}}$. The formula for Ψ_*^{-1} is suggested by the equivariance condition and is shown to be correct by direct calculation.

To show that Ψ_* is a diffeomorphism we prove smoothness of Ψ_* and Ψ_*^{-1} separately. First we show continuity of Ψ_* which yields smoothness by linearity. We will use the universal property of the initial topology on $C^\infty(\mathbb{R}^{(\mathbb{N})},\mathbb{C})$ with respect to the family $(c^* : C^\infty(\mathbb{R}^{(\mathbb{N})},\mathbb{C}) \to C^\infty(\mathbb{R},\mathbb{C}))_{c\in C^\infty(\mathbb{R},\mathbb{R}^{(\mathbb{N})})}$. So let $c : \mathbb{R} \to \mathbb{R}^{(\mathbb{N})}$ be smooth and denote by $\tilde{c}$ the associated curve into $H \supseteq \{0\} \times i\mathbb{R}^{(\mathbb{N})} \times \{0\}$ according to the embedding $x \mapsto (0 + ix, 0)$. Then for all $\varphi \in C^\infty(H,\mathbb{C})^K$ one can write $(c^* \circ \Psi_*)(\varphi) = \tilde{c}^*(\varphi)$ which shows continuity by definition of the initial topology on $C^\infty(H,\mathbb{C})^K$. Smoothness of Ψ_*^{-1} is seen directly from the formula since smooth curves $t \mapsto f_t$ are only multiplied by a factor independent of t. ■

7.3. Proposition: Define the representation $S : H \to \mathrm{GL}(C^\infty(\mathbb{R}^{(\mathbb{N})},\mathbb{C}))$ by means of the following diagram ($h \in H, \pi = \mathrm{ind}_K^H U$):

$$
\begin{array}{ccc}
C^\infty(\mathbb{R}^{(\mathbb{N})},\mathbb{C}) & \xrightarrow{S(h)} & C^\infty(\mathbb{R}^{(\mathbb{N})},\mathbb{C}) \\
\uparrow{\scriptstyle \Psi_*} & & \uparrow{\scriptstyle \Psi_*} \\
C^\infty(H,\mathbb{C})^K & \xrightarrow[\pi(h)]{} & C^\infty(H,\mathbb{C})^K
\end{array}
$$

Then we obtain a smooth action $H \times C^\infty(\mathbb{R}^{(\mathbb{N})},\mathbb{C}) \to C^\infty(\mathbb{R}^{(\mathbb{N})},\mathbb{C})$ given explicitly by the formula

$$(S(x,r)f)(z) = e^{i\hbar r - \frac{i\hbar}{2}\langle x_1|x_2\rangle + i\langle u^*,x_1\rangle} e^{i\hbar\langle x_1|z\rangle} f(z-x_2)\ . \tag{II.15}$$

Proof: Take $f \in C^\infty(\mathbb{R}^{(\mathbb{N})},\mathbb{C})$ arbitrarily and compute

$$\begin{aligned}(S(x,r)f)(z) &= (\Psi_* \circ \pi(x,r) \circ \Psi_*^{-1})(f)(z) = \\ &= (\Psi_* \circ \pi(x,r))((y,s) \mapsto e^{-i\langle u^*,y_1\rangle - i\hbar s - \frac{i\hbar}{2}\langle y_1|y_2\rangle} f(y_2))(z) = \\ &= \Psi_*((y,s) \mapsto e^{-i\langle u^*,y_1-x_1\rangle - i\hbar(s-r-\frac{1}{2}\sigma(x,y)) - \frac{i\hbar}{2}\langle y_1-x_1|y_2-x_2\rangle} f(y_2-x_2))(z) = \\ &= e^{-i\langle u^*,0-x_1\rangle - i\hbar(0-r-\frac{1}{2}\sigma(x,iz)) - \frac{i\hbar}{2}\langle 0-x_1|z-x_2\rangle} f(z-x_2) = \\ &= e^{i\hbar r + \langle u^*,x_1\rangle - \frac{i\hbar}{2}\langle x_1|x_2\rangle} e^{i\hbar\langle x_1|z\rangle} f(z-x_2)\end{aligned}$$

To show smoothness of $S : H \to \mathrm{GL}(C^\infty(\mathbb{R}^{(\mathbb{N})},\mathbb{C}))$ let $c \in C^\infty(\mathbb{R},H)$ and consider $S \circ c : \mathbb{R} \to \mathrm{GL}(C^\infty(\mathbb{R}^{(\mathbb{N})},\mathbb{C}))$. It suffices to show that for all $f \in C^\infty(\mathbb{R}^{(\mathbb{N})},\mathbb{C})$ the mapping $t \mapsto S(c(t))f$ is smooth $\mathbb{R} \to C^\infty(\mathbb{R}^{(\mathbb{N})},\mathbb{C})$. But $S(c(t))f = (\Psi_* \circ \pi(c(t)) \circ \Psi_*^{-1})(f) = \Psi_*(\pi(c(t))\Psi_*^{-1}(f))$ is smooth since Ψ_* is and $\Psi_*^{-1}(f)$ is a smooth vector by step 2 in the proof of Proposition 6.4 on page 23. ■

7.4. Remark:

(i) Proposition 7.3 enables us to define the derived Lie algebra representation $S' = T_{(0,0)}S : \mathfrak{h} \to \mathrm{End}(C^\infty(\mathbb{R}^{(\mathbb{N})},\mathbb{C}))$ and therefore to compute the physical field operators $\Phi(x) = -iS'(x,0)$ (for$x \in E$) which are now bounded operators — in contrast to their "brothers" arising from unitary representations in Hilbert spaces. The formula for the action of $\mathfrak{h}$ is derived (in both senses) by direct computation and reads

$$(S'(x,r)f)(z) = (i\hbar r + i\langle u^*,x_1\rangle + i\hbar\langle x_1|z\rangle)f(z) - \langle df_{(z)},x_2\rangle\ . \tag{II.16}$$

Especially by setting $Q_j = \Phi(\delta_j,0), P_k = \Phi(i\delta_k,0)$ one can recover the position and momentum operators acting according to

$$Q_j f = (\mathrm{Re}\, u_j^* + \hbar\delta_j^*)f, \quad P_k f = i\langle df,\delta_k\rangle\ . \tag{II.17}$$

(ii) The dependence of the representation S upon the orbit $\mathcal{O}$ and the functional $(u^*, \hbar) \in \mathcal{O}$ is reflected by the appearance of $\hbar$ and the "phase correction" $\langle u^*, x_1 \rangle$. A straightforward calculation shows that the choice of a special representative $(u^*, \hbar) \in \mathcal{O}$ is not essential: translating the functional to $(u^* + v, \hbar)$ with $v = v_1 + iv_2 \in \mathbb{C}^{(\mathbb{N})}$ yields exactly the same representation as intertwining S with translation by $\frac{1}{\hbar} v_1$ on the function space $C^\infty(\mathbb{R}^{(\mathbb{N})}, \mathbb{C})$.

(iii) The dependence of S upon the choice of the maximally isotropic subspace $L \subseteq E_{\mathbb{C}}$ is non-transparent and in section 8 we will notice that at least in further construction of unitary representations it is essential – in contrast to the finite dimensional case.

(iv) The non-equivalence of representations constructed with the same admissible subalgebra given in II.12 but corresponding to orbits at different heights is seen by restricting S to the centre of H: the parameter $\hbar$ characterizes the representation $r \mapsto e^{i\hbar r}$ uniquely.

7.5. Other admissible real subalgebras:

Only trivial changes are to be made if we choose $L = \mathrm{Im}\,(\mathbb{C}^{(\mathbb{N})}) \otimes_{\mathbb{R}} \mathbb{C}$. The resulting representation is just the above S where the rôles of x_1 and x_2 are changed.

Another simple choice is to take $L = \mathrm{span}\{x + ix \mid x \in \mathbb{R}^{(\mathbb{N})}\} \otimes_{\mathbb{R}} \mathbb{C} =: E_+ \otimes_{\mathbb{R}} \mathbb{C} \subseteq E_{\mathbb{C}}$ leading to the subgroup $K = \{(x + ix, r) \mid x \in \mathbb{R}^{(\mathbb{N})}, r \in \mathbb{R}\}$ with one-dimensional representation $U : K \to S^1, \quad U(x + ix, r) = e^{i\langle u^*, x+ix\rangle + i\hbar r}$. The quotient group is then given by

$$H/K \cong \{(x - ix, 0) \mid x \in \mathbb{R}^{(\mathbb{N})}\} =: E_- \tag{II.18}$$

according to the decomposition

$$y = \frac{y_1 + y_2}{2} + i\frac{y_1 + y_2}{2} + \frac{y_1 - y_2}{2} - i\frac{y_1 - y_2}{2} \quad \forall y \in \mathbb{C}^{(\mathbb{N})} \tag{II.19}$$

or in short notation $E = E_+ \oplus E_-, y = y_+ + y_-$. The trivialization of the associated vector bundle is now given by

$$\Psi([x, r, \lambda]) = (x_-, e^{i\langle u^*, x_+\rangle - \frac{i\hbar}{2}\sigma(x_-, x_+) + i\hbar r}\lambda)$$

yielding the representation $T : H \to \mathrm{GL}(C^\infty(E_-, \mathbb{C}),$

$$(T(x, r)f)(y_-) = e^{i\hbar r + i\langle u^*, x_+\rangle - \frac{i\hbar}{2}\sigma(x_+, x_-)} e^{i\hbar\sigma(x_+, y_-)} f(y_- - x_-) \ . \tag{II.20}$$

The derived Lie algebra representation $T' : \mathfrak{h} \to \mathrm{End}(C^\infty(E_-, \mathbb{C}))$ is therefore given by

$$T'(x, r)f = (i\hbar r + i\langle u^*, x_+\rangle + i\hbar\check{\sigma}(x_+))f - \langle df, x_-\rangle \ . \tag{II.21}$$

7.6. A "totally complex" admissible subalgebra:

It is straightforward to show that the subspace $L_+ = \mathrm{span}_{\mathbb{C}}\{x \otimes 1 + ix \otimes i \mid x \in \mathbb{C}^{(\mathbb{N})}\}$ is maximally isotropic. Hence $\mathfrak{n} = L_+ \oplus \mathbb{C}$ defines an admissible subalgebra which has the property

$$\mathfrak{n} \cap \bar{\mathfrak{n}} = \{0\} \oplus \mathbb{C} \ .$$

The corresponding real subalgebra is therefore very small: $\mathfrak{k} = \mathfrak{n} \cap \mathfrak{h} = \{0\} \oplus \mathbb{R}$. So the further construction starts with the character representation $U(0, r) = e^{i\hbar r}$ defining the associated vector bundle $H[\mathbb{C}]$ over the base manifold

$$H/K \cong \mathbb{C}^{(\mathbb{N})} \ . \tag{II.22}$$

Trivialization is now obtained by the mapping $\Psi : H[\mathbb{C}] \to \mathbb{C}^{(\mathbb{N})} \times \mathbb{C}$, $\Psi([x, r, \lambda]) = (x, e^{i\hbar r}\lambda)$ leading to the following representation $V : H \to \mathrm{GL}(C^\infty(\mathbb{C}^{(\mathbb{N})}, \mathbb{C}))$,

$$(V(x, r)f)(y) = e^{i\hbar r} e^{\frac{i\hbar}{2}\sigma(x, y)} f(y - x) \tag{II.23}$$

Note that L_+ contains no real subspace of E and therefore "gets lost" after restriction to the real Lie algebra. The next section will show how the situation changes if one stays in the complex world during Kirillov's construction and restricts afterwards to the real Heisenberg group.

7.7. Summary of the common features:

To carry out the trivialization of the associated line bundle $H[\mathbb{C}]$ in the general case the only thing to do is to describe the choice of a representative for each orbit under the right action $R : H\times\mathbb{C}\times K \to H\times\mathbb{C}$, $R((h,\lambda),k) = (hk, U(k^{-1})\lambda)$. This is easily done using the special structure $K = L\times\mathbb{R}$ and $E = L\oplus\bar{L}$, where $\bar{L}$ is a complementary σ-Lagrangian subspace (note that all subspaces of E are closed, so the decomposition is also topological). Therefore $H/K \cong \bar{L}$ and decomposing $x = x_L + \bar{x_L}$ we can define the mapping $\Psi : H[\mathbb{C}] \to \bar{L}\times\mathbb{C}$,

$$[(x,r),\lambda]_R \mapsto (\bar{x_L}, e^{i\hbar r+\frac{i\hbar}{2}\sigma(\bar{x_L},x_L)+i\langle u^*,x_L\rangle}\lambda) \tag{II.24}$$

which is seen to be one-to-one and onto exactly as in the above examples. This trivialization defines an isomorphism $C^\infty(H,\mathbb{C})^K \cong C^\infty(\bar{L},\mathbb{C})$ (denoted by Ψ_*) and the representation π on K-equivariant functions is turned into the action $\Theta : H \to \mathrm{GL}(C^\infty(\bar{L},\mathbb{C}))$, $\Theta(h) := \Psi_*\circ\pi(h)\circ\Psi_*^{-1}$,

$$(\Theta(x,r)f)(\bar{z}) = e^{i\hbar r+i\langle u^*,x_L\rangle-\frac{i\hbar}{2}\sigma(\bar{x_L},x_L)}e^{i\hbar\sigma(x_L,\bar{z})}f(\bar{z}-\bar{x_L}) \; . \tag{II.25}$$

8 Holomorphic Induction and Fock Representations

We will now make use of the general notion of admissible subalgebra as complex subalgebra of the complex hull $\mathfrak{h}_\mathbb{C}$ of the Heisenberg algebra. There is a natural complexification of the Heisenberg group $H(E,\sigma)$, namely $H_\mathbb{C} = E_\mathbb{C}\times\mathbb{C}$ with $\sigma_\mathbb{C}$ as defined in 6.1, which corresponds to the Lie algebra $\mathfrak{h}_\mathbb{C}$. So the procedure described in section 6 can be carried out with the natural changes of notions like smooth to holomorphic (again in the sense of [K/M, Global Analysis]) yielding holomorphic representations of the complex Heisenberg group $H_\mathbb{C}$ in spaces of holomorphic sections. Trivializations of the appearing vector bundles will again transform them into representations acting on “ordinary” function spaces (now holomorphic functions of countably many variables). The advantage of doing all this once again now looking through "complex glasses" (to say it in Harald Rindler's words) is the possibility of developing some of the arising representations further and turning them into certain regular representations of the Weyl relations (cf. I.2 and I.3) well known to physicists as *Fock representations.*

Since the details of the constructions are much the same as in the real case we will only sketch the story until the point of restriction to the "original" (real) Heisenberg group is reached.

8.1. The standard example:

We take up the situation of 7.6 beginning with the admissible subalgebra

$$\mathfrak{n} = L_+ \oplus \mathbb{C} = \mathrm{span}\{x \otimes 1 + ix \otimes i \mid x \in \mathbb{C}^{(\mathbb{N})}\} \oplus \mathbb{C} \tag{II.26}$$

subordinated to the functional $(0, \hbar) \in \mathfrak{h}^*$ (the elements of $\mathfrak{h}^*$ are thought as being extended to $\mathfrak{h}_{\mathbb{C}}$ by the usual $\mathbb{C}$-linear extension). We obtain a holomorphic representation of the corresponding subgroup $N = L_+ \times \mathbb{C}$ by

$$U_{\mathbb{C}}(\xi_+, \alpha) = e^{i\hbar\alpha} \qquad \forall (\xi_+, \alpha) \in N\ . \tag{II.27}$$

Note that with $L_- = \mathrm{span}\{x \otimes 1 - ix \otimes i \mid x \in \mathbb{C}^{(\mathbb{N})}\}$ we have the decomposition

$$E_{\mathbb{C}} = L_+ \oplus L_- \tag{II.28}$$

and we will use the notation $\xi = \xi_+ + \xi_-$, $\xi_{\underline{+}} \in L_{\underline{+}}$. Furthermore we have $H_{\mathbb{C}}/N \cong L_-$ and the corresponding vector bundle $H_{\mathbb{C}}[\mathbb{C}]$ over this base manifold is trivialized by the diffeomorphism $\Psi : H_{\mathbb{C}}[\mathbb{C}] \to L_- \times \mathbb{C}$,

$$\Psi([\xi, \alpha, \lambda]) = (\xi_-, e^{i\hbar\alpha + \frac{i\hbar}{2}\sigma_{\mathbb{C}}(\xi_+, \xi_-)}\lambda)\ . \tag{II.29}$$

Exactly as in the real case the representation $\pi_{\mathbb{C}} = \mathrm{ind}_N^{H_{\mathbb{C}}} U_{\mathbb{C}}$ is carried over to a representation $\tilde{F} : H_{\mathbb{C}} \to \mathrm{GL}(\mathrm{Hol}(L_-))$ given explicitly by

$$(\tilde{F}(\xi, \alpha)f)(z_-) = e^{i\hbar\alpha - \frac{i\hbar}{2}\sigma_{\mathbb{C}}(\xi_+, \xi_-)} e^{i\hbar\sigma_{\mathbb{C}}(\xi_+, z_-)} f(z_- - \xi_-) \tag{II.30}$$

8.2. Restriction to the real Heisenberg group:

H is embedded as a subgroup into $H_{\mathbb{C}}$ by $(x,r) \mapsto (x \otimes 1, r)$ (note that the embedding has to respect the symplectic structure on $E_{\mathbb{C}}$ and E) hence $F := \tilde{F}|_H$ yields a representation of H acting on a space of holomorphic functions:

to put the resulting action into a convenient form we note first that

$$x \otimes 1 = (\frac{x}{2} \otimes 1 + i\frac{x}{2} \otimes i) + (\frac{x}{2} \otimes 1 - i\frac{x}{2} \otimes i) = (x \otimes 1)_+ + (x \otimes 1)_-$$

and $\sigma_{\mathbb{C}}((x \otimes 1)_+, (y \otimes 1)_-) = -\frac{i}{2}h(x,y)$ where $h(.,.)$ denotes the standard hermitian form on $\mathbb{C}^{(\mathbb{N})}$ (cf. 1.1 (ii)), so $\beta(x,y) = i\hbar\sigma_{\mathbb{C}}((x \otimes 1)_+, (y \otimes 1)_-)$ defines a complex inner product on $\mathbb{C}^{(\mathbb{N})}$; finally we identify L_- with $\mathbb{C}^{(\mathbb{N})}$ by $z_- \mapsto z_- + \overline{z_-} =: z \otimes 1$ (where $\overline{x \otimes 1 - ix \otimes i} = x \otimes 1 + ix \otimes i$); then the representation $F : H \to \mathrm{GL}(\mathrm{Hol}(\mathbb{C}^{(\mathbb{N})}))$ can be written as follows

$$(F(x,r)f)(z) = e^{i\hbar - \frac{1}{2}\beta(x,x)} e^{\beta(x,z)} f(z-x) \tag{II.31}$$

8.3. The Fock space:

The following is essentially an elaboration of constructions given in [P/S, 9.5]. Different approaches in the sense of C^*-algebras can be found in [B/R, 5.2], [P, ch.4] or [T, 1.3].

Let $S(\mathbb{C}^{(\mathbb{N})})$ be the symmetric algebra over $\mathbb{C}^{(\mathbb{N})}$, i.e.

$$S(\mathbb{C}^{(\mathbb{N})}) = \bigoplus_{n \geq 0} ⓢ^n \mathbb{C}^{(\mathbb{N})} , \tag{II.32}$$

where $ⓢ^n \mathbb{C}^{(\mathbb{N})}$ denotes the n-fold symmetric tensor product of $\mathbb{C}^{(\mathbb{N})}$. $S(\mathbb{C}^{(\mathbb{N})})$ can be equipped with an inner product induced from $(\mathbb{C}^{(\mathbb{N})}, \beta(.,.))$, defined on the generating subset of monomials $c_1 \cdots c_m$ $(m \in \mathbb{N}, c_i \in \mathbb{C}^{(\mathbb{N})})$, as follows:

$$\langle a_1 \cdots a_k | b_1 \cdots b_l \rangle := \begin{cases} 0 & \text{if } k \neq l \\ \sum_{\tau \in \mathcal{S}_k} \beta(a_1, b_{\tau(1)}) \cdots \beta(a_k, b_{\tau(k)}) & \text{if } k = l \end{cases} \tag{II.33}$$

where $\mathcal{S}_k$ denotes the group of permutations of k elements. The Hilbert space V obtained by completion with respect to the corresponding norm $\|a\| = \sqrt{\langle a|a \rangle}$ is called *(bosonic) Fock space*.

Lemma: For $a \in \mathbb{C}^{(\mathbb{N})}$ let $f_a \in \mathrm{Hol}(\mathbb{C}^{(\mathbb{N})})$ be defined by $f_a(z) = e^{\beta(a,z)}$. Then on $\mathrm{span}\{f_a \mid a \in \mathbb{C}^{(\mathbb{N})}\} \subseteq \mathrm{Hol}(\mathbb{C}^{(\mathbb{N})})$ we have the inner product $(f_a|f_b) := e^{\beta(a,b)}$ and the completion W with respect to the associated norm is canonically isomorphic to V.

Proof:

Step 1 embedding $S(\mathbb{C}^{(\mathbb{N})})$ into $\mathrm{Hol}(\mathbb{C}^{(\mathbb{N})})$

define $\iota : S(\mathbb{C}^{(\mathbb{N})}) \to \mathrm{Hol}(\mathbb{C}^{(\mathbb{N})})$ to be $\iota(a_1 \cdots a_n)(z) := \beta(a_1, z) \cdots \beta(a_n, z)$, i.e. ι is the natural extension of $\check{\beta} : \mathbb{C}^{(\mathbb{N})} \to (\mathbb{C}^{(\mathbb{N})})^*$; ι is clearly one-to-one since for $a_1 \cdots a_n \neq 0 \in S(\mathbb{C}^{(\mathbb{N})})$ arbitrary one can find a $z \in \mathbb{C}^{(\mathbb{N})}$ such that $\beta(a_j, z) \neq 0,\ j = 1, \ldots, n$; now the inner product II.33 (and hence the norm) can be carried over to $\iota(S(\mathbb{C}^{(\mathbb{N})}))$; we identify the arising Hilbert space with V

Step 2 $W \hookrightarrow \overline{\iota(S(\mathbb{C}^{(\mathbb{N})}))}^{\|\cdot\|}$ isometrically

- for $a \in \mathbb{C}^{(\mathbb{N})}$ consider the power series expansion

$$f_a(z) = \sum_{n=0}^{\infty} \frac{\beta(a,z)^n}{n!} = \sum_{n=0}^{\infty} \frac{\iota(a^n)(z)}{n!} \ ;$$

  since

$$\|\sum_{n=0}^{N} \frac{\iota(a^n)}{n!}\| \leq \sum_{n=0}^{N} \frac{\|\iota(a^n)\|}{n!} = \sum_{n=0}^{N} \frac{\|a\|^n}{\sqrt{n!}}$$

  the series $\sum \frac{\iota(a^n)}{n!}$ is absolutely convergent and defines f_a within $\overline{\iota(S(\mathbb{C}^{(\mathbb{N})}))}^{\|\cdot\|}$

- we compute straightforward

$$\langle f_a | f_b \rangle = \sum_{n,m} \frac{\langle \iota(a^n) | \iota(b^m) \rangle}{n!m!} = \sum_{n} \frac{n!\beta(a,b)^n}{n!^2} = e^{\beta(a,b)}$$

  (using definition II.33) which is equal to $(f_a|f_b)$

Step 3 W is dense in V

the curve $t \mapsto f_{ta} = \sum \frac{t^n \iota(a^n)}{n!}$ is smooth $\mathbb{R} \to V$ since the series converges absolutely; so differentiating at $t = 0$ yields

$$\begin{aligned} \frac{d^k}{dt^k}|_0 \; f_{ta} &= \iota(a)^k \qquad \forall k \in \mathbb{N}, \\ \frac{d^2}{dt^2}|_0 \; f_{t(a+b)} &= (\iota(a) + \iota(b))^2 \end{aligned}$$

and so on ... showing $\iota(S(\mathbb{C}^{(\mathbb{N})})) \subseteq W$; ■

8.4. Proposition: The action of H according to II.31 defines a unitary representation $F : H \to \mathcal{U}(W)$ (we keep to the symbol F).

Proof: the action of H on a function f_a is given by

$$\begin{aligned} &(F(x,r)f_a)(z) = e^{i\hbar r - \frac{1}{2}\beta(x,x)} e^{\beta(x,z)} f_a(z-x) = \\ &\quad = e^{i\hbar r - \frac{1}{2}\beta(x,x)} e^{\beta(x,z)+\beta(a,z-x)} = e^{i\hbar r - \frac{1}{2}\beta(x,x)} e^{-\beta(a,x)} f_{a+x}(z) \; , \end{aligned}$$

which means

$$F(x,r)f_a = e^{i\hbar r - \frac{1}{2}\|x\|^2 - \beta(a,x)} f_{a+x} \; . \tag{II.34}$$

So W is F-invariant and unitarity of the action is proved by

$$\begin{aligned} &\langle F(x,r)f_a | F(x,r)f_b \rangle = e^{-\|x\|^2 - \beta(a,x) + \beta(b,x)} \langle f_{a+x} | f_{b+x} \rangle = \\ &\quad = e^{-\beta(x,x) - \beta(a,x) - \beta(x,b)} e^{\beta(a+x,b+x)} = e^{\beta(a,b)} = \langle f_a | f_b \rangle \end{aligned}$$

■

8.5. Remark:

(i) F is continuous with respect to the topology of pointwise convergence on $\mathcal{U}(W)$:

$$\begin{aligned} &\|F(x,r)f_a - F(0,0)f_a\|^2 = \langle F(x,r)f_a - f_a | F(x,r)f_a - f_a \rangle = \\ &\qquad = 2(e^{\beta(a,a)} - \mathrm{Re}\,(e^{i\hbar r - \beta(x,x)/2 - \beta(a,x)} e^{\beta(a+x,a)})) \end{aligned}$$

which tends to 0 if $(x,r) \to (0,0)$ in H; hence by application of the Banach-Steinhaus theorem (note that the set of unitary operators is bounded) and the fact that F is a group homomorphism we conclude that $F(x,r)f \to F(y,s)f$ whenever $(x,r) \to (y,s)$.

(ii) From equation II.34 we deduce that $\mathbb{1} := f_0 \in W$ is a cyclic vector for the representation F since $F(x,r)\mathbb{1} = \text{const} \cdot f_x \quad \forall (x,r) \in H$. Therefore F induces a cyclic representation of the Weyl algebra (cf. Appendix to chapter I, p. 17) with *vacuum state*

$$\omega(F(x,0)) := \langle \mathbb{1} | F(x,0)\mathbb{1} \rangle = e^{-\frac{\hbar}{4}\|x\|_2^2} \tag{II.35}$$

which is exactly the well known *Fock state*. Hence F is equivalent to the Gelfand-Neumark-Segal representation associated to the Fock state, i.e. F is equivalent to the irreducible *Fock representation* (cf. [P, 3 and 4]).

(iii) In the history of representation theory of the Weyl relations Fock representations always served as an explicit playground for development of new constructions. The first hint to an enormous number of (pairwise) inequivalent representations arising on Fock space is given in [F, 1953] and was made precise in [Seg, 1956,1958]. In our context the construction can be sketched as follows:

if $\alpha \in \mathrm{Aut}(H)$ we can define the new representation $F \circ \alpha$ acting again on Fock space; especially for automorphisms α_T induced from symplectic transformations on $(\mathbb{C}^{(\mathbb{N})}, \sigma)$ (cf. 3.1 (ii)) a remarkable phenomenon can be observed. The representations arising from "mixing" with the transformations $x_1 + ix_2 \mapsto \lambda x_1 + i\frac{1}{\lambda}x_2$ $(\lambda > 0)$ on $\mathbb{C}^{(\mathbb{N})}$ fall into (pairwise) disjoint equivalence classes as λ varies in the positive real numbers. Since irreducibility is not effected the above family gives uncountably many different irreducible (regular) representations. The result could be sharpened in [Sha, 1962] as stated in Theorem 8.6. So the essential contrast to the theory of finitely many degrees of freedom was obvious. It became clear that a special representation corresponds to an investigation of the physical system in a fixed state. This was the (or one?) starting point for the development of C^*-algebraic Quantum Field Theory.

We will state without proof the result of D.Shale on the deformed Fock representations $F_T := F \circ \alpha_T \quad (T \in \mathrm{Sp}(\mathbb{C}^{(\mathbb{N})}, \sigma))$, where α_T denotes the automorphism $(x, r) \mapsto (Tx, r)$ (cf. 3.1 (ii)).

8.6. Theorem (D.Shale 1962): If $S, T \in \mathrm{Sp}(\mathbb{C}^{(\mathbb{N})}, \sigma)$ and F_T, F_S are the associated Fock representations then the following are equivalent:

(i) $S - T \in \mathrm{End}(\mathbb{C}^{(\mathbb{N})})$ extends to a Hilbert-Schmidt operator on $\ell^2_{\mathbb{C}}(\mathbb{N})$

(ii) The representations F_T and F_S are unitarily equivalent

8.7. Remark:

(i) Since on the finite dimensional Hilbert space $(\mathbb{C}^n, h(.,.))$ all operators are trivially Hilbert-Schmidt the above theorem includes the von Neumann uniqueness result from 1931 (cf. [vN]).

(ii) Each $S \in \mathrm{Sp}(E, \sigma)$ defines a decomposition of $E_{\mathbb{C}}$ into maximally isotropic subspaces $L_+ \oplus L_-$ by simply setting

$$L_{\pm} = \text{eigenspace of } S_{\mathbb{C}} \text{ corresponding to eigenvalue } \pm i$$

where $S_{\mathbb{C}}$ denotes the usual complex linear extension of S. On the other hand if $E_{\mathbb{C}} = L_+ \oplus L_-$ with isotropic subspaces such that the hermitian form

$$\langle \xi | \eta \rangle := -i\hbar(\sigma_{\mathbb{C}}(\xi_+, \eta_-) + \sigma_{\mathbb{C}}(\xi_-, \eta_+))$$

is positive definite, then $S_{\mathbb{C}}$ defined by $S_{\mathbb{C}}\xi := i\xi_+ - i\xi_-$ corresponds to a symplectic operator S on E (for details see [Woo, 4.9]).

Now by 6.1 in turn such polarizations of $E_{\mathbb{C}}$ define admissible subalgebras for the Kirillov construction. In this way we obtain some insight into the sensible dependence of unitary equivalence classes of the representations F_T ($F \in \mathrm{Sp}(E, \sigma)$ on the "initial data", namely the choice of an orbit and a subordinated subalgebra of $\mathfrak{h}_{\mathbb{C}}$.

9 The Moment Mapping for the Fock Representation

The unitary Fock representation F of Proposition 8.4 defines by regularity (see Remark 8.5 (i)) a Lie algebra homomorphism from $\mathfrak{h}$ into a set of unbounded self-adjoint operators (the generators of the one-parameter subgroups) on a certain (common) dense domain W_∞, the smooth vectors in W. There is a way of going back from the representation to the "source" orbit. It is modelled on the classical situation of a symplectic action of a Lie group on a symplectic manifold and was carried over to unitary representations by Peter Michor (cf. [M, 1990] or [Wik] and [B] for more relations to physics and orbit theory). This "inversion" $\mu : W_\infty \to \mathfrak{h}^*$, the *moment mapping*, is defined as follows.

9.1. Definition of the moment mapping:

Equip W_∞ with the weak symplectic form $\Omega(.,.) = \mathrm{Im}\,\langle .|.\rangle$. Let

$$F'(v,a)f := \frac{d}{dt}|_0\ (F(tv,ta)f) \quad \forall (v,a) \in \mathfrak{h}, f \in W_\infty$$

be the Lie algebra representation associated to F. Then the *moment mapping* $\mu : W_\infty \to \mathfrak{h}^*$ is given by

$$\langle \mu(f), (v,a)\rangle := \Omega(F'(v,a)f, f) \qquad \forall (v,a) \in \mathfrak{h}, f \in W_\infty\ . \tag{II.36}$$

We will study the moment mappings associated to the deformed Fock representations $F_T \quad (T \in \mathrm{Sp}(\mathbb{C}^{(\mathbb{N})}, \sigma))$ and state a result which is near to the finite dimensional analogue.

9.2. Proposition: If F_T denotes the deformed Fock representation corresponding to $T \in \mathrm{Sp}(\mathbb{C}^{(\mathbb{N})}, \sigma)$ and μ_T is the associated moment mapping then we have (we write F, μ instead of $F_{\mathrm{Id}}, \mu_{\mathrm{Id}}$):

(i) $W_\infty \supseteq \{f_a \mid a \in \mathbb{C}^{(\mathbb{N})}\}$ independent of T

(ii) $\operatorname{im} \mu_T = \operatorname{im} \mu \quad \forall T$ and $\mu(\bigcirc W_\infty) \supseteq \mathcal{O}_{(0,\hbar)}$, where $\bigcirc W_\infty$ denotes the unit sphere.

Proof:

(i) Since α_T is smooth $H \to H$ and $F_T = F \circ \alpha_T$ it is enough to show smoothness of $(x, r) \to F(x, r) f_a$, $H \to W \quad \forall a \in \mathbb{C}^{(\mathbb{N})}$. Let $(x(t), r(t))$ be a smooth curve in H; since

$$F(x(t), r(t)) f_a = \underbrace{e^{i\hbar r(t) - \frac{1}{2}(\beta(x(t),x(t)) - \beta(a,a))} e^{i\hbar\sigma(x(t),a)}}_{\text{smooth}} \cdot F(a, 0) f_{x(t)}$$

it is enough to show that $t \mapsto f_{x(t)}$ is smooth $\mathbb{R} \to W$.

To show first weak differentiability we will apply the Banach-Steinhaus theorem to the net $(g_{s,t} := \frac{f_{x(t)} - f_{x(s)}}{t-s})_{t \neq s \in \mathbb{R}}$ considered as a net in $W^* \cong W$ (note that we use the one implication of the classical theorem which is valid for nets):

- pointwise convergence on a dense subset:

  $$\langle \frac{f_{x(t)} - f_{x(s)}}{t-s} | f_y \rangle = \frac{\langle f_{x(t)} | f_y \rangle - \langle f_{x(s)} | f_y \rangle}{t-s} = \frac{e^{\beta(x(t),y)} - e^{\beta(x(s),y)}}{t-s}$$

  tends to $\beta(\dot{x}(s), y) e^{\beta(x(s),y)}$ as $t \to s$;

- boundedness in norm:
  note that for elements in W^* the operator norm is equal to the Hilbert space norm; now

  $$\begin{aligned} &\| \frac{f_{x(t)} - f_{x(s)}}{t-s} \|^2 = \\ &= \quad \frac{1}{(t-s)^2}(e^{\beta(x(t),x(t)} - e^{\beta(x(t),x(s))} - e^{\beta(x(s),x(t))} + e^{\beta(x(s),x(s))}) = \\ &\quad \ldots \text{set } \psi(t,s) = e^{\beta(x(t),x(s))} \ldots \\ &= \quad \frac{1}{(t-s)^2}(\psi(t,t) - \psi(t,s) - (\psi(s,t) - \psi(s,s))) \end{aligned}$$

  which is bounded on a neighborhood of $(s, s) \in \mathbb{R}^2$ since it converges to $\partial_{12}\psi(s, s)$ as $t \to s$ by smoothness of ψ (write the difference quotients in integral form; see [F/K, 1.3] for more general results).

So in a first step we may conclude that $g_{s,t} \to \iota(\dot{x}(s))f_{x(s)} =: g$ weakly as $t \to s$. But using power series expansion one can easily compute

$$\|\iota(\dot{x}(s))f_{x(s)}\|^2 = (\beta(\dot{x}(s), \dot{x}(s)) + \beta(\dot{x}(s), x(s)))e^{\beta(x(s),x(s))}$$

which is exactly the expression for $\partial_{12}\psi(s,s)$ as straightforward differentiation shows. So we have a weakly convergent net $(g_{s,t})_{s\neq t}$ of vectors in W such that the net $(\|g_{s,t}\|)_{s\neq t}$ of norms converges to the norm $\|g\|$ of its weak limit. Hence the strong (norm) convergence follows now from the standard equation

$$\|g_{s,t} - g\|^2 = \|g_{s,t}\|^2 - 2\mathrm{Re}\,\langle g_{s,t}|g\rangle + \|g\|^2\ .$$

The investigation of higher derivatives can be reduced to the above case since by induction the n^{th} derivative of $f_{x(t)}$ with respect to t is of the form $p(t)f_{x(t)}$, where $p(t)$ is a polynomial in $\iota(\dot{x}(t)), \ldots, \iota(x^{(n)}(t))$. So each derivative is again a product of differentiable functions. So smoothness of $t \mapsto f_{x(t)}$ is proved.

(ii) We first compute (using the above result for the derivative of $f_{x(t)}$)

$$\begin{aligned} {F_T}'(v,r)f_a &= \frac{d}{dt}|_0\ (e^{i\hbar tr - t^2\frac{1}{2}\beta(Tv,Tv) - t\beta(a,Tv)}f_{a+tTv}) \\ &= (i\hbar r - \beta(a,Tv) + \iota(Tv))f_a \end{aligned}$$

and get further by

$$\begin{aligned} &\langle \mu_T(f_a), (v,r)\rangle = \mathrm{Im}\,\langle {F_T}'(v,r)f_a|f_a\rangle = \\ &\quad = \mathrm{Im}\,(i\hbar r\langle f_a|f_a\rangle - \beta(a,Tv)\langle f_a|f_a\rangle + \langle \iota(Tv)f_a|f_a\rangle) = \\ &\quad \ldots \text{now use again power series to compute } \langle bf_a|f_a\rangle \ldots \\ &\quad = e^{\beta(a,a)}\mathrm{Im}\,(i\hbar r - \beta(a,Tv) + \beta(Tv,a)) = e^{\beta(a,a)}(\hbar r + 2\mathrm{Im}\,\beta(Tv,a)) = \\ &\quad = \langle f_a|f_a\rangle(\hbar r + \hbar\mathrm{Im}\,h(Tv,a)) = \langle f_a|f_a\rangle(\hbar r - \hbar\sigma(a,Tv)) = \\ &\quad = \|f_a\|^2 \cdot \langle(-\hbar T^*\check{\sigma}(a), \hbar), (v,r)\rangle\ . \end{aligned}$$

Since $a \in \mathbb{C}^{(\mathbb{N})}$ was arbitrary and T is an isomorphism the result follows. ■

9.3. Remark: It is not obvious wether point (ii) of Proposition 9.2 should really read $\mu(\bigcirc W_\infty) = \mathcal{O}_{(0,\hbar)}$. Since $\mathcal{O}_{(0,\hbar)}$ is not closed in $\mathfrak{h}^* = \mathbb{C}^{\mathbb{N}} \oplus \mathbb{R}$ no continuity argument for μ can work.

Chapter III

GEOMETRIC QUANTIZATION

Emphasizing the geometrical and physical contents of the orbit method geometric quantization came into being as a tool for constructing unitary representations of Lie groups out of a symplectic action on classical phase space. A crucial point for the basis of the theory is that each such action can be considered in essence as coadjoint action on an orbit of the group or its central extension (cf. [Kir, 1976: 15.2,Th.1]). We will not present the theory here but mainly restrict to the aspects arising from working out its methods for our concrete infinite dimensional Heisenberg group. So the following story will hardly comment on the background of the ideas of geometric quantization and the requirements of *Dirac correspondence* between classical and quantum mechanical systems. For details of the (finite dimensional) theory we refer to [Kos] (the origin), [Wik] (describing connections to the moment mapping (compare 9)) and [Woo] (presenting physical applications).

In the following three sections we will compute and present explicitly the building blocks for geometric quantization of the coadjoint action of the Heisenberg group. These are the concepts of connection, curvature and the Dirac principle — an intuitively motivated correspondence of functions on phase space $\mathbb{C}^{(\mathbb{N})}$

and operators on sections of a line bundle over the orbit. The last section will then investigate restrictions of the arising representations suggested by certain polarizations of the underlying symplectic orbit manifold.

10 Orbits as Symplectic Manifolds

Once again we fix an (arbitrary) coadjoint orbit $\mathcal{O} = (u^* + \operatorname{im}\check{\sigma}) \times \{\hbar\}$ which will now be considered as manifold modelled on $\mathbb{C}^{(\mathbb{N})} \cong \operatorname{im}\check{\sigma}$ (note that the topology on $\operatorname{im}\check{\sigma} \subseteq \mathbb{C}^{\mathbb{N}}$ is therefore not the relative topology but strictly finer).

If $G_{(u^*,\hbar)}$ denotes the stabilizer of $(u^*,\hbar)$ with respect to the coadjoint action of H on $\mathcal{O}$ we have

$$G_{(u^*,\hbar)} = \{(x,r) \in H \mid u^* - \hbar\check{\sigma}(x) = u^*\} = \{(0,r) \mid r \in \mathbb{R}\} \qquad \text{(III.1)}$$

since $\check{\sigma}$ is one-to-one. Therefore we can identify $\mathcal{O}$ with $H/G_{(u^*,\hbar)}$ — a well known general fact for any finite dimensional Lie group.

10.1. Fundamental vector fields:

Using the kinematical definition of tangent vectors we can write $T\mathcal{O} = \mathcal{O} \times \operatorname{im}\check{\sigma}$ and therefore consider vector fields on $\mathcal{O}$ simply as smooth mappings $\mathcal{O} \to \operatorname{im}\check{\sigma}$. The fundamental vector field mapping $\zeta : \mathfrak{h} \to \mathfrak{X}(\mathcal{O})$ corresponding to $\mathrm{Ad}^* : H \times \mathcal{O} \to \mathcal{O}$ is defined as follows

$$\forall (v,a) \in \mathfrak{h} : \zeta_{(v,a)}(z^*,\hbar) = T_{(0,0)}\mathrm{Ad}^*(.,(v,a)) \cdot (v,a) \quad \forall (z^*,\hbar) \in \mathcal{O}\ .$$

Since by I.14 $\mathrm{Ad}^*(.,(z^*,\hbar)) : (x,r) \mapsto (z^* - \hbar\check{\sigma}(x),\hbar)$ is affine in the first component and constant in the second one we have

$$\zeta_{(v,a)}(z^*,\hbar) = -\hbar\check{\sigma}(v) \qquad \forall (z^*,\hbar), \qquad \text{(III.2)}$$

i.e. the fundamental vector fields are constant and reproduce their generators (up to the factor $-\hbar$). Furthermore we observe the familiar property that the tangent space $T_{(z^*,\hbar)}\mathcal{O}$ to each point $(z^*,\hbar) \in \mathcal{O}$ is spanned by the values of the fundamental vector fields.

10.2. Symplectic structure on $\mathcal{O}$:

We define a 2-form $\Omega \in \Omega^2(\mathcal{O})$ by the pointwise prescription

$$(\Omega(\zeta_{(v,a)}, \zeta_{(w,b)}))(z^*, \hbar) := \langle (z^*, \hbar), [(v,a),(w,b)] \rangle = \hbar\sigma(v,w) \qquad \text{(III.3)}$$

or equivalently pointwise as bilinear form on the tangent space im $\check{\sigma}$

$$\Omega_{(z^*,\hbar)}(\check{\sigma}(v), \check{\sigma}(w)) = \frac{1}{\hbar}\sigma(v,w) \; . \qquad \text{(III.4)}$$

Lemma: Ω is a non-degenerate closed 2-form on $\mathcal{O}$, i.e. $(\mathcal{O}, \Omega)$ is a weak symplectic manifold. The coadjoint action of H on $\mathcal{O}$ is a symplectic action (i.e. preserves the symplectic form).

Proof: one can reduce the proof to the finite dimensional case: each instance of fixed vector fields into Ω or $d\Omega$ keeps all computations in a finite dimensional subspace. Nevertheless we will give also a direct proof since one special part will be used later.

- nondegeneracy is clear from the corresponding property of σ
- we show that Ω is exact (and therefore closed):
  define a 1-form ω on $\mathcal{O}$ by putting

  $$\omega_{(z^*,\hbar)}(\check{\sigma}(v)) = \frac{1}{2\hbar}\langle z^*, v\rangle \qquad \forall (z^*,\hbar) \in \mathcal{O}, v \in \mathbb{C}^{(\mathbb{N})}; \qquad \text{(III.5)}$$

  now we directly calculate as follows:

  $$\begin{aligned} & d\omega(\zeta_{(v,a)}, (w,b))_{(z^*,\hbar)} = \\ & \quad = \zeta_{(v,a)}(\omega(\zeta_{(w,b)}))_{(z^*,\hbar)} - \zeta_{(w,b)}(\omega(\zeta_{(v,a)}))_{(z^*,\hbar)} - \omega([\zeta_{(v,a)}, \zeta_{(w,b)}]_{(z^*,\hbar)} = \\ & \quad \ldots \text{now the } \zeta\text{'s act by directional derivation and the commutator} \\ & \quad \text{of the constant vector fields inside the last term vanishes} \ldots \\ & \quad = \frac{d}{dt}|_0 \; (\omega_{(z^*-t\hbar\check{\sigma}(v),\hbar)}(-\hbar\check{\sigma}(w))) - \frac{d}{dt}|_0 \; (\omega_{(z^*-t\hbar\check{\sigma}(v),\hbar)}(-\hbar\check{\sigma}(v))) = \\ & \quad = \frac{1}{2\hbar}\frac{d}{dt}|_0 \; (\langle z^* - t\hbar\check{\sigma}(v), -\hbar w\rangle - \langle z^* - t\hbar\check{\sigma}(w), -\hbar v\rangle) = \\ & \quad = \frac{1}{2\hbar}(\hbar^2\langle\check{\sigma}(v), w\rangle - \hbar^2\langle\check{\sigma}(w), v\rangle) = \frac{\hbar}{2}(\sigma(v,w) - \sigma(w,v)) = \\ & \quad = \hbar\sigma(v,w) = \Omega(\zeta_{(v,a)}, \zeta_{(w,b)})_{(z^*,\hbar)} \end{aligned}$$

- H acts by symplectomorphisms: for fixed $(x,r) \in H$ the operator $\mathrm{Ad}^*(x,r)$ is just constant translation by $-\hbar\check{\sigma}(x)$, the corresponding tangent mapping is the identity and translating the foot point does not change the value of $\Omega(\zeta_{(v,a)}, \zeta_{(w,b)})$. ■

10.3. A character representation of $G_{(u^*,\hbar)}$:

If $\mathfrak{g}_{(u^*,\hbar)} \cong \mathbb{R}$ denotes the Lie algebra of $G_{(u^*,\hbar)}$ one can define a Lie algebra homomorphism $\chi' : \mathfrak{g}_{(u^*,\hbar)} \to \mathbb{R}$, $\chi'(0,a) = \langle (u^*,\hbar), (0,a)\rangle = \hbar a$ and integrate it to a character χ of $G_{(u^*,\hbar)} = \{0\} \times \mathbb{R}$

$$\chi(0,s) = e^{i\hbar s} \; . \tag{III.6}$$

Now as in chapter 2 the canonical surjection $H \twoheadrightarrow H/G_{(u^*,\hbar)}$ defines a principal fibre bundle with structure group $G_{(u^*,\hbar)} \cong \mathbb{R}$ and base manifold $H/G_{(u^*,\hbar)} \cong \mathcal{O}$. The character χ yields the associated (complex) line bundle $L = H[\mathbb{C}, \chi]$ over $\mathcal{O}$, which can be trivialized by the methods seen in chapter 2, 7.

Lemma: The mapping $q_0 : h \times \mathbb{C} \to \mathcal{O} \times \mathbb{C}$, $q_0((x,r),\lambda) = (x, e^{i\hbar r}\lambda)$, factors to an isomorphism $\psi : L \to \mathcal{O} \times \mathbb{C}$, i.e.

$$\begin{tikzcd} H \times \mathbb{C} \arrow[rr, "q_0"] \arrow[dr, "q"'] & & \mathcal{O} \times \mathbb{C} \\ & L = H[\mathbb{C}] \arrow[ur, "\psi"'] & \end{tikzcd}$$

Proof: compare with the situation in 7.1; the right action is now given by $R(((x,r),\lambda),(0,s)) = (x, r+s, e^{-i\hbar s}\lambda)$ ■

10.4. Remark: Let $L_1 := \psi^{-1}(\mathcal{O} \times \mathbb{C}^+)$ where $\mathbb{C}^+ = \mathbb{C}\backslash\{0\} \cong \mathrm{GL}(\mathbb{C})$. Then L_1 is the *linear frame bundle* $\mathrm{GL}(\mathbb{C}, L)$ — a $\mathrm{GL}(\mathbb{C})$-principal bundle associated to L (see [M, Lecture notes 91/92: 15.11] for a general context). Let $\psi_1 = \psi|_{L_1}$: $L_1 \to \mathcal{O} \times \mathbb{C}^+$ then $T\psi_1$ defines an isomorphism of tangent bundles and

$$TL_1 \cong T(\mathcal{O} \times \mathbb{C}^+) = (\mathcal{O} \times \mathbb{C}^+) \times (\mathrm{im}\, \check{\sigma} \oplus \mathbb{C}) \,, \tag{III.7}$$

where $\mathrm{im}\, \check{\sigma}$ represents the horizontal vectors and $\mathbb{C}$ constitutes the vertical part. Since the right action on $\mathcal{O} \times \mathbb{C}^+$ is just scalar multiplication in the $\mathbb{C}^+$-component we can define a connection form $\alpha \in \Omega^1(\mathcal{O} \times \mathbb{C}^+, \mathbb{C})$ by

$$\alpha = \omega_{\mathbb{C}} \oplus \frac{1}{i}\frac{dz}{z} \tag{III.8}$$

where $\omega_{\mathbb{C}}$ denotes the complex extension of the symplectic potential ω given in equation III.5. Obviously one obtains $d\alpha = \mathrm{pr}_1^* \Omega$. Therefore if $p : L_1 \to \mathcal{O}$ is the projection of the bundle L_1 one can define the *connection form*

$$\alpha_1 := \psi_1^* \alpha$$

on L_1. The next section will deal with the corresponding covariant differentiation and we will show by direct computation that its curvature is exactly Ω.

11 Covariant Derivative and Curvature on the Associated Line Bundle

In the following the conventions about the sign of the factor i appearing in definitions are due to [Woo].

11.1. Covariant derivative:

Identifying $\Gamma(\mathcal{O} \times \mathbb{C})$ with $C^\infty(\mathcal{O}, \mathbb{C})$ we define for each $\xi \in \mathfrak{X}_{\mathbb{C}}(\mathcal{O})$ an endomorphism $\nabla_\xi : \Gamma(\mathcal{O} \times \mathbb{C}) \to \Gamma(\mathcal{O} \times \mathbb{C})$ as follows

$$\nabla_\xi(f) = \xi(f) - i f \omega_{\mathbb{C}}(\xi) \qquad \forall f \in C^\infty(\mathcal{O}, \mathbb{C}) \tag{III.9}$$

(we will neglect the subscript $\mathbb{C}$ of ω from now on). So $\xi \mapsto \nabla_\xi$ yields a linear mapping $\nabla : \mathfrak{X}(\mathcal{O}) \to \mathrm{End}(\Gamma(\mathcal{O} \times \mathbb{C}))$ with the properties

(i) $\nabla_{\varphi\xi}(f) = \varphi\xi(f) - if\omega(\varphi\xi) = \varphi(\xi(f) - if\omega(\xi)) = \varphi\nabla_\xi(f)$

(ii) $\nabla_\xi(fg) = \xi(fg) - ifg\omega(\xi) = \xi(f)g + f\xi(g) - ifg\omega(\xi) = (g\xi)(f) + f\nabla_\xi(g)$,

i.e. ∇ is a *connection* or *covariant differentiation* on L.

For later use we compute the action of $\nabla_{\zeta_{(v,a)}}$ on functions $f \in C^\infty(\mathcal{O}, \mathbb{C})$:

$$\begin{aligned}\nabla_{\zeta_{(v,a)}}(f)(z^*, \hbar) &= \zeta_{(v,a)}(f)(z^*, \hbar) - if(z^*, \hbar)\omega(\zeta_{(v,a)})_{(z^*,\hbar)} = \\ &= \langle df_{(z^*,\hbar)}, -\hbar\check{\sigma}(v)\rangle - if(z^*, \hbar)\frac{1}{2\hbar}\langle z^*, -\hbar v\rangle = \\ &= -\hbar\langle df, \check{\sigma}(v)\rangle_{(z^*,\hbar)} + \frac{i}{2}f(z^*, \hbar)\langle z^*, v\rangle\end{aligned}$$

or in more compact form, if $\iota : \mathbb{C}^{(\mathbb{N})} \to (\mathbb{C}^{(\mathbb{N})})^{**} \cong \mathbb{C}^{(\mathbb{N})}$ denotes the canonical embedding into the bidual,

$$\nabla_{\zeta_{(v,a)}(f)} = -\hbar\langle df, \check{\sigma}(v)\rangle + \frac{i}{2}f\iota(v) \quad \forall (v, a) \in \mathfrak{h}, f \in C^\infty(\mathcal{O}, \mathbb{C}) \; . \qquad \text{(III.10)}$$

11.2. The symplectic form as curvature:

The *curvature* of the connection ∇ is by definition the 2-form $R \in \Omega^2(\mathcal{O}, \mathbb{C})$ satisfying

$$R(\xi, \eta)f = i([\nabla_\xi, \nabla_\eta] - \nabla_{[\xi,\eta]})(f) \qquad \forall \xi, \eta \in \mathfrak{X}_\mathbb{C}, f \in C^\infty(\mathcal{O}, \mathbb{C}) \; , \qquad \text{(III.11)}$$

where the first Lie bracket on the right denotes the commutator of operators in $\mathrm{End}(C^\infty(\mathcal{O}, \mathbb{C}))$ and the subscript Lie bracket is the Lie bracket of vector fields.

Lemma: $\forall (v, a), (w, b) \in \mathfrak{h} : \quad R(\zeta_{(v,a)}, \zeta_{(w,b)}) = \Omega(\zeta_{(v,a)}, \zeta_{(w,b)})$

Proof: let $f \in C^\infty(\mathcal{O}, \mathbb{C}); (v, a), (w, b) \in \mathfrak{h}$:

- first note that $[\zeta_{(v,a)}, \zeta_{(w,b)}] = 0$ hence the term $\nabla_{[\zeta_{(v,a)},\zeta_{(w,b)}]}$ can be neglected.
- we divide the computation of $[\nabla_{\zeta_{(v,a)}}, \nabla_{\zeta_{(w,b)}}]$ into investigation of $\nabla_{\zeta_{(v,a)}} \circ \nabla_{\zeta_{(w,b)}}$ and $-\nabla_{\zeta_{(w,b)}} \circ \nabla_{\zeta_{(v,a)}}$ separately:

$$
\begin{aligned}
&\nabla_{\zeta_{(v,a)}} \circ \nabla_{\zeta_{(w,b)}}(f) = \\
&\quad = \nabla_{\zeta_{(v,a)}}(-\hbar\langle df, \check{\sigma}(w)\rangle + \frac{i}{2} f\iota(w)) = \\
&\quad = -\hbar\nabla_{\zeta_{(v,a)}}(\langle df, \check{\sigma}(w)\rangle) + \frac{i}{2}\nabla_{\zeta(v,a)}(f\iota(w)) = \\
&\quad = -\hbar\big(- \hbar\langle d\langle df, \check{\sigma}(w)\rangle, \check{\sigma}(v)\rangle + \frac{i}{2}\langle df, \check{\sigma}(w)\rangle\iota(v)\big) + \\
&\quad\quad + \frac{i}{2}\big(- \hbar\langle d(f\iota(w)), \check{\sigma}(v)\rangle + \frac{i}{2} f\iota(w)\iota(v)\big) = \\
&\quad = \hbar^2 d^2 f(\check{\sigma}(w), \check{\sigma}(v)) - \frac{i\hbar}{2}\langle df, \check{\sigma}(w)\rangle\iota(v) - \\
&\quad\quad - \frac{i\hbar}{2}\big(\langle df, \check{\sigma}(v)\rangle\iota(w) + f\langle\iota(w), \check{\sigma}(v)\rangle\big) - \frac{1}{4} f\iota(w)\iota(v) = \\
&\quad = \hbar^2 d^2 f(\check{\sigma}(w), \check{\sigma}(v)) - \frac{i\hbar}{2}\big(\langle df, \check{\sigma}(v)\rangle\iota(w) + \langle df, \check{\sigma}(w)\rangle\iota(v)\big) - \\
&\quad\quad - f\big(\frac{i\hbar}{2}\sigma(v,w) + \frac{1}{4}\iota(v)\iota(w)\big)
\end{aligned}
$$

changing the rôles of v and w and putting a minus in front yields the expression

$$
\begin{aligned}
&-\nabla_{\zeta_{(w,b)}} \circ \nabla_{\zeta_{(v,a)}} = -\hbar^2 d^2 f(\check{\sigma}(v), \check{\sigma}(w)) + \\
&\quad + \frac{i\hbar}{2}\big(\langle df, \check{\sigma}(w)\rangle\iota(v) + \langle df, \check{\sigma}(v)\rangle\iota(w)\big) + f\big(\frac{i\hbar}{2}\sigma(w,v) + \frac{1}{4}\iota(w)\iota(v)\big) .
\end{aligned}
$$

Now by symmetry of $d^2 f$ we get

$$
\begin{aligned}
[\nabla_{\zeta_{(v,a)}}, \nabla_{\zeta_{(w,b)}}] &= f\frac{i\hbar}{2}(-\sigma(v,w) + \sigma(w,v)) = \\
&= -i\hbar\sigma(v,w) f = \frac{1}{i} f\Omega(\zeta_{(v,a)}, \zeta_{(w,b)})
\end{aligned}
$$

■

Since the differential operators ∇_ξ and the 2-forms are local (i.e. they are defined on germs) and $\{\zeta_{(v,a)} \mid (v,a) \in \mathfrak{h}\}$ spans the tangent space to each point we can state the following

Corollary: The curvature of ∇ is exactly the symplectic form Ω.

12 The Dirac Correspondence

An attempt to formalize an intuitive correspondence (if there is any ?) of quantum systems with classical systems was made by Dirac in 1926 (cf. [Woo, ch. 5] and the references given there). This so-called *Quantum Condition* or *Dirac Correspondence Principle* is formulated in terms of unitary representations roughly as follows:
let the classical phase space be represented by the symplectic manifold (M, Ω); then the classical observables are just the functions $\varphi \in C^\infty(M)$ and the Poisson bracket $\{.,.\}$ defines the structure of a real Lie algebra; a quantization should be a mapping $\varphi \mapsto \widehat{\varphi}$ of $C^\infty(M)$ into the set of self-adjoint (unbounded) operators on a Hilbert space such that

(D1) $\varphi \mapsto \widehat{\varphi}$ is $\mathbb{R}$-linear

(D2) $[\widehat{\varphi}, \widehat{\psi}] = -i\widehat{\{\varphi, \psi\}}$

(D3) if $\mathbb{1}(x) = 1 \quad \forall x \in M$ then $\widehat{\mathbb{1}} = \mathrm{Id}$

In our context, i.e. $M = \mathcal{O}$, we generalize the Dirac principle to the properties (D1)–(D3) being required for a mapping $C^\infty(\mathcal{O}) \to \mathrm{End}(C^\infty(\mathcal{O}, \mathbb{C}))$. We will explicitly construct such a mapping using Hamiltonian vector fields $X_\varphi \in \mathfrak{X}$ generated by (real) functions $\varphi \in C^\infty(\mathcal{O})$ and defining $\widehat{\varphi} = -i\nabla_{X_\varphi} + \varphi$.

12.1. Hamiltonian vector fields on $\mathcal{O}$:

Let $\varphi \in C^\infty(\mathcal{O})$, we define $X_\varphi \in \mathfrak{X}(\mathcal{O})$ by the equation

$$\Omega(X_\varphi, \xi) + \langle d\varphi, \xi \rangle = 0 \qquad \forall \xi \in \mathfrak{X}(\mathcal{O}) \tag{III.12}$$

We can compute X_φ by explicit solutions of the above equation for $\xi = \zeta_{(v,a)}$ (note that $\mathfrak{X}(\mathcal{O}) \cong C^\infty(\mathcal{O}, \operatorname{im} \check{\sigma})$).
Notation: introducing the partial derivatives

$$\begin{aligned} \partial_{q_j}\varphi &:= \langle d\varphi, \delta_j \rangle && \text{position variation in j. component} \\ \partial_{p_k}\varphi &:= \langle d\varphi, i\delta_k \rangle && \text{momentum variation in k. component} \end{aligned}$$

we can write $d\varphi = \partial_q\varphi + i\partial_p\varphi := (\partial_{q_j}\varphi)_{j\in\mathbb{N}} + i(\partial_{p_j}\varphi)_{j\in\mathbb{N}}$.

Lemma: If $\varphi \in C^\infty(\mathcal{O})$ then $X_\varphi = -\hbar\partial_p\varphi + i\hbar\partial_q\varphi$.

Proof: evaluate equation III.12 pointwise for $\xi = \zeta_{(v,a)}$:

$$-\langle d\varphi_{(z^*,\hbar)}, -\hbar\check{\sigma}(v)\rangle = \Omega_{(z^*,\hbar)}(X_phi(z^*,\hbar), -\hbar\check{\sigma}(v))$$

now let $X_\varphi(z^*,\hbar) = \check{\sigma}(w)$ for suitable $w = w_1 + iw_2 \in \mathbb{C}^{(\mathbb{N})}$ then

$$\hbar\langle d\varphi_{(z^*,\hbar)}, \check{\sigma}(v)\rangle = -\sigma(w,v)$$

- $v = \delta_j$ yields $\hbar\langle d\varphi_{(z^*,\hbar)}, i\delta_j\rangle = w_2^j$ or simply $w_2^j = \hbar\partial_{p_j}\varphi(z^*,\hbar)$
- $v = i\delta_k$ yields $\hbar\langle d\varphi_{(z^*,\hbar)}, -\delta_k\rangle = -w_1^k$ or simply $w_1^k = \hbar\partial_{q_k}\varphi(z^*,\hbar)$

and hence

$$X_\varphi(z^*,\hbar) = \check{\sigma}(w) = -w_2 + iw_1 = (-\hbar\partial_p\varphi + i\hbar\partial_q\varphi)(z^*,\hbar)$$

■

12.2. Remark:

(i) By the above Lemma decomposition into real and imaginary part of the differential equation $\dot{c} = X_\varphi \circ c$ for a curve $c : \mathbb{R} \to \mathcal{O}$, $c(t) = q(t) + ip(t)$, yields the following form of Hamiltonian equations:
$\dot{q} = -\hbar\partial_p\varphi$ and $\dot{p} = \hbar\partial_q\varphi$

(ii) Let $q_j, p_k \in C^\infty(\mathcal{O})$ be defined as follows:

$$\begin{aligned} q_j &:= \iota(\delta_j), \quad \text{i.e.} \qquad q_j(z^*,\hbar) = \langle z^*, \delta_j\rangle \\ p_k &:= \iota(i\delta_k), \quad \text{i.e.} \qquad p_k(z^*,\hbar) = \langle z^*, i\delta_k\rangle \end{aligned}$$

Now the above Lemma tells us

$$X_{q_j} = i\hbar\delta_j = \hbar\check{\sigma}(\delta_j) \quad \text{and} \quad X_{p_k} = -\hbar\delta_k = \hbar\check{\sigma}(i\delta_k)\ .$$

Therefore if $v \in \mathbb{C}^{(\mathbb{N})}$ then the Hamiltonian vector field associated to the function $\varphi_v = -\iota(v)$ reproduces the fundamental vector field $\zeta_{(v,0)}$.

12.3. Poisson bracket on $C^\infty(\mathcal{O})$:

There are no problems in defining the Poisson bracket $\{.,.\}$ on $C^\infty(\mathcal{O})$ exactly as in the finite dimensional case by

$$\{\varphi,\psi\} = \Omega(X_\varphi, X_\psi) \qquad \forall \varphi,\psi \in C^\infty(\mathcal{O}) \; . \tag{III.13}$$

Standard arguments show that $(C^\infty(\mathcal{O}), \{.,.\})$ is a real Lie algebra and the mapping $\varphi \mapsto X_\varphi$ defines a Lie algebra homomorphism $(C^\infty(\mathcal{O}), \{.,.\}) \to (\mathfrak{X}(\mathcal{O}), [.,.])$.

12.4. The Quantization map:

As announced at the beginning of this section we define the Dirac correspondence $\varphi \mapsto \widehat{\varphi}$, $C^\infty(\mathcal{O}) \to \mathrm{End}(C^\infty(\mathcal{O},\mathbb{C}))$, by

$$\widehat{\varphi}(f) = -i\nabla_{X_\varphi}(f) + \varphi f \qquad \forall f \in C^\infty(\mathcal{O},\mathbb{C}) \; . \tag{III.14}$$

Proposition: The mapping $\varphi \mapsto \widehat{\varphi}$ satisfies the requirements (D1)–(D3) of Dirac's quantum condition.

Proof:

(D1) $\mathbb{R}$-linearity is guaranteed by $\mathbb{R}$-linearity of ∇ and $\varphi \mapsto X_\varphi$

(D2) $[\widehat{\varphi},\widehat{\psi}](f) = if\Omega(X_\varphi, X_\psi) - \nabla_{[X_\varphi,X_\psi]}(f)$ by straightforward computation (using 11.2) and the result follows from 12.3

(D3) $\widehat{\mathbb{1}}(f) = -i\nabla_{X_{\mathbb{1}}}(f) + f = f$ since the Hamiltonian vector fields generated by constant functions are 0 ■

12.5. Position and momentum operator:

Using the functions q_j, p_k introduced in 12.2 (ii) together with equation III.10 for the vector fields $X_{q_j} = \zeta_{(-\delta_j,0)}$ and $X_{p_k} = \zeta_{(-i\delta_k,0)}$ we recover the well known position and momentum operators:

$$\begin{aligned}\widehat{q_j}(f) &= -i\nabla_{-\zeta_{(\delta_j,0)}}(f) + q_j f = -i\hbar\langle df, \check{\sigma}(\delta_j)\rangle - \frac{1}{2} f\iota(\delta_j) + q_j f = \\ &= -i\hbar\partial_{p_j} f + \frac{1}{2} q_j f\end{aligned}$$

and similarly

$$\widehat{p_k}(f) = i\hbar\partial_{q_k}(f) + \frac{1}{2} p_k f \ ,$$

which reads in short notation as follows

$$\widehat{q_j} = -i\hbar\partial_{p_j} + \frac{1}{2} q_j \qquad \widehat{p_k} = i\hbar\partial_{q_k} + \frac{1}{2} p_k \ . \tag{III.15}$$

12.6. Remark: The quantization map yields a Lie algebra representation ρ of $\mathfrak{h}$ by setting

$$\rho(v,a) = i\widehat{\iota(v) + a}\hbar \ . \tag{III.16}$$

Exponentiating this Lie algebra representation would give a representation of the Heisenberg group H on the whole space $C^\infty(\mathcal{O},\mathbb{C})$. But as in the finite dimensional case we will first restrict to subrepresentations arising from polarizations of the symplectic manifold $(\mathcal{O},\Omega)$. This is the subject of the last section.

13 Recovering the Generalized Schrödinger Representations

Polarizations of the symplectic manifold $(\mathcal{O},\Omega)$ arising from special Lagrange distributions in $T\mathcal{O}$ will lead to certain subrepresentations of the Lie algebra $\rho : \mathfrak{h} \to \mathrm{End}(C^\infty(\mathcal{O},\mathbb{C}))$ defined in 12.6. With the aid of a concrete diffeomorphism $\mathcal{O} \cong \mathbb{C}^{(\mathbb{N})}$ we will be able to identify these subrepresentations as derivatives of the generalized Schrödinger representations corresponding to the initial choice of the polarization by Lagrange subspaces.

13.1. Polarizations and subrepresentations:

A submanifold $\mathcal{L} \subseteq \mathcal{O}$ will be called a *Lagrange submanifold* if the tangent space $T_{(z^*,\hbar)}\mathcal{L}$ at each point of $\mathcal{L}$ is a Lagrange subspace of $T_{(z^*,\hbar)}\mathcal{O}$.

Especially

$$\mathcal{L} = (u^* + \check{\sigma}(L)) \times \{\hbar\}$$

where L is a Lagrangian subspace of $\mathbb{C}^{(\mathbb{N})}$ can be used to define the subspace

$$V = \{f \in C^\infty(\mathcal{O},\mathbb{C}) \mid \nabla_{\zeta_{(l,0)}}(f) = 0 \quad \forall l \in L\} \subseteq C^\infty(\mathcal{O},\mathbb{C}) \tag{III.17}$$

of "sections" parallel along $\mathcal{L}$. V defines a subrepresentation of ρ since for $f \in V, l \in L, (v,a) \in \mathfrak{h}$:

$$\begin{aligned}
\nabla_{\zeta_{(l,0)}}(\rho(v,a)f) &= \nabla_{\zeta_{(l,0)}}(-\nabla_{\zeta_{(v,a)}}(f) + if(-\iota(v) + a\hbar)) = \\
&= -\nabla_{\zeta_{(l,0)}} \circ \nabla_{\zeta_{(v,a)}}(f) + if\zeta_{(l,0)}(-\iota(v) + a\hbar) + (-\iota(v) + a\hbar)\nabla_{\zeta_{(l,0)}}(f) = \\
&= -[\nabla_{\zeta_{(l,0)}}, \nabla_{\zeta_{(v,a)}}](f) + \nabla_{[\zeta_{(l,0)},\zeta_{(v,a)}]} - if\zeta_{(l,0)}(\iota(v)) = \\
&= -if\Omega(\zeta_{(l,0)}, \zeta_{(v,a)}) - if\langle\iota(v), -\hbar\check{\sigma}(l)\rangle = i\hbar f(-\sigma(l,v) + \sigma(l,v)) = 0\ ,
\end{aligned}$$

so $\rho(v,a)f$ is again in V.

13.2. Lemma: The diffeomorphism $\alpha : \mathbb{C}^{(\mathbb{N})} \to \mathcal{O}$, $x \mapsto (u^* - \hbar\check{\sigma}(x), \hbar)$, induces a linear diffeomorphism $\alpha^* : C^\infty(\mathcal{O},\mathbb{C}) \to C^\infty(\mathbb{C}^{(\mathbb{N})},\mathbb{C})$, $f \mapsto f \circ \alpha$, with the following properties:

(i) $\alpha^*(q_j) = \operatorname{Re} u_j^* + \hbar\check{\sigma}(\delta_j)$, $\alpha^*(p_k) = \operatorname{Im} u_k^* + \hbar\check{\sigma}(i\delta_k)$

(ii) $\alpha^*(\partial_{q_j} f) = \frac{1}{\hbar}\partial_{p_j}\alpha^*(f)$, $\alpha^*(\partial_{p_k} f) = \frac{-1}{\hbar}\partial_{q_k}\alpha^*(f)$

Proof: smoothness in both directions is clear from smoothness of $\check{\sigma}$ and its inverse on $\operatorname{im}\check{\sigma}$; the above equations are obtained by direct computation:

$$\begin{aligned}
\text{(i)}\quad \alpha^*(q_j)(z) &= q_j(u^* - \hbar\check{\sigma}(z)) = \langle u^* - \hbar\check{\sigma}(z), \delta_j\rangle = \\
&= \langle u^*, \delta_j\rangle - \hbar\langle\check{\sigma}(z), \delta_j\rangle = \operatorname{Re} u_j^* - \hbar\sigma(z,\delta_j) = \operatorname{Re} u_j^* + \hbar\langle\check{\sigma}(\delta_j), z\rangle
\end{aligned}$$

and similar for $\alpha^*(p_k)$

$$\text{(ii)}\quad \partial_{p_j}\alpha^*(f) = \partial_{p_j}(f\circ\alpha) = \langle d(f\circ\alpha), i\delta_j\rangle = \langle (df)\circ\alpha, d\alpha(i\delta_j)\rangle = \\ = -\langle (df)\circ\alpha, \hbar\check{\sigma}(i\delta_j)\rangle = \hbar\langle df,\delta_j\rangle\circ\alpha = \hbar\alpha^*(\partial_{q_j}f)$$

and similar for $\partial_{q_k}\alpha^*(f)$. ∎

13.3. Remark:

(i) Note that the diffeomorphism α is suggested by the free action of $H/G_{(u^*,\hbar)}$ on $\mathcal{O}$ since $\alpha(z) = \mathrm{Ad}^*_{(z,0)}(u^*,\hbar)$.

(ii) By construction α respects the symplectic structures on $\mathbb{C}^{(\mathbb{N})}$ and $\mathcal{O}$, i.e. σ-Lagrange subspaces of $\mathbb{C}^{(\mathbb{N})}$ are mapped into Ω-Lagrange submanifolds of $\mathcal{O}$.

13.4. The position-momentum polarization:

Let $\mathcal{L} = (u^* + \check{\sigma}(\mathrm{Re}\,\mathbb{C}^{(\mathbb{N})}))\times\{\hbar\}\subseteq\mathcal{O}$ be the Lagrange submanifold arising from the decomposition $\mathbb{C}^{(\mathbb{N})} = \mathbb{R}^{(\mathbb{N})}\oplus i\mathbb{R}^{(\mathbb{N})}$ into position and momentum subspaces. The representation space $V\subseteq C^\infty(\mathcal{O},\mathbb{C})$ is then characterized by the equations

$$0 = \nabla_{\zeta_{(\delta_j,0)}}(f) = -\hbar\partial_{p_j}f + \frac{i}{2}q_j f \quad j\in\mathbb{N}. \tag{III.18}$$

If we define $q(z^*,\hbar) := (q_j(z^*,\hbar))_{j\in\mathbb{N}}$ and $p(z^*,\hbar) := (p_j(z^*,\hbar))_{j\in\mathbb{N}}$ we can write

$$(z^*,\hbar) = (q(z^*,\hbar) + ip(z^*,\hbar),\hbar)$$

and consider $f(z^*,\hbar)$ as a function of (q,p) (for fixed $\hbar$).

Lemma: The representation space V characterized by equation III.18 is isomorphic to the space $C^\infty(\mathcal{O}_q,\mathbb{C})$ of functions depending only on q, i.e. $\mathcal{O}_q$ is the submanifold of $\mathcal{O}$ given by the equation $p(z^*,\hbar) = 0$. The action of the position and momentum operators on $C^\infty(\mathcal{O}_q,\mathbb{C})$ is given by the following formulas ($g\in C^\infty(\mathcal{O}_q,\mathbb{C})$):

$$\widehat{q_j}(g) = q_j g \qquad \widehat{p_k}(g) = i\hbar\partial_{q_k}g \tag{III.19}$$

Proof: the differential equations III.18 can be written

$$\partial_{p_j} f(q,p) = \frac{i}{2\hbar} q_j f(q,p) \quad (j \in \mathbb{N})$$

and have the unique solution

$$f(q,p) = g(q) e^{\frac{i}{2\hbar}\langle q|p\rangle}$$

where $g(q) = f(q,0) \in C^\infty(\mathcal{O}_q, \mathbb{C})$; conversely each $g \in C^\infty(\mathcal{O}_q, \mathbb{C})$ defines a function $f \in V$ by the above equation.

We compute the actions of $\widehat{q_j}$ and $\widehat{p_k}$ on f according to III.15:

$$\begin{aligned}
\widehat{q_j}(f) &= -i\hbar\partial_{p_j} f + \frac{1}{2} q_j f = \frac{1}{2} q_j f + \frac{1}{2} q_j f = q_j g e^{\frac{i}{2\hbar}\langle q|p\rangle} \\
\widehat{p_k}(f) &= i\hbar\partial_{q_k} f + \frac{1}{2} p_k f = i\hbar\partial_{q_k}(g e^{\frac{i}{2\hbar}\langle q|p\rangle}) + \frac{1}{2} p_k g e^{\frac{i}{2\hbar}\langle q|p\rangle} = \\
&= i\hbar(\partial_{q_k} g e^{\frac{i}{2\hbar}\langle q|p\rangle} + \frac{i}{2\hbar} p_k g e^{\frac{i}{2\hbar}}) + \frac{1}{2} p_k g e^{\frac{i}{2\hbar}\langle q|p\rangle} = \\
&= (i\hbar\partial_{q_k} g) e^{\frac{i}{2\hbar}\langle q|p\rangle}
\end{aligned}$$

hence $\widehat{q_j}$ and $\widehat{p_k}$ can be considered as acting on g. ■

Now exactly as in Remark 12.6 we can define a Lie algebra representation of $\mathfrak{h}$ on $C^\infty(\mathcal{O}_q, \mathbb{C})$ by linear extension. We will again denote this representation by $\rho : \mathfrak{h} \to \mathrm{End}(C^\infty(\mathcal{O}, \mathbb{C}))$. Before integrating this Lie algebra representation we investigate its relation to derivations of previously constructed representations. Fortunately this will release us from working out the exponentiation.

Proposition: The representation $\rho : \mathfrak{h} \to \mathrm{End}(C^\infty(\mathcal{O}, \mathbb{C}))$ is equivalent to the derived Schrödinger representation S' on $C^\infty(\mathrm{Im}\,\mathbb{C}^{(\mathbb{N})}, \mathbb{C})$ described in 7.3 and 7.4. In other words there exists an isomorphism $\Phi : C^\infty(\mathcal{O}_q, \mathbb{C}) \to C^\infty(\mathrm{Im}\,\mathbb{C}^{(\mathbb{N})}, \mathbb{C})$ such that

$$S'(v,a) \circ \Phi = \Phi \circ \rho(v,a) \qquad \forall (v,a) \in \mathfrak{h}. \tag{III.20}$$

Proof: since $\alpha(\mathrm{Im}\,\mathbb{C}^{(\mathbb{N})}) = \mathcal{O}_q$ (with α from Lemma 13.2) we can set $\Phi = (\alpha|_{\mathrm{Im}\,\mathbb{C}^{(\mathbb{N})}})^*$ to get an isomorphism of the involved function spaces; it is sufficient to prove equation III.20 for the generators $(\delta_j, 0), (i\delta_k, 0)$ of $\mathfrak{h}$, i.e. if $Q_j = -iS'(\delta_j, 0)$ and $P_k = -iS'(i\delta_k, 0)$ (cf. 7.4) we have to show

$$Q_j \circ \Phi = \Phi \circ \widehat{q_j} \quad \text{and} \quad P_k \circ \Phi = \Phi \circ \widehat{p_k}\ .$$

Take $g \in C^\infty(\mathcal{O}_q, \mathbb{C})$ and compute (using the two previous Lemmata)

$$\begin{aligned}
\Phi(\widehat{q_j}(g))(y) &= \alpha^*(q_j g)(0+iy) = \alpha^*(q_j)(iy)\alpha^*(g)(iy) = \\
&= (\operatorname{Re} u_j^* + \hbar\sigma(\delta_j, iy))\Phi(g)(y) = (\operatorname{Re} u_j^* + \hbar y^j)\Phi(g)(y) = \\
&= ((\operatorname{Re} u_j^* + \hbar\delta_j^*)\Phi(g))(y) = (Q_j\Phi(g))(y)
\end{aligned}$$

and

$$\begin{aligned}
\Phi(\widehat{p_k}(g)) &= \Phi(i\hbar\partial_{q_k} g) = i\hbar\alpha^*(\partial_{q_k} g) = \\
&= i\partial_{p_k}\alpha^*(g) = i\langle d\Phi(g), \delta_k\rangle = P_k\Phi(g)
\end{aligned}$$

■

13.5. Balanced polarization:

Consider $E_+ = \operatorname{span}\{x + ix \mid x \in \mathbb{R}^{(\mathbb{N})}\} \subseteq \mathbb{C}^{(\mathbb{N})}$ as in 7.5 and the Lagrange submanifold $\mathcal{L} = (u^* + \check{\sigma}(E_+)) \times \{\hbar\}$. Proceeding as in 13.4 we begin with an investigation of the space V characterized by the equations

$$0 = \nabla_{\zeta_{(\delta_j + i\delta_j, 0)}} f = \hbar(\partial_{q_j} - \partial_{p_j})f + \frac{i}{2}(q_j - p_j)f\ . \tag{III.21}$$

Defining $\partial_j^{\pm} = \partial_{q_j} \pm \partial_{p_j}$ and $a_j^{\pm} = q_j \pm p_j$ ("creation and annihilation") we can write this in the form

$$\partial_j^- f = -\frac{i}{2\hbar} a_j^+ f$$

which is uniquely solved by

$$f(z^+, z^-) = g(z^+) e^{-\frac{i}{2\hbar}\langle z^+ | z^- \rangle}$$

with $g(z^+) = f(z^+, 0) \in C^\infty(\mathcal{O}_+, \mathbb{C})$, where we used the decomposition $q + ip = z^+ + z^-$ given in 7.5 and the notation $\mathcal{O}_+$ for the submanifold given by $z^- = 0$.

Lemma: The representation space V can be identified with $C^\infty(\mathcal{O}_+, \mathbb{C})$ and the position and momentum operators become

$$\widehat{q_j} = -\frac{i}{2\hbar}\partial_j^+ + \frac{1}{2}a_j^+ \qquad \widehat{p_k} = \frac{i}{2\hbar}\partial_k^+ + \frac{1}{2}a_k^+ \tag{III.22}$$

Proof: the above solution of the equation III.21 yields the isomorphism; it remains to compute the action of $\widehat{q_j}$ and $\widehat{p_k}$; rewriting the defining equations

III.15 in new coordinates $q_j = (a_j^+ + a_j^-)/2$, $p_j = (a_j^+ - a_j^-)/2$ and differentials $\delta_{q_j} = (\partial_j^+ + \partial_j^-)/2$, $\partial_{p_j} = (\partial_j^+ - \partial_j^-)/2$ yields

$$\begin{array}{rcl} \widehat{q_j} & = & -\frac{i\hbar}{2}(\partial_j^+ - \partial_j^-) + \frac{1}{4}(a_j^+ + a_j^-) \\ \text{and} & & \\ \widehat{p_k} & = & \frac{i\hbar}{2}(\partial_k^+ + \partial_k^-) + \frac{1}{4}(a_k^+ - a_k^-)\ ; \end{array}$$

now applying these to a function $f = ge^{-\frac{i}{2\hbar}\langle a^+|a^-\rangle}$ with $g \in C^\infty(\mathcal{O}_+, \mathbb{C})$ ends up after straightforward calculations in

$$\begin{array}{rcl} \widehat{q_j}(ge^{-\frac{i}{2\hbar}\langle a^+|a^-\rangle}) & = & (-\frac{i\hbar}{2}\partial_j^+ g + \frac{1}{2}a_j^+ g)e^{-\frac{i}{2\hbar}\langle a^+|a^-\rangle} \\ \widehat{p_k}(ge^{-\frac{i}{2\hbar}\langle a^+|a^-\rangle}) & = & (\frac{i\hbar}{2}\partial_k^+ g + \frac{1}{2}a_k^+ g)e^{-\frac{i}{2\hbar}\langle a^+|a^-\rangle} \end{array}$$

■

Once again we actually recovered here a representation which appeared first in section 7.

Proposition: The Lie algebra representation of $\mathfrak{h}$ arising from the operators in equation III.22 on the space $C^\infty(\mathcal{O}_+, \mathbb{C})$ is equivalent to the derived Lie algebra representation T' constructed in 7.5 corresponding to the admissible subalgebra $E_+ \oplus \mathbb{R}$.

Proof: let $\Phi = (\alpha|_{E_-})^* : C^\infty(\mathcal{O}_+, \mathbb{C}) \to C^\infty(E_-, \mathbb{C})$ be the connecting isomorphism; we have to show

$$-iT'(\delta_j, 0) \circ \Phi = \Phi \circ \widehat{q_j}, \quad -iT'(i\delta_k, 0) \circ \Phi = \Phi \circ \widehat{p_k}$$

Take $g \in C^\infty(\mathcal{O}_+, \mathbb{C})$ and go on with

$$\begin{aligned} &\Phi(\widehat{q_j}(g)) = \\ &\quad = \Phi(-\frac{i\hbar}{2}\partial_j^+ g + \frac{1}{2}a_j^+ g) = -\frac{i\hbar}{2}\alpha^*(\partial_{q_j} g + \partial_{p_j} g) + \frac{1}{2}\alpha^*(q_j + p_j)\Phi(g) = \\ &\quad = -\frac{i\hbar}{2}(\frac{1}{\hbar}\partial_{p_j}\Phi(g) - \frac{1}{\hbar}\partial_{q_j}\Phi(g)) + \frac{1}{2}(\mathrm{Re}\ u_j^* + \mathrm{Im}\ u_j^*)\Phi(g) + \frac{\hbar}{2}\check{\sigma}(\delta_j + i\delta_j)\Phi(g) = \\ &\quad = \frac{i}{2}(\partial_{q_j} - \partial_{p_j})\Phi(g) + \frac{1}{2}(\mathrm{Re}\ u_j^* + \mathrm{Im}\ u_j^*)\Phi(g)\frac{\hbar}{2}\check{\sigma}(\delta_j + i\delta_j)\Phi(g) \end{aligned}$$

which gives the correct expression for $-iT'(\delta_j, 0)$ acting on $\Phi(g)$ according to formula II.21. The computation of $\Phi(\widehat{p_k}(g))$ is much the same apart from one minus sign. $\blacksquare$

Bibliography

[B/R] BRATTELI/ROBINSON, Ola and Derek W.:
Operator Algebras and Quantum Statistical Mechanics II,
Springer 1981

[B] BREIN, Laurenz:
Convexity and Unitary Representations of Nilpotent Lie Groups,
Diplomarbeit, University of Vienna 1991

[D] DIEUDONNÉ, Jean:
Eléments d'Analyse II,
Gauthier-Villars 1982

[F] FRIEDRICHS, Kurt O.:
Aspects of the Quantum Theory of Fields,
Interscience Publishers 1953

[F/K] FRÖLICHER/KRIEGL, Alfred and Andreas:
Linear Spaces and Differentiation Theory,
John Wiley 1988

[G/W] GÅRDING/WIGHTMAN, L. and A.:
Representations of the Commutation Relations,
Proc. Nat. Acad. Sci. 40 (1954),622-626

[J] JARCHOW, Hans:
Locally Convex Spaces,
Teubner 1981

[Kir] KIRILLOV, Aleksandr A.:

Unitary Representations of Nilpotent Lie Groups,
Russian Math. Survey 17 (1962),53-104

Elements of the Theory of Representations,
Springer 1976

[Kos] KOSTANT, Bertram:
Quantization and Unitary Representations, Part I,
Springer Lecture Notes in Math. 170 (1970),87-208

[K/M] KRIEGL/MICHOR, Andreas and Peter W.:

Aspects of the Theory of Infinite Dimensional Manifolds,
Differential Geometry and its Appl. 1 (1991),159-176

Foundations of Global Analysis,
in preparation

[M] MICHOR, Peter W.:

Differential Geometry and Lie Groups,
Lecture Notes, University of Vienna 1991/92

The Moment Mapping for Unitary Representations,
Ann. Global Anal. Geom. 8 (1990),299-313

[P] PETZ, Dénes:
An Invitation to the Algebra of Canonical Commutation Relations,
Leuven University Press 1990

[P/S] PRESSLEY/SEGAL, Andrew and Graeme:
Loop Groups,
Oxford University Press 1986

[Sch] SCHAEFER, Helmut H.:
Topological Vector Spaces,
Springer GTM 3, fifth printing 1986

[Seg] SEGAL, Irving E.:

Foundations of the Theory of Dynamical Systems of Infinitely Many Degrees of Freedom,
Part I: Mat. Fys. Medd. Dan. Vid. Selsk. 31 (1959),no. 12
Part II: Canad. J. Math. 13 (1961),1-18
Part III: Illinois Jour. Math. 6 (1962),500-523

Tensor Algebras over Hilbert Spaces I,
Trans. Amer. Math. Soc. 81 (1956),106-134

Distributions in Hilbert Space and Canonical Systems of Operators,
Trans. Amer. Math. Soc. 88 (1958),12-41

[Sha] SHALE, D.:
Linear Symmetries of Free Boson Fields,
Trans. Amer. Math. Soc. 103 (1962),149-167

[Sla] SLAWNY, Joseph:
On Factor Representations and the $C^* -$ Algebra of Canonical Commutation Relations,
Comm. Math. Phys. 24 (1972),151-170

[T] THIRRING, Walter:
Quantum Mechanics of Large Systems,
Springer 1980

[vN] von NEUMANN, John:
Die Eindeutigkeit der Schrödingerschen Operatoren,
Math. Ann. 104 (1931),570-578

[Wik] WIKLICKY, Herbert:
Group Representations, Geometric Quantization and the Momentum Mapping,
Diplomarbeit, University of Vienna 1989

[Woo] WOODHOUSE, Nicholas:
Geometric Quantization,
Oxford University Press 1980

Curriculum Vitae

Ich wurde am 24.August 1967 in Wien geboren. Nach dem Besuch der Volksschule von 1973-77 begann in Wien Floridsdorf, Franklinstraße 21, meine Gymnasialzeit. Die Reifeprüfung legte ich am 17.Juni 1985 "mit ausgezeichnetem Erfolg" ab.

Nachdem das österreichische Bundesheer auf meine Dienste verzichtet hatte, immatrikulierte ich im darauffolgenden Herbst an der Technischen Universität Wien und belegte die Fächer Informations- und Datenverarbeitung (bis 1986) sowie Elektrotechnik (bis 1987 und wieder ab 1992). Im Wintersemester 1986/87 begann ich an der Universität Wien mit dem Studium der Mathematik, das ich mit einer Diplomarbeit am Institut für Theoretische Physik und der Diplomprüfung im Juni 1991 "mit Auszeichnung" abschloß.

Seit 1990 wirke ich im Lehrbetrieb des mathematischen Instituts in Form von Tutoriums- und Lehraufträgen (Übungen und Proseminare aus Analysis, Linearer Algebra, Topologie und Mathematik für Physiker) mit.