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n-ARY LIE AND ASSOCIATIVE ALGEBRAS

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To Wlodek Tulczyjew, on the occasion of his 65th birthday.

Abstract. With the help of the multigraded Nijenhuis– Richardson bracket and the multigraded Gerstenhaber bracket from [7] for every $n \geq 2$ we define n-ary associative algebras and their modules and also n-ary Lie algebras and their modules, and we give the relevant formulas for Hochschild and Chevalley cohomogy.

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1. Introduction

In 1985 V. Filipov [3] proposed a generalization of the concept of a Lie algebra by replacing the binary operation by an n-ary one. He defined an n-ary Lie algebra structure on a vector space V as an operation which associates with each n-tuple $(u_1, \dots, u_n)$ of elements in V another element $[u_1, \dots, u_n]$ which is n-linear, skew symmetric, and satisfies the n-Jacobi identity:

$$[u_1, \dots, u_{n-1}, [v_1, \dots, v_n]] = \sum [v_1, \dots, v_{i-1}[u_1, \dots, u_{n-1}, v_i], \dots, v_n]. \tag{1}$$

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Apparently Filippov was motivated by the fact that with this definition one can delelop a meaningful structure theory, in accordance with the aim of Malcev's school: To look for algebraic structures that manifest good properties.

On the other hand, in 1973 Y. Nambu [13] proposed an n-ary generalization of Hamiltonian dynamics by means of the n-ary 'Poisson bracket'

$$\{f_1, \dots, f_n\} = \det\left(\frac{\partial f_i}{\partial x_j}\right). \tag{2}$$

Apparently he looked for a simple model which explains the unseparability of quarks. Much later, in the early 90's, it was noticed by M. Flato, C. Fronsdal, and others, that the n-bracket (2) satisfies (1). On this basis L. Takhtajan [17] developed sytematically the foundations of of the theory of n-Poisson or Nambu-Poisson manifolds. It seems that the work of Filippov was unknown then; in particular Takhtajan reproduces some results from [3] without refereing to it.

Recently Alekseevsky and Guha [1] and later Marmo, Vilasi, and Vinogradov [9] proved that n-Poisson structures of the kind above are extremely rigid: Locally they are given by n commuting vector fields of rank n, if $n > 2$; in other words, n-Poisson structures are locally given by (2). This rigidity suggests that one should look for alternative n-ary analogs of the concept of a Lie algebra. One of them is proposed below in this paper. It is based on the completely skew symmetrized version of Filippov's Jacobi identity (2). It is shown in [20] that this approach leads to richer and more diverse structures which seem to be more useful for purposes of dynamics. In fact, we were lead in 1990-92 to the constructions of this paper by some expectations about n-body mechanics and the naturality of the machinary developed in [7]. So, our motives were quite different from that by Filippov, Nambu and Takhtajian. This paper is essentailly based on our unpublished notes from 1990-92. In view of the recent developments we decided to publish them now. In this paper we consider G-graded n-ary generalizations of the concept of associative algebras, of Lie algebras, their modules, and their cohomologies; all this is produced by the algebraic machinery of [7]. Related (but not graded) concepts are discussed in [4] in terms of operads and their Koszul duality. The recent preprints [2] and [5] propose dynamical models which correspond to the not graded case with even n in our construction.

2. Review of binary algebras and bimodules

In this section we review the results from the paper [7] in a slightly different point of view.

2.1. Conventions and definitions. By a *grading group* we mean a commutative group $(G, +)$ together with a $\mathbb{Z}$-bilinear symmetric mapping (bicharacter) $\langle\quad,\quad\rangle : G \times G \to \mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$. Elements of G will be called degrees, or G-degrees if more precision is necessary. A standard example of a grading group is $\mathbb{Z}^m$ with $\langle x, y\rangle = \sum_{i=1}^m x^i y^i (\mod 2)$. If G is a grading group we will consider the grading group $\mathbb{Z} \times G$ with $\langle (k, x), (l, y)\rangle = kl(\mod 2) + \langle x, y\rangle$.

A *G-graded vector space* is just a direct sum $V = \bigoplus_{x\in G} V^x$, where the elements of V^x are said to be homogeneous of G-degree x. We assume that vector spaces are defined over a field $\mathbb{K}$ of characteristic 0. In the following X, Y, etc will always denote homogeneous elements of some G-graded vector space of G-degrees x, y, etc.

By an G-graded algebra $\mathcal{A} = \bigoplus_{x\in G} \mathcal{A}^x$ we mean an G-graded vector space which is also a $\mathbb{K}$ algebra such that $\mathcal{A}^x \cdot \mathcal{A}^y \subseteq \mathcal{A}^{x+y}$.

(1) The G-graded algebra $(\mathcal{A}, \cdot)$ is said to be *G-graded commutative* if for homogeneous elements X, $Y \in \mathcal{A}$ of G-degree x, y, respectively,we have $X \cdot Y = (-1)^{\langle x,y\rangle} Y \cdot X$.

(2) If $X \cdot Y = -(-1)^{\langle x,y\rangle} Y \cdot X$ holds it is called *G-graded anticommutative*.

(3) By an *G-graded Lie algebra* we mean a G-graded anticommutative algebra $(\mathcal{E}, [\quad,\quad])$ for which the *G-graded Jacobi identity* holds:

$$[X, [Y, Z]] = [[X, Y], Z] + (-1)^{\langle x,y\rangle} [Y, [X, Z]]$$

Obviously the space $\mathrm{End}(V) = \bigoplus_{\delta\in G} \mathrm{End}^\delta(V)$ of all endomorphisms of a G-graded vector space V is a G-graded algebra under composition, where $\mathrm{End}^\delta(V)$ is the space of linear endomorphisms D of V of G-degree δ, i.e. $D(V^x) \subseteq V^{x+\delta}$. Clearly $\mathrm{End}(V)$ is a G-graded Lie algebra under the G-graded commutator

$$[D_1, D_2] := D_1 \circ D_2 - (-1)^{\langle \delta_1,\delta_2\rangle} D_2 \circ D_1. \tag{4}$$

If $\mathcal{A}$ is a G-graded algebra, an endomorphism $D : \mathcal{A} \to \mathcal{A}$ of G-degree δ is called a *G-graded derivation*, if for X, $Y \in \mathcal{A}$ we have

$$D(X \cdot Y) = D(X) \cdot Y + (-1)^{\langle \delta,x\rangle} X \cdot D(Y). \tag{5}$$

Let us write $\mathrm{Der}^\delta(\mathcal{A})$ for the space of all G-graded derivations of degree δ of the algebra $\mathcal{A}$, and we put

$$\mathrm{Der}(\mathcal{A}) = \bigoplus_{\delta\in G} \mathrm{Der}^\delta(\mathcal{A}). \tag{5}$$

The following lemma is standard:

Lemma. *If $\mathcal{A}$ is an G-graded algebra, then the space $\mathrm{Der}(\mathcal{A})$ of G-graded derivations is an G-graded Lie algebra under the G-graded commutator.*

2.2 Graded associative algebras. Let $V = \bigoplus_{x\in G} V^x$ be an G-graded vector space. We define

$$M(V) := \bigoplus_{(k,\kappa)\in\mathbb{Z}\times G} M^{(k,\kappa)}(V),$$

where $M^{(k,\kappa)}(V)$ is the space of all $k+1$-linear mappings $K : V \times \ldots \times V \to V$ such that $K(V^{x_0} \times \ldots \times V^{x_k}) \subseteq V^{x_0+\cdots+x_k+\kappa}$. We call k the *form degree* and κ the *weight degree* of K. We define for $K_i \in M^{(k_i,\kappa_i)}(V)$ and $X_j \in V^{x_j}$

$$(j(K_1)K_2)(X_0, \ldots, X_{k_1+k_2}) :=$$

$$= \sum_{i=0}^{k_2} (-1)^{k_1 i+\langle \kappa_1,\kappa_2+x_0+\cdots+x_{i-1}\rangle} K_2(X_0, \ldots, K_1(X_i, \ldots, X_{i+k_1}), \ldots, X_{k_1+k_2}),$$

$$[K_1, K_2]^\Delta = j(K_1)K_2 - (-1)^{k_1k_2+\langle \kappa_1,\kappa_2\rangle} j(K_2)K_1.$$

Theorem. *Let V be an G-graded vector space. Then we have:*

(1) $(M(V), [\quad,\quad]^\Delta)$ *is a $(\mathbb{Z}\times G)$-graded Lie algebra.*
(2) *If $\mu \in M^{(1,0)}(V)$, so $\mu : V \times V \to V$ is bilinear of weight $0 \in G$, then μ is an associative G-graded multiplication if and only if $j(\mu)\mu = 0$.*
(3) *If $\nu \in M^{(1,n)}(V)$, so $\nu : V\times V \to V$ is bilinear of weight $n \in G$, then $j(\nu)\nu = 0$ is equivalent to*

$$\nu(\nu(X_0, X_1), X_2) - (-1)^{\langle n,n\rangle}\nu(X_0, \nu(X_1, X_2)) = 0$$

which is the natural notion of an associative multiplication of weigth $n \in G$.

Proof. The first assertion is from [7]. The second and third assertion follows by writing out the definitions. □

In [7] the formulation was as follows: $\mu \in M^{(1,0)}(V)$ is an associative G-graded algebra structure if and only if $[\mu, \mu]^\Delta = 2j(\mu)\mu = 0$. For $\nu \in M^{(1,n)}(V)$ we have $[\nu, \nu]^\Delta = (1 + (-1)^{\langle n,n\rangle})j(\nu)\nu$.

2.3. Multigraded bimodules. Let V and W be G-graded vector spaces and $\mu : V \times V \to V$ a G-graded algebra structure. A *G-graded bimodule* $\mathcal{M} = (W, \lambda, \rho)$ over $\mathcal{A} = (V, \mu)$ is given by $\lambda, \rho : V \to \operatorname{End}(W)$ of weight 0 such that

$$j(\mu)\mu = 0 \quad \text{so } \mathcal{A} \text{ is associative} \tag{1}$$

$$\lambda(\mu(X_1, X_2)) = \lambda(X_1) \circ \lambda(X_2) \tag{2}$$

$$\rho(\mu(X_1, X_2)) = (-1)^{\langle x_1,x_2\rangle}\rho(X_2) \circ \rho(X_1) \tag{3}$$

$$\lambda(X_1) \circ \rho(X_2) = (-1)^{\langle x_1,x_2\rangle}\rho(X_2) \circ \lambda(X_1) \tag{4}$$

where $X_i \in V^{x_i}$ and $\circ$ denotes the composition in $\operatorname{End}(W)$.

2.4. Theorem. *Let E be the $(\mathbb{Z} \times G)$-graded vector space defined by*

$$E^{(k,*)} = \begin{cases} V & \text{if } k = 0 \\ W & \text{if } k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then $P \in M^{(1,0)}(E)$ defines a bimodule structure on W if and only if $j(P)P = 0$.

Proof. We define

$$\begin{aligned} \mu(X_1, X_2) &:= P(X_1, X_2) \\ \lambda(X)Y &:= P(X, Y) \\ \rho(X)Y &:= (-1)^{\langle x,y\rangle}P(Y, X) \end{aligned}$$

where we suppose the X_i's $\in V$ and $Y \in W$ to be embedded in E. Then if $Z_i \in E$ is arbitrary we get

$$(j(P)P)(Z_0, Z_1, Z_2) = P((Z_0, Z_1), Z_2) - P(Z_0, (Z_1, Z_2)).$$

Now specify $Z_i \in V$ resp. W to get eight independent equations. Four of them vanish identically because of their degree of homogeneity, the others recover the defining equations for the G-graded bimodules. □

2.5 Corollary. *In the above situation we have the following decomposition of the $(\mathbb{Z}^2 \times G)$-graded space $M(E)$:*

$$M^{(k,q,*)}(E) = \begin{cases} 0 & \text{for } q > 1 \\ L^{(k+1,*)}(V,W) & \text{for } q = 1 \\ M^{(k,*)}(V) \bigoplus^{k+1} (L^{(k,*)}(V, \mathrm{End}(W)) & \text{for } q = 0 \end{cases}$$

where $L^{(k,*)}(V,W)$ denotes the space of k-linear mappings $V \times \ldots \times V \to W$. If P is as above, then $P = \mu + \lambda + \rho$ corresponds exactly to this decomposition. $\square$

2.6. Hochschild cohomology and multiplicative structures. Let V,W and P be as in Theorem 2.4 and let $\nu : W \times W \to W$ be a G-graded algebra structure, so $\nu \in M^{(1,-1,0)}(E)$. Then for $C_i \in L^{(k_i,c_i)}(V,W)$ we define

$$C_1 \bullet C_2 := [C_1, [C_2, \nu]^\Delta]^\Delta = \pm\nu(C_1, C_2).$$

Since $[C_1, C_2]^\Delta = 0$ it follows that $(L(V,W), \bullet)$ is $(\mathbb{Z} \times G)$-graded commutative.

Theorem.

1. The mapping $[P, \quad]^\Delta : M(E) \to M(E)$ is a differential. Its restriction δ_P to $L(V,W)$ is a generalization of the Hochschild coboundary operator to the G-graded case: If $C \in L^{(k,c)}(V,W)$, then we have for $X_i \in V^{x_i}$

$$\begin{aligned}(\delta_P C)(X_0, \ldots, X_k) &= \lambda(X_0)C(X_1, \ldots, X_k) \\ &\quad - \sum_{i=0}^{k-1} (-1)^i C(X_0, \ldots, \mu(X_i, X_{i+1}), \ldots, X_k) \\ &\qquad + (-1)^{k+1+\langle x_0+\cdots+x_{k-1}+c, x_k\rangle} \rho(X_k)C(X_0, \ldots, X_{k-1})\end{aligned}$$

The corresponding $(\mathbb{Z} \times G)$-graded cohomology will be denoted by $H(\mathcal{A}, \mathcal{M})$.

2. If $[P, \nu]^\Delta = 0$, then δ_P is a derivation of $L(V,W)$ of $(\mathbb{Z} \times G)$-degree $(1,0)$. In this case the product $\bullet$ carries over to a $(\mathbb{Z} \times G)$-graded (cup) product on $H(\mathcal{A}, \mathcal{M})$.

3. n-ARY G-GRADED ASSOCIATIVE ALGEBRAS AND n-ARY MODULES

3.1. Definition. Let V be a G-graded vector space. Let $\mu \in M^{(n-1,0)}(V)$, so $\mu : V^{\otimes n} \to V$ is n-linear of weight $0 \in G$.

We call μ an *n-ary associative G-graded multiplication* of weigth $0 \in G$ if $j(\mu)\mu = 0 \in M^{(2n-2,0)}(V)$.

Remark. We are forced to use $j(\mu)\mu = 0$ instead of $[\mu, \mu]^\Delta = 0$ since the latter condition is automatically satisfied for odd n.

3.2. Example. If V is 0-graded, then a ternary associative multiplication $\mu : V \times V \times V \to V$ satisfies

$$\begin{aligned}(j(\mu)\mu)(X_0, \ldots, X_5) &= \mu(\mu(X_0, X_1, X_2), X_3, X_4) + \\ &\quad + \mu(X_0, \mu(X_1, X_2, X_3), X_4) + \mu(X_0, X_1, \mu(X_2, X_3, X_4)) = 0.\end{aligned}$$

3.3. Definition. Let V and W be G-graded vector spaces. We consider the $(\mathbb{Z}\times G)$-graded vector space E defined by

$$E^{(k,*)} = \begin{cases} V & \text{if } k=0 \\ W & \text{if } k=1 \\ 0 & \text{otherwise.} \end{cases}$$

Then $P \in M^{(n-1,0,0)}(E)$ is called an *n-ary G-graded module structure* on W over an n-ary algebra structure on V if $j(P)P = 0$. Let us denote the resulting n-ary algebra by $\mathcal{A}$, and the n-ary module by $\mathcal{W}$.

The mapping P is the sum of partial mappings

$$\begin{aligned}
&\mu = P : V \times \ldots \times V \to V && \text{the } n\text{-ary algebra structure}\\
&P : W \times V \times \ldots \times V \to W && \text{the rightmost } n\text{-ary module structure}\\
&P : V \times W \times V \times \ldots \times V \to W &&\\
&\quad \ldots &&\\
&P : V \times \ldots \times V \times W \times V \to W &&\\
&P : V \times \ldots \times V \times W \to W && \text{the leftmost } n\text{-ary module structure}
\end{aligned}$$

This decomposition of P corresponds exactly to the last line in the decomposition of $M^{(n-1,0,*)}$ of 2.5.

The above definition is easily generalized by changing the form degree of W or/and by augmenting the number of W's. For simplicity we don't discuss this possibility here.

3.4. Example. If V and W are 0-graded then a ternary module satisfies the following conditions besides the one from 3.2 describing the ternary algebra structure on V:

$$\begin{aligned}
&P(P(w_0,v_1,v_2),v_3,v_4) + P(w_0,\mu(v_1,v_2,v_3),v_4) + P(w_0,v_1,\mu(v_2,v_3,v_4)) = 0\\
&P(P(v_0,w_1,v_2),v_3,v_4) + P(v_0,P(w_1,v_2,v_3),v_4) + P(v_0,w_1,\mu(v_2,v_3,v_4)) = 0\\
&P(P(v_0,v_1,w_2),v_3,v_4) + P(v_0,P(v_1,w_2,v_3),v_4) + P(v_0,v_1,P(w_2,v_3,v_4)) = 0\\
&P(\mu(v_0,v_1,v_2),w_3,v_4) + P(v_0,P(v_1,v_2,w_3),v_4) + P(v_0,v_1,P(v_2,w_3,v_4)) = 0\\
&P(\mu(v_0,v_1,v_2),v_3,w_4) + P(v_0,\mu(v_1,v_2,v_3),w_4) + P(v_0,v_1,P(v_2,v_3,w_4)) = 0
\end{aligned}$$

3.5. Hochschild cohomology for even n. Let V and W be G-graded vector spaces, and let $P \in M^{(n-1,0,0)}(E)$ be an n-ary module structure on W over an n-ary G-graded algebra structure on V as in definition 3.3.

Theorem. *Let $n = 2k$ be even. Then we have:*

The mapping $[P, \quad]^\Delta : M(E) \to M(E)$ is a differential. Its restriction δ_P to $L(V,W)$ is called the Hochschild coboundary operator. For a cochain $C \in M^{(k,1,c)} = L^{(k+1,c)}(V,W)$ and with $p = n-1$ we have for $X_i \in V^{x_i}$

$$\begin{aligned}
(\delta_P C)(X_0,\dots,X_{k+p}) &= \sum_{i=0}^{k} (-1)^{pi} C(X_0\dots,P(X_i,\dots,X_{i+p}),\dots,X_{k+p})\\
&\quad - \sum_{j=0}^{p} (-1)^{k(j+p)+\langle x_0+\dots+x_{j-1},c\rangle} P(X_0,\dots,C(X_j,\dots,X_{j+k}),\dots,X_{k+p}).
\end{aligned}$$

The corresponding $(\mathbb{Z} \times G)$*-graded cohomology will be denoted by* $H(\mathcal{A}, \mathcal{M})$.

Proof. We have by the $(\mathbb{Z}^2 \times G)$-graded Jacobi identity

$$[P, [P, Q]^\Delta]^\Delta = [[P, P]^\Delta, Q]^\Delta + (-1)^{(n-1)^2}[P, [P, Q]^\Delta]^\Delta$$

which implies that $[P, \quad]^\Delta$ is a differential since $n-1$ is odd and $[P, P]^\Delta = j(P)P - (-1)^{(n-1)^2} j(P)P = 2j(P)P = 0$. The rest follows from a computation. □

3.6. Remark. We get an easy extension of the Hochschild coboundary operator for n-ary algebra structures for odd n if we choose the weigth accordingly. Let $P \in M^{(n-1,0,p)}(E)$ be an n-ary module structure of weight p on W over an n-ary G-graded algebra structure of weight p on V, similarly as in definition 3.3: We require that $j(P)P = 0$. Let us suppose that $\|(n-1, 0, p)\|^2 = (n-1)^2 + \langle p, p\rangle$ is odd. Then by 2.2 we have

$$[P, P]^\Delta = \Big(1 - (-1)^{(n-1)^2 + \langle p,p\rangle}\Big) j(P)P = 2j(P)P = 0,$$
$$[P, [P, Q]^\Delta]^\Delta = [[P, P]^\Delta, Q]^\Delta + (-1)^{(n-1)^2+\langle p,p\rangle}[P, [P, Q]^\Delta]^\Delta = 0,$$

so that we get a differential. A dual version of this can be seen in 7.2.(3) below.

3.7. Ideals. Let (V, μ) be an n-ary G-graded associative algebra. An ideal I in (V, μ) is a linear subspace $I \subset V$ such that $\mu(X_1, \dots, X_n) \in I$ whenever one of the $X_i \in I$. Then μ factors to an n-ary associative multiplication on the quotient space V/I. This quotient space is again G-graded, if I is a G-graded subspace in the sense that $I = \bigoplus_{x\in G}(I \cap V^x)$.

Of course any ideal I is an n-ary module over (V, μ) which is G-graded if and only if I is G-graded. Conversely, any n-ary module W over (V, μ) is an ideal in the n-ary algebra $V \oplus W = E$ with the multiplication P from 3.3. Here $P(X_1, \dots, X_n) = 0$ if any two elements X_i lie in W, so that E may be regarded as an G-graded or as a $(\mathbb{Z} \times G)$-graded algebra. It could be called also the *semidirect product* of V and W.

3.8. Homomorphisms. A linear mapping $f : V \to W$ of degree 0 between two G-graded algebras (V, μ) and (W, ν) is called a *homomorphism of* G*-graded algebras* if it is compatible with the two n-ary multiplications:

$$f(\mu(X_1, \dots, X_n)) = \nu(f(X_1), \dots, f(X_n))$$

Then the kernel of f is an n-ary ideal in (V, μ) and the image of f is an n-ary subalgebra of (W, ν) which is isomorphic to $V/\ker(f)$.

Similarly we can define the notion of an n-ary V-module homomorphism between two V-modules W_0 and W_1. Then the category of all (G-graded) n-ary V-modules and of their homomorphisms is an abelian category. We did not investigate the relation to the embedding theorem of Freyd and Mitchell.

4. Review of G-graded Lie algebras and modules

In this section we sketch the theory from [7] for G-graded Lie algebras from a slightly different angle. In this section section we need that the ground field $\mathbb{K}$ has characteristic 0.

4.1. Multigraded signs of permutations. Let $\mathbf{x} = (x_1, \dots , x_k) \in G^k$ be a multi index of G-degrees $x_i \in G$ and let $\sigma \in \mathcal{S}_k$ be a permutation of k symbols. Then we define the *G-graded sign* $\operatorname{sign}(\sigma, \mathbf{x})$ as follows: For a transposition $\sigma = (i, i+1)$ we put $\operatorname{sign}(\sigma, \mathbf{x}) = -(-1)^{\langle x_i, x_{i+1}\rangle}$; it can be checked by combinatorics that this gives a well defined mapping $\operatorname{sign}(\quad, \mathbf{x}) : \mathcal{S}_k \to \{-1, +1\}$.

Let us write $\sigma x = (x_{\sigma 1}, \dots , x_{\sigma k})$, then we have the following

Lemma. $\operatorname{sign}(\sigma \circ \tau, \mathbf{x}) = \operatorname{sign}(\sigma, \mathbf{x}). \operatorname{sign}(\tau, \sigma\mathbf{x})$. $\square$

4.2 Multigraded Nijenhuis-Richardson algebra. We define the *G-graded alternator* $\alpha : M(V) \to M(V)$ by

$$(\alpha K)(X_0, \dots, X_k) = \frac{1}{(k+1)!} \sum_{\sigma \in \mathcal{S}_{k+1}} \operatorname{sign}(\sigma, \mathbf{x}) K(X_{\sigma 0}, \dots , X_{\sigma k}) \tag{1}$$

for $K \in M^{(k,*)}(V)$ and $X_i \in V^{x_i}$. By lemma 4.1 we have $\alpha^2 = \alpha$ so α is a projection on $M(V)$, homogeneous of $(\mathbb{Z} \times G)$-degree 0, and we set

$$A(V) = \bigoplus_{(k,\kappa)\in\mathbb{Z}\times G} A^{(k,\kappa)}(V) = \bigoplus_{(k,\kappa)\in\mathbb{Z}\times G} \alpha(M^{(k,\kappa)}(V)).$$

A long but straightforward computation shows that for $K_i \in M^{(k_i,\kappa_i)}(V)$

$$\alpha(j(\alpha K_1)\alpha K_2) = \alpha(j(K_1)K_2),$$

so the following operator and bracket is well defined:

$$\begin{aligned} i(K_1)K_2 :&= \frac{(k_1 + k_2 + 1)!}{(k_1 + 1)!(k_2 + 1)!} \alpha(j(K_1)K_2) \\ [K_1, K_2]^\wedge &= \frac{(k_1 + k_2 + 1)!}{(k_1 + 1)!(k_2 + 1)!} \alpha([K_1, K_2]^\Delta) \\ &= i(K_1)K_2 - (-1)^{\langle (k_1 \kappa_1),(k_2,\kappa_2)\rangle} i(K_2)K_1 \end{aligned}$$

The combinatorial factor is explained in [7], 3.4.

4.3. Theorem. *1. If K_i are as above, then*

$$\begin{aligned} &(i(K_1)K_2)(X_0, \dots , X_{k_1+k_2}) = \\ &\qquad = \frac{1}{(k_1 + 1)!k_2!} \sum_{\sigma\in\mathcal{S}_{k_1+k_2+1}} \operatorname{sign}(\sigma, \mathbf{x})(-1)^{\langle\kappa_1,\kappa_2\rangle} \cdot \\ &\qquad\qquad \cdot K_2((K_1(X_{\sigma 0}, \dots , X_{\sigma k_1}), \dots , X_{\sigma(k_1+k_2)}). \end{aligned}$$

2. $(A(V), [\quad,\quad]^\wedge)$ is a $(\mathbb{Z} \times G)$-graded Lie algebra.

3. If $\mu \in A^{(1,0)}(V)$, so $\mu : V \times V \to V$ is bilinear G-graded anticommutative mapping of weight $0 \in G$, then $i(\mu)\mu = 0$ if and only if (V, μ) is a G-graded Lie algebra.

Proof. For 1 and 2 see [7].

3. Let $\mu \in A^{(1,0)}(V)$, then from 1 we see that

$$(i(\mu)\mu)(X_0, X_1, X_2) = \tfrac{1}{2!} \sum_{\sigma\in\mathcal{S}_3} \operatorname{sign}(\sigma, \mathbf{x}) \cdot \mu(\mu(X_{\sigma 0}, X_{\sigma 1}), X_{\sigma 2}))$$

which is equivalent to the G-graded Jacobi expression of (V, μ). $\square$

$(A(V), [\quad,\quad]^\wedge)$ is called the *$(\mathbb{Z} \times G)$-graded Nijenhuis-Richardson algebra*, since $A(V)$ coincides for $G = 0$ with $Alt(V)$ of [14].

4.4. Theorem. *Let V and W be G-graded vector spaces. Let E be the $(\mathbb{Z}\times G)$-graded vector space defined by*

$$E^{(k,*)} = \begin{cases} V & \text{if } k = 0 \\ W & \text{if } k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $P \in A^{(1,0,0)}(E)$ then $i(P)P = 0$ if and only if

(a) $$i(\mu)\mu = 0$$

so $(V,\mu) = \mathfrak{g}$ is a G-graded Lie algebra, and

(b) $$\rho(\mu(X_1, X_2))Y = [\rho(X_1), \rho(X_2)]Y$$

where $\mu(X_1, X_2) = P(X_1, X_2) \in V$ and $\rho(X)Y = P(X, Y) \in W$ for X, $X_i \in V$ and $Y \in W$, and where $[\quad,\quad]$ denotes the G-graded commutator in $\mathrm{End}(W)$. So $i(P)P = 0$ is by definition equivalent to the fact that $\mathcal{M} := (W, \rho)$ is a G-graded Lie-$\mathfrak{g}$ module.

If P is as above the mapping $\partial_P := [P,\quad]^\wedge : A(E) \to A(E)$ is a differential and its restriction to

$$\bigoplus_{k\in\mathbb{Z}} \Lambda^{(k,*)}(\mathfrak{g}, \mathcal{M}) := \bigoplus_{k\in\mathbb{Z}} A^{(k,1,*)}(E)$$

generalizes the Chevalley-Eilenberg coboundary operator to the G-graded case:

$$\begin{aligned}(\partial_P C)(X_0, \dots, X_k) &= \sum_{i=0}^{k} (-1)^{\alpha_i(\mathbf{x})+\langle x_i, c\rangle} \rho(X_i) C(X_0, \dots, \widehat{X_i}, \dots, X_k) \\ &\quad + \sum_{i<j} (-1)^{\alpha_{ij}(\mathbf{x})} C(\mu(X_i, X_j), \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots)\end{aligned}$$

where

$$\begin{cases} \alpha_i(\mathbf{x}) = \langle x_i, x_0 + \cdots + x_{i-1}\rangle + i \\ \alpha_{ij}(\mathbf{x}) = \alpha_i(\mathbf{x}) + \alpha_i(\mathbf{x}) + \langle x_i, x_j\rangle \end{cases}$$

We denote the corresponding $(\mathbb{Z} \times G)$-graded cohomology space by $H(\mathfrak{g}, \mathcal{M})$.

If $\nu : W \times W \to W$ is G-graded symmetric (so $\nu \in A^{(1,-1,*)}(E)$) and $[P, \nu]^\wedge = 0$ then ∂_P acts as derivation of G-degree $(1, 0)$ on the $(\mathbb{Z} \times G)$-graded commutative algebra $(\Lambda(\mathfrak{g}, \mathcal{M}), \bullet)$, where

$$C_1 \bullet C_2 := [C_1, [C_2, \nu]^\wedge]^\wedge \quad C_i \in \Lambda^{(k_i,c_i)}(\mathfrak{g}, \mathcal{M}).$$

In this situation the product $\bullet$ carries over to a $(\mathbb{Z} \times G)$-graded symmetric (cup) product on $H(\mathfrak{g}, \mathcal{M})$.

Proof. Apply the G-graded alternator α to the results of 2.3, 2.4, 2.5, and 2.6. $\square$

5. n-ary G-graded Lie algebras and their modules

5.1. Definition. Let V be a G-graded vector space. Let $\mu \in A^{(n-1,0)}(V)$, so $\mu : V^n \to V$ is a G-graded skew symmetric n-linear mapping.

We call μ an *n-ary G-graded Lie algebra structure* on V if $i(\mu)\mu = 0$.

5.2. Example. If V is 0-graded, then a ternary Lie algebra structure on V is a skew symmetric trilinear mapping $\mu : V \times V \times V \to V$ satisfying

$$\begin{aligned}
0 = (i(\mu)\mu)(X_0, \dots , X_4) &= \frac{1}{3!\,2!} \sum_{\sigma\in\mathcal{S}_3} \operatorname{sign}(\sigma)\, \mu(\mu(X_{\sigma 0}, X_{\sigma 1}, X_{\sigma 2}), X_{\sigma 3}, X_{\sigma 4}) \\
&= +\mu(\mu(X_0, X_1, X_2), X_3, X_4) - \mu(\mu(X_0, X_1, X_3), X_2, X_4) \\
&\quad + \mu(\mu(X_0, X_1, X_4), X_2, X_3) + \mu(\mu(X_0, X_2, X_3), X_1, X_4) \\
&\quad - \mu(\mu(X_0, X_2, X_4), X_1, X_3) + \mu(\mu(X_0, X_3, X_4), X_1, X_2) \\
&\quad - \mu(\mu(X_1, X_2, X_3), X_0, X_4) + \mu(\mu(X_1, X_2, X_4), X_0, X_3) \\
&\quad - \mu(\mu(X_1, X_3, X_4), X_0, X_2) + \mu(\mu(X_2, X_3, X_4), X_0, X_1)
\end{aligned}$$

5.3. Definition. Let V and W be G-graded vector spaces. We consider the $(\mathbb{Z}\times G)$-graded vector space E defined by

$$E^{(k,*)} = \begin{cases} V & \text{if } k = 0 \\ W & \text{if } k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then $P \in A^{(n-1,0,0)}(E)$ is called an *n-ary G-graded Lie module structure* on W over an n-ary Lie algebra structure on V if $i(P)P = 0$. Let us denote the resulting n-ary Lie algebra by $\mathfrak{g}$, and the n-ary module by $\mathcal{W}$.

Ordering by degree and using the G-graded skew symmetry we see that P is now the sum of only two partial n-linear mappings

$$\begin{aligned}
&\mu = P : V \times \ldots \times V \to V && \text{the } n\text{-ary Lie algebra structure} \\
&\rho = P : V \times \ldots \times V \times W \to W && \text{the } n\text{-ary Lie module structure}
\end{aligned}$$

5.4. Example. If V and W are 0-graded, then a ternary Lie module satisfies the following condition besides the one from 5.2 describing the ternary Lie algebra structure on V:

$$\begin{aligned}
0 &= \rho(\mu(v_0, v_1, v_2), v_3, w) - \rho(\mu(v_0, v_1, v_3), v_2, w) + \rho(v_2, v_3, \rho(v_0, v_1, w)) \\
&\quad + \rho(\mu(v_0, v_2, v_3), v_1, w) - \rho(v_1, v_3, \rho(v_0, v_2, w)) + \rho(v_1, v_2, \rho(v_0, v_3, w)) \\
&\quad - \rho(\mu(v_1, v_2, v_3), v_0, w) + \rho(v_0, v_3, \rho(v_1, v_2, w)) - \rho(v_0, v_2, \rho(v_1, v_3, w)) \\
&\quad + \rho(v_0, v_1, \rho(v_2, v_3, w)).
\end{aligned}$$

5.5. Theorem. *If P is as in 5.3 above and if n is even then the mapping $\partial_P := [P, \quad]^\wedge : A(E) \to A(E)$ is a differential. Its restriction to*

$$\bigoplus_{k\in\mathbb{Z}} \Lambda^{(k,*)}(V, W) := \bigoplus_{k\in\mathbb{Z}} A^{(k,1,*)}(E)$$

generalizes the Chevalley-Eilenberg coboundary operator to the G-graded case: For $C \in A^{(c,1,\gamma)}(E) = \Lambda^{(c,\gamma)}(V,W)$ we have

$$\begin{aligned}(\partial_P C)(X_1,\dots,X_{k+n}) &= [P,C]^\wedge(X_1,\dots,X_{k+n}) =\\ &= \tfrac{-1}{(n-1)!(k+1)!}\sum_{\sigma\in\mathcal{S}_{k+n}} \operatorname{sign}(\sigma,\mathbf{x})(-1)^{\langle x_{\sigma 1}+\dots+x_{\sigma(n-1)},\gamma\rangle}\\ &\qquad\qquad \rho(X_{\sigma 1},\dots,X_{\sigma(n-1)}).C(X_{\sigma n},\dots,X_{\sigma(k+n)})+\\ &\quad+\tfrac{1}{n!k!}\sum_{\sigma\in\mathcal{S}_{k+n}} \operatorname{sign}(\sigma,\mathbf{x})C(\mu(X_{\sigma 1},\dots,X_{\sigma(n)}),X_{\sigma(n+1)},\dots,X_{\sigma(k+n)})\end{aligned}$$

We denote the corresponding cohomology space by $H(\mathfrak{g},\mathcal{M})$.

If $\nu : W\times W\to W$ is G-graded symmetric (so $\nu\in A^{(1,-1,*)}(E)$) and $[P,\nu]^\wedge = 0$ then ∂_P acts as derivation of $(\mathbb{Z}\times G)$-degree $(1,0)$ on the $(\mathbb{Z}\times G)$-graded commutative algebra $(\Lambda(\mathfrak{g},\mathcal{M}),\bullet)$, where

$$C_1\bullet C_2 := [C_1,[C_2,\nu]^\wedge]^\wedge \quad C_i\in\Lambda^{(k_i,c_i)}(\mathfrak{g},\mathcal{M}).$$

In this situation the product $\bullet$ carries over to a $(\mathbb{Z}\times G)$-graded symmetric (cup) product on $H(\mathfrak{g},\mathcal{M})$.

Proof. We have by the $(\mathbb{Z}^2\times G)$-graded Jacobi identity

$$[P,[P,Q]^\wedge]^\wedge = [[P,P]^\wedge,Q]^\wedge + (-1)^{(n-1)^2}[P,[P,Q]^\wedge]^\wedge$$

which implies that $[P,\quad]^\wedge$ is a differential since $n-1$ is odd and $[P,P]^\wedge = j(P)P - (-1)^{(n-1)^2}j(P)P = 2j(P)P = 0$.

The rest follows from a computation. $\square$

5.6. Ideals. Let (V,μ) be an n-ary G-graded Lie algebra. An ideal I in (V,μ) is a linear subspace $I\subset V$ such that $\mu(X_1,\dots,X_n)\in I$ whenever one of the $X_i\in I$. Then μ factors to an n-ary Lie algebra structure on the quotient space V/I. This quotient space is again G-graded, if I is a G-graded subspace in the sense that $I = \bigoplus_{x\in G}(I\cap V^x)$.

Of course, any ideal I is an n-ary module over (V,μ) which is G-graded if and only if I is G-graded. Conversely, any n-ary module W over (V,μ) is an ideal in the n-ary algebra $V\oplus W = E$ with the multiplication P from 5.3. Here $P(X_1,\dots,X_n) = 0$ if any two elements X_i lie in W, so that E may be regarded as an G-graded or as a $(\mathbb{Z}\times G)$-graded Lie algebra. It could be called also the *semidirect product* of V and W.

5.7. Homomorphisms. A linear mapping $f : V\to W$ of degree 0 between two G-graded algebras (V,μ) and (W,ν) is called a *homomorphism of G-graded Lie algebras* if it is compatible with the two n-ary multiplications:

$$f(\mu(X_1,\dots,X_n)) = \nu(f(X_1),\dots,f(X_n))$$

Then the kernel of f is an n-ary ideal in (V,μ) and the image of f is an n-ary subalgebra of (W,ν) which is isomorphic to $V/\ker(f)$.

Similarly, we can define the notion of an n-ary V-module homomorphism between two V-modules W_0 and W_1.

6. Relations between n-ary algebras and Lie algebras

6.1. The n-ary commutator. Let $\mu \in M^{(n-1,0)}(V)$, so $\mu : V \times \ldots \times V \to V$ is an n-ary multiplication. The G-graded alternator α from 4.2 transforms μ into an element

$$\gamma\mu := n!\,\alpha\mu \in A^{(n,0)}(V),$$

which we call the *n-ary commutator* of μ. From 4.2 we also have:

If μ is n-ary associative, then $\gamma\mu$ is an n-ary Lie algebra structure on V.

Definition. An n-ary $(\mathbb{Z}\times G)$-graded multiplication $\mu \in M^{(n-1,0)}(V)$ is called *n-ary Lie admissible* if $\gamma\mu$ is an n-ary $(\mathbb{Z}\times G)$-graded Lie algebra structure. By 5.1 this is the case if and only if $i(\gamma\mu)(\gamma\mu) = \frac{(2n-1)!}{(n!)^2}\alpha(j(\mu)\mu) = 0$; i. e. the alternation of the n-ary associator $j(\mu)(\mu)$ vanishes. For the binary version of this notion see [12] and [11].

An n-ary multiplication μ is called *n-ary commutative* if $\gamma\mu = 0$.

6.2. Induced mapping in cohomology. Let V and W be G-graded vector spaces and let E be the $(\mathbb{Z}\times G)$-graded vector space

$$E^{(k,*)} = \begin{cases} V & \text{if } k = 0 \\ W & \text{if } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

as in 3.3. Let $P \in M^{(n-1,0)}(E)$ be an n-ary G-graded module structure on W over an n-ary algebra structure on V, i. e. $j(P)P = 0$.

Then $\gamma P = n!\,\alpha P \in A^{(n-1,0)}(E)$ is an n-ary G-graded Lie module structure on W over V and some multiple of α defines a homomorphism from the Hochschild cohomology of (V,μ) with values in W into the Chevalley cohomology of $(V,\gamma\mu)$ with values in the Lie module V.

7. Hochschild operations and non commutative differential calculus

7.1. Let V be a G-graded vector space. We consider the tensor algebra $V^{\otimes} = \bigoplus_{k=0}^{\infty} V^{\otimes k}$ which is now $(\mathbb{Z}\times G)$-graded such that the degree of $X_1 \otimes \cdots \otimes X_i$ is $(i, x_1 + \cdots + x_i)$. Put also $V_n^{\otimes} = \bigoplus_{k\geq n}^{\infty} V^{\otimes k}$. Obviously, $V_o^{\otimes} = V^{\otimes}$.

The *Hochschild operator* δ_K associated with $K \in M^{(k,\kappa)}(V)$ (as in 2.2) is a map $\delta_K : V_k^{\otimes} \to V_1^{\otimes}$ given by

$$\delta_K = 0 \quad \text{on} \quad V^{\otimes k} \quad \text{and}$$

$$\begin{aligned} &\delta_K(X_0 \otimes \cdots \otimes X_l) := \\ &\quad = \sum_{i=0}^{l-k} (-1)^{ki + \langle \kappa, x_0 + \cdots + x_{i-1}\rangle} X_0 \otimes \cdots \otimes X_{i-1} \otimes K(X_i \otimes \cdots \otimes X_{i+k}) \otimes \cdots \otimes X_l \end{aligned}$$

In the natural $(\mathbb{Z}\times G)$-grading of $L(V^{\otimes}, V^{\otimes})$ the operator δ_K has degree $(-k,\kappa)$. The mapping δ is called the *Hochschild operation* since for an associative multiplication $\mu : V \times V \to V$ the operator δ_μ is the differential of the Hochschild homology.

For $K_i \in M^{(k_i,\kappa_i)}(V)$ with $k_i > 0$ the composition $\delta_{K_1} \circ \delta_{K_2}$ is well–defined as a map from $V_{k_1+k_2}^{\otimes}$ to $V_1^{\otimes}$.

7.2. Proposition. *For $K_i \in M^{(k_i,\kappa_i)}(V)$ we have*

(1) *in general* $\delta_{K_1} \circ \delta_{K_2} \neq \delta_{j(K_1)K_2}$,
(2) $[\delta_{K_1}, \delta_{K_2}] = \delta_{K_1} \circ \delta_{K_2} - (-1)^{k_1k_2+\langle\kappa_1,\kappa_2\rangle}\delta_{K_2} \circ \delta_{K_1} = \delta_{[K_1,K_2]^\triangle}$,
(3) $[\delta_K, \delta_K] = 2\delta_K \circ \delta_K = 2\delta_{j(K)K}$ *if and only if* $\|\deg(\delta_K)\|^2 = k^2 + \langle\kappa,\kappa\rangle \equiv 1 \mod 2$.

Proof. We get

$$
\begin{aligned}
&\delta_{K_1} \circ \delta_{K_2}(X_1 \otimes \cdots \otimes X_s) = \\
&= \sum_{j+k_2<i} (-1)^{k_1 i+\langle\kappa_1,x_0+\cdots+x_{i-1}\rangle+k_2 j+\langle\kappa_2,x_0+\cdots+x_{i-1}\rangle} \\
&\qquad X_0 \otimes \cdots \otimes K_2(X_j \otimes \cdots \otimes X_{j+k_2}) \otimes \cdots \otimes K_1(X_i \otimes \cdots \otimes X_{i+k_1}) \otimes \cdots \otimes X_s \\
&\quad + \sum_{i-k_2\le j\le i} (-1)^{k_1 i+\langle\kappa_1,x_0+\cdots+x_{i-1}\rangle+k_2 j+\langle\kappa_2,x_0+\cdots+x_{i-1}\rangle} \\
&\qquad X_0 \otimes \cdots \otimes K_2(X_j \otimes \cdots \otimes K_1(X_i \otimes \cdots \otimes X_{i+k_1}) \otimes \cdots \otimes X_{j+k_1+k_2}) \otimes \cdots \otimes X_s \\
&\quad + \sum_{j>i} (-1)^{k_1 i+\langle\kappa_1,x_0+\cdots+x_{i-1}\rangle+k_2 j+\langle\kappa_2,x_0+\cdots+x_{i-1}\rangle+k_1k_2+\langle\kappa_1,\kappa_2\rangle} \\
&\qquad X_0 \otimes \cdots \otimes K_1(X_i \otimes \cdots \otimes X_{i+k_1}) \otimes \cdots \otimes K_2(X_j \otimes \cdots \otimes X_{j+k_2}) \otimes \cdots \otimes X_s.
\end{aligned}
$$

From this all assertions follow. □

7.3. Rudiments of a non commutative differential calculus. An intrinsic characterization of the Hochschild operators can be given as follows. For $X \in V^x$ we consider the left and right multiplication operators $X^l, X^r \in L(V_m^\otimes, V_n^\otimes)^{(1,x)}$ which are given by

$$
\begin{aligned}
X^l(X_1 \otimes \cdots \otimes X_k) &:= X \otimes X_1 \otimes \cdots \otimes X_k, \\
X^r(X_1 \otimes \cdots \otimes X_k) &:= (-1)^{k+\langle x,x_1+\cdots+x_k\rangle} X_1 \otimes \cdots \otimes X_k \otimes X.
\end{aligned}
$$

Then we have $[X^l, Y^r] = 0$ in $L(V_m^\otimes, V_n^\otimes)$ for all $X, Y \in V$.

Proposition. *An operator $A \in L(V_k^\otimes, V_1^\otimes)$ is of the form $A = \delta_K$ for an uniquely defined $K \in M(V)^{(k,\kappa)}$ if and only if $A|V^{\otimes k} = O$ and $[X_0^l, [X_1^r, A]] = 0$ in $L(V_k^\otimes, V_1^\otimes)$ for all $X_i \in V$.*

Proof. A computation. □

In view of the theory developed in [18] (see also [6], [19]) the Hochschild operators δ_K can be naturaly interpreted as the first order differential operators in the current non–commutative context.

7.4. Example. An element $e \in V$ is the left (resp., right) unit of a binary multiplication μ on V if and only if $[\delta_\mu, e^l] = id$ (on $V_1^\otimes$) (resp., $[\delta_\mu, e^r] = id$). Differential calculus touched in 7.3 can be put in the following general cadre.

7.5. Definition. Let $\mathbf{A}$ be a G-graded associative (binary) algebra. For $A, B \in \mathbf{A}$ let $A^l, B^r : \mathbf{A} \to \mathbf{A}$ be the left and (signed) right multiplications, $A^l(B) = (-1)^{\langle a,b\rangle} B^r(A) = AB$. Then we have

$$[A^l, B^r] = A^l \circ B^r - (-1)^{\langle a,b\rangle} B^r \circ A^l = 0.$$

A differential operator $\mathbf{A} \to \mathbf{A}$ of order (p,q) is an element $\Delta \in L(\mathbf{A},\mathbf{A})$ such that

$$[X_1^l, [\dots, [X_p^l, [Y_1^r, [\dots, [Y_q^r, \Delta] \dots] = 0 \quad \text{for all } X_i, Y_j \in \mathbf{A},$$

which we also denote by the shorthand $l^p r^q \Delta = 0$. Obviously this definition also makes sense for mappings $\mathbf{M} \to \mathbf{N}$ between G-graded $\mathbf{A}$-bimodules, where now A^l is left multiplication of $A \in \mathbf{A}$ on any G-graded $\mathbf{A}$-bimodule, etc.

7.6. Example. $\mathbf{A} = L(V,V)$ Let V be a finite dimensional vector space, ungraded for simplicity's sake, and let us consider the associative algebra $\mathbf{A} = L(V,V)$.

Proposition. *If $\Delta : L(V,V) \to L(V,V)$ is a differential operator of order (p,q) with $(p,q>0)$, then*

$$\Delta = \begin{cases} P^r, & \text{if} \quad l^p\Delta = 0 \\ Q^l, & \text{if} \quad r^q\Delta = 0 \\ P^r + Q^l, & \text{if} \quad l^p r^q \Delta = 0 \end{cases}$$

where P and Q are in $L(V,V)$.

Proof. We shall use the notation $l_Y\Delta := [Y^l, \Delta]$ and similarly $r_Y\Delta = [Y^r, \Delta]$, for $Y \in L(V,V)$. We start with the following

Claim. *If $l_Y\Delta = P_Y^l + Q_Y^r$ for each $Y \in L(V,V)$ and suitable $P = P_Y, Q = Q_Y : L(V,V) \to L(V,V)$, then we have $\Delta = A^l + B^r$ where $A = 0$ if $P = 0$. If on the other hand $r_Y\Delta = P_Y^l + Q_Y^r$ for each Y then we have $\Delta = A^l + B^r$ where $B = 0$ if $Q = 0$.*

Let us assume that $l_Y\Delta = P_Y^l + Q_Y^r$ for each Y. By replacing Δ by $\Delta - \Delta(1)^r$ we may assume without loss that $\Delta(1) = 0$. We have $(l_Y\Delta)(X) = PX + XQ = (P+Q)X - [Q,X] =: [R,X] + SX$; if we assume that R is traceless then $R = -Q$ and $S = P + Q$ are uniquely determined, thus linear in Y. Thus

$$Y\Delta(X) - \Delta(YX) = [R_Y, X] + S_Y X$$

Insert $X = 1$ and use $\Delta(1) = 0$ to obtain $\Delta(Y) = -S_Y$, hence

$$[R_Y, X] = Y\Delta(X) + \Delta(Y)X - \Delta(YX) \tag{1}$$

Replacing Y by YZ and applying the equation (1) repeatedly we obtain

$$\begin{aligned}
[R_{YZ}, X] &= YZ\Delta(X) + \Delta(YZ)X - \Delta(YZX) \\
&= YZ\Delta(X) + Y\Delta(Z)X + \Delta(Y)ZX - [R_Y, Z]X \\
&\quad - Y\Delta(ZX) - \Delta(Y)ZX + [R_Y, ZX] \\
&= YZ\Delta(X) + Y\Delta(Z)X - YZ\Delta(X) - Y\Delta(Z)X + Y[R_Z, X] + Z[R_Y, X] \\
&= Y[R_Z, X] + Z[R_Y, X].
\end{aligned}$$

The right hand side is symmetric in Y and Z, thus $[R_{[Y,Z]}, X] = 0$; inserting $Y = Z = 1$ we get also $[R_1, X] = 0$, hence $R = 0$. From (1) we see that $\Delta : L(V,V) \to L(V,V)$ is a derivation, thus of the form $\Delta(X) = [A,X] = (A^l - A^r)(X)$. If $P = 0$ then $\Delta = -S = R - P = 0$. So the first part of the claim follows since we already substracted $\Delta(1)^r$ from the original Δ.

The second part of the claim follows by mirroring the above proof.

Now we prove the proposition itself. If $l^p\Delta = 0$ then by induction using the first part of the claim with $P = 0$ we have $\Delta = B^r$. Similarly for $r^q\Delta = 0$ we get $\Delta = A^l$.

If $l^p r^q \Delta = 0$ with $p, q > 0$, by induction on $p + q \geq 2$, using the claim, the result follows. $\square$

The obtained result is parallel to the obvious fact that differential operators over 0–dimensional manifolds are of zero order.

8. Remarks on Filipov's n-ary Lie algebras

Here we show how Filoppov's concept of an n–Lie algebra is related with that of 5.1 and sketch a similar framework for it. For simplicity's sake no grading on the vector space is assumed.

8.1. Let V be a vector space. According to [3], an n-linear skew symmetric mapping $\mu : V \times \ldots \times V \to V$ is called an *F-Lie algebra structure* if we have

(1)

$$\mu(\mu(Y_1, \ldots, Y_n), X_2, \ldots, X_n) = \sum_{i=1}^{n} \mu(Y_1 \ldots, Y_{i-1}, \mu(Y_i, X_2, \ldots, X_n), Y_{i+1}, \ldots, Y_n)$$

The idea is that $\mu(\quad, X_2, \ldots, X_n)$ should act as derivation with respect to the 'multiplication' $\mu(Y_1, \ldots, Y_n)$.

8.2. The dot product. For $P \in L^p(V; L(V, V))$ and $Q \in L^q(V; L(V, V))$ let us consider the first entry as the distinguished one (belonging to $L(V, V)$, so that $P(\quad, X_1, \ldots, X_p) \in L(V, V))$ and then let us define $P \cdot Q \in L^{p+q}(V; L(V, V))$ by

$$\begin{aligned}(P \cdot Q)&(Z, Y_1, \ldots, Y_q, X_1, \ldots, X_p) := \\ &= P(Q(Z, Y_1, \ldots, Y_q), X_1, \ldots, X_p) - Q(P(Z, X_1, \ldots, X_p), Y_1, \ldots, Y_q) - \\ &- \sum_{i=1}^{q} Q(Z, Y_1, \ldots, P(Y_i, X_1, \ldots, X_p), \ldots, Y_q)\end{aligned}$$

Then $\mu \in L^{n-1}(V; L(V, V))$ which is skew symmetric in all arguments, is an F-Lie algebra structure if and only if $\mu \cdot \mu = 0$.

8.3. Lemma. *We have*

$$\operatorname{Alt}(P \cdot Q) = (p+1)!(q+1)!(\tfrac{1}{p+1} i_{\operatorname{Alt} Q} \operatorname{Alt} P - (-1)^{pq} i_{\operatorname{Alt} P} \operatorname{Alt} Q),$$

where $\operatorname{Alt} : L^p(V, L(V, V)) \to L^{p+1}_{\text{skew}}(V; V) = A^p(V)$ *is the alternator in all appearing variables.*

In particular, if μ is an n-ary F-Lie algebra structure, then $\operatorname{Alt}\mu$ *is a Lie algebra structure in the sense of 5.1.*

Proof. An easy computation. $\square$

8.4. The grading operator. For a permutation $\sigma \in \mathcal{S}_p$ and $\mathbf{a} = (a_1, \dots, a_p) \in \mathbb{N}_0^p$ let the *grading operator* or *(generalized) sign operator* be given by

$$S^{\mathbf{a}}_\sigma : L^{a_1+\dots+a_p}(V;W) \to L^{a_1+\dots+a_p}(V;W),$$
$$(S^{\mathbf{a}}_\sigma P)(X^1_1, \dots, X^1_{a_1}, \dots, X^p_1, \dots, X^p_{a_p}) = P(X^{\sigma 1}_1, \dots, X^{\sigma 1}_{a_{\sigma 1}}, \dots, X^{\sigma p}_1, \dots, X^{\sigma p}_{a_{\sigma p}}),$$

which obviously satisfies

$$S^{\mathbf{a}}_{\mu\sigma} = S^{\sigma(\mathbf{a})}_\mu \circ S^{\mathbf{a}}_\sigma.$$

We shall use the simplified version $S^{a_1,a_2} = S^{a_1,a_2,*}_{(12)}$ for the permutation of the first two blocks of arguments of lenght a_1 and a_2. Note that also $S^{a,b}(\alpha\otimes\beta\otimes\gamma) = \beta\otimes\alpha\otimes\gamma$.

If P is skew symmetric on V, then $S^{\mathbf{a}}_\sigma P = \operatorname{sign}(\sigma, \mathbf{a})P$, the sign from [7] or 4.1.

8.5. Lemma. *For $P \in L^p(V; L(V,V))$ and $\psi \in L^q(V,W)$ let*

$$(\rho(P)\psi)(X_1, \dots, X_p, Y_1, \dots, Y_q) := -\sum_{i=1}^q \psi(Y_1, \dots, P(Y_i, X_1, \dots, X_p), \dots, Y_q)$$

then we have for $\omega \in L^*(V;\mathbb{R})$

$$\rho(P)(\psi\otimes\omega) = (\rho(P)\psi)\otimes\omega + S^{q,p}\psi\otimes\rho(P)\omega.$$

Proof. A straightforward computation. $\square$

8.6. Lemma 8.5 suggests that $\rho(P)$ behaves like a derivation with coefficients in a trivial representation of $\mathfrak{gl}(V)$ with respect to the sign operators from 8.4. The corresponding derivation with coefficients in the adjoint representation of $\mathfrak{gl}(V)$ then is given by the formula which follows directly from the definitions:

$$P\cdot Q = [P,Q]_{\mathfrak{gl}(V)} + \rho(P)Q,$$

where $[P,Q]_{\mathfrak{gl}(V)}$ is the pointwise bracket

$$[P,Q]_{\mathfrak{gl}(V)}(X_1, \dots) = [P(X_1, \dots), Q(X_{p+1}, \dots)].$$

Moreover we have the following result

8.7. Proposition. *For $P \in L^p(V; L(V,V))$ and $Q \in L^q(V; L(V,V))$ we have*

$$P\cdot(Q\cdot R) - S^{q,p}(Q\cdot(P\cdot R)) = [P,Q]\cdot R,$$

where

$$[P,Q]^S = [P,Q]_{\mathfrak{gl}(V)} + \rho(P)Q - S^{q,p}\rho(Q)P$$

is a graded Lie bracket in the sense that

$$[P,Q]^S = -S^{q,p}[Q,P]^S,$$
$$[P,[Q,R]^S]^S = [[P,Q]^S,R]^S + S^{q,p}[Q,[P,R]^S]^S.$$

Also the derivation ρ is well behaved with respect to this bracket,

$$\rho(P)\rho(Q) - S^{q,p}\rho(Q)\rho(P) = \rho([P,Q]^S).$$

Proof. For decomposable elements like in the proof of lemma 8.5 this is a long but straightforward computation. $\square$

9. Dynamical aspects

It is natural to expect an eventual dynamical realization of algebraic constructions discussed above when the underlying vector space V is the algebra of observables of a mechanical or physical system. In the classical approach it should be an algebra of the form $V = \mathcal{C}^\infty(M)$ with M being the space–time, configuration or phase space of a system, etc. The localizability principle forces us to limit the considerations to n–ary operations which are given by means of multi–fferential operators. The following list of definitions is in conformity with these remarks.

9.1 Definition. An n–Lie algebra structure $\mu(f_1, \dots, f_n)$ on $\mathcal{C}^\infty(M)$ is called

(1) *local*, if μ is a multi–differential operator
(2) *n– Jacobi*, if μ is a first–order differential operator with respect to any its argument
(3) *n–Poisson* if μ is an n–derivation.

(M, μ) is called an n–Jacobi or n–Poisson manifold if μ is an n–Jacobi or, respectively, n–Poisson structure on $\mathcal{C}^\infty(M)$.

It seemes plausible that Kirillov's theorem is still valid for the proposed n–ary generalization. It so, n–Jacobi structures exhaust all local ones.

9.2 Examples. Any k–derivation μ on a manifold M is of the form

$$\mu(f_1, \dots, f_k) = P(df_1, \dots, df_k)$$

where $P = P_\mu$ is a k–vector field on M and vice versa. If k is even, then μ is an n–Poisson structure on M iff $[P_\mu, P_\mu]_{Schouten} = 0$. In particular, μ is a k–Poisson structure in each of below listed cases:

(1) P_μ is of constant coefficients on $M = \mathbb{R}^m$
(2) $P_\mu = X \wedge Q$ where X is a vector field on M such that $L_X(Q) = 0$
(3) $P_\mu = Q_1 \wedge \cdots \wedge Q_r$ where all multi–vector fields Q_i's are of even degree and such that $[Q_i, Q_j]_{Schouten} = 0, \quad \forall\, i, j$.

These examples are taken from [20] where the reader will find a systematical exposition and further structural results.

References

[1] Alekseevsky, D. V.; Guha, P., *On Decomposability of Nambu-Poisson Tensor*, to appear, Acta Mathematica Universitatis Comenianae **65, 1** (1996), 1–9.

[2] Azcárraga, J.A. de; Perelomov, A.M.; Pérez Bueno, J.C., *New generalized Poisson structures*, Preprint FTUV 96-1, IFIC 96-1.

[3] Filippov, V. T., *n-ary Lie algebras*, Sibirskii Math. J. **24, 6** (1985), 126–140. (Russian)

[4] Gnedbaye, A. V., *Les algébres k-aires et leurs opérades*, C. R. Acad. Sci. Paris, Série I **321** (1995), 147–142.

[5] Ibáñez, R.; Leon, M. de; Marrero, J.C.; Martin de Diego, D., *Dynamics of generalized Poisson and Nambu-Poisson brackets*, Preprint July 26, 1996.

[6] Krasil'shchik, I. S.; Lychagin, V. V.; Vinogradov, A. M., *Geometry of jet spaces and nonlinear partial differential equations*, Gordon and Breach, New York, 1986.

[7] Lecomte, Pierre; Michor, Peter W.; Schicketanz, Hubert, *The multigraded Nijenhuis-Richardson Algebra, its universal property and application*, J. Pure Applied Algebra **77** (1992), 87–102.

[8] Lecomte, P. B. A.; Roger, C., *Modules et cohomologies des bigebres de Lie*, C. R. Acad. Sci. Paris **310** (1990), 405–410; *(Note rectificative)*, C. R. Acad. Sci. Paris **311** (1990), 893–894.

[9] Marmo, G.; Vilasi, G.; Vinogradov, A., *The local structure of n- Poisson and n-Jacobi manifolds and some applications*, submitted to J.Geom.Phys..

[10] Michor, Peter W., *Knit products of graded Lie algebras and groups*, Suppl. Rendiconti Circolo Matematico di Palermo, Ser. II **22** (1989), 171–175.

[11] Michor, Peter W.; Ruppert, Wolfgang; Wegenkittl, Klaus, *A connection between Lie algebras and general algebras*, Suppl. Rendiconti Circolo Matematico di Palermo, Serie II, **21** (1989), 265–274.

[12] Myung, H. C., *Malcev-admissible algebras*, Progress in Mathematics Vol. 64, Birkhäuser, Basel – Boston, 1986.

[13] Nambu, Y., *Generalized Hamiltonian dynamics*, Phys. Rev. **D7** (1973), 2405–2412.

[14] Nijenhuis, A.; Richardson, R., *Deformation of Lie algebra structures*, J. Math. Mech. **17** (1967), 89–105.

[15] Nijenhuis, A., *On a class of common properties of some different types of algebras I, II*, Nieuw Archief voor Wiskunde (3) **17** (1969), 17–46, 87–108.

[16] Roger, C., *Algebres de Lie graduees et quantification*, Symplectic Geometry and Mathematical Physics (P. Donato et al., eds.), Progress in Math. 99, Birkhäuser, 1991.

[17] Takhtajan, Leon, *On foundation of generalized Nambu mechanics*, Comm. Math. Physics **160** (1994), 295–315.

[18] Vinogradov, A.M., *The logic algebra for the theory of linear differential operators*, Sov. Math. Dokl. **13** (1972), 1058–1062.

[19] Vinogradov, A. M., *The C-spectral sequence, Lagrangian formalism and conservation laws; I. The linear theory; II. The non-linear theory*, J. Math. Anal. and Appl. **100** (1984), 1–40, 41–129.

[20] Vinogradov, A.M.; Vinogradov, M., *Alternative n-Poisson manifolds*, in progress.

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