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MANIFOLDS OF SMOOTH MAPS III: THE PRINCIPAL BUNDLE OF EMBEDDINGS OF A NON-COMPACT SMOOTH MANIFOLD

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dedicated to Charles Ehresmann

ABSTRACT. It is shown that the manifold $E(X, Y)$ of all smooth embeddings from a manifold X in a manifold Y is a smooth principal $Diff(X)$-bundle, where $Diff(X)$ is the smooth Lie-group of all diffeomorphisms of X.

This paper is a sequel to [9] which is again a sequel to [7]; in [7] it was shown that the space $C^\infty(X, Y)$ of all smooth mappings $X \to Y$ for arbitrary non-compact smooth finite dimensional manifolds X and Y is again a smooth manifold in a natural way, using the notion $C^\infty_\pi = C^\infty_c$ of Keller [6]. In [9] it was shown that the open subset $Diff(X) \subset C^\infty(X, Y)$ of all C^∞-diffeomorphisms is a smooth Lie group in the same notion of differentiability C^∞_π. In this paper we show that the open subset

$$E(X, Y) \subset C^\infty(X, Y)$$

of all smooth embeddings $X \to Y$ is a smooth principal $Diff(X)$-bundle with base space $U(X, Y) = E(X, Y)/Diff(X)$, the space of all «submanifolds of Y of type X», which is again a smooth C^∞_π-manifold. We remark that all proofs of [7, 9] and this paper may easily be adapted to furnish the results in the notion of differentiability C^∞_Γ used by Fischer, Gutknecht, Yamamuro, Omori and others. We prefer the notion $C^\infty_\pi = C^\infty_c$ for esthetical reasons: it is only slightly weaker than the notion C^∞_Γ but much simpler and it does not need fumbling around with explicit systems of seminorms on the model spaces of the manifolds.

The construction of the principal $Diff(X)$-bundle given here is adapted from Binz and Fischer [1], who proved this result in case that X is compact. [1] also contains some discussions about applications of the result to general relativity in the form of «hyperspace» or «superspace». We refer the reader to this paper for references and information about this application.

This paper contains Sections 9 and 10, continuing the counting of [7] (1-4) and [9] (5-8). References like 6.3 refer to one of these papers without further notice.

9. LOCAL ADDITIONS ON VECTOR BUNDLES.

9.1. We begin by repeating a definition (6.3; 3.3, 3.4):

DEFINITION. A *local addition* τ on a locally compact smooth manifold M is a mapping $\tau : TM \to M$ with the following properties:

(A1) $(\tau, \pi_M): TM \to M \times M$ is a diffeomorphism onto an open neighborhood of the diagonal in $M \times M$, where $\pi_M: TM \to M$ is the projection.

(A2) $\tau(0_m) = m$, where $0_m \in T_m M$ is the zero element.

A local addition has the following (weaker) property:

For any $m \in M$ the mapping $\tau_m = \tau|_{T_m M}: T_m M \to M$ is a diffeomorphism onto an open neighborhood of m.

Local additions can be constructed by using exponential maps (which are canonically associated to sprays) and pulling them back over the whole tangent bundle, using a fibre respecting diffeomorphism of TM on the appropriate neighborhood of the zero section. The notion of local addition is more general than that of an exponential map: in general there is no «local flow property along curves».

DEFINITION. Let $L \subset M$ be a submanifold and let τ be a local addition on M. L is said to be *additively closed* with respect to τ if $\tau|_{TL}$ takes its values in L (and so defines a local addition on L). Compare with the notion of a geodesically closed submanifold.

9.2. PROPOSITION. *Let $p: E \to B$ be a vector bundle. Then there exists a local addition $\tau : TE \to E$ with the following properties:*

1° B, identified with the zero section in E, is an additively closed submanifold of E with respect to τ.

2° Each vector subspace of each fibre $p^{-1}(b)$, $b \in B$, is additively closed with respect to τ.

PROOF. Let $p': E' \to B$ be another vector bundle such that $E \oplus E'$ is trivial (see [4], page 76, or [5], page 100), i.e. $E \oplus E'$ is isomorphic as a vector bundle over B to $B \times R^n$ for some n. Let τ_1 be a local addition on B and let τ_2 be the affine local addition on R^n, i.e. $\tau_2 : TR^n \to R^n$ is given by:

$$\tau_2(v_x) = x + v_x \quad \text{for} \quad v_x \in T_x R^n .$$

Then

$$\tau_1 \times \tau_2 : TB \times TR^n = T(B \times R^n) \to B \times R^n$$

is a local addition on $B \times R^n$ satisfying 1 and 2. Now transport $\tau_1 \times \tau_2$ back to $E \oplus E'$ via the isomorphism and restrict it to the subbundle E of $E \oplus E'$. This gives the desired local addition. QED

9.3. By a submanifold A of an infinite dimensional C^∞_π-manifold B we mean of course a subset $A \subset B$ such that for each $a \in A$ there is a chart $\phi : U \to V$ centered at a (i.e. $\phi(a) = 0$ in V, where V is the complete locally convex vector space modelling B near a) and a topological linear direct summand W in V with $\phi^{-1}(W) = A \cap U$.

9.4. COROLLARY. *Let X, Y be smooth locally compact manifolds and let L be a submanifold of Y. Then $C^\infty(X, L)$ is a C^∞_π-submanifold of $C^\infty(X, Y)$ via $i_* : C^\infty(X, L) \to C^\infty(X, Y)$, where $i : L \hookrightarrow Y$ is the embedding.*

PROOF. Let $p : W \to L$ be a tubular neighborhood of L in Y, i.e. W is an open neighborhood of L in Y and $p : W \to L$ is a vector bundle whose zero section is given by the embedding $L \hookrightarrow W$. Let τ be a local addition on the vector bundle W satisfying 9.2.1 and 9.2.2. Let

$$f \in C^\infty(X, Y) \quad \text{with} \quad f(X) \subset L .$$

Choose the canonical chart of $C^\infty(X, W)$ centered at f coming from the local addition τ on W (cf. 6.3), i.e.

$$\phi_f : U_f \to \mathfrak{D}(f^* TW) = \mathfrak{D}(f^* TY)$$

is given by

$$\phi_f(g)(x) = \tau_{f(x)}^{-1} g(x) = (\tau, \pi_W)^{-1}(g, f)(x)$$

and

$$U_f = \{ g \in C^\infty(X, W) \mid (g, f)(X) \subset (\tau, \pi_W)(TW),\ \ g \sim f \}.$$

The inverse $\phi_f^{-1} = \psi_f$ is given by $\psi_f(s) = \tau \circ s$.
Now if $g \in U_f \cap C^\infty(X, L)$, then

$$\phi_f(g)(x) = \tau_{f(x)}^{-1} g(x) \in T_{f(x)}L \subset T_{f(x)}W,$$

since L is an additively closed submanifold of W by 9.2.1, so $\phi_f(g)$ is in $\mathfrak{D}(f^*TL)$. Clearly for each $s \in \mathfrak{D}(f^*TL)$ the mapping $\psi_f(s) = \tau \circ s$ takes its values in L. So $\phi_f^{-1}(\mathfrak{D}(f^*TL)) = U_f \cap C^\infty(X, L)$.
It remains to show that $\mathfrak{D}(f^*TL)$ is a topological direct summand in $\mathfrak{D}(f^*TW)$. Since $p: W \to L$ is a vector bundle we have

$$TW|_L = TL \oplus W, \text{ so } f^*TW = f^*(TW|_L) = f^*(TL \oplus W) = f^*TL \oplus f^*W$$

and consequently $\mathfrak{D}(f^*TW) = \mathfrak{D}(f^*TL) \times \mathfrak{D}(f^*W)$ is a topological direct sum.
So we have proved that $C^\infty(X, L)$ is a C^∞_π-submanifold of $C^\infty(X, W)$, which is again an open submanifold of $C^\infty(X, Y)$. QED

10. THE PRINCIPAL BUNDLE OF EMBEDDINGS.

10.1. Let X, Y be smooth locally compact manifolds; neither is assumed to be compact, but we assume that $\dim X < \dim Y$. There are two spaces of smooth embeddings $X \to Y$: let $E(X, Y)$ denote the space of all smooth embeddings, which is an open subset of $C^\infty(X, Y)$ (see [5] page 37); and let $E_{prop}(X, Y)$ denote the space of all proper embeddings $X \to Y$. It is an open subset of $E(X, Y)$ since the set of proper maps is open (1.9). The proper embeddings coincide with the closed embeddings, since a proper map is closed if Y is locally compact ([3] page 47). The open subsets $E(X, Y)$ and $E_{prop}(X, Y)$ thus inherit C^∞_π-manifold structures from $C^\infty(X, Y)$ (cf. 6.3 or 3.3, 3.4, 3.6).

10.2. We consider the following C^∞_π-mappings (7.2):

$$\rho : Diff(X) \times E(X,Y) \to E(X,Y), \quad \rho(g,i) = i \circ g ;$$
$$\rho : Diff(X) \times E_{prop}(X,Y) \to E_{prop}(X,Y).$$

ρ stands for the right action of $Diff(X)$ on $E(X,Y)$ and $E_{prop}(X,Y)$. Each $g \in Diff(X)$ induces a C^∞_π-diffeomorphism $\rho(g,.)$ on $E(X,Y)$ and $E_{prop}(X,Y)$ respectively; the inverse is given by $\rho(g^{-1},.)$. The injectivity of the elements of $E(X,Y)$ implies that the right action ρ of $Diff(X)$ on $E(X,Y)$ is free:

$$i \circ g_1 = i \circ g_2 \text{ implies } g_1 = g_2 \text{ in } Diff(X).$$

Thus the mapping $\rho(.,i): Diff(X) \to E(X,Y)$ gives a bijection of $Diff(X)$ onto the orbit $i \circ Diff(X)$ of i ; if i is proper, then the whole orbit of i is contained in $E_{prop}(X,Y)$. We will see later on that $\rho(.,i)$ is even a C^∞_π-diffeomorphism onto the orbit.

10.3. DEFINITION. Let $U(X,Y) = E(X,Y)/Diff(X)$ denote the orbit space, equipped with the quotient topology; let $u: E(X,Y) \to U(X,Y)$ denote the canonical quotient mapping.

$U(X,Y)$ is, heuristically speaking, just the space of all submanifolds of type X in Y.

10.4. PROPOSITION. *Let $i \in E(X,Y)$ and write $L = i(X)$.*

1° The orbit $i \circ Diff(X)$ of i is the space $Diff(X,L)$.

2° The inclusion $Diff(X,L) \to E(X,Y)$ is a C^∞_π-submanifold embedding.

3° The mapping $\rho(.,i): Diff(X) \to i \circ Diff(X) = Diff(X,L)$ is a C^∞_π-diffeomorphism.

4° If i is proper, then the orbit of i is closed in $E_{prop}(X,Y)$.

5° If X has finitely many connected components, then

$$Diff(X,L) = E_{prop}(X,L).$$

PROOF. 1 is clear. 2 follows from 9.4 since $Diff(X,L)$ is open in $C^\infty(X,L)$ (cf. 5.2 or [5] page 38) and $C^\infty(X,L)$ is a C^∞_π-submanifold of $C^\infty(X,Y)$.

3 is clear by the Ω-Lemma 3.7 or by 7.1.

4. Let (g_α) be a net in $Diff(X)$ such that $i \circ g_\alpha = \rho(g_\alpha, i)$ converges to f, say, in $E_{prop}(X, Y)$. Then

$$i \circ g_\alpha(X) = i(X) = L \quad \text{for all } \alpha,$$

and since L is closed (i is closed) in Y, we conclude that $f(X) \subset L$. Now let X_j be one of the connected components of X, so X_j is open and closed in X, so $f(X_j)$ is open in L (since f is an immersion) and closed in L (since f is proper); thus $f(X_j)$ is one of the connected components of L. Now let L_j be one of the connected components of L. If α_0 is big enough, then $i \circ g_\alpha(X_k) = L_j$ for some component X_k of X and all $\alpha \geq \alpha_0$. So $f(X_k) = L_j$ and f is surjective, thus $f \in Diff(X, L)$.

5. Let $X_1, \ldots, X_k$ be the connected components of X, let $f \in E_{prop}(X, L)$. As above one sees that $f(X_1), \ldots, f(X_k)$ are different connected components of L; since L has as many components as X (for $i: X \to L$ is a diffeomorphism), the assertion follows. QED

10.5. Fix $i \in E(X, Y)$ and denote $i(X)$ by L. Let $p_L: W_L \to L$ be a tubular neighborhood of L in Y.

LEMMA. *If $j \in C^\infty(X, W_L)$ is such that $p_L \circ j \in E(X, Y)$, then j is an embedding with inverse*

$$(p_L \circ j)^{-1} \circ (p_L|_{j(X)}): j(X) \to X.$$

Moreover for $x \in X$:

$$(T_x j)(T_x X) \oplus T_{j(x)}(p_L^{-1}(p_L j(x))) = T_{j(x)} W_L = T_{j(x)} Y.$$

PROOF. $p_L \circ j$ is injective so $j: X \to W_L$ is injective, so $j: X \to j(X)$ is invertible with inverse $(p_L \circ j)^{-1} \circ (p_L|_{j(X)})$. This inverse is continuous, so j is a topological embedding. For $x \in X$ we have

$$(T_{j(x)} p_L)(T_x j)(T_x X) = T_x(p_L \circ j)(T_x X) =$$
$$T_{p_L j(x)} L = (T_{j(x)} p_L)(T_{j(x)} W_L),$$

so

$$dim(T_x j)(T_x X) \geq dim\, T_{p_L j(x)} L = dim\, T_x X \geq dim(T_x j)(T_x X),$$

so j is an immersion, thus $j \in E(X, W_L)$. Now the kernel of

$$T_{j(x)}p_L : T_{j(x)}W_L \to T_{p_L j(x)}L$$

is $T_{j(x)}(p_L^{-1}(p_L j(x)))$, the tangent space to the fibre through $j(x)$ so the second assertion follows. QED

10.6. DEFINITION. In the setup of 10.5 let us denote

$$\begin{aligned} Q_i &= \{ j \in C^\infty(X, W_L) \mid p_L \circ j = i \text{ and } j \sim i \} \\ &= (p_L)_*^{-1}(i) \cap \{ j \mid j \sim i \}. \end{aligned}$$

By 10.5 we see that $Q_i \subset E(X, W_L)$. Remember that

$$u: E(X,Y) \to U(X,Y)$$

denotes the quotient map.

LEMMA. *1°* $u|_{Q_i} : Q_i \to U(X,Y)$ *is injective.*

2° $Q_i \circ V$ *is open in* $E(X,Y)$ *if* V *is open in* $Diff(X)$.

PROOF. 1° Let $j, j' \in Q_i$ and suppose $u(j) = u(j')$, i.e. $j = j' \circ g$ for some $g \in Diff(X)$, then

$$i = p_L \circ j = p_L \circ (j' \circ g) = i \circ g,$$

so $g = Id_X$ and $j = j'$.

2° Let us first assume that

$$V \subset \{ g \in Diff(X) \mid g \sim Id_X \},$$

the open subgroup of diffeomorphisms with compact support.
$(p_L)_*: E(X, W_L) \to C^\infty(X, L)$ is continuous, $i \circ Diff(X) = Diff(X, L)$ is open in $C^\infty(X, L)$, $\rho(\,.\,, i): Diff(X) \to Diff(X, L)$ is homeomorphic so we have in turn that $\rho(\,.\,, i)(V)$ is open in $Diff(X, L)$ and that the set $(p_L)_*^{-1}(\rho(\,.\,, i)(V))$ is open in $E(X, W_L)$ and in $E(X, Y)$. Now, we claim that

$$(p_L)_*^{-1}(\rho(\,.\,, i)(V)) \cap \{ j \in E(X,Y) \mid j \sim i \} = Q_i \circ V$$

which proves the assertion in this special case: If

$$j \in (p_L)_*^{-1}(\rho(\,.\,, i)(V)) \text{ and } j \sim i,$$

then $p_L \circ j \in i \circ V$, so $p_L \circ j = i \circ g$ for some $g \in V$ with $g \sim Id_X$. But then $j \circ g^{-1} \in E(X, W_L)$ and $p_L \circ (j \circ g^{-1}) = i \circ g \circ g^{-1} = i$ and $j \circ g^{-1} \sim i$, so

$$j \circ g^{-1} \in Q_i \quad \text{and} \quad j = (j \circ g^{-1}) \circ g \in Q_i \circ V.$$

Now suppose conversely that $j \in Q_i$, $g \in V$. Then $p_L \circ j = i$, $j \sim i$, so

$$p_L \circ (j \circ g) = i \circ g \in \rho(.,i)(V) \quad \text{and} \quad j \circ g \sim i,$$

hence

$$j \circ g \in (p_L)_*^{-1}(\rho(.,i)(V)) \cap \{ j \mid j \sim i \}.$$

Now let V be an arbitrary open subset of $Diff(X)$. Decompose V into the disjoint union of all nonempty intersections of V with the open equivalence classes of $\sim$ in $Diff(X)$, which we call V_α. For each α, take $g_\alpha \in V_\alpha$, then $V_\alpha \circ g_\alpha^{-1}$ is an open subset of the subgroup of diffeomorphisms with compact support, so $Q_i \circ (V_\alpha \circ g_\alpha^{-1})$ is open in $E(X,Y)$ by the first part of the proof. But then

$$Q_i \circ V_\alpha = \rho(g_\alpha, .)(Q_i \circ (V_\alpha \circ g_\alpha^{-1}))$$

is open too and thus $Q_i \circ V = \bigcup_\alpha Q_i \circ V_\alpha$ is open. QED

10.7. COROLLARY. *With the above notation, $u(Q_i)$ is open in the quotient topology in $U(X,Y) = E(X,Y)/Diff(X)$.*

PROOF. By Lemma 10.6 (for $V = Diff(X)$) we see that $Q_i \circ Diff(X)$, the full inverse image of $u(Q_i)$ under u, is open in $E(X,Y)$. So $u(Q_i)$ is open in $U(X,Y)$ in the quotient topology. QED

10.8. As in 10.5 let $i \in E(X,Y)$, $L = i(X)$ and let $p_L : W_L \to L$ be a tubular neighborhood of L in Y; furthermore let $\tau_L : TW_L \to W_L$ be a local addition for the vector bundle W_L as constructed in 9.2. Since $p_L : W_L \to L$ is a vector bundle, we may decompose it: $TW_L|_L = TL \oplus W_L$, where we have identified

$$(W_L)_l = p_L^{-1}(l) \quad \text{with} \quad T_l(p^{-1}(l)),$$

the tangent space to the fibre through l.

For reasons of clarity we will not identify as radically as we have done above: Let $V_L \to L$ denote the subbundle of $TW_L|_L$ consisting of the vertical elements of $TW_L|_L$, those tangent to the fibres of W_L. Then the decomposition mentioned above may be written as $TW_L|_L = TL \oplus V_L$.

By 9.2 we have the following:

$$\tau_L|_{(V_L)_l}: (V_L)_l = T_l(p_L^{-1}(l)) \to (W_L)_l = p_L^{-1}(l)$$

is a diffeomorphism onto for each $l \in L$. So $\tau_L|_{V_L}: V_L \to W_L$ is a diffeomorphism onto.

10.9. PROPOSITION. *In the setting of 10.8, the subset Q_i of 10.6 is a C^∞_π-submanifold of $E(X, Y)$.*

PROOF. We will show that Q_i is a C^∞_π-submanifold of the open subset $E(X, W_L)$ of $E(X, Y)$. Let

$$\phi_i: U_i \to \mathfrak{D}(i^*TW_L) = \mathfrak{D}(i^*(TW_L|_L))$$

be the canonical chart coming from the local addition τ_L on W_L, i.e.

$$U_i = \{ j \in E(X, W_L) \mid (j, i)(X) \subset (\tau_L, \pi_{W_L})(TW_L) \text{ and } j \sim i \} =$$
$$= \{ j \in E(X, W_L) \mid j \sim i \}$$

since τ_L is onto W_L by construction. So $Q_i \subset U_i$, and Q_i carries a global chart.

Now $j \in Q_i$ means that $j \sim i$ and $p_L \circ j = i$, so $j(x) \in p_L^{-1}(i(x))$ and

$$\phi_i(j)(x) = (\tau_L)^{-1}_{i(x)}(j(x)) \in (V_L)_{i(x)},$$

since the fibre $p_L^{-1}(i(x))$ is additively closed with respect to τ_L. By the same reason we see that for any $s \in \mathfrak{D}(i^*V_L)$,

$$\phi_i^{-1}(s) = \psi_i(s) = \tau_L \circ s \in Q_i .$$

So $\phi_i|_{Q_i}: Q_i \to \mathfrak{D}(i^*V_L)$ is a bijection, and $\mathfrak{D}(i^*V_L)$ is a topological direct summand in $\mathfrak{D}(i^*(TW_L|_L))$, since

$$\mathfrak{D}(i^*(TW_L|_L)) = \mathfrak{D}(i^*(TL \oplus V_L)) = \mathfrak{D}(i^*TL \oplus i^*V_L) =$$
$$= \mathfrak{D}(i^*TL) \times \mathfrak{D}(i^*V_L)$$

(cf. 9.4). QED

10.10. Now we can show that $u: E(X, Y) \to U(X, Y)$ is a principal $Diff(X)$-bundle. Let $i \in E(X, Y)$, denote $\hat{i} = u(i) \in U(X, Y)$, then $\hat{Q}_i = u(Q_i)$ is an open neighborhood of $\hat{i}$ in $U(X, Y)$, which we will show to be trivializing.

1° DEFINITION. Let $s_i : \hat{Q}_i \to E(X,Y)$ be given by $s_i = (u|_{Q_i})^{-1}$, which is well defined, since $u|_{Q_i}$ is injective by 10.6.1.

2° The fibres of $u: E(X,Y) \to U(X,Y)$ (which are the $Diff(X)$-orbits) over the points of $\hat{Q}_i$ meet Q_i at exactly one point each by construction; since moreover the action of $Diff(X)$ is free we see that the mapping

$$\rho|_{Diff(X)\times Q_i} : Diff(X)\times Q_i \to u^{-1}(\hat{Q}_i)$$

is bijective, so there is an inverse

$$(\rho|_{Diff(X)\times Q_i})^{-1} = (\mu_i, \delta_i) : u^{-1}(\hat{Q}_i) \to Diff(X)\times Q_i .$$

So

$$\mu_i : u^{-1}(\hat{Q}_i) \to Diff(X), \quad \delta_i : u^{-1}(\hat{Q}_i) \to Q_i$$

and $\delta_i(j)\circ\mu_i(j) = j$ for each $j \in u^{-1}(\hat{Q}_i)$, furthermore

$$\delta_i(j) \sim i \quad \text{and} \quad p_L \circ \delta_i(j) = i .$$

3° We claim that μ_i is C^∞_π-differentiable:

$$i \circ \mu_i(j) = p_L \circ \delta_i(j) \circ \mu_i(j) = p_L \circ j$$

(this implies that $p_L \circ j$ is defined too), so

$$\mu_i(j) = \rho(.,i)^{-1} \circ (p_L)_*(j),$$

or

$$\mu_i = \rho(.,i)^{-1} \circ (p_L)_* : u^{-1}(\hat{Q}_i) \to Diff(X).$$

By 10.4 and the Ω-Lemma 3.7, we see that μ_i is C^∞_π-differentiable.

4° We claim that δ_i is C^∞_π-differentiable too:

$$\delta_i(j)\circ\mu_i(j) = j, \quad \text{so} \quad \delta_i(j) = j \circ \mu_i(j)^{-1}$$

or

$$\delta_i = \rho \circ (Inv \circ \mu_i, Id) : u^{-1}(\hat{Q}_i) \to Q_i .$$

By 10.2 and 8.1 we see that δ_i is C^∞_π-differentiable.

5° So $\rho : Diff(X)\times Q_i \to u^{-1}(Q_i)$ is a C^∞_π-diffeomorphism. This will provide the trivializing map.

6° We claim that $s_i : \hat{Q}_i \to Q_i$ (from 1) is continuous (so $\hat{Q}_i$ is homeomorphic to Q_i): For $y \in \hat{Q}_i$ we have $\{ s_i(y)\} = \delta_i(u^{-1}(y))$ by construction. Let V be open in Q_i, then $\delta_i^{-1}(V)$ is open in $u^{-1}(\hat{Q}_i)$ by 4, $u^{-1}(\hat{Q}_i)$ is itself open in $E(X,Y)$, so

$$u^{-1}(s_i^{-1}(V)) = u^{-1}(u(V)) = \delta_i^{-1}(V)$$

is open in $E(X, Y)$. This implies that $s_i^{-1}(V)$ is open in $U(X, Y)$ in the quotient topology.

10.11. We have proved the following theorem:

THEOREM. *$u: E(X, Y) \to U(X, Y)$ is a topological principal $Diff(X)$-bundle, trivial over the open neighborhood $\hat{Q}_i$ of $\hat{i}$ in $U(X, Y)$ for each $i \in E(X, Y)$, a trivializing map being given by:*

$$Diff(X) \times \hat{Q}_i \to u^{-1}(Q_i), \quad (g, y) \mapsto s_i(y) \circ g.$$

10.12. THEOREM. *$U(X, Y)$ is a C^∞_π-manifold.*

PROOF. For each $i \in E(X, Y)$ the open neighborhood $\hat{Q}_i$ is homeomorphic to the submanifold Q_i of $E(X, Y)$ (cf. 10.10.6 and 10.9) so we only have to check that these «fit together nicely». In other words, we use the mappings:

$$(\phi_i | Q_i) \circ s_i : \hat{Q}_i \to \mathfrak{D}(i^* V_L)$$

as charts. We have to check whether the chart-change is C^∞_π-differentiable. Let $i, k \in E(X, Y)$ be such that $\hat{Q}_i \cap \hat{Q}_k \neq \emptyset$ in $U(X, Y)$. Let us first assume that i and k lie on the same $Diff(X)$-orbit, then there is some $g \in Diff(X)$ with $i = k \circ g$. Then we have $L = i(X) = k(X)$ and

$$\begin{aligned} Q_i &= \{ j \in E(X, W_L) \mid p_L \circ j = i, \ j \sim i \} \\ &= \{ j \in E(X, W_L) \mid p_L \circ j = k \circ g, \ j \sim k \circ g \} \\ &= \{ j \circ g \mid j \in E(X, W_L), \ p_L \circ j = k, \ j \sim k \} \\ &= Q_k \circ g = \rho(g, .)(Q_k). \end{aligned}$$

So Q_i and Q_k are translates of each other, $\hat{Q}_i = \hat{Q}_k$ and

$$\begin{aligned} ((\phi_k | Q_k) \circ s_k) \circ ((\phi_i | Q_i) \circ s_i)^{-1} &= (\phi_k | Q_k) \circ s_k \circ s_i^{-1} \circ (\phi_i | Q_i)^{-1} \\ &= (\phi_k | Q_k) \circ (\rho(g, .)^{-1} | Q_i) \circ (\phi_i | Q_i)^{-1} \\ &= (\phi_k | Q_k) \circ (\rho(g^{-1}, .) | Q_i) \circ (\phi_i | Q_i)^{-1}, \end{aligned}$$

which is a C^∞_π-diffeomorphism by 10.2 and 10.9.

So let us now suppose that $i, k \in E(X,Y)$ with $\hat{Q}_i \cap \hat{Q}_k \neq \emptyset$, but that i and k do not lie on the same orbit. Let $L = i(X)$, $K = k(X)$. Then

$$s_k(\hat{Q}_i \cap \hat{Q}_k) = s_k(\hat{Q}_k) \cap u^{-1}(\hat{Q}_i) = Q_k \cap u^{-1}(\hat{Q}_i).$$

For $j \in Q_k$ we have $p_K \circ j = k$ and $j \sim k$, so

$$j = \tau_K \circ t = \psi_k(t) \quad \text{for some} \quad t \in \mathfrak{D}(k^* V_K).$$

If moreover $j \in u^{-1}(\hat{Q}_i)$, then

$$j = \delta_i(j) \circ \mu_i(j) \quad \text{for} \quad \delta_i(j) \in Q_i \quad \text{and} \quad \mu_i(j) \in Diff(X).$$

So if $t \in (\phi_k|_{Q_k}) \circ s_k(\hat{Q}_i \cap \hat{Q}_k) \subset \mathfrak{D}(k^* V_K)$, then

$$\begin{aligned}(\phi_i|_{Q_i}) \circ s_i \circ ((\phi_k|_{Q_k}) \circ s_k)^{-1}(t) &= (\phi_i|_{Q_i}) \circ s_i \circ s_k^{-1} \circ (\phi_k|_{Q_k})^{-1}(t) \\ &= (\phi_i|_{Q_i}) \circ s_i \circ u(j) = (\phi_i|_{Q_i})(s_i(\hat{j})) \\ &= (\phi_i|_{Q_i})(\delta_i(j)) = (\phi_i|_{Q_i}) \circ \delta_i \circ (\phi_k|_{Q_k})^{-1}(t),\end{aligned}$$

where we have used again the argument of 10.10.6. This last mapping is C^∞_π-differentiable in t by 10.10.4 and 10.9. QED

10.13. PROPOSITION. *$u: E(X,Y) \to U(X,Y)$ is a submersion, i.e., for each $i \in E(X,Y)$, the mapping*

$$T_i u: T_i E(X,Y) = \mathfrak{D}(i^* TY) \to T_{\hat{i}} U(X,Y) = \mathfrak{D}(i^* V_L)$$

is surjective, a topological quotient map, and the kernel

$$ker\, T_i u = T_i(i \circ Diff(X)) = \mathfrak{D}(i^* TL)$$

is a linear and topological direct summand.

PROOF. That the kernel of $T_i u$ is splitting has been proved in 10.9; that $T_i u$ is a quotient map follows directly from the construction of the charts for $U(X,Y)$. QED

10.14. THEOREM. *$u: E(X,Y) \to U(X,Y)$ is a C^∞_π-differentiable principal $Diff(X)$-bundle, trivial over the open neighborhoods $\hat{Q}_i$ of $\hat{i}$ in $U(X,Y)$ for each $i \in E(X,Y)$, a trivializing map being given by:*

$$Diff(X) \times \hat{Q}_i \to u^{-1}(Q_i), \quad (g, y) \mapsto s_i(y) \circ g.$$

PROOF. $s_i: \hat{Q}_i \to Q_i$ is a C^∞_π-diffeomorphism by the construction of the charts for $U(X, Y)$. QED

10.15. Let $U_{prop}(X, Y)$ denote the space of all proper orbits, i.e. (cf. 10.1, 10.2, 10.4) $U_{prop}(X, Y) = u(E_{prop}(X, Y))$.

COROLLARY. *$u: E_{prop}(X, Y) \to U_{prop}(X, Y)$ is a smooth principal $Diff(X)$-bundle. $E_{prop}(X, Y) = E(X, Y)|_{U_{prop}(X, Y)}$, the restriction of the bundle $E(X, Y) \to U(X, Y)$ to the open subset $U_{prop}(X, Y)$ of $E(X, Y)$.*

REFERENCES.

1. E. BINZ & H.R. FISCHER, *The manifold of embeddings of a closed manifold*, Mannheim, 1977.
2. H.G. GARNIR, M. DE WILDE & J. SCHMETS, *Analyse fonctionnelle*, Tome 1, Birkhäuser, Basel-Stuttgart, 1968.
3. M. GOLUBITSKY & V. GUILLEMIN, *Stable mappings and their singularities*, Springer, 1973, G.T.M. 14.
4. W. GREUB, S. HALPERIN & R. VANSTONE, *Connections, Curvature and Cohomology* I, Academic Press, 1972.
5. W.W. HIRSCH, *Differential topology*, Springer 1976, G.T.M. 33.
6. H.H. KELLER, Differential calculus in locally convex spaces, *Lecture Notes in Math.* 417, Springer (1974).
7. P. MICHOR, Manifolds of smooth maps, *Cahiers Topo. et Géom. Diff.* XIX-1 (1978), 47-78.
8. P. MICHOR, Manifolds of differentiable maps, *Proc. CIME Conf. on Diff. Topology*, Varenna, 1976.
9. P. MICHOR, Manifolds of smooth maps II: The Lie group of diffeomorphisms of a non-compact smooth manifold, *Cahiers Topo. et Géom. Diff.* XX-3 (1979).

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