

**POISSON STRUCTURES ON THE COTANGENT
BUNDLE OF A LIE GROUP OR A PRINCIPLE
BUNDLE AND THEIR REDUCTIONS**

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INTRODUCTION

The standard description of physical systems, both for particles and for fields, usually starts with an action principle. When particles are thought of as test particles one considers fields as given, i.e. as external fields, and only the point particle dynamics is considered. In this framework the Lagrangian function usually is the sum of three terms: A kinematic term which is quadratic in the velocities, a current-potential coupling term which is linear in the velocities, and a term which depends only on the positions like electrostatic potential.

When one passes to the Hamiltonian description in the symplectic or Poisson formalism one may think of the magnetic field absorbed in a change of coordinates, so p is replaced by $p + eA$, and the Hamiltonian takes into account the electrostatic potential or other effective potentials.

There are situations, however, like in the electric monopole system for instance, where the magnetic field cannot be absorbed in a change of coordinates. This is due to the fact that the associated symplectic structures are in different cohomology classes. From this point of view the Lagrangian formalism seems to be able to take into account a specific identification of physical variables which is not possible in the other descriptions. Nevertheless Poisson brackets seem to be an important

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starting point for various quantization procedures. More recently a special class of Poisson brackets has been considered to represent the classical limits of quantum groups. It seems therefore appropriate to look at Poisson brackets in a more direct way in order to learn how one may be able to ‘add interactions’ directly to the brackets. To this aim we would like to analyze the brackets that are appropriate to describe point particles carrying internal degrees of freedom, isospin-like variables, interacting with external Yang-Mills fields. Eventually one might be able to either deform these brackets or to learn how a given bracket can be read off as arising from reduction of a system described by an action principle. The present paper tries to provide partial answers to our questions while referring to future work for further developments.

The paper is organized as follows. In section 1 we define a Liouville form on a symplectic manifold (E, ω) fibered over a manifold M with Lagrangian fibers as a horizontal 1-form Θ with $d\Theta = \omega$. We denote by $\mathfrak{A}(\Theta)$ the Lie algebra of all projectable vector fields on E which preserve the Liouville form Θ . A right inverse to the projection $\mathfrak{A}(\Theta) \rightarrow \mathfrak{X}(M)$ is called the Liouville lift $\mathfrak{X}(M) \rightarrow \mathfrak{A}(\Theta)$. It assigns to a vector field $X \in \mathfrak{X}(M)$ the Hamiltonian vector field X^Θ on E for the function $-\Theta(X \circ p)$, where $p : E \rightarrow M$ is the projection. Using the Liouville lift we obtain the explicit expression for the Poisson structure $\Lambda = \omega^{-1}$. We also sketch a reverse of this construction.

In section 2 we apply these simple construction to the cotangent bundle $\pi : T^*G \rightarrow G$ of a Lie group G and we describe explicitly the standard symplectic form ω on T^*G as follows:

$$\omega = d\Theta = \frac{1}{2} (\langle d\zeta^l, \pi^* \kappa^r \rangle_{\mathfrak{g}}^\wedge + \langle d\zeta^r, \pi^* \kappa^l \rangle_{\mathfrak{g}}^\wedge),$$

where κ^l, κ^r are the left and right Maurer-Cartan forms on G , and where ζ^l, ζ^r are the momenta of the left and right action of G onto itself. The associated Poisson structure is given by

$$\Lambda = \frac{1}{2} \sum_i (R_i^* \wedge Z_i^r + L_i^* \wedge Z_i^l),$$

where L_i^* and R_i^* are the flow lifts to T^*G of the left and right invariant vector fields on G corresponding to the basis vectors X_i of the Lie algebra \mathfrak{g} , and where the vertical vector fields Z_j^l, Z_j^r are defined to be ω -dual to the 1-forms $-\pi^* \kappa_j^l, -\pi^* \kappa_j^r$ which are pullbacks of the components of the Maurer-Cartan forms: $\kappa^l = \sum \kappa_j^l \otimes X_j$ and $\kappa^r = \sum \kappa_j^r \otimes X_j$.

Some generalization of the above construction is presented in section 3. The starting point is that instead of the standard momentum mapping we consider an arbitrary smooth mapping $f : T^*G \rightarrow \mathfrak{g}^*$ to define a generalized Liouville form

$$\Theta_f = \langle f, \pi^* \kappa^l \rangle \in \Omega^1(T^*G)$$

and a derived closed 2-form $\omega_f = d\Theta_f$. In general the 2-form ω_f is degenerate and we need a reduction in order to obtain a symplectic form. We consider some examples. In particular, for a compact Lie group G , starting with a reduced momentum mapping we come after reduction to a symplectic form and associated

Poisson structure Λ on the manifold $G \times C$, where C is an open Weyl-chamber in the dual to a Cartan subalgebra. This Poisson structure was also described in [1].

In the last section we generalize the construction of section 3 to the case of a principal bundle $p : P \rightarrow M$ with structure group G . We choose a principal connection with connection form $\gamma : TP \rightarrow \mathfrak{g} = \text{Lie}(G)$ and note that its pullback to the cotangent bundle $\pi_P : T^*P \rightarrow P$ gives a \mathfrak{g} -valued G -equivariant π_P -horizontal 1-form $\pi^*\gamma : T(T^*P) \rightarrow \mathfrak{g}$. We define a generalized momentum mapping as an arbitrary G -equivariant \mathfrak{g}^* -valued function $f : T^*P \rightarrow \mathfrak{g}^*$ and consider the generalized Liouville form

$$\Theta_f = \langle f, \pi^*\gamma \rangle = \sum f_j \cdot \gamma^j \in \Omega^1(T^*P).$$

The corresponding 2-form $\omega_f = d\Theta_f$ on T^*P is degenerate in general. However, we can try to reduce it to a symplectic form on the reduced manifold $P/\ker \omega_f$. As a simple example we consider the form Θ_f associated to the standard momentum mapping $f : T^*P \rightarrow \mathfrak{g}^*$ associated to the Hamiltonian right action of G on T^*P . We call it the vertical Liouville form. We also calculate $d\Theta_f$ in coordinates. Using local trivializations of the bundle $p : P \rightarrow M$ we consider a Liouville form Θ_γ which is the sum of the vertical one and a horizontal one which is the pullback of the standard one on T^*M . The associated 2-form $\omega_\gamma = d\Theta_\gamma$ is non degenerate and we calculate the inverse Poisson structure Λ_γ in coordinates. Since Λ_γ is G -invariant we can factorize it and as a result we obtain a Poisson structure $\tilde{\Lambda}_\gamma$ on the orbit space T^*P/G , which is degenerate and does not come from a symplectic form. The Poisson bracket associated to $\tilde{\Lambda}_\gamma$ was considered in [9].

Then we consider the case of a trivial principal bundle $P = \mathbb{R}^n \times G$ with a compact structure group G and take as a momentum mapping f the projection of the canonical momentum mapping onto the dual \mathfrak{h}^* of a Cartan subalgebra \mathfrak{h} of $\mathfrak{g} = \text{Lie}(G)$. We show that in this case our construction leads to a Poisson structure on $T^*\mathbb{R}^n \times G \times \mathfrak{h}^*$ which has singularities on the walls of the Weyl chambers in \mathfrak{h}^* .

We consider also the case when a principal bundle $p : P \rightarrow M$ is equipped with a displacement form (soldering form) $\theta : TP \rightarrow \mathbb{R}^n$ and describes a G -structure on M . In this case any G -equivariant function $f : T^*P \rightarrow \mathbb{R}^n \times \mathfrak{g}^*$ defines a G -invariant generalized Liouville form on T^*P .

1. LIOUVILLE 1-FORMS ON FIBER BUNDLES AND THE LIFTING OF VECTOR FIELDS

1.1. Liouville forms on fiber bundles. Let $p : E \rightarrow M$ be a locally trivial smooth fiber bundle, and let $\omega \in \Omega^2(E)$ be a symplectic form on E . A 1-form $\Theta \in \Omega^1(E)$ is called a *Liouville form* if:

- (1) It is horizontal: $i_Y \Theta = 0$ for Y in the vertical bundle $VE := \ker(Tp : TE \rightarrow TM)$.
- (2) $\omega = d\Theta$.
- (3) The fibers of $p : E \rightarrow M$ are Lagrangian submanifolds.

Lemma. *In this situation, if a Liouville form Θ exists, (E, Θ) is locally fiber respecting diffeomorphic to (T^*M, Θ_M) , where Θ_M is the canonical Liouville form on T^*M .*

Proof. For a point $u \in E$, choose local coordinates q^i near $p(u) \in M$. Then since Θ is horizontal, near u we have $\Theta = \sum_i p_i dq^i$ for local smooth functions p_i on E . Since $\omega = d\Theta = \sum_i dp_i \wedge dq^i$ is symplectic, and since the fibers of $p : E \rightarrow M$ are Lagrangian submanifolds, $dq^1, \dots, dq^n, dp_1, \dots, dp_n$ is a local frame for T^*E , so $q^1, \dots, q^n, p_1, \dots, p_n$ is a coordinate system near u on E . \square

1.2. Liouville lift of a vector field. Let $p : E \rightarrow M$ be a fiber bundle with a symplectic form $\omega \in \Omega^2(E)$ and a Liouville form $\Theta \in \Omega^1(E)$. Since $\Theta : TE \rightarrow \mathbb{R}$ is horizontal it factors to a form on the quotient bundle $TE/VE \cong p^*TM \rightarrow \mathbb{R}$, and for each vector field $X \in \mathfrak{X}(M)$ on M we may consider $\Theta(X \circ p)$ as a function on E .

A vector field $X^\Theta \in \mathfrak{X}(E)$ is called the *Liouville lift* of a vector field $X \in \mathfrak{X}(M)$ if X^Θ is projectable onto X and preserves Θ , that is $Tp \circ X^\Theta = X \circ p$ and $\mathcal{L}_{X^\Theta} \Theta = 0$. But then

$$0 = \mathcal{L}_{X^\Theta} \Theta = i_{X^\Theta} \omega + di_{X^\Theta} \Theta = i_{X^\Theta} \omega + d(\Theta(X \circ p)).$$

Hence X^Θ is the Hamiltonian vector field $-H_{\Theta(X \circ p)}$ for the function $-\Theta(X \circ p) \in C^\infty(E, \mathbb{R})$.

Using the fact from lemma 1.1 that (E, Θ) is locally diffeomorphic to (T^*M, Θ_M) one can check easily that $X^\Theta = H_{-\Theta(X \circ p)}$ is indeed projectable onto $X \in \mathfrak{X}(M)$. It is also well known for cotangent bundles that the mapping $\mathfrak{X}(M) \rightarrow C^\infty(T^*M, \mathbb{R})$ given by $X \mapsto -\Theta(X)$ is an injective homomorphism of Lie algebras from the Lie algebra of vector fields to the Lie algebra of functions with the standard Poisson bracket, so this holds also for the general situation.

1.3. A standard frame of E . Let us assume now that the manifold M is parallelizable and that X_1, \dots, X_n form a global frame field for TM . Let us denote by $\alpha_1, \dots, \alpha_n \in \Omega^1(M)$ the dual coframe field. Then we consider the frame field

$$X_1^\Theta, \dots, X_n^\Theta, Z_1, \dots, Z_n,$$

where X_i^Θ is the Liouville lift from 1.2 and where

$$(1) \quad Z_j := \omega^{-1} p^* \alpha_j, \quad \text{or equivalently } i_{Z_j} \omega = p^* \alpha_j$$

is the vertical vector field dual to the 1-form $p^* \alpha_i \in \Omega^1(E)$. From lemma 1.1 and a computation in T^*M it follows that $[Z_i, Z_j] = 0$.

Then the Liouville form Θ and the symplectic form $\omega = d\Theta$ may be written as

$$(2) \quad \begin{aligned} \Theta &= \sum_i f_i p^* \alpha_i \quad \text{for } f_i \in C^\infty(E, \mathbb{R}) \\ \omega = d\Theta &= \sum_i df_i \wedge p^* \alpha_i + \sum_i f_i p^* d\alpha_i \\ &= \sum_i df_i \wedge p^* \alpha_i + \sum_{i,j,k} f_i c_{jk}^i p^* \alpha_j \wedge p^* \alpha_k, \end{aligned}$$

where the *torsion functions* $c_{jk}^i \in C^\infty(M, \mathbb{R})$ are given by $[X_j, X_k] = \sum_i c_{jk}^i X_i$. From the definition of X_i^Θ we have

$$i_{X_j^\Theta} \omega = -d\Theta(X_j \circ p) = -df_j,$$

thus the Poisson structure associated to ω is given by

$$(2) \quad \Lambda = \omega^{-1} = \sum_i X_j^\Theta \wedge Z_i + \sum_{i,j,k} f_i c_{jk}^i Z_j \wedge Z_k \in C^\infty(\Lambda^2 TE)$$

1.4. The reversed construction. Let $p : E \rightarrow M$ be a fiber bundle whose fiber dimension equals the dimension of the base space M . Let M be parallelizable and let

$$\alpha = (\alpha_1, \dots, \alpha_n) : TM \rightarrow \mathbb{R}^n =: V$$

is a coframe field for M . If we are given a V^* -valued function $f : E \rightarrow V^*$, we may consider the 1-form

$$\Theta_f := \langle f, p^* \alpha \rangle \in \Omega^1(E).$$

Proposition. *The 1-form Θ_f defines a symplectic structure $\omega_f = d\Theta_f$ on E if and only if $f|_{E_x} : E_x \rightarrow V^*$ is a local diffeomorphism for each fiber E_x , $x \in M$. In this case, Θ_f is a Liouville form for (E, ω_f) . The associated Poisson bivector field $\Lambda_f = \omega_f^{-1}$ is given by formula 1.3.(2), \square*

2. THE CANONICAL POISSON STRUCTURE ON T^*G

2.1 Products of differential forms. Let $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ be a representation of a Lie algebra \mathfrak{g} in a finite dimensional vector space V and let M be a smooth manifold.

For vector valued differential forms $\varphi \in \Omega^p(M; \mathfrak{g})$ and $\Psi \in \Omega^q(M; V)$ we define the form $\rho^\wedge(\varphi)\Psi \in \Omega^{p+q}(M; V)$ by

$$\begin{aligned} (\rho^\wedge(\varphi)\Psi)(X_1, \dots, X_{p+q}) &= \\ &= \frac{1}{p!q!} \sum_\sigma \text{sign}(\sigma) \rho(\varphi(X_{\sigma_1}, \dots, X_{\sigma_p})) \Psi(X_{\sigma_{p+1}}, \dots, X_{\sigma_{p+q}}). \end{aligned}$$

Then $\rho^\wedge(\varphi) : \Omega^*(M; \mathfrak{g}) \rightarrow \Omega^{*+p}(M; V)$ is a graded $\Omega(M)$ -module homomorphism of degree p .

Recall also that $\Omega(M; \mathfrak{g})$ is a graded Lie algebra with the bracket $[\ , \]^\wedge = [\ , \]_{\mathfrak{g}}^\wedge$ given by

$$\begin{aligned} [\varphi, \psi]^\wedge(X_1, \dots, X_{p+q}) &= \\ &= \frac{1}{p!q!} \sum_\sigma \text{sign}\sigma [\varphi(X_{\sigma_1}, \dots, X_{\sigma_p}), \psi(X_{\sigma_{p+1}}, \dots, X_{\sigma_{p+q}})]_{\mathfrak{g}}, \end{aligned}$$

where $[\ , \]_{\mathfrak{g}}$ is the bracket in \mathfrak{g} . One may easily check that for the graded commutator in $\text{End}(\Omega(M; V))$ we have

$$\rho^\wedge([\varphi, \psi]^\wedge) = [\rho^\wedge(\varphi), \rho^\wedge(\psi)] = \rho^\wedge(\varphi) \circ \rho^\wedge(\psi) - (-1)^{pq} \rho^\wedge(\psi) \circ \rho^\wedge(\varphi)$$

so that $\rho^\wedge : \Omega^*(M; \mathfrak{g}) \rightarrow \text{End}^*(\Omega(M; V))$ is a homomorphism of graded Lie algebras.

For any vector space V let $\otimes V$ be the tensor algebra generated by V . For $\Phi, \Psi \in \Omega(M; \otimes V)$ we will use the associative bigraded product

$$\begin{aligned} (\Phi \otimes_{\wedge} \Psi)(X_1, \dots, X_{p+q}) &= \\ &= \frac{1}{p!q!} \sum_{\sigma} \text{sign}(\sigma) \Phi(X_{\sigma_1}, \dots, X_{\sigma_p}) \otimes \Psi(X_{\sigma_{p+1}}, \dots, X_{\sigma_{p+q}}) \end{aligned}$$

In the same spirit we will use the following product: Let V be a finite dimensional vector space with dual V^* , and let $\langle \cdot, \cdot \rangle : V^* \times V \rightarrow \mathbb{R}$ be the duality pairing. For $\Phi \in \Omega^p(M; V^*)$ and $\Psi \in \Omega^q(M; V)$ we consider the ‘product’ $\langle \Phi, \Psi \rangle^{\wedge} \in \Omega(M)$ which is given by

$$\begin{aligned} \langle \Phi, \Psi \rangle^{\wedge}(X_1, \dots, X_{p+q}) &= \\ &= \frac{1}{p!q!} \sum_{\sigma} \text{sign}(\sigma) \langle \Phi(X_{\sigma_1}, \dots, X_{\sigma_p}), \Psi(X_{\sigma_{p+1}}, \dots, X_{\sigma_{p+q}}) \rangle. \end{aligned}$$

2.2. Notation for Lie groups. Let G be a Lie group with Lie algebra $\mathfrak{g} = T_e G$, multiplication $\mu : G \times G \rightarrow G$, and for $g \in G$ let $\mu_g, \mu^g : G \rightarrow G$ denote the left and right translation, $\mu(g, h) = g.h = \mu_g(h) = \mu^h(g)$.

Let $L, R : \mathfrak{g} \rightarrow \mathfrak{X}(G)$ be the left and right invariant vector field mappings, given by $L_X(g) = T_e(\mu_g).X$ and $R_X = T_e(\mu^g).X$, respectively. They are related by $L_X(g) = R_{\text{Ad}(g)X}(g)$. Their flows are given by

$$\text{Fl}_t^{L_X}(g) = g.\exp(tX) = \mu^{\exp(tX)}(g), \quad \text{Fl}_t^{R_X}(g) = \exp(tX).g = \mu_{\exp(tX)}(g).$$

Let $\kappa^l, \kappa^r : \Omega^1(G, \mathfrak{g})$ be the left and right Maurer-Cartan forms, given by $\kappa_g^l(\xi) = T_g(\mu_{g^{-1}}).\xi$ and $\kappa_g^r(\xi) = T_g(\mu^{g^{-1}}).\xi$, respectively. These are the inverses to L, R in the following sense: $L_g^{-1} = \kappa_g^l : T_g G \rightarrow \mathfrak{g}$ and $R_g^{-1} = \kappa_g^r : T_g G \rightarrow \mathfrak{g}$. They are related by $\kappa_g^r = \text{Ad}(g)\kappa_g^l : T_g G \rightarrow \mathfrak{g}$ and they satisfy the Maurer-Cartan equations $d\kappa^l + \frac{1}{2}[\kappa^l, \kappa^l]^{\wedge} = 0$ and $d\kappa^r - \frac{1}{2}[\kappa^r, \kappa^r]^{\wedge} = 0$.

The (exterior) derivative of the function $\text{Ad} : G \rightarrow GL(\mathfrak{g})$ can be expressed by

$$d\text{Ad} = \text{Ad}.(\text{ad} \circ \kappa^l) = (\text{ad} \circ \kappa^r). \text{Ad},$$

which follows from $d\text{Ad}(T\mu_g.X) = \frac{d}{dt}|_0 \text{Ad}(g.\exp(tX)) = \text{Ad}(g).\text{ad}(\kappa^l(T\mu_g.X))$.

We also consider the left moment mapping $\zeta^l : T^*G \rightarrow \mathfrak{g}^*$ which is related to the left action of G on T^*G by left translations, given by $\zeta_g^l = ((\kappa_g^r)^{-1})^*$. Note here the transition from left to right: right invariant vector fields generate left translations. Similarly we consider the right moment mapping $\zeta^r : T^*G \rightarrow \mathfrak{g}^*$ which is related to the right action of G on T^*G by right translations, given by $\zeta_g^r = ((\kappa_g^l)^{-1})^*$. They are related by $\zeta_g^l = \text{Ad}(g^{-1})^* \zeta_g^r = \text{Ad}^*(g).\zeta_g^r : T_g^*G \rightarrow \mathfrak{g}^*$.

2.3. The canonical symplectic structure on T^*G . We consider now the tangent bundle $\pi_G : TG \rightarrow G$ and the cotangent bundle $\pi = \pi_G : T^*G \rightarrow G$ of the Lie group G . We use π for the projection of each bundle which is derived

from the tangent bundle in a direct way. On T^*G we consider the Liouville form $\Theta : T(T^*G) \rightarrow \mathbb{R}$, $\Theta \in \Omega^1(T^*G)$, which is given by

$$(1) \quad \Theta(\Xi) = \langle \pi_{T^*G}(\Xi), T(\pi_G) \cdot \Xi \rangle_{TG},$$

where we use the following commutative diagram

$$\begin{array}{ccc} TT^*G & \xrightarrow{T(\pi_G)} & TG \\ \pi_{T^*G} \downarrow & & \downarrow \pi_G \\ T^*G & \xrightarrow{\pi} & G. \end{array}$$

Considering the momentum mapping $\zeta^l : T^*G \rightarrow \mathfrak{g}^*$ as a function and the pullback $\pi_{T^*G}^* \kappa^r = \pi^* \kappa^r \in \Omega^1(T^*G, \mathfrak{g})$ of the right Maurer-Cartan form, we have for $\Xi \in TT^*G$ with 'lowest footprint' $g \in G$:

$$\begin{aligned} \langle \zeta^l, \pi^* \kappa^r \rangle_{\mathfrak{g}}(\Xi) &= \langle ((\kappa_g^r)^{-1})^* (\pi_{T^*G}(\Xi)), \kappa_g^r(T(\pi_G) \cdot \Xi) \rangle_{\mathfrak{g}} \\ &= \langle \pi_{T^*G}(\Xi), (\kappa_g^r)^{-1} \kappa_g^r(T(\pi_G) \cdot \Xi) \rangle_{\mathfrak{g}} \\ &= \langle \pi_{T^*G}(\Xi), T(\pi_G) \cdot \Xi \rangle_{\mathfrak{g}} = \Theta(\Xi). \end{aligned}$$

Similarly we have $\Theta = \langle \zeta^r, \pi^* \kappa^l \rangle_{\mathfrak{g}}$ and thus also

$$(2) \quad \Theta = \frac{1}{2} (\langle \zeta^l, \pi^* \kappa^r \rangle_{\mathfrak{g}} + \langle \zeta^r, \pi^* \kappa^l \rangle_{\mathfrak{g}}).$$

Let us now compute the exterior derivative of the one form $\Theta = \langle \zeta^l, \pi^* \kappa^r \rangle_{\mathfrak{g}}$ in order to get an expression for the symplectic structure $\omega \in \Omega^2(T^*G)$:

$$\begin{aligned} \omega = d\Theta &= d(\langle \zeta^l, \pi^* \kappa^r \rangle_{\mathfrak{g}}) = \langle d\zeta^l, \pi^* \kappa^r \rangle_{\mathfrak{g}}^{\wedge} + \langle \zeta^l, \pi^* d\kappa^r \rangle_{\mathfrak{g}} \\ &= \langle d\zeta^l, \pi^* \kappa^r \rangle_{\mathfrak{g}}^{\wedge} - \frac{1}{2} \langle \zeta^l, \pi^* [\kappa^r, \kappa^r]_{\mathfrak{g}}^{\wedge} \rangle_{\mathfrak{g}}. \end{aligned}$$

by the Maurer-Cartan equation. Similarly we get

$$\omega = d\Theta = d(\langle \zeta^r, \pi^* \kappa^l \rangle_{\mathfrak{g}}) = \langle d\zeta^r, \pi^* \kappa^l \rangle_{\mathfrak{g}}^{\wedge} + \frac{1}{2} \langle \zeta^r, \pi^* [\kappa^l, \kappa^l]_{\mathfrak{g}}^{\wedge} \rangle_{\mathfrak{g}}.$$

For $\Xi_i \in T_{\xi}(T^*G)$ with $\pi(\xi) = g \in G$ we have

$$\begin{aligned} \langle \zeta^r, \pi^* [\kappa^l, \kappa^l]_{\mathfrak{g}}^{\wedge} \rangle_{\mathfrak{g}}(\Xi_1, \Xi_2) &= \langle \zeta^r(\xi), [\pi^* \kappa^l(\Xi_1), \pi^* \kappa^l(\Xi_2)]_{\mathfrak{g}} \rangle_{\mathfrak{g}} \\ &= \langle \text{Ad}(g^{-1})^* \zeta^r(\xi), \text{Ad}(g)[\pi^* \kappa^l(\Xi_1), \pi^* \kappa^l(\Xi_2)]_{\mathfrak{g}} \rangle_{\mathfrak{g}} \\ &= \langle \text{Ad}^*(g) \zeta^r(\xi), [\text{Ad}(g) \cdot \kappa^l(T\pi \cdot \Xi_1), \text{Ad}(g) \cdot \kappa^l(T\pi \cdot \Xi_2)]_{\mathfrak{g}} \rangle_{\mathfrak{g}} \\ &= \langle \zeta^l(\xi), [\kappa^r(T\pi \cdot \Xi_1), \kappa^r(T\pi \cdot \Xi_2)]_{\mathfrak{g}} \rangle_{\mathfrak{g}} \\ &= \langle \zeta^l, \pi^* [\kappa^r, \kappa^r]_{\mathfrak{g}} \rangle_{\mathfrak{g}}(\Xi_1, \Xi_2). \end{aligned}$$

From (2) we get the following final formula for the canonical symplectic structure ω on T^*G :

$$(3) \quad \omega = d\Theta = \frac{1}{2} (\langle d\zeta^l, \pi^* \kappa^r \rangle_{\mathfrak{g}}^{\wedge} + \langle d\zeta^r, \pi^* \kappa^l \rangle_{\mathfrak{g}}^{\wedge}).$$

2.4. The canonical symplectic structure on T^*G in coordinates. We fix now a basis X_1, \dots, X_n of the Lie algebra \mathfrak{g} , with dual basis ξ_1, \dots, ξ_n in \mathfrak{g}^* , and with structure constants $[X_i, X_j] = \sum_k c_{ij}^k X_k$. We may expand the Maurer-Cartan forms and the moment mappings in terms of these bases, as

$$\begin{aligned}\kappa^l &= \sum_i \kappa_i^l \cdot X_i, & \kappa_i^l &:= \langle \xi_i, \kappa^l \rangle \in \Omega^1(G), \\ \kappa^r &= \sum_i \kappa_i^r \cdot X_i, & \kappa_i^r &:= \langle \xi_i, \kappa^r \rangle \in \Omega^1(G), \\ \zeta^l &= \sum_i \zeta_i^l \cdot \xi_i, & \zeta_i^l &:= \langle \zeta^l, X_i \rangle \in C^\infty(T^*G, \mathbb{R}), \\ \zeta^r &= \sum_i \zeta_i^r \cdot \xi_i, & \zeta_i^r &:= \langle \zeta^r, X_i \rangle \in C^\infty(T^*G, \mathbb{R}).\end{aligned}$$

The Maurer-Cartan equations then become

$$\begin{aligned}d\kappa_i^l + \frac{1}{2} \sum_{jk} c_{jk}^i \kappa_j^l \wedge \kappa_k^l &= 0, \\ d\kappa_i^r - \frac{1}{2} \sum_{jk} c_{jk}^i \kappa_j^r \wedge \kappa_k^r &= 0.\end{aligned}$$

The Liouville form then becomes

$$\Theta = \sum_i \zeta_i^l \cdot \pi^* \kappa_i^r = \sum_i \zeta_i^r \cdot \pi^* \kappa_i^l = \frac{1}{2} \sum_i (\zeta_i^l \cdot \pi^* \kappa_i^r + \zeta_i^r \cdot \pi^* \kappa_i^l)$$

and the symplectic form looks like

$$\begin{aligned}\omega &= \sum_i \left(d\zeta_i^l \wedge \pi^* \kappa_i^r + \frac{1}{2} \zeta_i^l \sum_{jk} c_{jk}^i \pi^* \kappa_j^r \wedge \pi^* \kappa_k^r \right) \\ &= \sum_i \left(d\zeta_i^r \wedge \pi^* \kappa_i^l - \frac{1}{2} \zeta_i^r \sum_{jk} c_{jk}^i \pi^* \kappa_j^l \wedge \pi^* \kappa_k^l \right) \\ &= \frac{1}{2} \sum_i (d\zeta_i^l \wedge \pi^* \kappa_i^r + d\zeta_i^r \wedge \pi^* \kappa_i^l).\end{aligned}$$

Next we consider for the basis vectors X_i of \mathfrak{g} the left invariant vector fields $L_i := L_{X_i}$ and the right invariant vector fields $R_i = R_{X_i}$ on G and their ‘flow lifts’ to T^*G which are given by $L_i^* := \frac{\partial}{\partial t} \Big|_0 T^*(\text{Fl}_t^{L_i})$ and by $R_i^* := \frac{\partial}{\partial t} \Big|_0 T^*(\text{Fl}_t^{R_i})$. Then $L_i^* \in \mathfrak{X}(T^*G)$ is π -related to $L_i \in \mathfrak{X}(G)$, similarly $T(\pi) \circ R_i^* = R_i \circ \pi : T^*G \rightarrow TG$. Thus

$$\begin{aligned}\pi^* \kappa_i^l(L_j^*) &= \kappa_i^l(T(\pi) \cdot L_j^*) = \kappa_i^l(L_j) \circ \pi = \delta_{ij}, \\ \pi^* \kappa_i^r(R_j^*) &= \delta_{ij}.\end{aligned}$$

By general principles their flows preserve the Liouville form Θ ,

$$0 = \mathcal{L}_{L_j^*} \Theta = i_{L_j^*} d\Theta + di_{L_j^*} \Theta = i_{L_j^*} d\Theta + d\zeta_j^r,$$

so we conclude that L_j^* is the hamiltonian vector field for the generating function ζ_j^r . Similarly $-i_{R_j^*} \omega = d\zeta_j^l$.

We consider the vector fields $Z_j^l, Z_j^r \in \mathfrak{X}(T^*G)$ which are given by

$$-i_{Z_j^l} \omega = \pi^* \kappa_j^l, \quad -i_{Z_j^r} \omega = \pi^* \kappa_j^r.$$

The fields Z_j^l, Z_j^r are vertical, i.e. in the kernel of $T(\pi)$. Then the Poisson structure $\Lambda = \omega^{-1}$ is given by

$$\Lambda = \frac{1}{2} \sum_i (R_i^* \wedge Z_i^r + L_i^* \wedge Z_i^l)$$

2.5. The tangent group of a Lie group. As motivation for the following we recall that for a Lie group G the tangent group TG is also a Lie group with multiplication $T\mu$ and inversion $T\nu$, the tangent mapping of the inversion ν on G , given by $T_{(a,b)}\mu \cdot (\xi_a, \eta_b) = T_a(\mu^b) \cdot \xi_a + T_b(\mu_a) \cdot \eta_b$ and $T_a\nu \cdot \xi_a = -T_e(\mu_{a^{-1}}) \cdot T_a(\mu^{a^{-1}}) \cdot \xi_a$.

Lemma. *In the right trivialization, i.e. via the isomorphism $(\kappa^r, \pi) : TG \rightarrow \mathfrak{g} \times G$, the group structure on TG looks as follows:*

$$(X, a) \cdot (Y, b) = (X + \text{Ad}(a)Y, a \cdot b), \quad (X, a)^{-1} = (-\text{Ad}(a^{-1})X, a^{-1}).$$

In the left trivialization, i.e. via the isomorphism $(\pi, \kappa^l) : TG \rightarrow G \times \mathfrak{g}$, the group structure on TG looks as follows:

$$(a, X) \cdot (b, Y) = (ab, \text{Ad}(b^{-1})X + Y), \quad (a, X)^{-1} = (a^{-1}, -\text{Ad}(a)X). \quad \square$$

2.6. The Lie group T^*G as semidirect product. The Lie group G acts on the dual of its Lie algebra by the coadjoint representation $\text{Ad}^*(g) = \text{Ad}(g^{-1})^*$. So we can consider on $\mathfrak{g}^* \times G$ the structure of a Lie group given by the semidirect product

$$(a, \xi) \cdot (b, \eta) = (ab, \text{Ad}^*(b^{-1})\xi + \eta), \quad (a, \xi)^{-1} = (a^{-1}, -\text{Ad}^*(a)\xi).$$

We will consider on T^*G the Lie group structure induced from this semidirect product by the diffeomorphism $(\zeta^l, \pi) : T^*G \rightarrow \mathfrak{g}^* \times G$, whose inverse is given by $(\zeta^l, \pi)^{-1}(\alpha, g) = (T_g(\mu^{g^{-1}}))^* \alpha = T^*(\mu^g)\alpha$. This Lie group structure on T^*G has no obvious intrinsic meaning, but it has nice relations to some structures which are naturally given on T^*G . One of them is conjugation by elements of G . For $g \in G$ we consider $\text{Conj}_g : G \rightarrow G$, given by $\text{Conj}_g(a) = g \cdot a \cdot g^{-1}$, and the induced mapping $T^* \text{Conj}_g = (T \text{Conj}_{g^{-1}})^* : T^*G \rightarrow T^*G$. Then we have

$$\begin{aligned} (\zeta^l, \pi) T^*(\text{Conj}_g)(\xi_a) &= ((T\mu^{gag^{-1}})^* (T\mu^g \cdot T\mu_{g^{-1}}))^* \cdot \xi_a, g \cdot a \cdot g^{-1} \\ &= (\text{Ad}(g^{-1})^* (T\mu^a)^* \cdot \xi_a, g \cdot a \cdot g^{-1}) = (\text{Ad}^*(g)\zeta^l(\xi_a), g \cdot a \cdot g^{-1}) \\ &= (0, g) \cdot (\zeta^l(\xi_a), a) \cdot (0, g)^{-1}. \end{aligned}$$

Obviously $\zeta^l \circ (\zeta^l, \pi)^{-1} = \text{pr}_1 : \mathfrak{g}^* \times G \rightarrow \mathfrak{g}^*$, and from

$$\zeta^r | T_g^* G = \text{Ad}^*(g^{-1}) \cdot \zeta^l | T_g^* G = \text{Ad}(g)^* \cdot \zeta^l | T_g^* G$$

we get that

$$\zeta_i^r (\zeta^l, \pi)^{-1}(\alpha, g) = \langle \alpha, \text{Ad}(g) \cdot X_i \rangle.$$

Since $\pi \circ (\zeta^l, \pi)^{-1} = \text{pr}_2 : \mathfrak{g}^* \times G \rightarrow G$ we get for both Maurer-Cartan forms κ^l, κ^r that

$$\pi^* \kappa^{l,r} \circ T(\zeta^l, \pi)^{-1} = \kappa^{l,r} \circ T\pi \circ T(\zeta^l, \pi)^{-1} = \kappa^{l,r} \circ T\text{pr}_2.$$

So using 2.3, we get for the symplectic form

$$\begin{aligned} \omega^r &:= ((\zeta^l, \pi)^{-1})^* \omega = d((\zeta^l, \pi)^{-1})^* \langle \zeta^l, \pi^* \kappa^r \rangle = d\langle \text{pr}_1, \text{pr}_2^* \kappa^r \rangle \\ &= \langle d\text{pr}_1, \text{pr}_2^* \kappa^r \rangle + \langle \text{pr}_1, \text{pr}_2^* d\kappa^r \rangle \\ (1) \quad &= \langle d\text{pr}_1, \text{pr}_2^* \kappa^r \rangle + \langle \text{pr}_1, \frac{1}{2} \text{pr}_2^* [\kappa^r, \kappa^r]_{\mathfrak{g}} \rangle. \end{aligned}$$

The other expressions of 2.2 for the Liouville form Θ quickly lead again to (1).

We look for an explicit expression of the Poisson structure $\Lambda^r := ((\zeta^l, \pi)^{-1})^* \Lambda \in \Gamma(\Lambda^2 T^*(\mathfrak{g}^* \times G))$. For that we fix a basis X_1, \dots, X_n of the Lie algebra \mathfrak{g} with structure constants $[X_i, X_j] = \sum_k c_{ij}^k X_k$ as in 2.4. We consider the dual basis ξ_1, \dots, ξ_n in \mathfrak{g}^* with coordinate functions $x^1, \dots, x^n : \mathfrak{g}^* \rightarrow \mathbb{R}$, so that $Id_{\mathfrak{g}^*} = \sum x^i \xi_i$. In these coordinates the symplectic structure is given by

$$(2) \quad \omega^r = ((\zeta^l, \pi)^{-1})^* \omega = \sum_i dx^i \wedge \kappa_i^r + \frac{1}{2} \sum_{ijk} x^i c_{jk}^i \kappa_j^r \wedge \kappa_k^r$$

By computing $\frac{\partial}{\partial t} \Big|_0 (\zeta^l, \pi) T^*(\text{Fl}_t^{L_i})(\xi_g)$ one easily checks that the vector field $L_i^* \in \mathfrak{X}(T^*G)$ is (ζ^l, π) -related to $0 \times L_i \in \mathfrak{X}(\mathfrak{g}^* \times G)$; for the prolongations of the right invariant vector fields one obtains that R_i^* is (ζ^l, π) -related to $\text{ad}(-X_i)^* \times R_i = \text{ad}^*(X_i) \times R_i \in \mathfrak{X}(\mathfrak{g}^* \times G)$. Note that $\text{ad}(X_i)^*(\alpha) = \alpha \circ \text{ad}(X_i) = \sum_j x^j(\alpha) \cdot (\xi_j \circ \text{ad}(X_i)) = \sum_{jk} x^j(\alpha) \cdot c_{ik}^j \xi_k$, so that the vector field $\text{ad}(X_i)^* \in \mathfrak{X}(\mathfrak{g}^*)$ is given by $\sum_{jk} x^j c_{ik}^j \frac{\partial}{\partial x^k}$. Thus we get $i(\text{ad}(-X_i)^* \times R_i) \omega^r = -dx^i$ and $i(-\frac{\partial}{\partial x^i} \times 0) \omega^r = -\kappa_i^r$. The Poisson structure is given by

$$\begin{aligned} \Lambda^r &= ((\zeta^l, \pi)^{-1})^* \Lambda \\ &= \sum_i \text{ad}(X_i)^* \wedge \frac{\partial}{\partial x^i} - \sum_i R_i \wedge \frac{\partial}{\partial x^i} + \frac{1}{2} \sum_{ijk} x^i c_{jk}^i \frac{\partial}{\partial x^j} \wedge \frac{\partial}{\partial x^k} \\ &= \sum_{ijk} x^j c_{ik}^j \frac{\partial}{\partial x^k} \wedge \frac{\partial}{\partial x^i} - \sum_i R_i \wedge \frac{\partial}{\partial x^i} + \frac{1}{2} \sum_{ijk} x^i c_{jk}^i \frac{\partial}{\partial x^j} \wedge \frac{\partial}{\partial x^k} \\ (3) \quad &= - \sum_i R_i \wedge \frac{\partial}{\partial x^i} - \frac{1}{2} \sum_{ijk} x^i c_{jk}^i \frac{\partial}{\partial x^j} \wedge \frac{\partial}{\partial x^k} \end{aligned}$$

The Poisson bracket of two functions $f, g \in C^\infty(\mathfrak{g}^* \times G)$ is then given by

$$(4) \quad -\{f, g\} = \sum_i (0 \times R_i)(f) \cdot \frac{\partial g}{\partial x^i} - \sum_i (0 \times R_i)(g) \cdot \frac{\partial f}{\partial x^i} + \sum_{ijk} x^i c_{jk}^i \frac{\partial f}{\partial x^j} \frac{\partial g}{\partial x^k}$$

3. GENERALIZING MOMENTUM MAPPINGS

3.1. A more general construction. For a Lie group G with Lie algebra \mathfrak{g}^* we generalize the formulae obtained in section 2. Instead of the left or right momentum let us start with an arbitrary smooth mapping $f : T^*G \rightarrow \mathfrak{g}^*$ and in view of 2.3 let us consider the 1-form

$$\Theta_f = \langle f, \pi^* \kappa^l \rangle \in \Omega^1(T^*G).$$

The 2-form $\omega_f = d\Theta_f$ is closed but it may be degenerate. We try reduction by the kernel $\ker \omega_f$ of ω_f in order to get the symplectic form $\tilde{\omega}_f$ on $T^*G/\ker \omega_f$. This quotient space is a smooth manifold only if we restrict ourselves to such open subsets of T^*G on which $\ker \omega_f$ is a smooth distribution of constant rank.

The most important case is when f is invariant with respect to the left action of G on T^*G . Then f may be viewed as a function in the coordinates ζ_i^r of the right momentum ζ^r , so $f = f(\zeta_1^r, \dots, \zeta_n^r)$, and as in 2.4 we get

$$\begin{aligned} f &= \sum_{i=1}^n f_i \cdot \xi_i, \\ \Theta_f &= \sum_i f_i \cdot \pi^* \kappa_i^l = \sum_i \tilde{f}_i \cdot \pi^* \kappa_i^r, \\ \omega_f &= \sum_i \left(df_i \wedge \pi^* \kappa_i^l + \frac{1}{2} \sum_{j,k} c_{jk}^i f_i \cdot \kappa_j^l \wedge \kappa_k^l \right), \\ -i_{R_j^*} \omega_f &= d\tilde{f}_j. \end{aligned}$$

The kernel of ω_f depends on the df_i 's and also on the structure constants, and it is difficult to describe in this generality.

3.2. Coadjoint orbits. We show here that each orbit of the coadjoint action with its symplectic structure can be obtained by the method in 3.1. For $\xi \in \mathfrak{g}^*$ let G_ξ be the isotropy group with Lie algebra $\mathfrak{g}_\xi \subset \mathfrak{g}$. We let $f = \xi$, the constant mapping $T^*G \rightarrow \mathfrak{g}^*$. Then

$$\begin{aligned} \Theta_\xi &= \langle \xi, \pi^* \kappa^l \rangle = \pi^* \langle \xi, \kappa^l \rangle = \pi^* \kappa_\xi^l, \\ \omega_\xi &= \pi^* \langle \xi, d\kappa^l \rangle = -\frac{1}{2} \langle \xi, [\pi^* \kappa^l, \pi^* \kappa^l]^\wedge \rangle \\ &= \frac{1}{2} \langle \text{ad}(\pi^* \kappa^l)^* \xi, \pi^* \kappa^l \rangle, \end{aligned}$$

so the kernel of ω_ξ is spanned by the vertical vector fields and by the L_X^* for $X \in \mathfrak{g}_\xi$. Thus $T^*G/\ker \omega_\xi = G/G_\xi$, and the last formula for ω_ξ shows that the induced symplectic structure on G/G_ξ corresponds to the canonical symplectic structure on the coadjoint orbit through ξ .

3.3. Example. The reduction from 3.2 is used in mechanics with respect to gauge invariant Lagrangians. Consider for instance the Lagrangian of the free spinning particle [3], [5]

$$L = \frac{1}{2} \sum_{k=1}^3 m \dot{q}_k^2 + \lambda \text{tr}(\sigma_3 \cdot g^{-1} \cdot \dot{g})$$

where σ_3 is the Pauli matrix and $g \in SU(2)$, so that the configuration space is $\mathbb{R}^3 \times SU(2)$.

The 1-form Θ_L of this Lagrangian can be written in the form

$$\Theta_L = \sum_{k=1}^3 p_k dq_k + \lambda \operatorname{tr}(\sigma_3 \cdot g^{-1} \cdot dg),$$

where $p_k = m\dot{q}_k$ and where $SU(2)$ is just the matrix group. Using the standard basis $X_k = \frac{1}{2}i\sigma_k$ in $\mathfrak{su}(2)$ and the dual basis ξ_k we may write in our previous notation

$$\begin{aligned} \Theta_L &= \sum_{k=1}^3 p_k dq_k + \lambda \pi^* \kappa_3^l, \\ \omega_L = d\Theta_L &= \sum_{k=1}^3 dp_k \wedge dq_k + \lambda \pi^* (\kappa_1^l \wedge \kappa_2^l), \end{aligned}$$

and the kernel of ω_L is spanned by the vertical vector fields on $T^*SU(2)$ and by L_3^* . The reduced space is therefore

$$T^*\mathbb{R}^3 \times SU(2)/SU(1) \cong T^*\mathbb{R}^3 \times S^2,$$

where $SU(1)$ is the subgroup in $SU(2)$ generated by L_3 . This is exactly the phase space for the free spinning particle.

3.4. Reduction to a Cartan subalgebra. Let now G be a compact Lie group with Lie algebra \mathfrak{g} , and choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Let Δ be the system of roots, choose a positive root system Δ_+ and denote by $\Pi = \{\alpha_1, \dots, \alpha_k\}$ the associated simple roots. Let

$$\{E_\alpha, E_{-\alpha} : \alpha \in \Delta_+\} \cup \{H_1, \dots, H_k\}$$

be the corresponding Cartan basis in \mathfrak{g} and let

$$\{\xi_\alpha, \xi_{-\alpha} : \alpha \in \Delta_+\} \cup \{\xi_1, \dots, \xi_k\}$$

be the dual basis in \mathfrak{g}^* . Let us expand the right momentum in it,

$$\zeta^r = \sum_{j=1}^k \zeta_j^r \xi_j + \sum_{\alpha \in \Delta} \zeta_\alpha^r \xi_\alpha,$$

and let us consider

$$f := \sum_{j=1}^k \zeta_j^r \xi_j.$$

The corresponding 2-form ω_f is then given by

$$\begin{aligned}\omega_f &= \sum_{j=1}^k d\zeta_j^r \wedge \pi^* \kappa_j^l + \sum_{j=1}^k \zeta_j^r \wedge \pi^* d\kappa_j^l \\ &= \sum_{j=1}^k d\zeta_j^r \wedge \pi^* \kappa_j^l - \frac{1}{2} \sum_{\alpha \in \Delta_+} \frac{2B(f, \alpha)}{B(\alpha, \alpha)} \pi^* \kappa_\alpha^l \wedge \pi^* \kappa_{-\alpha}^l,\end{aligned}$$

where $B(\cdot, \cdot)$ is the Cartan Killing form on \mathfrak{g}^* . Since $\bigcap_{j=1}^k \ker d\zeta_j^r \subset \ker \omega_f$ we can reduce ω_f to a 2-form $\tilde{\omega}_f$ on the respective quotient manifold which can be identified with $G \times \mathfrak{h}^*$:

$$\tilde{\omega}_f = \sum_{j=1}^k d\zeta_j^r \wedge \kappa_j^l - \sum_{\alpha \in \Delta_+} \frac{B(f, \alpha)}{B(\alpha, \alpha)} \kappa_\alpha^l \wedge \kappa_{-\alpha}^l,$$

where ζ_j^r is regarded as a coordinate just in \mathfrak{h}^* . The 2-form $\tilde{\omega}_f$ is now non-degenerate and hence symplectic, on $G \times C$ for each open Weyl-chamber C in \mathfrak{h}^* . We also have the following correspondence (by dual bases)

$$L_j \leftrightarrow d\zeta_j^r, \quad L_\alpha \leftrightarrow \frac{B(f, \alpha)}{B(\alpha, \alpha)} \kappa_{-\alpha}^l \text{ for } \alpha \in \Delta, \quad Z_j^r \leftrightarrow -\kappa_j^l,$$

so the associated Poisson structure may be written as

$$\Lambda = \sum_{j=1}^k Z_j^r \wedge L_j - \sum_{\alpha \in \Delta_+} \frac{B(\alpha, \alpha)}{B(f, \alpha)} L_\alpha \wedge L_{-\alpha}.$$

For $G = SU(2)$ for example, we get the Poisson structure $\Lambda = \frac{\partial}{\partial p} \wedge L_3 - \frac{1}{p} L_1 \wedge L_2$ on $\mathbb{R}^* \times SU(2)$, where $L_j = i\sigma_j$, $j = 1, \dots, 3$ is the standard basis of $\mathfrak{su}(2)$.

We observe finally, that by fixing a value ξ of f we obtain a submanifold $\Sigma \subset G \times \mathfrak{h}^*$ on which $\tilde{\omega}_f$ has kernel generated by all left invariant vector fields on G corresponding to the isotropy Lie algebra \mathfrak{g}_ξ , and the reduction gives again the canonical symplectic structure on the orbit G/G_ξ .

A similar formula as the last one is contained in [1], without geometric interpretation, in the context of quantum groups.

4. SYMPLECTIC STRUCTURES ON COTANGENT BUNDLES OF PRINCIPAL BUNDLES

4.1. Symplectic forms on T^*P . Let $p : P \rightarrow M^n$ be a principal G -bundle, i.e. there is a free right action $r : P \times G \rightarrow P$ of G on P and M is the orbit space, and we suppose that p is a locally trivial fiber bundle. For $X \in \mathfrak{g}$, the Lie algebra of G , we denote by $\zeta_X \in \mathfrak{X}(P)$ the fundamental vector field of the principal right action, $\zeta_X(u) = \left. \frac{\partial}{\partial t} \right|_0 u \cdot \exp(tX)$. Then $\zeta : \mathfrak{g} \rightarrow \mathfrak{X}(P)$ is an injective homomorphism of Lie algebras and it gives us an isomorphism $\zeta_u : \mathfrak{g} \rightarrow V_u P$ onto the vertical space $V_u P := \ker(T_u p : T_u P \rightarrow T_{p(u)} M) = T_u(u \cdot G)$ for any $u \in P$. The inverse isomorphisms $\kappa_u^v = \zeta_u^{-1} : V_u P \rightarrow \mathfrak{g}$ form a mapping $\kappa^v : VP \rightarrow \mathfrak{g}$ which we call the

vertical parallelism. Note for further use that from $T(r^g) \cdot \zeta_X(u) = \zeta_{\text{Ad}(g^{-1})X}(u \cdot g)$ we have

$$(0) \quad \text{Ad}(g) \circ \kappa_{u \cdot g}^v = \kappa_u^v \circ T(r^{g^{-1}}).$$

Let now $\gamma \in \Omega^1(P; \mathfrak{g})$ be a principal connection in P , i.e. a G -equivariant \mathfrak{g} -valued 1-form on P that prolongs the vertical parallelism κ^v . So we have:

- (1) $(r^g)^* \gamma = \text{Ad}(g^{-1}) \cdot \gamma$ for each $g \in G$, where $r^g : P \rightarrow P$ is the right action by g .
- (2) $\gamma|_{VP} = \kappa^v$.

If X_1, \dots, X_k is a basis of the Lie algebra \mathfrak{g} , we can expand γ by

$$(3) \quad \gamma = \sum_{j=1}^k \gamma^j \cdot X_j.$$

Let $\pi = \pi_P : T^*P \rightarrow P$ be the canonical projection and consider the pullback $\pi^* \gamma \in \Omega^1(T^*P; \mathfrak{g})$. Since $\pi : T^*P \rightarrow P$ is G -equivariant, the pullback $\pi^* \gamma$ is a G -equivariant \mathfrak{g} -valued 1-form on T^*P , so that $(T^*r^g)^* \pi^* \gamma = ((Tr^{g^{-1}})^*)^* \pi^* \gamma = \pi^*(r^g)^* \gamma = \text{Ad}(g^{-1}) \pi^* \gamma$.

For any smooth mapping $f : T^*P \rightarrow \mathfrak{g}^*$, in coordinates $f = \sum_{j=1}^k f_j \cdot \xi_j$, where ξ_1, \dots, ξ_k is the basis of \mathfrak{g}^* dual to (X_i) , we may then consider the 1-form

$$(4) \quad \Theta_f = \Theta_{f, \gamma} = \langle f, \pi^* \gamma \rangle_{\mathfrak{g}} = \sum f_j \cdot \gamma^j \in \Omega^1(T^*P).$$

We consider now Θ_f as a generalized Liouville form on T^*P , corresponding to the moment mapping $f : T^*P \rightarrow \mathfrak{g}^*$, and we consider the 2-form

$$(5) \quad \begin{aligned} \omega_f &:= d\Theta_f = \langle df, \pi^* \gamma \rangle^\wedge + \langle f, \pi^* d\gamma \rangle \\ &= \langle df, \pi^* \gamma \rangle^\wedge + \langle f, \pi^*(\Omega - \frac{1}{2}[\gamma, \gamma]^\wedge) \rangle \\ &= \sum_{j=1}^k df_j \wedge \pi^* \gamma^j + \sum_{j=1}^k f_j \cdot \pi^* \Omega^j - \frac{1}{2} \sum_{j,k,l} c_{kl}^j \cdot f_j \cdot \pi^* \gamma^k \wedge \gamma^l, \end{aligned}$$

where $\Omega = d\gamma + \frac{1}{2}[\gamma, \gamma]^\wedge = \sum_j \Omega^j X_j$ is the curvature form of the principal connection γ . This 2-form is closed but no longer non degenerate and we try to reduce it to a symplectic form on $P/\ker \omega_f$. This cannot be done in general and we will discuss now some particular choices of f .

4.3. The canonical momentum. There is a canonical choice of the momentum $f : T^*P \rightarrow \mathfrak{g}^*$. Namely, the action of G on T^*P is Hamiltonian with respect to the canonical symplectic structure $\omega_P = d\Theta_P$ on T^*P . The associated canonical momentum mapping is given by

$$\begin{aligned} f_{\text{can}} &: T^*P \rightarrow \mathfrak{g}^*, \\ \langle f_{\text{can}}(\varphi), X \rangle &= \langle \varphi, \zeta_X \rangle, \quad \varphi \in T^*P, X \in \mathfrak{g}, \end{aligned}$$

and the associated mapping $(\kappa^v)_{f_{\text{can}}}^* : T^*P \rightarrow V^*P$ is the fiberwise adjoint of the inclusion $VP \rightarrow TP$. So this moment mapping is invariant under all gauge transformations, by 4.2. For the associated Liouville form and its derivative we have

$$\Theta^v = \Theta_{f_{\text{can}}} = \langle f_{\text{can}}, \pi^* \gamma \rangle,$$

which we will call the *vertical Liouville form*.

4.4. The canonical momentum in coordinates. We continue our investigation in a principal bundle chart now, so we assume that $P = \mathbb{R}^n \times G$ and $T^*P = T^*\mathbb{R}^n \times T^*G$. We will use coordinates q_i in \mathbb{R}^n and (q_i, p_i) in $T^*\mathbb{R}^n$. We use again a basis X_i of the Lie algebra \mathfrak{g} with dual basis ξ_i in \mathfrak{g}^* . Then $f_{\text{can}} : T^*P \rightarrow \mathfrak{g}^*$ is of the form $f_{\text{can}}(\alpha', \alpha'') = \zeta^r(\alpha'')$, where $\alpha' \in T^*\mathbb{R}^n$, $\alpha'' \in T^*G$, and where $\zeta^r : T^*G \rightarrow \mathfrak{g}^*$ is the right momentum mapping from 2.2.

The principal connection $\gamma \in \Omega^1(P; \mathfrak{g})$ with coordinate expression $\gamma = \sum_j \gamma^j X_j$ may then be written in terms of the vector potential A as

$$\begin{aligned} \gamma_{(q,g)} &= \kappa_g^l + \text{Ad}(g^{-1})A_q, \\ A &= \sum_j A^j X_j = \sum_{i,j} A_i^j dq_i \otimes X_j \in \Omega^1(\mathbb{R}^n; \mathfrak{g}). \end{aligned}$$

The curvature form Ω of γ is then expressed in terms of the curvature F by

$$\begin{aligned} \Omega &= d\gamma + \frac{1}{2}[\gamma, \gamma]^\wedge, & \Omega_{(q,g)} &= \text{Ad}(g^{-1})F_q, \\ F &= dA + \frac{1}{2}[A, A]^\wedge = \sum_i F^i X_i = \sum_{i,j,k} F_{jk}^i dq_j \wedge dq_k \otimes X_i \in \Omega^2(\mathbb{R}^n, \mathfrak{g}). \end{aligned}$$

From 4.3 we have for the vertical Liouville form

$$\begin{aligned} \Theta^v &= \langle f_{\text{can}}, \pi^* \gamma \rangle = \sum_i \zeta_i^r \cdot \pi^* \gamma^i, \\ \Theta_{(\alpha'_q, \alpha''_g)}^v &= \langle \zeta^r(\alpha''_g), \kappa_g^l T\pi_G + \text{Ad}(g^{-1})A_q T\pi_{\mathbb{R}^n} \rangle \\ &= \langle \text{Ad}^*(g)\zeta^r(\alpha''_g), \text{Ad}(g)\kappa_g^l T\pi_G + A_q T\pi_{\mathbb{R}^n} \rangle \\ &= \langle \zeta^l(\alpha''_g), \kappa_g^r T\pi_G + A_q T\pi_{\mathbb{R}^n} \rangle, \\ (1) \quad \Theta^v &= \Theta_{T^*G} + \langle \zeta^l, \pi^* A \rangle, \end{aligned}$$

since we have $\kappa_g^l = \text{Ad}(g^{-1})\kappa_g^r$ and $\text{Ad}(g^{-1})\zeta^r = \zeta^l$, see 2.2. The 1-form Θ^v in (1) was considered in [3], [8], [10], and [11]. Let us call $\eta = \pi^* \kappa^r + \pi^* A = \sum_j \eta_j \otimes X_j \in \Omega^1(T^*P; \mathfrak{g})$, then using this trick once more we get the exterior derivative of Θ^v as

$$\begin{aligned} (2) \quad \omega^v &= d\Theta^v = d\langle \zeta^r, \pi^* \gamma \rangle \\ &= \langle d\zeta^r, \pi^* \gamma \rangle^\wedge - \langle \zeta^r, \pi^* [\gamma, \gamma]^\wedge \rangle + \langle \zeta^r, \pi^* \Omega \rangle \\ &= \langle d\zeta^r, \pi^* \gamma \rangle^\wedge - \langle \zeta^l, [\eta, \eta]^\wedge \rangle + \langle \zeta^l, \pi^* F \rangle \\ &= \sum_i d\zeta_i^r \wedge \pi^* \gamma^i - \frac{1}{2} \sum_{ijk} \zeta_i^l c_{jk}^i \eta_j \wedge \eta_k + \sum_i \zeta_i^l F_{jk}^i dq_j \wedge dq_k. \end{aligned}$$

Equation (1) shows that Θ^v is invariant with respect to the right action of G on T^*P induced from the principal right action on T^*P . For the infinitesimal generators L_j^* of this action (see 2.4) we have then for the Lie derivative

$$(3) \quad \mathcal{L}_{L_j^*} \Theta^v = 0.$$

We add now to the vertical Liouville form a horizontal one, for which we choose the pullback of the canonical Liouville form $\Theta_{\mathbb{R}^n} \in \Omega^1(T^*\mathbb{R}^n)$ (We could choose a more general 1-form here):

$$\Theta_\gamma = \Theta^v + \text{pr}^* \Theta_{\mathbb{R}^n} = \Theta^v + \sum_i p_i dq_i, \quad \omega_\gamma = d\Theta_\gamma,$$

where (q_i, p_j) are the standard coordinates on $T^*\mathbb{R}^n$. This depends on the choice of the trivialization since we have no canonical projection $T^*P \rightarrow T^*M$. The 2-form ω_γ is a symplectic form on $T^*P \cong T^*\mathbb{R}^n \times T^*G$, and since $\mathcal{L}_{L_j^*} \Theta_\gamma = 0$, we get $-i_{L_j^*} \omega_\gamma = d\zeta_j^r$. So the isomorphism between 1-forms and vector fields on T^*P induced by ω_γ gives us the correspondence $L_j^* \leftrightarrow d\zeta_j^r$. Similarly as in 2.4 we find vector fields $\partial_{\zeta_j^r} \in \mathfrak{X}(T^*P)$ such that $\partial_{\zeta_j^r} \leftrightarrow -\pi^* \gamma^j$. Equation (1) shows that we can find $\partial_{\zeta_j^l} \leftrightarrow -\eta_j$. It is easy to see that $\partial_{p_j} \leftrightarrow -dq_j$. From equation (2) and the fact that $\langle \partial_{q_j}, \eta_k \rangle = \langle \partial_{q_j}, A^k \rangle = A_j^k$ we see that

$$\partial_{q_j} \longleftrightarrow \sum_i b_j^i d\zeta_i^r - \sum_{i,k,s} \zeta_i^l c_{ks}^i A_j^s \eta_k + 2 \sum_{i,k} \zeta_i^l F_{kj}^i dq_k + dp_j,$$

where $b_j^i = \langle \gamma_i, \partial_{q_j} \rangle$. So if we put

$$\widetilde{\partial}_{q_j} := \partial_{q_j} - \sum_i b_j^i L_i^* - \sum_{i,k,s} \zeta_i^l c_{ks}^i A_j^s \partial_{\zeta_k^l} + 2 \sum_{i,k} \zeta_i^l F_{kj}^i \partial_{p_k},$$

we get $\widetilde{\partial}_{q_j} \leftrightarrow dp_j$. Therefore we can write the Poisson structure Λ_γ corresponding to ω_γ in the form

$$\begin{aligned} (4) \quad \Lambda_\gamma &= \sum_i \partial_{\zeta_i^r} \wedge L_i^* - \frac{1}{2} \sum_{i,j,k} \zeta_i^l c_{jk}^i \partial_{\zeta_j^l} \wedge \partial_{\zeta_k^l} + \sum_{i,j,k} \zeta_i^l F_{jk}^i \partial_{p_j} \wedge \partial_{p_k} \\ &\quad + \sum_j \partial_{p_j} \wedge \left(\partial_{q_j} - \sum_i b_j^i L_i^* - \sum_{i,k,s} \zeta_i^l c_{ks}^i A_j^s \partial_{\zeta_k^l} + 2 \sum_{i,k} \zeta_i^l F_{kj}^i \partial_{p_k} \right) \\ &= \sum_i \partial_{\zeta_i^r} \wedge L_i^* - \sum_{i,j} b_j^i \partial_{p_j} \wedge L_i^* - \frac{1}{2} \sum_{i,j,k} \zeta_i^l c_{jk}^i \partial_{\zeta_j^l} \wedge \partial_{\zeta_k^l} \\ &\quad - \sum_{i,j,k} \zeta_i^l F_{jk}^i \partial_{p_j} \wedge \partial_{p_k} - \sum_{i,j,k,s} \zeta_i^l c_{ks}^i A_j^s \partial_{p_j} \wedge \partial_{\zeta_k^l} + \sum_j \partial_{p_j} \wedge \partial_{q_j}. \end{aligned}$$

Since Λ_γ is invariant with respect to the G -action we can reduce Λ_γ to $\widetilde{\Lambda}_\gamma$ on $T^*P/G \cong T^*\mathbb{R}^n \times (T^*G/G) \cong T^*\mathbb{R}^n \times \mathfrak{g}^*$ by putting $L_j^* = 0$ and considering $I_k := \zeta_k^l$ as coordinate in \mathfrak{g}^* :

$$\begin{aligned} (5) \quad \widetilde{\Lambda}_\gamma &= -\frac{1}{2} \sum_{i,j,k} I_i c_{jk}^i \partial_{I_j} \wedge \partial_{I_k} - \sum_{i,j,k} I_i F_{jk}^i \partial_{p_j} \wedge \partial_{p_k} \\ &\quad - \sum_{i,j,k,s} I_i c_{ks}^i A_j^s \partial_{p_j} \wedge \partial_{I_k} + \sum_j \partial_{p_j} \wedge \partial_{q_j}. \end{aligned}$$

It should be noticed that this bivector field is degenerate, therefore it does not have a symplectic description on T^*P/G . In this respect our result is quite different from those available in the literature. Our description deals with several particles instead of a given one with fixed isospin or color. That latter is obtained by fixing a value for the Casimir functions of our brackets. In the generic case we find a symplectic level set diffeomorphic to $T^*\mathbb{R}^n \times \text{Ad}^*(G)c$, where $\text{Ad}^*(G)c$ is the coadjoint orbit through c in \mathfrak{g}^* . The Poisson structure $\tilde{\Lambda}_\gamma$ depends on the choice of the connection γ and its curvature Ω , so in the local trivialization on the vector potential A and its curvature F . If for example instead of equation (1) we consider $\Theta^v = \Theta_{T^*G} + \lambda\langle \zeta^l, \pi^*A \rangle$ (where λ could be absorbed into the choice of A , so it is not more general) then in the expression (4) for Λ_γ the constant λ would appear as a coupling constant. The Poisson bracket associated to (5) appears already in [9].

Let us find the Hamiltonian vector field Γ corresponding to the free Hamiltonian

$$H = \frac{1}{2} \sum_j p_j^2.$$

It turns out to be

$$\Gamma = 2 \sum_{i,k,j} I_i F_{kj}^i p_j \partial_{p_k} - \sum_{i,j,k,s} I_i c_{ks}^i A_j^s p_j \partial_{I_k} + \sum_j p_j \partial_{q_j},$$

which describes the motion of a Yang-Mills particle which carries a ‘charge’ given by the spin-like variable I_k . In the free case, $A = 0$, we would have

$$\begin{aligned} \tilde{\Lambda}_\gamma &= -\frac{1}{2} \sum_{i,j,k} I_i c_{jk}^i \partial_{I_j} \wedge \partial_{I_k} + \sum_j \partial_{p_j} \wedge \partial_{q_j}, \\ H &= \frac{1}{2} \sum_j p_j^2. \end{aligned}$$

It is interesting to notice that the presence of the vector potential changes the Poisson bracket without changing the Hamiltonian.

When G is the group $U(1)$ the Yang-Mills field (curvature F) reduces to electromagnetism and the vector field Γ contains the standard Lorentz force expression:

$$\Gamma = 2 \sum_{k,j} e F_{kj} p_j \partial_{p_k} + \sum_j p_j \partial_{q_j},$$

where e is the electric charge and the equation of motion in the internal variables reduces to $\frac{\partial e}{\partial t} = 0$.

4.2. Behavior under gauge transformations. Let $\text{Gau}(P)$ denote the group of all gauge transformations of the principal bundle $p : P \rightarrow M$. So elements of $\text{Gau}(P)$ are diffeomorphisms $\varphi : P \rightarrow P$ which respect fibers ($p \circ \varphi = p$) and which commute with the principal right action r of G on P .

If $\gamma \in \Omega^1(P; \mathfrak{g})$ is a principal connection in P , let $\zeta_\gamma : TP \rightarrow VP$ be the G -equivariant projection onto the vertical bundle induced by γ , i.e. $\zeta_\gamma(\xi_u) = \zeta_{\gamma(\xi_u)}(u)$,

where $\xi_u \in T_u P$ and $\zeta : \mathfrak{g} \rightarrow \mathfrak{X}(P)$ is the fundamental vector field mapping of the principal right action. A gauge transformation φ act naturally on a connection γ :

$$(1) \quad \varphi^*(\zeta_\gamma) := T\varphi^{-1} \circ \zeta_\gamma \circ T\varphi = \zeta_{\varphi^*\gamma} = \zeta_{\gamma \circ T\varphi},$$

where the second equation follows from properties of ζ . We may describe a gauge transformation $\varphi \in \text{Gau}(P)$ either as a section $s_\varphi \in \Gamma(P \times_G G)$ of the associated bundle $P \times_G G$, where the structure group G acts on the fiber G by conjugation, or equivalently as a G -equivariant mapping $f_\varphi \in C^\infty(P, (G, \text{Conj}))^G$. Then $\varphi(u) = u \cdot f_\varphi(u)$, and the isomorphism $\Gamma(P[G, \text{Conj}]) \cong C^\infty(P, (G, \text{Conj}))^G$ is standard, see e.g. [4], 10.14. Then we get for the action of φ on a connection γ

$$(2) \quad \varphi^*\gamma = \gamma \circ T\varphi = \text{Ad}(f_\varphi^{-1})\gamma + f_\varphi^*\kappa^l,$$

where $f_\varphi^*\kappa^l$ is the the pullback of the left Maurer-Cartan form on G . For the curvature forms one has

$$\Omega_{\varphi^*\gamma} = \Omega_{\text{Ad}(f_\varphi^{-1})\gamma + f_\varphi^*\kappa^l} = \varphi^*\Omega_\gamma = \text{Ad}(f_\varphi^{-1})\Omega_\gamma.$$

For a smooth mapping $f : T^*P \rightarrow \mathfrak{g}^*$ the action of a gauge transformation $\varphi \in \text{Gau}(P)$ on the 1-form $\Theta_{f,\gamma} = \langle f, \pi^*\gamma \rangle$ from 4.1, (4) turns out as

$$(3) \quad (T^*\varphi)^*\Theta_{f,\gamma} = \langle f \circ T^*\varphi, \pi^*(\text{Ad}(f_\varphi^{-1})\gamma + f_\varphi^*\kappa^l) \rangle = \Theta_{f \circ T^*\varphi, \varphi^*\gamma}.$$

Using the vertical parallelism κ^l we can associate to $f : T^*P \rightarrow \mathfrak{g}^*$ the fiber respecting smooth mapping $(\kappa^v)_f^* : T^*P \rightarrow V^*P$ which is given by $((\kappa^v)_f^*)|_{T_u^*P} = (\kappa_u^v)^* \circ (f|_{T_u^*P}) : T_u^*P \rightarrow \mathfrak{g}^* \rightarrow V_u^*P$. Note that $(\kappa^v)_f^*$ is G -equivariant, i.e. $(k^v)_f^* \circ (T^*r^g) = (T^*r^g) \circ (k^v)_f^*$ for all $g \in G$, if and only if $f \circ (T^*r^g) = \text{Ad}^*(g^{-1}) \circ f = \text{Ad}(g)^* \circ f$. We can then express $\Theta_{f,\gamma}$ also by

$$(4) \quad \Theta_{f,\gamma} = \langle f, \pi^*\gamma \rangle_{\mathfrak{g}} = \langle (\kappa^v)_f^*, \zeta_\gamma \circ T\pi \rangle_{VP}$$

from which it follows very easily that $\Theta_{f,\gamma}$ is G -invariant if and only if the mapping $(\kappa^v)_f^* : T^*P \rightarrow V^*P$ is G -equivariant, i.e. $f \circ (T^*r^g) = \text{Ad}^*(g^{-1}) \circ f = \text{Ad}(g)^* \circ f$.

Also from $\zeta_X \circ \varphi = T\varphi \circ \zeta_X$ it follows in turn that

$$\begin{aligned} (\kappa_{\varphi(u)}^v)^* &= T^*\varphi \circ (\kappa_u^v)^* \\ (T^*\varphi)^*((\kappa^v)_f^*) &= T^*\varphi^{-1} \circ (\kappa^v)_f^* \circ T^*\varphi = (\kappa^v)_{f \circ T^*\varphi}^*, \end{aligned}$$

so that again all possible actions of gauge transformations on moment mappings $f : T^*P \rightarrow \mathfrak{g}^*$ coincide. Note that the canonical momentum f from 4.3 is invariant under all gauge transformations.

Let us continue now in a local trivialization as in 4.4, so that we assume $P = \mathbb{R}^n \times G$. In this case a gauge transformation $\varphi \in \text{Gau}(\mathbb{R}^n \times G)$ is given by $\varphi(q, g) = (q, s_\varphi(q) \cdot g)$ for $s = s_\varphi \in C^\infty(\mathbb{R}^n, G)$. If a connection γ is given in terms of a vector

potential A by $\gamma_{(q,g)} = \kappa_g^l + \text{Ad}(g^{-1})A_q$ as in 4.4, then the action of the gauge transformation φ on γ is given by

$$(\varphi^*\gamma)_{(q,g)} = \kappa_g^l + \text{Ad}(g^{-1})(s^*\kappa^l + \text{Ad}(s(q)^{-1})A_q),$$

so that s acts on the vector potential and the curvature by

$$\begin{aligned} A &\mapsto s^*\kappa^l + \text{Ad}(s^{-1})A \\ F &\mapsto \text{Ad}(s^{-1})F \end{aligned}$$

Now it remains the task to write down the behavior of the main expression in 4.4 under gauge transformations. For that we have to use $\text{Ad}(s^{-1}).X_i = \sum_j s_i^j X_j$ and to use a matrix representation of s . Note that this is not the usual way to write gauge transformations in a linear group $G \subset GL(N, \mathbb{R})$. So the resulting formulas will look quite unfamiliar. But using matrix representations of all objects would hide the intrinsic symmetry of our approach and will lead to a sea of indices.

4.5. Further reduction to a Cartan subalgebra. Let us consider again the trivial principal bundle $P = \mathbb{R}^n \times G$ with compact structure group G , and choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. In the notation of 3.4, instead of the momentum $T^*P \rightarrow \mathfrak{g}^*$ pulled back from the right momentum

$$\zeta^r = \sum_{j=1}^k \zeta_j^r \xi_j + \sum_{\alpha \in \Delta} \zeta_\alpha^r \xi_\alpha$$

on G , we use again the reduced momentum

$$f := \sum_{j=1}^k \zeta_j^r \xi_j.$$

For a principal connection γ on P we consider the following Liouville form and its derivative

$$\begin{aligned} \Theta_\gamma^{\mathfrak{h}} &= \langle f, \pi^*\gamma \rangle + \Theta_{\mathbb{R}^n} \\ &= \sum_{j=1}^k \zeta_j^r \gamma^j + \sum_{i=1}^n p_i dq_i, \\ \omega_\gamma^{\mathfrak{h}} &= d\Theta_\gamma^{\mathfrak{h}}. \end{aligned}$$

But now $\Theta_\gamma^{\mathfrak{h}}$ is no longer invariant with respect to the whole of G , but it remains invariant with respect to the Cartan subgroup H corresponding to \mathfrak{h} , so

$$\mathcal{L}_{L_j^*} \Theta_\gamma^{\mathfrak{h}} = 0, \quad -i_{L_j^*} \omega_\gamma^{\mathfrak{h}} = d\zeta_j^r.$$

Similarly as in 3.4 we get

$$\begin{aligned} \omega_\gamma^{\mathfrak{h}} &= \sum_{j=1}^k d\zeta_j^r \wedge \pi^*\gamma^j - \sum_{\alpha \in \Delta_+} \frac{B(f, \alpha)}{B(\alpha, \alpha)} \pi^*\gamma^\alpha \wedge \pi^*\gamma^{-\alpha} + \sum_{i,j,k} \zeta_i^r \Omega_{k,j}^i dq_k \wedge dq_j \\ &\quad + \sum_j dp_j \wedge dq_j, \end{aligned}$$

which after reduction can be regarded as a 2-form on $T^*\mathbb{R}^n \times G \times \mathfrak{h}^*$. It is non degenerate on $T^*\mathbb{R}^n \times G \times C$ for any open Weyl chamber C . There this symplectic form gives us the correspondence

$$\begin{aligned} L_j &\leftrightarrow d\zeta_j^r, & L_\alpha &\leftrightarrow \frac{B(f, \alpha)}{B(\alpha, \alpha)} \gamma^{-\alpha}, \\ \partial_{p_i} &\leftrightarrow -dq_i, & \partial_{\zeta_j^r} &\leftrightarrow -\gamma^j, \\ \partial_{q_i} &\leftrightarrow \sum_i b_j^i d\zeta_i^r + \sum_{\alpha \in \Delta} \frac{B(f, \alpha)}{B(\alpha, \alpha)} b_j^\alpha \gamma^{-\alpha} + \sum_{i,k} \zeta_i^r \Omega_{k,j}^i dq_k + dp_j. \end{aligned}$$

Hence we have

$$dp_j \leftrightarrow \partial_{q_j} - \sum_i b_j^i L_i - \sum_{\alpha \in \Delta} b_j^\alpha L_\alpha + 2 \sum_{i,j,k} \zeta_i^r \Omega_{k,j}^i \partial_{p_k},$$

where $b_j^\alpha = \langle \gamma^\alpha, \partial_{q_j} \rangle$, and we get the corresponding Poisson structure $\Lambda_\gamma^{\mathfrak{h}}$ on $T^*\mathbb{R}^n \times G \times \mathfrak{h}^*$ in the form

$$\begin{aligned} \Lambda_\gamma^{\mathfrak{h}} &= \sum_i \partial_{\zeta_i^r} \wedge L_i - \sum_{\alpha \in \Delta_+} \frac{B(\alpha, \alpha)}{B(f, \alpha)} L_\alpha \wedge L_{-\alpha} \\ &\quad - \sum_{i,k} \zeta_i^r \Omega_{k,j}^i \partial_{p_k} \wedge \partial_{p_j} - \sum_i b_j^i \partial_{p_j} \wedge L_i - \sum_{\alpha \in \Delta} b_j^\alpha \partial_{p_j} \wedge L_\alpha + \sum_j \partial_{p_j} \wedge \partial_{q_j}. \end{aligned}$$

This Poisson structure is L_i -invariant, so we can do the reduction once more and get the Poisson structure

$$\begin{aligned} \tilde{\Lambda}_\gamma^{\mathfrak{h}} &= - \sum_{\alpha \in \Delta_+} \frac{B(\alpha, \alpha)}{B(f, \alpha)} L_\alpha \wedge L_{-\alpha} - \sum_{i,k} \zeta_i^r \Omega_{k,j}^i \partial_{p_k} \wedge \partial_{p_j} \\ &\quad - \sum_{\alpha \in \Delta} b_j^\alpha \partial_{p_j} \wedge L_\alpha + \sum_j \partial_{p_j} \wedge \partial_{q_j}. \end{aligned}$$

on $T^*\mathbb{R}^n \times G/H \times \mathfrak{h}^*$, which also gives a Poisson structure on $T^*\mathbb{R}^n \times G/H$ for any fixed value of f . This generalizes 3.4.

4.6. Generalization to Cartan connections. Assume that we have now not only a connection γ but also a displacement form θ on the principal bundle $p : P \rightarrow M$, that is a G -equivariant form $\theta : TP \rightarrow V = \mathbb{R}^n$ with $\ker \theta = VP$, where $\dim M = n$. Then

$$\kappa = \theta + \gamma : TP \rightarrow V \oplus \mathfrak{g}$$

is a Cartan connection, i.e. a G -equivariant absolute parallelism on P . So we obtain a $(V \oplus \mathfrak{g})$ -valued 1-form on T^*P :

$$\pi^* \kappa = \kappa \circ T\pi : T(T^*P) \rightarrow TP \rightarrow V \oplus \mathfrak{g}.$$

Now any smooth mapping

$$f : T^*P \rightarrow V^* \oplus \mathfrak{g}^*$$

defines a 1-form

$$\Theta_f = \langle f, \pi^* \kappa \rangle \in \Omega^1(T^*P)$$

which is G -invariant if f is G -equivariant.

There is a canonical function f as above: namely for $u \in P$ the Cartan connection $\kappa_u : T_u P \rightarrow V \oplus \mathfrak{g}$ is a linear isomorphism and

$$f_u := (\kappa_u^{-1})^* : T_u^* P \rightarrow V^* \oplus \mathfrak{g}^*$$

is G -equivariant. Then $\Theta_f = \Theta_P$ is the canonical Liouville form on the cotangent bundle T^*P .

REFERENCES

1. Alekseev, A.; Todorov, I., *Quadratic brackets from symplectic forms*, Preprint ESI **33** (1993).
2. Balachandran, A. P.; Borchardt; Stern, A., *Lagrangian and Hamiltonian description of Yang Mills particles*, Phys. Rev. **D17** (1978), 3247–3256.
3. Balachandran, A. P.; Marmo, G.; Skagerstam, B. S.; Stern, A., *Gauge symmetries and fiber bundles*, Lecture Notes in Physics 188, Springer-Verlag, 1983.
4. Kolář, I.; Michor, P. W.; Slovák, J., *Natural operations in differential geometry*, Springer-Verlag, Berlin Heidelberg New York, 1993.
5. Marmo, G.; Rubano, C., *Particle dynamics on fiber bundles*, Monographs and Textbooks in the Physical Sciences 8, Bibliopolis, Napoli, 1988.
6. Marsden, J.; Ratiu, T.; Weinstein, A., *Semidirect products and reduction in mechanics*, Trans. Amer. Math. Soc. **281** (1984), 147–177.
7. Marsden, J.; Ratiu, T.; Weinstein, A., *Reduction and Hamiltonian structures on duals of semidirect product Lie algebras*, Contemp. Math. **28** (1984), 55–100.
8. Montgomery, R., *Canonical formulation of a classical particle in a Yang-Mills field and Wong's equations*, Lett. Math. Phys. **8** (1984), 59–67.
9. Montgomery, R.; Marsden, J.; Ratiu, T., *Gauged Lie-Poisson structures*, Contemp. Math. **28** (1984), 101–114.
10. Sternberg, S., *On minimal coupling and the symplectic mechanics of a classical particle in the presence of a Yang-Mills field*, Proc. Nat. Acad. Sci. **74** (1977), 5253–5254.
11. Weinstein, A., *A universal phase space for particles in Yang-Mills fields*, Letters in Math. Phys. **2** (1978), 417–420.
12. Wong, S. K., *Field and particle equations for the classical Yang Mills field and particles with isotopic spin*, Nuovo Cimento **65A** (1970), 689–694.

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