CONNECTIONS ON CENTRAL BIMODULES

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Abstract

We define and study the theory of derivation-based connections on a recently introduced class of bimodules over an algebra which reduces to the category of modules whenever the algebra is commutative. This theory contains, in particular, a noncommutative generalization of linear connections. We also discuss the different noncommutative versions of differential forms based on derivations. Then we investigate reality conditions and a noncommutative generalization of pseudo-riemannian structures.

1 Introduction and notations

There are several noncommutative generalizations of the calculus of differential forms and, more generally, of the differential calculus of classical differential geometry, e.g. [2 to 11]. As stressed in [3], the extension of classical tools to the noncommutative setting is never straightforward. This means that, in order to produce relevant objects, one must have in mind a lot of examples coming both from mathematics and from physics. In this paper, we concentrate on the differential calculus based on derivations as generalization of vector fields, [4]. It was shown in [5] that this differential calculus is natural for quantum mechanics in the sense that with it, quantum mechanics has the same relation to noncommutative symplectic geometry as classical mechanics to classical symplectic geometry. For finite quantum spin systems this was already pointed out in [6]. Furthermore it is also worth noticing that this differential calculus was used in [7] (see also in [5] for a review) to produce a first non-commutative extension of classical gauge theory in which the Higgs fields appear as the components of the gauge potential (i.e. the connection) in the "noncommutative directions".

In this paper, A is an associative algebra over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ with a unit $\mathbb{1}$. The algebra A is to be considered as the generalization of the algebra of smooth

functions and the Lie algebra $\mathrm{Der}(A)$ of all derivations of A as the generalization of the Lie algebra of smooth vector fields. The Lie algebra $\mathrm{Der}(A)$ is also a module over the center $Z(A)$ of A and furthermore $Z(A)$ is stable by the action of $\mathrm{Der}(A)$. The corresponding Lie algebra homomorphism of $\mathrm{Der}(A)$ into the Lie algebra $\mathrm{Der}(Z(A))$ factorizes through the Lie algebra $\mathrm{Out}(A)$ of all derivations of A modulo the ideal $\mathrm{Int}(A)$ of all inner derivations of A; the Lie algebra $\mathrm{Out}(A)$ is also a $Z(A)$-module. Notice that if A is commutative, $A = Z(A)$ and $\mathrm{Der}(A) = \mathrm{Out}(A)$; so $\mathrm{Out}(A)$ is also a generalization of the Lie algebra of vector fields and this is a good generalization for a theory of "invariants". Indeed in general one has $H^0(A, A) = Z(A)$ and $H^1(A, A) = \mathrm{Out}(A)$, (whereas $\mathrm{Der}(A) = Z^1(A, A)$), where $H(A, A)$ is the Hochschild cohomology of A with value in A. So $Z(A)$ and $\mathrm{Out}(A)$ are Morita invariant as well as the homomorphism of $\mathrm{Out}(A)$ into $\mathrm{Der}(Z(A))$. We now recall the relevant generalizations of differential forms in this context [4], [10]. As for the commutative case [12], the notions of differential forms can be extracted from the differential algebra $C(\mathrm{Der}(A), A)$ of Chevalley-Eilenberg cochains of the Lie algebra $\mathrm{Der}(A)$ with values in the $\mathrm{Der}(A)$-module A. There are two natural generalizations of the graded differential algebra of differential forms which use $\mathrm{Der}(A)$ as generalization of vector fields : A minimal one, $\Omega_{\mathrm{Der}}(A)$, which is the smallest differential subalgebra of $C(\mathrm{Der}(A), A)$ which contains A and a maximal one, $\underline{\Omega}_{\mathrm{Der}}(A)$, which consists of all cochains in $C(\mathrm{Der}(A), A)$ which are $Z(A)$-multilinear.

As mentioned above, it is also useful to use $\mathrm{Out}(A)$ as generalization of vector fields. The corresponding generalizations of differential forms $\Omega_{\mathrm{Out}}(A)$ and $\underline{\Omega}_{\mathrm{Out}}(A)$ are respectively graded differential subalgebras of $\Omega_{\mathrm{Der}}(A)$ and $\underline{\Omega}_{\mathrm{Der}}(A)$. To obtain them, one notices that there is a canonical operation, in the sense of H. Cartan [1], $X \mapsto i_X$ for $X \in \mathrm{Der}(A)$, of the

Lie algebra $\mathrm{Der}(A)$ in the graded differential algebra $C(\mathrm{Der}(A), A)$ defined by $i_X\alpha(X_1, \ldots, X_{n-1}) = \alpha(X, X_1, \ldots, X_{n-1})$ for $X_k \in \mathrm{Der}(A)$ and $\alpha \in C^n(\mathrm{Der}(A), A)$. Both $\Omega_{\mathrm{Der}}(A)$ and $\underline{\Omega}_{\mathrm{Der}}(A)$ are stable by the $i_X, X \in \mathrm{Der}(A)$, and $\Omega_{\mathrm{Out}}(A)$ and $\underline{\Omega}_{\mathrm{Out}}(A)$ are defined to be the respective differential subalgebras which are basic with respect to the corresponding operation of $\mathrm{Int}(A)$, i.e. one has:

$$\Omega_{\mathrm{Out}}(A) = \{\alpha \in \Omega_{\mathrm{Der}}(A) | i_X\alpha = 0 \text{ and } L_X\alpha = 0,\ \forall X \in \mathrm{Int}(A)\}$$

$$\underline{\Omega}_{\mathrm{Out}}(A) = \{\alpha \in \underline{\Omega}_{\mathrm{Der}}(A) | i_X\alpha = 0 \text{ and } L_X\alpha = 0,\ \forall X \in \mathrm{Int}(A)\},$$

where $L_X = di_X + i_Xd$ as usual. One has the inclusions of graded differential algebras

$$\begin{array}{ccc} \Omega_{\mathrm{Der}}(A) & \subset & \underline{\Omega}_{\mathrm{Der}}(A) \\ \bigcup & & \bigcup \\ \Omega_{\mathrm{Out}}(A) & \subset & \underline{\Omega}_{\mathrm{Out}}(A) \end{array}$$

In the case where A is the algebra of smooth functions on a finite-dimensional paracompact smooth manifold, all these graded differential algebras coincide with the graded differential algebra of differential forms. In general, there is a differential calculus for A in $\Omega_{\mathrm{Der}}(A)$ and in $\underline{\Omega}_{\mathrm{Der}}(A)$. However if A is not commutative, i.e. $A \neq Z(A)$, then $\Omega_{\mathrm{Out}}(A)$ and $\underline{\Omega}_{\mathrm{Out}}(A)$ do not contain A and are not A-modules. So they do not carry a differential calculus for A. The differential algebra $\underline{\Omega}_{\mathrm{Out}}(A)$ can be identified with the differential algebra $C_{Z(A)}(\mathrm{Out}(A), Z(A))$ of $Z(A)$-linear cochains of the Lie algebra $\mathrm{Out}(A)$ with values in $Z(A)$. So $\underline{\Omega}_{\mathrm{Out}}(A)$ is a Morita invariant generalization of differential forms. We shall use $\underline{\Omega}_{\mathrm{Der}}(A)$ for the differential calculus and then, the "invariants" will be closed elements in the subalgebra $\underline{\Omega}_{\mathrm{Out}}(A)$ leading to Morita-invariants in the cohomology $\underline{H}_{\mathrm{Out}}(A)$.

In this paper, we wish to extend, for A noncommutative, the theory of connections (derivation laws) on A-modules for A commutative as formulated

in [12]. There are several noncommutative generalizations of the notion of module over a commutative algebra. First one can consider the notion of right (or left) A-module, that was in particular the point of view adopted in [7] . Alternatively, one can remember that a module over a commutative algebra is canonically a bimodule of a very specific kind and we speak of the *induced structure of bimodule*. There are several reasons to prefer a notion of bimodule rather than that of right or left module. The first one is that, for any differential calculus for A, the one-forms constitute such a bimodule and that we wish to be able to define linear connections (e.g. to produce non-commutative versions of general relativity). A second very general reason, which is connected with reality in noncommutative geometry, is explained in Section 8. In [9], we introduced the notion of *central bimodule*: This is just a A-bimodule such that the underlying structure of $Z(A)$-bimodule is induced by a structure of $Z(A)$-module, i.e. multiplication by elements of $Z(A)$ on both sides coincide. This notion is stable by arbitrary projective and inductive limits and by tensor products over A or over $Z(A)$. When A is commutative, a central bimodule is just a module (for the induced bimodule structure). It is for this notion that we define and study connections in this paper. In [9] and [10] we also introduced the more restrictive notion of *diagonal bimodule*: This is a bimodule isomorphic to a subbimodule of A^I, for some set I, where A is equipped with its canonical structure of A-bimodule. A diagonal bimodule is central and, if A is commutative, a diagonal bimodule is just a module such that the canonical mapping into its bidual is injective. Both $\Omega_{\mathrm{Der}}(A)$ and $\underline{\Omega}_{\mathrm{Der}}(A)$ are diagonal and therefore central; this is why the notion of connection considered here includes a generalization of the notion of linear connection. Furthermore, and this was the very reason diagonal bimodules were introduced, it was shown in [9] that the derivation (differential)

$d : A \to \Omega^1_{\mathrm{Der}}(A)$ is universal for derivations of A into diagonal bimodules: i.e. for any derivation $\delta : A \to M$ of A into a diagonal bimodule M, there is a unique bimodule homomophism $i_\delta : \Omega^1_{\mathrm{Der}}(A) \to M$ such that $\delta = i_\delta \circ d$.
Finally we shall need, to describe torsion for instance, the generalization of vector valued differential forms. It was shown in [10] that the right spaces to generalize the Frölicher-Nijenhuis bracket were the space $\mathrm{Der}(A, \Omega_{\mathrm{Der}}(A))$ of derivations of A into $\Omega_{\mathrm{Der}}(A)$ if one uses $\Omega_{\mathrm{Der}}(A)$ as generalization of differential forms and the space $\mathrm{Der}(A, \underline{\Omega}_{\mathrm{Der}}(A))$ if one uses $\underline{\Omega}_{\mathrm{Der}}(A)$ as generalization of differential forms. In this paper it is this latter generalization that will be considered. If N and M are A-bimodules, we use the notation $\mathrm{Hom}^A_A(N, M)$ to denote the space of bimodule homomorphisms of N into M. This is a $Z(A)$-bimodule which is in fact a $Z(A)$-module whenever M is central.
The plan of the paper is the following. In Section 2 we define the notion of derivation-based connection on central bimodules. In Section 3 we describe some constructions which allow to produce new connections from given connections. In Section 4 we define linear connections and their torsions. In Section 5 we give some basic examples. In Section 6 we introduce and study a duality between bimodules and modules over the center. In Section 7 we apply this duality to the one-forms showing, in particular, that $\underline{\Omega}^1_{\mathrm{Der}}(A)$ is the bidual of $\Omega^1_{\mathrm{Der}}(A)$ for this duality. In Section 8 we study reality conditions for the case of $*$-algebras. Finally, in Section 9 we investigate a noncommutative generalization of pseudo-riemannian structures in our framework.

2 Connections on central bimodules

Let M be a central bimodule over A, *a connection on* M is a linear mapping ∇, $X \mapsto \nabla_X$, of $\mathrm{Der}(A)$ into the linear endomorphisms of M such that one

has

$$\begin{cases} \nabla_{zX}(m) = z\nabla_X(m) \\ \nabla_X(amb) = X(a)mb + a\nabla_X(m)b + amX(b) \end{cases}$$

$\forall m \in M, \forall X \in \mathrm{Der}(A), \forall z \in Z(A)$ and $\forall a, b \in A$.

Given ∇ as above, *the curvature R of* ∇ is the bilinear antisymmetric mapping $(X, Y) \mapsto R_{X,Y}$ of $\mathrm{Der}(A) \times \mathrm{Der}(A)$ into the linear endomorphisms of M defined by

$$R_{X,Y}(m) = \nabla_X(\nabla_Y(m)) - \nabla_Y(\nabla_X(m)) - \nabla_{[X,Y]}(m),$$

$$\forall X, Y \in \mathrm{Der}A, \ \forall m \in M.$$

One has the following properties

$$\begin{cases} R_{zX,Y}(m) = zR_{X,Y}(m), \\ R_{X,Y}(amb) = aR_{X,Y}(m)b \end{cases}$$

$\forall m \in M, \forall X, Y \in \mathrm{Der}(A), \forall z \in Z(A), \forall a, b \in A$.

Thus R is an antisymmetric $Z(A)$-bilinear mapping of $\mathrm{Der}(A) \times \mathrm{Der}(A)$ into the $Z(A)$-module $\mathrm{Hom}_A^A(M, M)$ i.e.

$$R \in \mathrm{Hom}_{Z(A)} \ (\Lambda^2_{Z(A)}\mathrm{Der}(A), \mathrm{Hom}_A^A(M, M)).$$

From its very definition and from the Jacobi identity, it follows that R satisfies the *Bianchi identity*

$$[\nabla_X, R_{Y,Z}] + [\nabla_Y, R_{Z,X}] + [\nabla_Z, R_{X,Y}] = R_{[X,Y],Z} + R_{[Y,Z],X} + R_{[Z,X],Y}.$$

There is another way to describe all that. Let $\underline{\Omega}^n_{\mathrm{Der}}(A, M)$ be the space (in fact the $Z(A)$-module) of antisymmetric $Z(A)$-multilinear mappings of $(\mathrm{Der}(A))^n$ into M, i.e. one has

$$\underline{\Omega}^n_{\mathrm{Der}}(A, M) = \mathrm{Hom}_{Z(A)}(\Lambda^n_{Z(A)}\mathrm{Der}(A), M).$$

The spaces $\underline{\Omega}^n_{\mathrm{Der}}(A,M)$ as well as

$$\underline{\Omega}_{\mathrm{Der}}(A,M) = \bigoplus_n \underline{\Omega}^n_{\mathrm{Der}}(A,M)$$

are canonically A-bimodules which are central bimodules. Then a connection ∇ as above on M is simply a linear mapping of M into $\underline{\Omega}^1_{\mathrm{Der}}(A,M)$ which satisfies

$$\nabla(amb) = da \underset{A}{\otimes} mb + a\nabla(m)b + am \underset{A}{\otimes} db, \ \forall a,b \in A \text{ and } \forall m \in M,$$

where the canonical bimodule homomorphisms

$$\underline{\Omega}^1_{\mathrm{Der}}(A) \underset{A}{\otimes} M \to \underline{\Omega}^1_{\mathrm{Der}}(A,M) \text{ and } M \underset{A}{\otimes} \underline{\Omega}^1_{\mathrm{Der}}(A) \to \underline{\Omega}^1_{\mathrm{Der}}(A,M)$$

have been used.

More generally, by using the canonical bimodule homomorphisms

$$\underline{\Omega}^m_{\mathrm{Der}}(A) \underset{A}{\otimes} \underline{\Omega}^n_{\mathrm{Der}}(A,M) \to \underline{\Omega}^{m+n}_{\mathrm{Der}}(A,M)$$

and

$$\underline{\Omega}^n_{\mathrm{Der}}(A,M) \underset{A}{\otimes} \underline{\Omega}^m_{\mathrm{Der}}(A) \to \underline{\Omega}^{m+n}_{\mathrm{Der}}(A,M),$$

one equips $\underline{\Omega}_{\mathrm{Der}}(A,M)$ with a structure of graded $\underline{\Omega}_{\mathrm{Der}}(A)$-bimodule. Let us extend $\nabla : \underline{\Omega}^0_{\mathrm{Der}}(A,M) \to \underline{\Omega}^1_{\mathrm{Der}}(A,M)$ to an endomorphism, again denoted by ∇, of $\underline{\Omega}_{\mathrm{Der}}(A,M)$ with $\nabla(\underline{\Omega}^n_{\mathrm{Der}}(A,M)) \subset \underline{\Omega}^{n+1}_{\mathrm{Der}}(A,M)$ by the following definition

$$\begin{aligned}(\nabla\varphi)(X_0,\ldots,X_n) &= \sum_{0\le k\le n} (-1)^k \nabla_{X_k}\varphi(X_0,\ldots\overset{\overset{\mathrm{k}}{\vee}}{\ldots},,X_n) \\ &+ \sum_{0\le r<s\le n} (-1)^{r+s}\varphi([X_r,X_s],X_0,\ldots\overset{\overset{\mathrm{r}}{\vee}}{\ldots}\ldots\overset{\overset{\mathrm{s}}{\vee}}{\ldots},X_n)\end{aligned}$$

for $\varphi \in \underline{\Omega}^n_{\mathrm{Der}}(A, M)$ and $X_k \in \mathrm{Der}(A)$, where $\overset{\mathrm{k}}{\vee}$ means omission of X_k. One has, for $\alpha \in \underline{\Omega}^a_{\mathrm{Der}}(A)$, $\beta \in \underline{\Omega}^b_{\mathrm{Der}}(A)$ and $\varphi \in \underline{\Omega}^n_{\mathrm{Der}}(A, M)$:

$$\nabla(\alpha\varphi\beta) = (d\alpha)\varphi\beta + (-1)^a\alpha\nabla(\varphi)\beta + (-1)^{a+n}\alpha\varphi d\beta.$$

It follows that ∇^2 which is the canonical extension of the curvature satisfies $\nabla^2(\alpha\varphi\beta) = \alpha\nabla^2(\varphi)\beta$, i.e. it is a homomorphism of $\underline{\Omega}_{\mathrm{Der}}(A)$-bimodules (and of graded $\underline{\Omega}_{\mathrm{Der}}(A)$-bimodules) of $\underline{\Omega}_{\mathrm{Der}}(A, M)$ into itself, (the Bianchi identity now reads $\nabla(\nabla^2) = (\nabla^2)\nabla$).

3 Associated connections

There exist central bimodules which do not admit connections. For instance, in [12], J.L. Koszul gives the following example: take $A = \mathbb{K}[\approx]$, i.e. the commutative algebra of polynomials in t, and $M = A/N$ where N is the ideal of polynomials without constant term; then M is a central bimodule since it is an A-module with A commutative and there is no connection on M because if ∇ is such a connection and if e denotes the class of $\mathbb{1}$ in A/N, one must have

$$0 = \nabla_{\partial/\partial t}(te) = e + t\nabla_{\partial/\partial t}(e) = e,$$

i.e. a contradiction. However, if $X \mapsto \nabla_X$ is a connection on a central bimodule M and if $X \mapsto \Gamma_X$ is a $Z(A)$-linear mapping of $\mathrm{Der}(A)$ into $\mathrm{Hom}^A_A(M, M)$ then $X \mapsto \nabla_X + \Gamma_X$ is also a connection on M and all connections on M are of this form; i.e. if the set of connections on a central bimodule M is not empty, it is an affine space modelled on $\mathrm{Hom}_{Z(A)}(\mathrm{Der}(A), \mathrm{Hom}^A_A(M, M))$. Notice that, for $M = A$, $\nabla_X(a) = X(a)$ ($\forall a \in A$, $\forall X \in \mathrm{Der}(A)$) is a connection on A with vanishing curvature which will be referred to as the *canonical connection on A*. In this section, we will describe connections on central

bimodules associated with bimodules which admit connections. These connections will be accordingly called *associated connections*.

Let M be a central bimodule equipped with a connection ∇ and let N be a subbimodule of M. Assume that $\nabla_X N \subset N$ for any $X \in \mathrm{Der}(A)$. Then the restriction of ∇ to N, (i.e. of the $\nabla_X,\ X \in \mathrm{Der}(A)$), is a connection on N and ∇ induces a connection on the quotient bimodule M/N. In both cases, we shall speak of the *induced connections* by ∇ to design these connections on N and on M/N.

Let $(M_i)_{i\in I}$ be a family of central bimodules equipped with connections ∇^i. Then $\nabla_X((m_i)_{i\in I}) = (\nabla^i_X(m_i))_{i\in I}$, for $m_i \in M_i$ and $X \in \mathrm{Der}(A)$, defines a connection on the product $\prod\limits_{i\in I} M_i$. By restriction, one obtains a connection on the direct sum $\underset{i\in I}{\oplus} M_i$, $\nabla_X(\sum_i m_i) = \sum_i \nabla^i_X m_i$, since $\nabla_X(\underset{i\in I}{\oplus} M) \subset \underset{i\in I}{\oplus} M_i$ for any $X \in \mathrm{Der}(A)$. These connections will be called *product* and *direct sum* of the connections ∇^i. One defines similarily *projective limits* and *inductive limits* of connections when the appropriate stability conditions are satisfied.

Let M and M' be two central bimodules equipped with connections ∇ and ∇'. For $X \in \mathrm{Der}(A)$, consider the linear endomorphisms $\nabla_X \otimes id_{M'} + id_M \otimes \nabla'_X$ of $M \otimes M'$. The bimodule $M \otimes M'$ is not central in general, however the subbimodules generated, respectively by the

$$ma \otimes m' - m \otimes am',\ a \in A,\ m \in M,\ m' \in M'$$

and by the

$$mz \otimes m' - m \otimes zm',\ z \in Z(A),\ m \in M,\ m' \in M$$

are stable by the above endomorphisms (remembering that $\mathrm{Der}(A)(Z(A)) \subset Z(A)$), so they define endomorphisms of $M \underset{A}{\otimes} M'$ and of

$M \underset{Z(A)}{\otimes} M'$ which are easily seen to be connections on $M \underset{A}{\otimes} M'$ and $M \underset{Z(A)}{\otimes} M'$, respectively. These connections will be called *tensor product* of ∇ and ∇' over A and over $Z(A)$, respectively. By induction, one defines the tensor product (over A or over $Z(A)$) of a finite family of connections on a finite family of central bimodules. This tensor product is associative in an obvious sense.

In particular, if M is a central bimodule with a connection ∇, then by applying the above construction, one obtains a connection $\nabla^{\otimes}$ on the tensor algebra of M over A, $T_A(M) = \oplus_n(\otimes_A^n M)$, satisfying $\nabla_X^{\otimes}(a) = X(a)$ for $a \in A = T_A^0(M)$ and $X \in \mathrm{Der}(A)$. One has $\nabla_X^{\otimes}(tt') = \nabla_X^{\otimes}(t)t' + t\nabla_X^{\otimes}(t')$ for $t, t' \in T_A(M),\ X \in \mathrm{Der}(A)$.

Let M be a central bimodule, then $\mathrm{Hom}_A^A(M, M)$ is an algebra over $Z(A)$. The group of invertible elements of $\mathrm{Hom}_A^A(M, M)$ will be called *the group of gauge transformations of* M. Given a connection $X \mapsto \nabla_X$ and a gauge transformation g on M, $X \mapsto g \circ \nabla_X \circ g^{-1}$ is again a connection which will be referred to as *the gauge transform of* ∇ *by* g. Two connections belonging to the same orbit will be called *gauge equivalent connections*.

4 The case $M = \underline{\Omega}^1_{\mathrm{Der}}(A)$: Linear connections

The bimodule $\underline{\Omega}^1_{\mathrm{Der}}(A)$ is diagonal and therefore central. A connection ∇ on $\underline{\Omega}^1_{\mathrm{Der}}(A)$ will be called a *linear connection*. There is a canonical bimodule homomorphism $\mu : \underline{\Omega}^1_{\mathrm{Der}}(A, \underline{\Omega}^1_{\mathrm{Der}}(A)) \to \underline{\Omega}^2_{\mathrm{Der}}(A)$ which extends the product $\underline{\Omega}^1_{\mathrm{Der}}(A) \underset{A}{\otimes} \underline{\Omega}^1_{\mathrm{Der}}(A) \to \underline{\Omega}^2_{\mathrm{Der}}(A)$, namely $\mu(\varphi)(X, Y) = \varphi_X(Y) - \varphi_Y(X)$ for $X, Y \in \mathrm{Der}(A)$ and $\varphi \in \underline{\Omega}^1_{\mathrm{Der}}(A, \underline{\Omega}^1_{\mathrm{Der}}(A))$. Given a linear connection ∇, one defines a linear mapping T of A into $\underline{\Omega}^2_{\mathrm{Der}}(A)$ by setting $T(a) = -\mu \circ \nabla(da)$ for $a \in A$. One has $T(ab) = T(a)b + aT(b)$ for $a, b \in A$, therefore T is

an element of $\mathrm{Der}(A, \underline{\Omega}^2_{\mathrm{Der}}(A))$ which will be called *the torsion* of the linear connection ∇. Since $\underline{\Omega}^2_{\mathrm{Der}}(A)$ is a diagonal bimodule, it follows from the universal property of the derivation $d : A \to \Omega^1_{\mathrm{Der}}$ that there is a unique bimodule homomorphism $i_T : \Omega^1_{\mathrm{Der}}(A) \to \underline{\Omega}^2_{\mathrm{Der}}(A)$ such that $T = i_T \circ d$. The explicit form of i_T is easy to write, one has $i_T = d - \mu \circ \nabla$ which extends as a bimodule homomorphism of $\underline{\Omega}^1_{\mathrm{Der}}(A)$ into $\underline{\Omega}^2_{\mathrm{Der}}(A)$. We shall frequently identify the torsion $T \in \mathrm{Der}(A, \underline{\Omega}^2_{\mathrm{Der}}(A))$ with this element of $\mathrm{Hom}^A_A(\underline{\Omega}^1_{\mathrm{Der}}(A), \underline{\Omega}^2_{\mathrm{Der}}(A))$.

5 Examples

5.1 The case where A is commutative

In the case where A is commutative, a central bimodule is simply an A-module and the notion of connection defined here reduces to the usual one, i.e. to the notion of derivation laws of [12]. One obtains the classical notion of connection on a smooth vector bundle E of finite rank over a smooth finite-dimensional paracompact manifold V by taking the algebra $C^\infty(V)$ of smooth functions on V for A and by taking the module $\Gamma(E)$ of smooth sections of E, i.e. a typical finite projective module over $A = C^\infty(V)$. Since the canonical mapping of $\Gamma(E)$ into its bidual is injective, the underlying bimodule is not only central but it is also a diagonal bimodule.
Now we investigate cases which are of "opposite side".

5.2 The case where $\mathbf{Out}(A) = 0$

Let us now assume that A is a noncommutative algebra which has only inner derivations, i.e. $\mathrm{Int}(A) = \mathrm{Der}(A)$ or, equivalently $\mathrm{Out}(A) = 0$. In this case, every central bimodule M admits a canonical connection $\overset{c}{\nabla}$ with vanishing

curvature defined by: $\overset{c}{\nabla}_{ad(x)}(m) = xm - mx$, $\forall x \in A$ and $\forall m \in M$. The other connections on M are of course of the form $\nabla_{ad(x)} = \overset{c}{\nabla}_{ad(x)} + \Gamma_{ad(x)}$ where $\Gamma \in \mathrm{Hom}_{Z(A)}(\mathrm{Int}(A), \mathrm{Hom}_A^A(M, M))$. Since the curvature of $\overset{c}{\nabla}$ vanishes one cannot have a non trivial theory of characteristic classes using the above notion of connection for such algebras. This also partly explains why, in the general case, one has to factorize the inner derivations out in order to get a good theory of invariants.

For $M = \underline{\Omega}^1_{\mathrm{Der}}(A)$, $\overset{c}{\nabla}$ is a linear connection. Its torsion T is given by $T(a)(ad(x), ad(y)) = -ad[x, y](a) = -[[x, y], a]$, or $i_T(\omega)(ad(x), ad(y)) = -\omega(ad([x, y]))$, for $x, y, a \in A$, $\omega \in \underline{\Omega}^1_{\mathrm{Der}}(A)$.

5.3 The case where A has a trivial center $Z(A) = \mathbb{K}.\mathbb{1}$

In this case, $Z(A)$-linearity reduces to $\mathbb{K}$-linearity, so in particular the Lie derivative $X \mapsto L_X = i_X d + d i_X$ is a connection on any of the central bimodules $\Omega^n_{\mathrm{Der}}(A)$ and $\underline{\Omega}^n_{\mathrm{Der}}(A)$. These connections have vanishing curvatures since the Lie derivative is a homomorphism of Lie algebras. Acting on $\underline{\Omega}^1_{\mathrm{Der}}(A)$ the Lie derivative is then a linear connection with a torsion T given by $T(a)(X, Y) = [X, Y](a)$, or $i_T(\omega)(X, Y) = \omega([X, Y])$, for $a \in A$, $X, Y \in \mathrm{Der}(A)$, $\omega \in \underline{\Omega}^1_{\mathrm{Der}}(A)$.

Notice that if one has also $\mathrm{Out}(A) = 0$, then both $\overset{c}{\nabla}$ and L are connections with zero curvature on the $\Omega^n_{\mathrm{Der}}(A)$ and $\underline{\Omega}^n_{\mathrm{Der}}(A)$ but in general they are not gauge equivalent, except for $n = 0$ where they coincide with the canonical connection on A. In particular, on $\underline{\Omega}^1_{\mathrm{Der}}(A)$ they are linear connections with opposite torsion and therefore $\frac{1}{2}(\overset{c}{\nabla} + L)$ is (on $\underline{\Omega}^1_{\mathrm{Der}}(A)$) torsion-free.

Remarks

A priori, examples 5.2 and 5.3 are independent (Morita invariant) classes of algebras. For instance if C is a unital commutative algebra which is different from $\mathbb{K}.\mathbb{1}$ and which has no nonzero derivation, e.g. $C = \mathbb{K}^n$ with $n \geq 2$, then the matrix algebra $M_N(C)$ has a non-trivial center, C, and all its derivations are inner; on the other hand, if E is a vector space of dimension ≥ 2, the tensor algebra $T(E)$ of E has a trivial center but any non vanishing endomorphism of E extends uniquely as a derivation of $T(E)$ which is never inner. However since here A is the analog of the algebra of smooth functions, one could prefer to choose A in such a way that it has "many" derivations. From this point of view, it is natural to introduce the following class $\mathcal{C}^{\infty,0}$: A belongs to the class $\mathcal{C}^{\infty,o}$ if $X(a) = 0$, $\forall X \in \mathrm{Der}(A)$ for $a \in A$ implies $a \in \mathbb{K}.\mathbb{1}$. It is worth noticing here that this condition might not be sufficient to ensure the existence of "many" derivations: For instance let $A = \oplus A^n$ be a $\mathbb{Z}$-graded algebra with $A^0 = \mathbb{K}.\mathbb{1}$, then the degree derivation defined by $deg(a) = na$ if $a \in A^n$ is such that $deg(a) = 0$ implies $a \in \mathbb{K}.\mathbb{1}$, so A is in $\mathcal{C}^{\infty,0}$ but it is easy to construct examples such that the only derivations are the multiple of deg. In any case, any A in $\mathcal{C}^{\infty,0}$ such that $\mathrm{Out}(A) = 0$ has a trivial center (i.e. examples 5.2 in $\mathcal{C}^{\infty,0}$ are contained in examples 5.3).

6 Duality and diagonal bimodules

Let M be a central bimodule over A, then the space $\mathrm{Hom}_A^A(M, A)$ of all bimodule homomorphisms of M into A is a module over the center $Z(A)$ of A, i.e. it is a $Z(A)$-module which will be denoted by M^{*_A} and called the *dual of the bimodule* M when no confusion arises. The reader must be aware of

the fact that M^{*_A} *is not* the dual of M as $A \otimes A^{op}$-module or as $A \underset{Z(A)}{\otimes} A^{op}$-module. Conversely, let N be a $Z(A)$-module then the space $\mathrm{Hom}_{Z(A)}(N, A)$ is canonically a bimodule over A which is diagonal, and therefore central, since it is a subbimodule of A^N. This diagonal bimodule will be denoted by N^{*_A} and called *the dual of the* $Z(A)$-module N. Thus one has a duality between central bimodules over A and modules over $Z(A)$ which obviously refers to A; this duality is similar to the duality between left and right A-modules. In fact, when A is commutative *all these four notions coincide* with the notion of A-module. Notice that if M is a central bimodule, *the duality* (M, M^{*_A}) *is separated if and only if* M *is diagonal*; another way to say the same thing is to remark that there is a canonical bimodule homomorphism $c_M : M \to M^{*_A *_A}$ and that this canonical homomorphism is injective if and only if M is diagonal. Dually, if N is a $Z(A)$-module, then there is a canonical $Z(A)$-module homomorphism $c_N : N \to N^{*_A *_A}$ which is in general not injective nor surjective; a sufficient condition for the injectivity of c_N is that the canonical mapping of N into its $Z(A)$-module bidual $N^{*_{Z(A)} *_{Z(A)}}$ is injective. A $Z(A)$-module N will be said to be *A-diagonal*, or simply *diagonal* if no confusion arises, whenever the canonical mapping c_N is injective or, which is the same, whenever it is separated by $N^{*_A} = \mathrm{Hom}_{Z(A)}(N, A)$; this means that it is isomorphic to a $Z(A)$-submodule of A^I for some set I. Thus the dual M^{*_A} of any central bimodule M is diagonal. More generally, *a duality* between a central bimodule M and a $Z(A)$-module N will be a bimodule homomorphism $\langle , \rangle$ of $M \underset{Z(A)}{\otimes} N$ into A, $(m, n) \mapsto \langle m, n \rangle$; the duality $\langle , \rangle$ *is separated in* M if and only if $\langle m, n \rangle = 0 \ \forall n \in N$ implies $m = 0$, it is *separated in* N if and only if $\langle m, n \rangle = 0 \ \forall m \in M$ implies $n = 0$ and it is *separated* if and only if it is separated both in M and in N. We already

know that if $\langle , \rangle$ is separated in M, then M is diagonal and if $\langle , \rangle$ is separated in N then N is diagonal.

Finally a central bimodule M will be said to be *reflexive* whenever $M = M^{*_A *_A}$, which implies that M is diagonal, and a $Z(A)$-module N will be said to be *A-reflexive*, or simply *reflexive*, whenever $N = N^{*_A *_A}$, which implies that N is diagonal. If M is reflexive then M^{*_A} is reflexive and if N is reflexive then N^{*_A} is reflexive.

Remark

In fact the duality between central bimodules and $Z(A)$-modules comes from a duality between bimodules and $Z(A)$-modules. Indeed, if M is an arbitrary bimodule over A, then $M^{*_A} = \mathrm{Hom}_A^A(M, A)$ is again canonically a module over the center $Z(A)$ of A. Furthermore $M^{*_A *_A} = \mathrm{Hom}_{Z(A)}(M^{*_A}, A)$ is still a diagonal bimodule and one has again a canonical bimodule homomorphism $c_M : M \to M^{*_A *_A}$ which is, as a homomorphism of M onto $c_M(M)$, the functor Diag defined and studied in [9] and [10] of the category of bimodules into the category of diagonal bimodules. The very reason why we here restrict attention to central bimodules is that only central bimodules reduce canonically to modules whenever A is commutative. From the point of view of the above duality, the diagonal bimodules and the A-diagonal $Z(A)$-modules are favoured and of course, even more favoured are the reflexive bimodules and the A-reflexive $Z(A)$-modules.

After having introduced a notion of connection for central bimodules, it is natural to define a dual notion for $Z(A)$-modules. Let N be a $Z(A)$-module, *a connection on N related to A*, or simply *a connection on N* when no confusion arises, is a linear mapping $\nabla, X \mapsto \nabla_X$, of $\mathrm{Der}(A)$ into the linear

endomorphisms of N such that one has

$$\begin{cases} \nabla_{zX}(n) & = z\nabla_X(n) \\ \nabla_X(zn) & = X(z)n + z\nabla_X(n) \end{cases}$$

$\forall n \in N,\ \forall X \in \mathrm{Der}(A)$ and $\forall z \in Z(A)$.
One defines, as in §2, the curvature R of ∇ by $R_{X,Y} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ and R is now an antisymmetric $Z(A)$-bilinear mapping of $\mathrm{Der}(A) \times \mathrm{Der}(A)$ into the $Z(A)$-module $\mathrm{Hom}_{Z(A)}(N, N)$. The set of connections on N is, if not empty, an affine space modelled on

$$\mathrm{Hom}_{Z(A)}(\mathrm{Der}(A), \mathrm{Hom}_{Z(A)}(N, N)).$$

The above definition is justified by the following lemma.

LEMMA 1 *Let M be a central bimodule with a connection ∇. Then, there is a unique connection, again denoted by ∇, on the $Z(A)$-module M^{*_A} which satisfies*

$$X(\mu(m)) = \nabla_X(\mu)(m) + \mu(\nabla_X(m)),\ \forall X \in \mathrm{Der}(A),\ \forall \mu \in M^{*_A}, \forall m \in M.$$

Dually, let N be a $Z(A)$-module with a connection ∇. Then there is a unique connection, again denoted by ∇, on the central bimodule N^{*_A} which satisfies

$$X(\nu(n)) = \nabla_X(\nu)(n) + \nu(\nabla_X(n)),\ \forall X \in \mathrm{Der}(A),\ \forall \nu \in N^{*_A}, \forall n \in N.$$

<u>Proof</u>. Define $\nabla_X(\mu)$ for $X \in \mathrm{Der}(A)$ and $\mu \in M^{*_A}$ by $\nabla_X(\mu)(m) = X(\mu(m)) - \mu(\nabla_X(m))$, then it is easy to show that $\nabla_X(\mu) \in M^{*_A}$ and that ∇ is a connection on M^{*_A} in the above sense. On the other hand ∇ is obviously unique under the condition of the lemma. The proof of the dual statement is similar. $\square$

In the case where M (resp. N) is reflexive then the affine space of all connections on M (resp. N) and the affine space of all connections on M^{*_A} (resp.N^{*_A}) are isomorphic under the above mapping.
More generally, let $\langle , \rangle$ be a duality between a central bimodule M and a $Z(A)$-module N, then a pair (∇, ∇') of a connection ∇ on M and a connection ∇' on N will be said to be *compatible with the duality* $\langle , \rangle$ if one has $X(\langle m, n\rangle) = \langle \nabla_X(m), n\rangle + \langle m, \nabla'_X(n)\rangle$, $\forall X \in \mathrm{Der}(A)$, $\forall m \in M$ and $\forall n \in N$. If the duality is separated in M (resp. N) then given ∇' (resp. ∇), if ∇ (resp. ∇') exists it is unique.

7 Derivations and forms

As an illustration of the notions introduced in the latter section, let us investigate the duality, between $\Omega^1_{\mathrm{Der}}(A)$ and $\mathrm{Der}(A)$ and between $\mathrm{Der}(A)$ and $\underline{\Omega}^1_{\mathrm{Der}}(A)$. We summarize the result in the following theorem.

THEOREM 1 *One has* $\underline{\Omega}^1_{\mathrm{Der}}(A) = (\Omega^1_{\mathrm{Der}}(A))^{*_A *_A}$. *More precisely, one has canonically:*
a) $\Omega^1_{\mathrm{Der}}(A))^{*_A} = \mathrm{Der}(A)$ *and the duality is separated,*
b) $(\mathrm{Der}(A))^{*_A} = \underline{\Omega}^1_{\mathrm{Der}}(A)$ *and the duality is separated.*

Proof. By the universal property of $d : A \to \Omega^1_{\mathrm{Der}}(A)$, [9], we know that we have canonically $\mathrm{Hom}^A_A(\Omega^1_{\mathrm{Der}}(A), M) = \mathrm{Der}(A, M)$ for any diagonal bimodule M; so the equality of a) follows by taking $M = A$. The corresponding duality is separated since $\Omega^1_{\mathrm{Der}}(A)$ is diagonal (in fact this follows directly from the definitions). On the other hand, the equality b) is just the definition of $\underline{\Omega}^1_{\mathrm{Der}}(A)$ and the corresponding duality is separated because a) implies that the $Z(A)$-module $\mathrm{Der}(A)$ is A-diagonal. (Actually this last

statement also follows directly from the fact that if $X \in \mathrm{Der}(A)$ is such that $\omega(X) = 0,\ \forall \omega \in \underline{\Omega}^1_{\mathrm{Der}}(A)$, then $da(X) = X(a) = 0,\ \forall a \in A$, which means $X = 0$). $\square$

This theorem shows exactly in what sense the minimal bimodule of derivation-based one-forms $\Omega^1_{\mathrm{Der}}(A)$ is "dense" in the maximal one $\underline{\Omega}^1_{\mathrm{Der}}(A)$. Applied to the case where A is the Heisenberg algebra, it implies that the algebra $\hat{\Omega}_{\mathrm{Der}}(A)$ introduced in [5] in connection with the noncommutative symplectic structure of quantum mechanics is just $\underline{\Omega}_{\mathrm{Der}}(A)$ (and in fact all the cochains in this case).

In Section 4, we have defined a linear connection to be a connection on $\underline{\Omega}^1_{\mathrm{Der}}(A)$. Part b) of the theorem shows that there is a more restrictive notion of linear connection, namely a connection relative to A on the $Z(A)$-module $\mathrm{Der}(A)$ because by applying the second part of lemma 1, to such a connection corresponds a unique connection on $\underline{\Omega}^1_{\mathrm{Der}}(A)$ and this mapping is affine and injective. In fact, given a connection ∇ on $\mathrm{Der}(A)$ the torsion of the corresponding linear connection can be identified with the element T of $\mathrm{Hom}_{Z(A)}(\Lambda^2_{Z(A)}\mathrm{Der}(A), \mathrm{Der}(A))$ defined by

$$T_{X,Y} = \nabla_X(Y) - \nabla_Y(X) - [X, Y],\ \forall X, Y \in \mathrm{Der}(A).$$

Part a) of the theorem combined with lemma 1 shows that there is an even more restrictive notion of linear connection, namely a connection on $\Omega^1_{\mathrm{Der}}(A)$.

8 Reality and hermitian structures

In this section A is a unital $*$-algebra over $\mathbb{C}$. An *involutive bimodule* or a $*$-*bimodule* over A is a bimodule M equipped with an antilinear involution $m \mapsto m^*$ such that $(amb)^* = b^* m^* a^*,\ \forall m \in M$ and $\forall a, b \in A$. Dually an

involutive $Z(A)$-module is a $Z(A)$-module N equipped with an antilinear involution $n \mapsto n^*$ such that $(zn)^* = z^* n^*$, $\forall n \in N$ and $\forall z \in Z(A)$. Given an involutive bimodule M then the $Z(A)$-module $\mathrm{Hom}_A^A(M, A)$ is an involutive $Z(A)$-module with involution $\mu \mapsto \mu^*$ given by $\mu^*(m) = (\mu(m^*))^*$, $\forall \mu \in \mathrm{Hom}_A^A(M, A)$ and $\forall m \in M$. Given an involutive $Z(A)$-module N then the diagonal bimodule $N^{*A} = \mathrm{Hom}_{Z(A)}(N, A)$ is an involutive bimodule with involution $\nu \mapsto \nu^*$ given by $\nu^*(n) = (\nu(n^*))^*$, $\forall \nu \in N^{*A}$ and $\forall n \in N$. Elements of such sets satisfying $\lambda = \lambda^*$ are called *hermitian* or *real*. The $Z(A)$-module $\mathrm{Der}(A)$ is an involutive $Z(A)$-module with involution $X \mapsto X^*$ defined by $X^*(a) = (X(a^*))^*$, $\forall X \in \mathrm{Der}(A)$ and $\forall a \in A$. $\underline{\Omega}^1_{\mathrm{Der}}(A)$ and $\Omega^1_{\mathrm{Der}}(A)$ are therefore involutive bimodules. More generally one extends the involution to $\underline{\Omega}_{\mathrm{Der}}(A)$ and $\Omega_{\mathrm{Der}}(A)$ by setting $\omega^*(X_1, \ldots, X_k) = (\omega(X_1^*, \ldots, X_k^*))^*$ for $\omega \in \underline{\Omega}^k_{\mathrm{Der}}(A)$, (or $\Omega^k_{\mathrm{Der}}(A)$) and $X_i \in \mathrm{Der}(A)$. With this involution $\underline{\Omega}_{\mathrm{Der}}(A)$ is a differential graded $*$-algebra in the sense that one has $d(\omega^*) = (d\omega)^*$ and $(\alpha\beta)^* = (-1)^{k\ell}\beta^*\alpha^*$ for $\omega \in \underline{\Omega}_{\mathrm{Der}}(A)$ and $\alpha \in \underline{\Omega}^k_{\mathrm{Der}}(A)$, $\beta \in \underline{\Omega}^\ell_{\mathrm{Der}}(A)$; the subspace $\Omega_{\mathrm{Der}}(A)$ is a differential graded $*$-subalgebra.

It is more or less well known that from the point of view of quantum theory as well as from the point of view of spectral theory the good generalization of the notion of algebra of real functions *is not* the notion of real associative algebra but is the notion of the real Jordan algebra of all hermitian elements of an involutive complex algebra, i.e. $*$-algebra, which plays the role of the noncommutative generalization of the algebra of complex functions. It follows that what must generalize the module of sections of a real vector bundle for instance, or more generally the notion of module over an algebra of real functions is not the notion of right or left module or a notion of bimodules over a real noncommutative algebra but the set of real (i.e. hermitian)

elements of a $*$-bimodule over a $*$-algebra which plays the role of the sections of the complexified vector bundle. Thus the natural category at hand is the category of involutive central bimodules over a $*$-algebra, and even more, if one thinks of real vector bundles for instance, the category of involutive diagonal bimodules and for the finite case the category of involutive reflexive bimodules over a $*$-algebra, (with some other conditions replacing projectivity). Notice also that one can alternatively use the dual notion of the real elements of an involutive $Z(A)$-module or of an involutive diagonal or involutive reflexive $Z(A)$-module. In fact, there is a more restrictive notion of involutive diagonal and involutive reflexive which we call *diagonal involutive* and *reflexive involutive* which we now define. For any $*$-algebra A and any set I, A^I is canonically an involutive bimodule. A *diagonal involutive* bimodule over A, (resp. a *A-diagonal involutive* $Z(A)$-module), is a A-bimodule (resp. a $Z(A)$-module) which is isomorphic to an involutive subbimodule (resp. sub-$Z(A)$-module) of A^I for some set I. These notions are A-dual and therefore if M is diagonal involutive M^{*A*A} is also so, and if furthermore $M = M^{*A*A}$ we say that M is *reflexive involutive*. Notice that $\Omega_{\mathrm{Der}}(A), \mathrm{Der}(A)$ and $\underline{\Omega}_{\mathrm{Der}}(A)$ are diagonal involutive.

Recall that a *hermitian form* on a right A-module E, [2], [3], is a sesquilinear mapping $h : E \times E \to A$ such that $h(\varphi a, \psi b) = a^* h(\varphi, \psi) b$ and $(h(\varphi, \psi))^* = h(\psi, \varphi)$, $\forall \varphi, \psi \in E$ and $\forall a, b \in A$.
For a bimodule M, [15], *a right-hermitian form on M*, or simply a *hermitian* form on M when no confusion arises, will be a sesquilinear mapping $h : M \times M \to A$ such that $h(ma, nb) = a^* h(m, n) b$ and $(h(m, n))^* = h(n, m)$, $\forall m, n \in M$ and $\forall a, b \in A$, as above, *and* $h(m, cn) = h(c^* m, n)$, $\forall m, n \in M$ and $\forall c \in A$. The reason why the latter condition has been included is

that it allows to compose hermitian forms on right modules with (right-) hermitian forms on bimodules. Namely if E is a right module with a hermitian form h_E and if M is a bimodule with a right-hermitian form h_M then one defines a hermitian form h on the right module $E \underset{A}{\otimes} M$ by setting $h(\varphi \otimes m, \psi \otimes n) = h_M(m, h_E(\varphi, \psi)n)(= h_M(h_E(\psi, \varphi)m, n)),\ \forall \varphi, \psi \in E$ and $\forall m, n \in M$. It is also clear that if E is a bimodule and if h_E is a right-hermitian form then the above definition gives a right-hermitian form h on the bimodule $E \underset{A}{\otimes} M$. Furthermore, this composition of (right-) hermitian forms is associative in an obvious sense. Assume now that the positive cone $A^+ = \{\sum_i a_i^* a_i | a_i \in A\}$ of A is strict i.e. that one has $A^+ \bigcap (-A^+) = \{0\}$, then a (right-) hermitian form h on a right module or a bimodule E is *positive* if $h(\varphi, \varphi) \in A^+,\ \forall \varphi \in E$ and *strictly positive* if furthermore $h(\varphi, \varphi) = 0$ implies $\varphi = 0$.

Let M be an involutive bimodule and let g be a bimodule homomorphism of $M \underset{A}{\otimes} M$ into A, i.e. $g \in \mathrm{Hom}_A^A(M \underset{A}{\otimes} M, A)$, such that $(g(m, n))^* = g(n^*, m^*)$ then $(m, n) \mapsto h(m, n) = g(m^*, n)$ is a right-hermitian form on M. Conversely, if h is a hermitian form on M then one defines a $g \in \mathrm{Hom}_A^A(M \underset{A}{\otimes} M, A)$ by setting $g(m, n) = h(m^*, n)$ and one has $(g(m, n))^* = g(n^*, m^*)$. Such a $g \in \mathrm{Hom}_A^A(M \underset{A}{\otimes} M, A)$ satisfying $(g(m, n))^* = g(n^*, m^*)$ will be called a *real inner product* on the involutive bimodule M; $g(m, m)$ is real whenever m is real. We shall say that g is *positive* (resp. *strictly positive*) whenever the corresponding hermitian form is so.

Let M be a bimodule and let $M' = \mathrm{Hom}^A(M, A)$ be the left A-module dual of M as a right A-module. The left module M' is in fact a bimodule if one defines $\alpha.a$ for $\alpha \in M'$ and $a \in A$ by $(\alpha.a)(m) = \alpha(am)$, $\forall m \in M$. If M is a central bimodule, then M' is also a central bimodule since, for $\alpha \in M'$, $m \in M$ and $z \in Z(A)$, one has $(z\alpha)(m) = z\alpha(m) = \alpha(m)z =$

$\alpha(mz) = \alpha(zm) = (\alpha z)(m)$. Assume now that M is an involutive bimodule equipped with a real inner product g. One defines a bimodule homomorphism $g^\sharp \in \mathrm{Hom}_A^A(M, M')$ by setting $g^\sharp(m)(n) = g(m,n)\ \forall m,n \in M$. The real inner product g on M will be said to be *nondegenerate* whenever $g^\sharp$ is injective. If g is strictly positive, then g is nondegenerate.
Given an involutive central bimodule M, a connection ∇ on M will be said to be *real* if $(\nabla_X(m))^* = \nabla_{X^*}(m^*)$. If g is a real inner product on M, a real connection ∇ on M will be said to be *compatible with* g if one has

$$X(g(m,n)) = g(\nabla_X m, n) + g(m, \nabla_X n),\ \forall m,n \in M, \forall X \in \mathrm{Der}(A).$$

With obvious notations the above condition also reads

$$Xg(m \underset{A}{\otimes} n) = g(\nabla_X^{\otimes^2}(m \underset{A}{\otimes} n)) \text{ or } X \circ g = g \circ \nabla_X^{\otimes^2}.$$

Notice that a nondegenerate real inner product g on $\underline{\Omega}^1_{\mathrm{Der}}(A)$ is not yet a complete noncommutative generalization of the notion of pseudo-riemannian structure (and of riemannian structure whenever g is strictly positive); indeed the noncommutative generalization of the symmetry is still missing.

9 Noncommutative (pseudo-)riemannian structures

In this section A is again a unital $*$-algebra over $\mathbb{C}$. We wish to investigate what kind of additional symmetry one has to impose on a nondegenerate real inner product on $\underline{\Omega}^1_{\mathrm{Der}}(A)$ in order that it can be considered as a noncommutative generalization of a pseudo-riemannian metric. Although the solution is quite obvious in simple situations, for instance if A is finite-dimensional,

this is not the case for a general $*$-algebra A as we shall see. Fortunately, by taking a dual point of view, there is a natural generalization of the notion of a pseudo-riemannian metric on the $Z(A)$-module $\mathrm{Der}(A)$. We define a *pseudo-metric* to be a symmetric $Z(A)$-bilinear mapping g_* of $\mathrm{Der}(A) \times \mathrm{Der}(A)$ into A, i.e. $g_* \in (S^2_{Z(A)}\mathrm{Der}(A))^{*_A}$, which is real, i.e. $(g_*(X,Y))^* = g_*(X^*, Y^*)$, and which is nondegenerate in the sense that the corresponding mapping $g_*^\sharp : \mathrm{Der}(A) \to \underline{\Omega}^1_{\mathrm{Der}}(A)$ defined by $g_*^\sharp(X)(Y) = g_*(X,Y)$ is injective. A connection ∇ relative to A on the $Z(A)$-module $\mathrm{Der}(A)$ which is torsion-free, i.e. which satisfies $\nabla_X(Y) - \nabla_Y(X) = [X,Y]$, and which is such that one has $Z(g_*(X,Y)) = g_*(\nabla_Z(X), Y) + g_*(X, \nabla_X(Y))$ for $X, Y, Z \in \mathrm{Der}(A)$ will be called a *Levi-Civita connection* for g_*. Summing over the cyclic permutations of the last equation with signs $++-$ and using the symmetry and the vanishing of the torsion one obtains

$$\begin{aligned} 2g_*(\nabla_X(Y), Z) = \; & X(g_*(Y,Z)) + Y(g_*(X,Z)) - Z(g_*(X,Y)) \\ & + g_*([X,Y],Z) - g_*([Y,Z],X) + g_*([Z,X],Y). \end{aligned}$$

So if there exists such a Levi-Civita connection for g_*, then it is unique since g_* is nondegenerate. It follows from the reality of g_* and from the uniqueness that a Levi-Civita connection for g_* is real, i.e. that one has $(\nabla_X(Y))^* = \nabla_{X^*}(Y^*)$. As pointed out in Section 7, such a connection can be identified with a connection on $\underline{\Omega}^1_{\mathrm{Der}}(A)$ (i.e. with a linear connection) which is torsion-free and the above reality condition implies that it is a real connection on $\underline{\Omega}^1_{\mathrm{Der}}(A)$ in the sense of Section 8. We are now in a position to discuss the additional symmetry that one has to impose on a nondegenerate real inner product on $\underline{\Omega}^1_{\mathrm{Der}}(A)$ in order that it generalize a pseudo-riemmannian metric. Both $\underline{\Omega}^1_{\mathrm{Der}}(A) \underset{A}{\otimes} \underline{\Omega}^1_{\mathrm{Der}}(A)$ and $(S^2_{Z(A)}\mathrm{Der}(A))^{*_A}$ are sub-bimodules of the diagonal bimodule $(\mathrm{Der}(A) \underset{Z(A)}{\otimes} \mathrm{Der}(A))^{*_A}$ of all $Z(A)$-bilinear mappings of $\mathrm{Der}(A) \times \mathrm{Der}(A)$ into A. One defines a bi-

module automorphism σ of $(\mathrm{Der}(A) \underset{Z(A)}{\otimes} \mathrm{Der}(A))^{*_A}$ by setting $\sigma(b)(X,Y) = b(Y,X)$ for $b \in (\mathrm{Der}(A) \underset{Z(A)}{\otimes} \mathrm{Der}(A))^{*_A}$ and $X, Y \in \mathrm{Der}(A)$. The set of all σ-invariant elements constitutes the bimodule $(S^2_{Z(A)}\mathrm{Der}(A))^{*_A}$ whereas $\underline{\Omega}^1_{\mathrm{Der}}(A) \underset{A}{\otimes} \underline{\Omega}^1_{\mathrm{Der}}(A)$ is not stable by σ in general. The latter point is the only draw back to writing the additional symmetry on the nondegenerate real inner product on $\underline{\Omega}^1_{\mathrm{Der}}(A)$. Indeed, suppose that A is such that $\underline{\Omega}^1_{\mathrm{Der}}(A) \underset{A}{\otimes} \underline{\Omega}^1_{\mathrm{Der}}(A)$ is stable by σ, for instance assume that

$$\underline{\Omega}^1_{\mathrm{Der}}(A) \underset{A}{\otimes} \underline{\Omega}^1_{\mathrm{Der}}(A) = (\mathrm{Der}(A) \underset{Z(A)}{\otimes} \mathrm{Der}(A))^{*_A}$$

which is the case when A is finite-dimensional, then one can take the pseudo-metrics in $\underline{\Omega}^1_{\mathrm{Der}}(A) \underset{A}{\otimes} \underline{\Omega}^1_{\mathrm{Der}}(A)$. One sees, by duality, that in order that a non-degenerate real inner product g be a generalization of a pseudo-riemannian metric, it must be σ-invariant, i.e. $g = g \circ \sigma$. In any case, in our framework, we can content ourself with the above definition of pseudo-metric. It is worth noticing that it has been suggested in [14] that one can generalize our definition of linear connections in the case where $\underline{\Omega}_{\mathrm{Der}}(A) \underset{A}{\otimes} \underline{\Omega}_{\mathrm{Der}}(A)$ is σ-invariant to other differential calculi (non derivation-based) by generalizing the bimodule homomorphism σ. This latter approach has been used in two simple cases [8],[13].

Conclusion

This paper is the first one of a series. Here we essentially introduce the basic definitions and motivations without paying attention to the existence problems. Also we have not introduced characteristic classes but we have contented ourself with some comments on what they cannot be, (factorization of inner derivations etc.). It must be clear that, in order to define such classes

as well as to develop a corresponding K-theory, one must restrict attention to a class of bimodules (and $Z(A)$-modules) which is smaller than the class of all central bimodules (and all $Z(A)$-modules). It is also obvious that the (finite projective) right and left modules together with their tensor products and their tensor products with the appropriate bimodules have to be taken into account. It is also worth noticing here that many notions introduced in this paper do not refer to the specific differential calculus (derivation-based) that we use and could be applied to other differential calculi. Finally here we have worked in the purely algebraic setting; but one can easily put everything in the setting of convenient vector spaces in order to eventually take into account topologies as in [10].

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