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MORE ON THE FRÖLICHER-NIJENHUIS BRACKET IN NON COMMUTATIVE DIFFERENTIAL GEOMETRY

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Abstract. In commutative differential geometry the Frölicher-Nijenhuis bracket computes all kinds of curvatures and obstructions to integrability. In [1] the Frölicher-Nijenhuis bracket was developed for universal differential forms of non-commutative algebras, and several applications were given. In this paper this bracket and the Frölicher-Nijenhuis calculus will be developed for several kinds of differential graded algebras based on derivations, which were introduced by [6].

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1. Introduction

There are several generalizations of the differential calculus of differential forms in the non-commutative setting [4], [13], [14], [6], [1]. We concentrate here on the differential calculus based on derivations as generalizations of vector fields, [6], and we develop the Frölicher-Nijenhuis bracket in this setting, following the lead of [1].

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Let us recall what are the relevant notions of differential forms in this context. Let A be an associative algebra with a unit 1. As usual we think of A as a generalization of an algebra of smooth functions. Then it is natural to consider the Lie algebra $\mathrm{Der}(A)$ of all derivations of A with values in A, i.e. of all infinitesimal automorphisms of A, as a generalization of the Lie algebra of vector fields. Alternatively, for an 'invariant' theory one may prefer to use the Lie algebra $\mathrm{Out}(A)$ of all derivations of A modulo the inner derivations. Then $\mathrm{Out}(A)$ is Morita invariant and it also coincides with the Lie algebra of all vector fields in case that A is the algebra of smooth functions on a manifold. As for the commutative case (see [15]) the notions of differential forms can be extracted from the differential algebra $C(\mathrm{Der}(A), A)$ of Chevalley-Eilenberg cochains of the Lie algebra $\mathrm{Der}(A)$ with values in the $\mathrm{Der}(A)$-module A. There are then two natural generalizations of the graded differential algebra of differential forms: A minimal one, $\Omega_{\mathrm{Der}}(A)$, which is the smallest differential subalgebra of $C(\mathrm{Der}(A), A)$ which contains $A = C^0(\mathrm{Der}(A), A)$. And a maximal one, $\underline{\Omega}_{\mathrm{Der}}(A) = C_{Z(A)}(\mathrm{Der}(A), A)$, which consists of all cochains in $C(\mathrm{Der}(A), A)$ which are module homomorphisms for the module structure of $\mathrm{Der}(A)$ over the center $Z(A)$ of A. In order to pass to the corresponding notions for $\mathrm{Out}(A)$ we notice that there is a canonical operations, in the sense of H. Cartan [2], [3], $X \mapsto i_X$ for $X \in \mathrm{Der}(A)$, of the Lie algebra $\mathrm{Der}(A)$ in the graded differential algebra $C(\mathrm{Der}(A), A)$. Both $\Omega_{\mathrm{Der}}(A)$ and $\underline{\Omega}_{\mathrm{Der}}(A) = C_{Z(A)}(\mathrm{Der}(A), A)$ are stable under this operation and we define $\Omega_{\mathrm{Out}}(A)$ and $\underline{\Omega}_{\mathrm{Out}}(A)$ to be the differential subalgebras of $\Omega_{\mathrm{Der}}(A)$ and $\underline{\Omega}_{\mathrm{Der}}(A)$ consisting of all elements which are basic with respect to the corresponding operation of the ideal $\mathrm{Int}(A)$ of inner derivations of $\mathrm{Der}(A)$; one defines similarly $C_{\mathrm{Out}(A)}(\mathrm{Der}(A), A)$. These graded differential algebras are also obvious generalizations of differential forms. Notice, however, that in contrast to $\Omega_{\mathrm{Der}}(A)$ and $\underline{\Omega}_{\mathrm{Der}}(A)$ there is in general no differential calculus starting with A in these algebras, since they do not contain A but merely its center $Z(A) = \Omega^0_{\mathrm{Out}}(A) = \underline{\Omega}^0_{\mathrm{Out}}(A) = C^0_{\mathrm{Out}(A)}(\mathrm{Der}(A), A)$. The differential algebra $C_{\mathrm{Out}(A)}(\mathrm{Der}(A), A)$ turns out to be the differential algebra $C(\mathrm{Out}(A), Z(A))$ of all cochains of the Lie algebra $\mathrm{Out}(A)$ with values in the center $Z(A)$, which is an $\mathrm{Out}(A)$-module. Under this identification $\underline{\Omega}_{\mathrm{Out}}(A)$ becomes the differential algebra $C_{Z(A)}(\mathrm{Out}(A), Z(A))$ of $Z(A)$-multilinear cochains. This implies that $\underline{\Omega}_{\mathrm{Out}}(A)$ is a Morita invariant generalization of the differential algebra of differential forms. $C(\mathrm{Out}(A), Z(A))$ is of course also Morita invariant.

We shall develop the theory in several directions. Firstly, we shall show that the derivation $d : A \to \Omega^1_{\mathrm{Der}}(A)$ is universal for the derivations of A in a category of bimodules containing all bimodules which are isomorphic to sub bimodules of arbitrary products of A considered as a bimodule: As suggested by A. Connes, we call these last bimodules *diagonal bimodules*. This means that when one restricts attention to the above category of bimodules containing the diagonal ones, the universal property of the derivation $d : A \to \Omega^1(A)$ factors through the canonical surjective bimodule homomorphism $\zeta : \Omega^1(A) \to \Omega^1_{\mathrm{Der}}(A)$.

Secondly, we shall generalize for these differential forms the Frölicher-Nijenhuis calculus for vector valued differential forms. As generalization of the space of vector valued differential forms we shall consider $\mathrm{Der}(A, \Omega_{\mathrm{Der}}(A))$ in the case of $\Omega_{\mathrm{Der}}(A)$, and $\mathrm{Der}(A, \underline{\Omega}_{\mathrm{Der}}(A))$ in the case of $\underline{\Omega}_{\mathrm{Der}}(A)$, etc. For the universal differential enveloping algebra $\Omega(A)$ of A, the generalization of the Frölicher-Nijenhuis bracket has already been introduced on $\operatorname{Der}(A, \Omega(A))$ in [1], and we shall also define such a generalization for $\operatorname{Der}(A, C(\operatorname{Der}(A), A)) \cong C(\operatorname{Der}(A), \operatorname{Der}(A))$. The generalizations proposed are natural, so that under the sequence of homomorphisms and inclusions of graded differential algebras

$$\Omega(A) \to \Omega_{\operatorname{Der}}(A) \subset \underline{\Omega}_{\operatorname{Der}}(A) \subset C(\operatorname{Der}(A), A)$$

the corresponding sequence

$$\operatorname{Der}(A, \Omega(A)) \to \operatorname{Der}(A, \Omega_{\operatorname{Der}}(A)) \subset \operatorname{Der}(A, \underline{\Omega}_{\operatorname{Der}}(A)) \subset \operatorname{Der}(A, C(\operatorname{Der}(A), A))$$

is a sequence of homomorphisms for the generalized Frölicher-Nijenhuis brackets. Moreover we present here a novel approach to the Frölicher-Nijenhuis bracket, which uses the Chevalley coboundary operator for the adjoint representation and which works also in other situations, see the proofs of 5.6 and 5.7.

Since it is useful to have a theory which is well suited to topological algebras we develop from the beginning the whole theory in the setting of convenient vector spaces as developed by Frölicher and Kriegl. The reasons for this are the following: If the non-commutative theory should contain some version of differential geometry, a manifold M should be represented by the algebra $C^\infty(M, \mathbb{R})$ of smooth functions on it. The simplest considerations of groups need products, and $C^\infty(M \times N, \mathbb{R})$ is a certain completion of the algebraic tensor product $C^\infty(M, \mathbb{R}) \otimes C^\infty(N, \mathbb{R})$. Now the setting of convenient vector spaces offers in its multilinear version a monoidally closed category, i.e. there is an appropriate tensor product which has all the usual (algebraic) properties with respect to bounded multilinear mappings. So multilinear algebra is carried into this kind of functional analysis without loss. The theory of convenient vector spaces is sketched in section 2.

We note that all results of this paper also hold in a purely algebraic setting: Just equip each vector space with the finest locally convex topology, then all linear mappings are bounded.

2. Convenient vector spaces

2.1. The notion of convenient vector spaces arose in the quest for the right setting for differential calculus in infinite dimensions: The traditional approach to differential calculus works well for Banach spaces, but for more general locally convex spaces there are difficulties. The main one is that the composition of linear mappings stops being jointly continuous at the level of Banach spaces, for any compatible topology, so that even the chain rule is not valid without further assumptions. In 1982, Alfred Frölicher and Andreas Kriegl presented independently the correct setting for differential calculus in infinite dimensions, see their joint book [10].

In addition to their importance for differential calculus convenient vector spaces together with bounded linear mappings and the appropriate tensor product form a monoidally closed category, the only useful one which functional analysis offers beyond Banach spaces. So this category is the right setting for us: we shall need 2.7 and 2.8 below.

In this section we will sketch the basic definitions and the most important results concerning convenient vector spaces and Frölicher-Kriegl calculus. All locally convex spaces will be assumed to be Hausdorff. Proofs for the results sketched here can be found in [10] (except for 2.8 which was proved in [1]). A complete coverage will be in the forthcoming book [18]; [16], [17], and [1] contain overviews.

2.2. The c^∞-topology. Let E be a locally convex vector space. A curve $c : \mathbb{R} \to E$ is called *smooth* or C^∞ if all derivatives exist (and are continuous) - this is a concept without problems. Let $C^\infty(\mathbb{R}, E)$ be the space of smooth curves. It can be shown that $C^\infty(\mathbb{R}, E)$ does depend on the locally convex topology of E only through its underlying bornology (system of bounded sets). The final topologies with respect to the following sets of mappings into E coincide:

(1) $C^\infty(\mathbb{R}, E)$.
(2) Lipschitz curves (so that $\{\frac{c(t)-c(s)}{t-s} : t \neq s\}$ is bounded in E).
(3) $\{E_B \to E : B$ bounded absolutely convex in $E\}$, where E_B is the linear span of B equipped with the Minkowski functional $p_B(x) := \inf\{\lambda > 0 : x \in \lambda B\}$.
(4) Mackey-convergent sequences $x_n \to x$ (there exists a sequence $0 < \lambda_n \nearrow \infty$ with $\lambda_n(x_n - x)$ bounded).

This topology is called the c^∞-topology on E and we write $c^\infty E$ for the resulting topological space. In general (on the space $\mathcal{D}$ of test functions for example) it is finer than the given locally convex topology; it is not a vector space topology, since addition is no longer jointly continuous. The finest among all locally convex topologies on E which are coarser than the c^∞-topology is the bornologification of the given locally convex topology. If E is a Fréchet space, then $c^\infty E = E$.

2.3. Convenient vector spaces. Let E be a locally convex vector space. E is said to be a *convenient vector space* if one of the following equivalent conditions is satisfied (called c^∞-completeness):

(1) Any Mackey-Cauchy-sequence (so that $(x_n - x_m)$ is Mackey convergent to 0) converges.
(2) If B is bounded closed absolutely convex, then E_B is a Banach space.
(3) Any Lipschitz curve in E is locally Riemann integrable.
(4) For any $c_1 \in C^\infty(\mathbb{R}, E)$ there is $c_2 \in C^\infty(\mathbb{R}, E)$ with $c_1 = c_2'$ (existence of antiderivative).
(5) If $f : \mathbb{R} \to E$ is scalarwise Lip^k, then f is Lip^k, for $k > 1$.
(6) If $f : \mathbb{R} \to E$ is scalarwise C^∞ then f is differentiable at 0.
(7) If $f : \mathbb{R} \to E$ is scalarwise C^∞ then f is C^∞.

Here a mapping $f : \mathbb{R} \to E$ is called Lip^k if all partial derivatives up to order k exist and are Lipschitz, locally on $\mathbb{R}$. f scalarwise C^∞ means that $\lambda \circ f$ is C^∞ for all continuous (equivalently: all bounded) linear functionals on E. Obviously c^∞-completeness is weaker than sequential completeness, so any sequentially complete locally convex vector space is convenient. From 2.2.4 one easily sees that c^∞-closed linear subspaces of convenient vector spaces are again convenient. We always assume that a convenient vector space is equipped with its bornological topology. For any locally convex space E there is a convenient vector space $\tilde{E}$ called the completion of E, and a bornological embedding $i : E \to \tilde{E}$, which is characterized by the property that any bounded linear map from E into an arbitrary convenient vector space extends to $\tilde{E}$.

2.4. Smooth mappings. Let E and F be locally convex vector spaces. A mapping $f : E \to F$ is called *smooth* or C^∞, if $f \circ c \in C^\infty(\mathbb{R}, F)$ for all $c \in C^\infty(\mathbb{R}, E)$; so $f_* : C^\infty(\mathbb{R}, E) \to C^\infty(\mathbb{R}, F)$ makes sense. Let $C^\infty(E, F)$ denote the space of all smooth mappings from E to F. For E and F finite dimensional this gives the usual notion of smooth mappings. Multilinear mappings are smooth if and only if they are bounded. We denote by $L(E, F)$ the space of all bounded linear mappings from E to F.

2.5. Differential calculus. We equip the space $C^\infty(\mathbb{R}, E)$ with the bornologification of the topology of uniform convergence on compact sets, in all derivatives separately. Then we equip the space $C^\infty(E, F)$ with the bornologification of the initial topology with respect to all mappings $c^* : C^\infty(E, F) \to C^\infty(\mathbb{R}, F)$, $c^*(f) := f \circ c$, for all $c \in C^\infty(\mathbb{R}, E)$. We have the following results:

(1) *If F is convenient, then also $C^\infty(E, F)$ is convenient, for any E. The space $L(E, F)$ is a closed linear subspace of $C^\infty(E, F)$, so it is convenient also.*
(2) *If E is convenient, then a curve $c : \mathbb{R} \to L(E, F)$ is smooth if and only if $t \mapsto c(t)(x)$ is a smooth curve in F for all $x \in E$.*
(3) *The category of convenient vector spaces and smooth mappings is cartesian closed. So we have a natural bijection*

$$C^\infty(E \times F, G) \cong C^\infty(E, C^\infty(F, G)),$$

which is even a diffeomorphism. Of course this statement is also true for c^∞-open subsets of convenient vector spaces. Note that this result, for $E = \mathbb{R}$, is the prime assumption of variational calculus. As a consequence evaluation mappings, insertion mappings, and composition are smooth.
(4) *The differential $d : C^\infty(E, F) \to C^\infty(E, L(E, F))$, given by $df(x)v := \lim_{t\to 0} \frac{1}{t}(f(x + tv) - f(x))$, exists and is linear and bounded (smooth). Also the chain rule holds: $d(f \circ g)(x)v = df(g(x))dg(x)v$.*

2.6. The category of convenient vector spaces and bounded linear maps is complete and cocomplete, so all categorical limits and colimits can be formed. In particular we can form products and direct sums of convenient vector spaces.

For convenient vector spaces E_1, ... ,E_n, and F we can now consider the space of all bounded n-linear maps, $L(E_1, \dots, E_n; F)$, which is a closed linear subspace of $C^\infty(\prod_{i=1}^n E_i, F)$ and thus again convenient. It can be shown that multilinear maps are bounded if and only if they are partially bounded, i.e. bounded in each coordinate and that there is a natural isomorphism (of convenient vector spaces) $L(E_1, \dots, E_n; F) \cong L(E_1, \dots, E_k; L(E_{k+1}, \dots, E_n; F))$

2.7. Result. *On the category of convenient vector spaces there is a unique tensor product $\tilde\otimes$ which makes the category symmetric monoidally closed, i.e. there are natural isomorphisms of convenient vector spaces $L(E_1; L(E_2, E_3)) \cong L(E_1 \tilde\otimes E_2, E_3)$, $E_1 \tilde\otimes E_2 \cong E_2 \tilde\otimes E_1$, $E_1 \tilde\otimes (E_2 \tilde\otimes E_3) \cong (E_1 \tilde\otimes E_2) \tilde\otimes E_3$ and $E \tilde\otimes \mathbb{R} \cong E$.*

2.8. Result. [1], 2.7. *Let A be a convenient algebra, M a convenient right A-module and N a convenient left A-module. This means that all structure mappings are bounded bilinear.*

(1) *There is a convenient vector space $M \tilde\otimes_A N$ and a bounded bilinear map $b : M \times N \to M \tilde\otimes_A N$, $(m, n) \mapsto m \otimes_A n$ such that $b(ma, n) = b(m, an)$ for all $a \in A$, $m \in M$ and $n \in N$ which has the following universal property:*

If E is a convenient vector space and $f : M \times N \to E$ is a bounded bilinear map such that $f(ma, n) = f(m, an)$ then there is a unique bounded linear map $\tilde{f} : M\tilde{\otimes}_A N \to E$ with $\tilde{f} \circ b = f$.

(2) *Let $L^A(M, N; E)$ denote the space of all bilinear bounded maps $f : M \times N \to E$ having the above property, which is a closed linear subspace of $L(M, N; E)$. Then we have an isomorphism of convenient vector spaces $L^A(M, N; E) \cong L(M\tilde{\otimes}_A N, E)$.*

(3) *If B is another convenient algebra such that N is a convenient right B-module and such that the actions of A and B on N commute, then $M\tilde{\otimes}_A N$ is in a canonical way a convenient right B-module.*

(4) *If in addition P is a convenient left B-module then there is a natural isomorphism of convenient vector spaces*

$$M\tilde{\otimes}_A(N\tilde{\otimes}_B P) \cong (M\tilde{\otimes}_A N)\tilde{\otimes}_B P$$

2.9. Remark. In the following all spaces will be convenient spaces, and all multilinear mappings will be bounded, even if it is not stated explicitly. So all algebras will be convenient algebras and all modules will be convenient modules. $L(E, F)$ etc. will always denote the space of bounded (multi)linear mappings. This setting includes the purely algebraic theory, where one just equips each vector space with its finest locally convex topology, because then each multilinear mapping is bounded automatically.

3. Preliminaries: graded differential algebras, derivations, and operations of Lie algebras

3.1. Graded derivations. Let $\mathfrak{A} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{A}^k$ be a graded associative algebra with unit, so that $\mathfrak{A}^k.\mathfrak{A}^l \subset \mathfrak{A}^{k+l}$. We denote by $\mathrm{Der}_k\,\mathfrak{A}$ the space of all *(graded) derivations* of degree k, i.e. all linear mappings $D : \mathfrak{A} \to \mathfrak{A}$ with $D(\mathfrak{A}^l) \subset \mathfrak{A}^{k+l}$ and $D(\varphi\dot{\psi}) = D(\varphi)\dot{\psi} + (-1)^{kl}\varphi\dot{D}(\psi)$ for $\varphi \in \mathfrak{A}^l$. Then the space $\mathrm{Der}\mathfrak{A} = \bigoplus_k \mathrm{Der}_k\mathfrak{A}$ is a graded Lie algebra with the graded commutator $[D_1, D_2] := D_1 \circ D_2 - (-1)^{k_1k_2} D_2 \circ D_1$ as bracket. This means that the bracket is graded anticommutative, $[D_1, D_2] = -(-1)^{k_1k_2}[D_2, D_1]$, and satisfies the graded Jacobi identity

$$[D_1, [D_2, D_3]] = [[D_1, D_2], D_3] + (-1)^{k_1k_2}[D_2, [D_1, D_3]]$$

(so that $ad(D_1) = [D_1, \quad]$ is itself a derivation of degree k_1).

Let $Z(\mathfrak{A})^q = \{a \in \mathfrak{A}^q : [a, b] = ab - (-1)^{ql}ba = 0 \text{ for all } b \in \mathfrak{A}^l \text{ and all } l \in \mathbb{Z}\}$ and consider the *graded center* $Z(\mathfrak{A}) = \bigoplus_{q\in\mathbb{Z}} Z(\mathfrak{A})^q$ of $\mathfrak{A}$, a graded commutative algebra with unit, which is stable under $\mathrm{Der}(\mathfrak{A})$. Then $\mathrm{Der}(\mathfrak{A})$ is a (left) graded $Z(\mathfrak{A})$-module and we have $[a.D_1, D_2] = a.[D_1, D_2] - (-1)^{(q+k_1)k_2} D_2(a).D_1$.

3.2. Graded differential algebras. In this paper, a graded differential algebra is a $\mathbb{Z}$-graded associative algebra $\mathfrak{A} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{A}^k$ equipped with a graded derivation d of degree 1 with $d^2 = 0$, called the *differential* of $\mathfrak{A}$.

3.3. The graded differential algebra of universal differential forms. Let A be a convenient associative algebra with unit, and let $(\Omega^*(A), d)$ be the convenient graded differential algebra of (universal) (Kähler) differential forms, see for example [1]. Let us briefly repeat its construction: $\Omega^1(A)$ is the kernel of the multiplication $\mu : A\tilde{\otimes}A \to A$, a convenient A-bimodule. The bounded linear mapping $d : A \to \Omega^1(A)$, given by $d(a) = 1 \otimes a - a \otimes 1$, has the following universal property, and the pair $(\Omega^1(A), d)$ is uniquely determined by it:

(1) For any bounded derivation $D : A \to N$ into a convenient A-bimodule N there is a unique bounded A-bimodule homomorphism $j_D : \Omega^1(A) \to N$ such that $D = j_D \circ d$.

We put $\Omega^0(A) := A$, and for $k \in \mathbb{Z}$ we define $\Omega^k(A) := \Omega^1(A)\tilde{\otimes}_A \dots \tilde{\otimes}_A \Omega^1(A)$ (k factors). There is a canonical extension of $d : A \to \Omega^1(A)$ to a bounded differential of the graded algebra $\Omega^*(A)$, i.e. a bounded derivation of degree 1 satisfying $d^2 = 0$.

$$A \xrightarrow{d} \Omega^1(A) \xrightarrow{d} \Omega^2(A) \xrightarrow{d} \Omega^3(A) \xrightarrow{d} \dots$$

and the resulting convenient graded differential algebra has the following universal property and is uniquely determined by it:

(2) For any bounded homomorphism $\varphi : A \to B$ of convenient algebras and for any convenient graded differential algebra $(\mathcal{B} = \bigoplus_{k=0}^{\infty} \mathcal{B}^k, d^{\mathcal{B}})$ with $\mathcal{B}_0 = B$ there exists a unique extension of φ to a homomorphism $\Omega(A) \to \mathcal{B}$ of graded differential algebras.

3.4 Chevalley-Eilenberg cochains. Let $\mathfrak{g}$ be a convenient Lie algebra and let E be a convenient vector space carrying a bounded representation of $\mathfrak{g}$, i.e. one has a bounded Lie algebra homomorphism $\mathfrak{g} \to L(E, E)$. For each positive integer k, let $C^k(\mathfrak{g}, E)$ denote the convenient vector space $L(\Lambda^k\mathfrak{g}, E)$ of all bounded k-linear skew symmetric mappings of $\mathfrak{g}$ into E. Notice that one has $C^0(\mathfrak{g}, E) = E$ canonically. The elements of $C^k(\mathfrak{g}, E)$ are called (bounded) *k-cochains of $\mathfrak{g}$ with values in E*. The space of all (bounded) cochains of $\mathfrak{g}$ with values in E is the convenient graded vector space $C(\mathfrak{g}, E) = \bigoplus_{k=0}^{\infty} C^k(\mathfrak{g}, E)$. One defines a linear mapping $d : C(\mathfrak{g}, E) \to C(\mathfrak{g}, E)$ of degree one by setting:

$$\begin{aligned} d\varphi(X_0, \dots, X_k) : &= \sum_{i=0}^{k} (-1)^i X_i(\varphi(X_0, \dots, \widehat{X_i}, \dots, X_k)) \\ &+ \sum_{i<j} (-1)^{i+j} \varphi([X_i, X_j], X_0, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots , X_k) \end{aligned}$$

for $\varphi \in C^k(\mathfrak{g}, E)$ and $X_i \in \mathfrak{g}$.

One has $d^2 = 0$ and therefore $(C(\mathfrak{g}, E), d)$ is a complex. This is the *Chevalley-Eilenberg complex* of E-valued cochains of $\mathfrak{g}$ and d is the *Chevalley coboundary*. In the case when E is a convenient unital algebra A, $C(\mathfrak{g}, A)$ is canonically a graded unital algebra with product defined by

$$(\varphi.\psi)(X_1, \dots, X_{k+l}) = \tfrac{1}{k!l!} \sum_{\sigma \in \mathcal{S}_{k+l}} \operatorname{sign} \sigma . \varphi(X_{\sigma 1}, \dots, X_{\sigma k}).\psi(X_{\sigma(k+1)}, \dots, X_{\sigma(k+l)})$$

for $\varphi \in C^k(\mathfrak{g}, A), \psi \in C^\ell(\mathfrak{g}, A)$ and $X_i \in \mathfrak{g}$.

If furthermore $\mathfrak{g}$ acts by derivations of A, i.e. if the Lie algebra homomorphism $\mathfrak{g} \to L(A, A)$ is valued in the Lie algebra

$$\mathrm{Der}(A) = \{X \in L(A, A) : X(ab) = X(a)b + aX(b) \text{ for all } a, b \in A\}$$

then d is an antiderivation of $C(\mathfrak{g}, A)$ and therefore, $(C(\mathfrak{g}, A), d)$ is a graded differential algebra.

In particular, $C(\mathrm{Der}(A), A)$ is a (convenient, unital) graded differential algebra. The convenient vector space $\mathrm{Der}(A)$ is not only a Lie algebra, its is also a bounded module over the center $Z(A)$ of A. Moreover, $Z(A)$ is stable by $\mathrm{Der}(A)$ and one has the usual formulas $[X, aY] = X(a)Y + a[X, Y]$, etc. Let $\underline{\Omega}_{\mathrm{Der}}(A) = C_{Z(A)}(\mathrm{Der}(A), A)$ denote the subspace of $C(\mathrm{Der}(A), A)$ of all cochains which are $Z(A)$-multilinear. Then, as easily verified, $\underline{\Omega}_{\mathrm{Der}}(A) = \bigoplus_{k\geq 0} \underline{\Omega}^k_{\mathrm{Der}}(A)$ is a graded differential subalgebra of $C(\mathrm{Der}(A), A)$. The graded differential algebra $\underline{\Omega}_{\mathrm{Der}}(A)$ is a noncommutative generalization of differential forms since it coincides with the graded differential algebra $\Omega(M)$ of differential forms on M whenever A is the algebra $C^\infty(M)$ of all smooth functions on a finite dimensional smooth manifold M.

3.5 H. Cartan's operations. If $\mathfrak{g}$ is a convenient Lie algebra and if $(\mathfrak{A}, d)$ is a graded differential algebra, an operation in the sense of Cartan of $\mathfrak{g}$ in $\mathfrak{A}$ is a linear mapping $\mathfrak{g} \to \mathrm{Der}_{-1}(\mathfrak{A})$, written $X \mapsto i_X$, such that for $\mathcal{L}_X := i_X\, d + d\, i_X = [i_X, d] \in \mathrm{Der}_0(\mathfrak{A})$ we have for all X, $Y \in \mathfrak{g}$:

$$i_X\, i_Y + i_Y\, i_X = [i_X, i_Y] = 0$$
$$\mathcal{L}_X\, i_Y - i_Y\, \mathcal{L}_X = [\mathcal{L}_X, i_Y] = i_{[X,Y]}.$$
$$\mathcal{L}_X\, \mathcal{L}_Y - \mathcal{L}_Y\, \mathcal{L}_X = [\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}.$$

An element $\alpha \in \mathfrak{A}$ is called *horizontal* with respect to $\mathfrak{g}$ whenever $i_X\alpha = 0$ for all $X \in \mathfrak{g}$; it is called *invariant* with respect to $\mathfrak{g}$ if $\mathcal{L}_X\alpha = 0$ for all $X \in \mathfrak{g}$; and α is called *basic* if it is both horizontal and invariant.

The convenient space $\mathfrak{A}_H$ of horizontal elements of $\mathfrak{A}$ is a graded subalgebra of $\mathfrak{A}$ which is stable under $\mathcal{L}_X$ for all $X \in \mathfrak{g}$. The convenient space $\mathfrak{A}_I$ of all invariant elements is a graded differential subalgebra of $(\mathfrak{A}, d)$, and the convenient space $\mathfrak{A}_B$ of all basic elements is a graded differential subalgebra of $\mathfrak{A}_I$, thus also of $\mathfrak{A}$.

4. Derivations on universal differential forms

4.1. Let A be a convenient associative algebra with unit, and let $(\Omega^*(A), d)$ be the convenient graded differential algebra of universal differential forms as described in 3.3. In this section we review from [1] the description of all graded derivations on the graded differential algebra $\Omega^*(A)$. This leads directly to what we like to call the 'calculus of Frölicher-Nijenhuis'.

4.2 Derivations vanishing on A. For every derivation $X \in \mathrm{Der}(A)$ there exists a bounded A-bimodule homomorphism $j_X : \Omega^1(A) \to A$, by the universal property 3.3.(1). It prolongs uniquely to a graded derivation $j(X) = j_X : \Omega(A) \to \Omega(A)$ of degree -1 which is called the *contraction operator* of the derivation X. By definition, j_X vanishes on A.

More generally, let us consider a graded derivation $D \in \mathrm{Der}_k(\Omega^*(A))$ of degree k which vanishes on A, $D \mid \Omega^0(A) = 0$. Such derivations are called *algebraic derivations*; they form a Lie subalgebra of $\mathrm{Der}(\Omega^*(A))$. Then $D(a\omega) = aD(\omega)$ and $D(\omega a) = D(\omega)a$ for $a \in A$, so D restricts to a bounded bimodule homomorphism, an element of $\mathrm{Hom}_A^A(\Omega^l(A), \Omega^{l+k}(A))$. Since $\Omega^l(A)$ for $l \geq 1$ is generated by $\Omega^1(A)$, the derivation D is uniquely determined by its restriction $D|\Omega^1(A) \in \mathrm{Hom}_A^A(\Omega^1(A), \Omega^{k+1}(A))$.

Let us denote by

$$K := (D|\Omega^1(A)) \circ d \in \mathrm{Der}(A, \Omega^{k+1}(A))$$

the corresponding derivation on A. We write $D = j(K) = j_K$ to express the unique dependence of D on K. Note the defining equation $j_K(a_1.d(a_2).a_3) = a_1.K(a_2).a_3$ for $a_i \in A$.

Conversely for $K \in \mathrm{Der}(A, \Omega^{k+1}(A))$ with corresponding homomorphism $j_K \in \mathrm{Hom}_A^A(\Omega^1(A), \Omega^{k+1}(A))$ and $\omega_i \in \Omega^1(A)$ the formula

$$j_K(\omega_0 \otimes_A \cdots \otimes_A \omega_\ell) = \sum_{i=0}^{\ell} (-1)^{ik} \omega_0 \otimes_A \cdots \otimes_A j_K(\omega_i) \otimes_A \cdots \otimes_A \omega_k$$

defines an algebraic graded derivation $j_K \in \mathrm{Der}_k\Omega(A)$ and any algebraic derivation is of this form. The mapping

$$j : \mathrm{Der}(A, \Omega^{k+1}(A)) \to \mathrm{Der}_k\Omega(A)$$

induces an isomorphism of convenient vector spaces onto the closed linear subspace of $\mathrm{Der}_k\Omega(A)$ consisting of all graded bounded derivations which vanish on A.

By stipulating $j([K, L]^\Delta) := [j_K, j_L]$ we get a bracket $[\ ,\ ]^\Delta$ on the space $\mathrm{Der}(A, \Omega^{*+1}(A))$ which defines a convenient graded Lie algebra structure with the grading as indicated, and for $K \in \mathrm{Der}(A, \Omega^{k+1}(A))$, and $L \in \mathrm{Der}(A, \Omega^{l+1}(A))$ we have

$$[K, L]^\Delta = j_K \circ L - (-1)^{k\ell} j_L \circ K.$$

$[\ ,\ ]^\Delta$ is a version of the bracket of Gerstenhaber, De Wilde - Lecomte, see [11], [12], and [5].

4.3. Lie derivations and the Frölicher-Nijenhuis bracket. The exterior derivative d is an element of $\mathrm{Der}_1\Omega(A)$. We define for $K \in \mathrm{Der}(A, \Omega^k(A))$ the *Lie derivation* $\mathcal{L}_K = \mathcal{L}(K) \in \mathrm{Der}_k\Omega(A)$ by

$$\mathcal{L}_K := [j_K, d] = j_K\, d - (-1)^{k-1} d\, j_K.$$

Then the mapping $\mathcal{L} : \mathrm{Der}(A, \Omega^*(A)) \to \mathrm{Der}_*\Omega(A)$ is obviously bounded and it is injective by the universal property of $\Omega^1(A)$, since $\mathcal{L}_K a = j_K da = K(a)$ for $a \in A$. Note that $\mathcal{L}_K$ is an extension of the derivation $K : A \to \Omega^k(A)$ to a graded derivation of the graded algebra $\Omega^*(A)$ of degree k.

Lemma. [1], 4.7. *For any graded derivation $D \in \mathrm{Der}_k\Omega(A)$ there are unique homomorphisms $K \in \mathrm{Der}(A, \Omega^k(A))$ and $L \in \mathrm{Der}(A, \Omega^{k+1}(A))$ such that*

$$D = \mathcal{L}_K + j_L.$$

We have $L = 0$ if and only if $[D, d] = 0$. D is algebraic if and only if $K = 0$. □

Note that $j_d\omega = k\omega$ for $\omega \in \Omega^k(A)$. Therefore we have $\mathcal{L}_d\omega = j_d\, d\omega - d\, j_d\omega = (k+1)d\omega - kd\omega = d\omega$. Thus $\mathcal{L}_d = d$.

Let $K \in \mathrm{Der}(A, \Omega^k(A))$ and $L \in \mathrm{Der}(A, \Omega^l(A))$. Then obviously $[[\mathcal{L}_K, \mathcal{L}_L], d] = 0$, so we have

$$[\mathcal{L}(K), \mathcal{L}(L)] = \mathcal{L}([K, L])$$

for a uniquely defined $[K, L] \in \mathrm{Der}(A, \Omega^{k+l}(A))$ which is called the *(abstract) Frölicher-Nijenhuis bracket* of K and L.

The space $\mathrm{Der}(A, \Omega^*(A)) = \bigoplus_k \mathrm{Der}(A, \Omega^k(A))$ with its usual grading and the Frölicher-Nijenhuis bracket is a convenient graded Lie algebra. $d \in \mathrm{Der}(A, \Omega^1(A))$ is in the center, i.e. $[K, d] = 0$ for all K, see [1], 4.9.

$\mathcal{L} : (\mathrm{Der}(A, \Omega^*(A)), [\ \ ,\ \ ]) \to \mathrm{Der}\Omega(A)$ is a bounded injective homomorphism of graded Lie algebras. For $K \in \mathrm{Der}(A, \Omega^k(A))$ and $L \in \mathrm{Der}(A, \Omega^{l+1}(A))$ we have

$$[\mathcal{L}_K, j_L] = j([K, L]) - (-1)^{kl}\mathcal{L}(j_L \circ K). \tag{1}$$

For $K_i \in \mathrm{Der}(A, \Omega^{k_i}(A))$ and $L_i \in \mathrm{Der}(A, \Omega^{k_i+1}(A))$ we have

$$\begin{aligned}
&[\mathcal{L}_{K_1} + j_{L_1}, \mathcal{L}_{K_2} + j_{L_2}] = \\
&\quad = \mathcal{L}\left([K_1, K_2] + j_{L_1} \circ K_2 - (-1)^{k_1k_2} j_{L_2} \circ K_1\right) \\
&\qquad + j\left([L_1, L_2]^\Delta + [K_1, L_2] - (-1)^{k_1k_2}[K_2, L_1]\right).
\end{aligned} \tag{2}$$

Each summand of this formula looks like a semidirect product of graded Lie algebras, but the mappings

$$\begin{aligned}
j &: \mathrm{Der}(A, \Omega^{*-1}(A)) \to \mathrm{End}_{\mathbb{K}}(\mathrm{Der}(A, \Omega^*(A)), [\ \ ,\ \ ]) \\
\mathrm{ad} &: \mathrm{Der}(A, \Omega^*(A)) \to \mathrm{End}_{\mathbb{K}}(\mathrm{Der}(A, \Omega^{*-1}(A)), [\ \ ,\ \ ]^\Delta), \\
&\mathrm{ad}_K L = [K, L],
\end{aligned}$$

do not take values in the subspaces of graded derivations. We have instead for $K \in \mathrm{Der}(A, \Omega^k(A))$ and $L \in \mathrm{Der}(A, \Omega^{l+1}(A))$ the following relations:

$$\begin{aligned}
j_L \circ [K_1, K_2] &= [j_L \circ K_1, K_2] + (-1)^{k_1 l}[K_1, j_L \circ K_2] \\
&\quad - \Big((-1)^{k_1 l} j(\mathrm{ad}_{K_1} L) \circ K_2 - (-1)^{(k_1+l)k_2} j(\mathrm{ad}_{K_2} L) \circ K_1\Big)
\end{aligned} \tag{3}$$

$$\begin{aligned}
\mathrm{ad}_K [L_1, L_2]^\Delta &= [\mathrm{ad}_K L_1, L_2]^\Delta + (-1)^{kk_1}[L_1, \mathrm{ad}_K L_2]^\Delta - \\
&\quad - \Big((-1)^{kk_1} \mathrm{ad}(j(L_1) \circ K)L_2 - (-1)^{(k+k_1)k_2} \mathrm{ad}(j(L_2) \circ K)L_1\Big)
\end{aligned} \tag{4}$$

4.4. Naturality of the Frölicher-Nijenhuis bracket. Let $f : A \to B$ be a bounded algebra homomorphism. Two elements $K \in \operatorname{Der}(A, \Omega^k(A))$ and $K' \in \operatorname{Der}(B, \Omega^k(B))$ are called *f-related* or *f-dependent*, if we have

$$K' \circ f = \Omega^k(f) \circ K : A \to \Omega^k(B), \tag{1}$$

where $\Omega^*(f) : \Omega^*(A) \to \Omega^*(B)$ is given by the universal property 3.3.(2). From [1], 4.12. we have the following results:

(2) If K and K' are f-related then $j_{K'} \circ \Omega(f) = \Omega(f) \circ j_K : \Omega(A) \to \Omega(B)$.
(3) If $j_{K'} \circ \Omega(f)|d(A) = \Omega(f) \circ j_K|d(A)$, then K and K' are f-related, where $d(A) \subset \Omega^1(A)$ denotes the space of exact 1-forms.
(4) If K_j and K'_j are f-related for $j = 1, 2$, then $j_{K_1} \circ K_2$ and $j_{K'_1} \circ K'_2$ are f-related, and also $[K_1, K_2]^\Delta$ and $[K'_1, K'_2]^\Delta$ are f-related.
(5) If K and K' are f-related then $\mathcal{L}_{K'} \circ \Omega(f) = \Omega(f) \circ \mathcal{L}_K : \Omega(A) \to \Omega(B)$.
(6) If $\mathcal{L}_{K'} \circ \Omega(f) \mid \Omega^0(A) = \Omega(f) \circ \mathcal{L}_K \mid \Omega^0(A)$, then K and K' are f-related.
(7) If K_j and K'_j are f-related for $j = 1, 2$, then their Frölicher-Nijenhuis brackets $[K_1, K_2]$ and $[K'_1, K'_2]$ are also f-related.

5. The Frölicher-Nijenhuis calculus on Chevalley type cochains

5.1. Insertion operators. Let A be a convenient associative algebra with unit. Using the setting from 3.4, for $X \in \operatorname{Der}(A)$ we consider the insertion operator

$$\begin{aligned} &i_X : C^*(\operatorname{Der}(A), A) \to C^{*-1}(\operatorname{Der}(A), A),\\ &(i_X\omega)(X_1, \dots, X_k) := \omega(X, X_1, \dots, X_k), \end{aligned}$$

which is a derivation of degree -1 satisfying $i_X|C^0(\operatorname{Der}(A), A) = i_X|A = 0$. Note that i_X maps the graded differential subalgebra $C_{Z(A)}(\operatorname{Der}(A), A)$ into itself.

More generally, let $K \in C^k(\operatorname{Der}(A), \operatorname{Der}(A)) := L^k_{\text{skew}}(\operatorname{Der}(A); \operatorname{Der}(A))$ be a bounded skew symmetric k-linear mapping $\operatorname{Der}(A)^{k+1} \to \operatorname{Der}(A)$. Then we may define:

$$\begin{aligned} &i_K : C^l(\operatorname{Der}(A), A) \to C^{l+k-1}(\operatorname{Der}(A), A)\\ &(i_K\omega)(X_1, \dots, X_{l+k-1}) =\\ &\quad = \tfrac{1}{k!\,(l-1)!} \sum_{\sigma \in \mathcal{S}_{k+l-1}} \operatorname{sign}(\sigma)\, \omega(K(X_{\sigma 1}, \dots, X_{\sigma k}), X_{\sigma(k+1)}, \dots) \end{aligned} \tag{1}$$

Note that $i_{Id_{\operatorname{Der}(A)}}\omega = l.\omega$ for $\omega \in C^l(\operatorname{Der}(A), A)$.

Lemma. *i_K is a derivation of degree $k-1$, $i_K \in \operatorname{Der}_{k-1}(C^*(\operatorname{Der}(A), A))$.*

Proof. Clearly one may write

$$\begin{aligned} &(i_K\omega)(X_1, \dots, X_{l+k-1}) =\\ &= \sum_{i_1 < \dots < i_k} (-1)^{i_1 + \dots + i_k - \frac{k(k+1)}{2}} \omega(K(X_{i_1}, \dots, X_{i_k}), X_1, \dots, \widehat{X_{i_1}}, \dots, \widehat{X_{i_2}}, \dots). \end{aligned} \tag{2}$$

Using expression (2) one may check directly that then we have

$$[i_X, i_K] = i_{K(X,\dots)} = i_{i_X K} \text{ for any } X \in \operatorname{Der}(A). \tag{3}$$

From (3) it is then easy to prove that for $\varphi \in C^p(\operatorname{Der}(A), A)$ and $\psi \in C^q(\operatorname{Der}(A), A)$ we have

$$i_K(\varphi.\psi) = (i_K\varphi).\psi + (-1)^{(k-1)p}\varphi.(i_K\psi),$$

by induction on $k + p + q$. □

5.2. Remark. In general there exist more bounded graded derivations D in the space $\operatorname{Der}(C(\operatorname{Der}(A), A))$ which vanish on A, than just those of the form i_K for some $K \in C(\operatorname{Der}(A), \operatorname{Der}(A))$. We shall give an explicit description of all graded derivations on $C(\operatorname{Der}(A), A)$ in section 6 below, but we will not extend the Frölicher-Nijenhuis calculus to this description: it is too complicated.

5.3. Now let $K \in C^k(\operatorname{Der}(A), \operatorname{Der}(A))$ and $L \in C^l(\operatorname{Der}(A), \operatorname{Der}(A))$. Then for the graded commutator we have

$$\begin{aligned}
&[i_K, i_L] = i_K\, i_L - (-1)^{(k-1)(l-1)} i_L\, i_K = i_{[K,L]^\wedge}, \text{ where}\\
&[K, L]^\wedge = i_K L - (-1)^{(k-1)(l-1)} i_L K \in C^{k+l-1}(\operatorname{Der}(A), \operatorname{Der}(A))
\end{aligned}$$

is the *Nijenhuis-Richardson bracket*, see [21], a graded Lie bracket on the graded vector space $C^{*+1}(\operatorname{Der}(A), \operatorname{Der}(A))$. Here i_L acts on $C(\operatorname{Der}(A), \operatorname{Der}(A))$ by the same formula as in 5.1.(1).

5.4. Lie derivations. For $K \in C^k(\operatorname{Der}(A), \operatorname{Der}(A))$ let us now define the *Lie derivation* $\mathcal{L}_K$ along K by the graded commutator

$$\mathcal{L}_K = [i_K, d] = i_K\, d - (-1)^{k-1} d\, i_K \in \operatorname{Der}_k(C(\operatorname{Der}(A), A)).$$

Note that $\mathcal{L}_{Id_{\operatorname{Der}(A)}} = d$.

5.5. Lemma. *The Lie derivative of $\omega \in C^l(\operatorname{Der}(A), A)$ along K is given by*

$$\begin{aligned}
&(\mathcal{L}_K\omega)(X_1, \dots, X_{k+l}) =\\
&= \tfrac{1}{k!\,l!} \sum_\sigma \operatorname{sign}\sigma\; \mathcal{L}_{K(X_{\sigma 1},\dots,X_{\sigma k})}(\omega(X_{\sigma(k+1)}, \dots, X_{\sigma(k+l)}))\\
&+ (-1)^k \Big(\tfrac{1}{k!\,(l-1)!} \sum_\sigma \operatorname{sign}\sigma\; \omega([X_{\sigma 1}, K(X_{\sigma 2}, \dots, X_{\sigma(k+1)})], X_{\sigma(k+2)}, \dots)\\
&\qquad - \tfrac{1}{(k-1)!\,(l-1)!\,2!} \sum_\sigma \operatorname{sign}\sigma\; \omega(K([X_{\sigma 1}, X_{\sigma 2}], X_{\sigma 3}, \dots), X_{\sigma(k+2)}, \dots) \Big)
\end{aligned}$$

Proof. This can be shown by a direct computation starting from formula 5.1.(2). □

5.6. Proposition. *Let* $K \in C^k(\mathrm{Der}(A), \mathrm{Der}(A))$ *and* $L \in C^l(\mathrm{Der}(A), \mathrm{Der}(A))$. *Then for the graded commutator we have*

$$[\mathcal{L}_K, \mathcal{L}_L] = \mathcal{L}_K\, \mathcal{L}_L - (-1)^{(k-1)(l-1)}\mathcal{L}_L\, \mathcal{L}_K = \mathcal{L}_{[K,L]},$$

where $[K, L] \in C^{k+l}(\mathrm{Der}(A), \mathrm{Der}(A))$ *is given by the following formula*

(1)
$$\begin{aligned}
&[K, L](X_1, \dots , X_{k+l}) = \\
&= \tfrac{1}{k!\, l!} \sum_\sigma \operatorname{sign}\sigma\ [K(X_{\sigma 1}, \dots , X_{\sigma k}), L(X_{\sigma(k+1)}, \dots , X_{\sigma(k+l)})] \\
&+ (-1)^k \Big(\tfrac{1}{k!\,(l-1)!} \sum_\sigma \operatorname{sign}\sigma\ L([X_{\sigma 1}, K(X_{\sigma 2}, \dots , X_{\sigma(k+1)})], X_{\sigma(k+2)}, \dots) \\
&\qquad - \tfrac{1}{(k-1)!\,(l-1)!\,2!} \sum_\sigma \operatorname{sign}\sigma\ L(K([X_{\sigma 1}, X_{\sigma 2}], X_{\sigma 3}, \dots), X_{\sigma(k+2)}, \dots) \Big) \\
&- (-1)^{kl+l} \Big(\tfrac{1}{(k-1)!\, l!} \sum_\sigma \operatorname{sign}\sigma\ K([X_{\sigma 1}, L(X_{\sigma 2}, \dots , X_{\sigma(l+1)}), X_{\sigma(l+2)}, \dots) \\
&\qquad - \tfrac{1}{(k-1)!\,(l-1)!\,2!} \sum_\sigma \operatorname{sign}\sigma\ K(L([X_{\sigma 1}, X_{\sigma 2}], X_{\sigma 3}, \dots), X_{\sigma(l+2)}, \dots) \Big).
\end{aligned}$$

The bracket $[K, L]$ *is called the Frölicher-Nijenhuis bracket. It is a graded Lie bracket on* $C^*(\mathrm{Der}(A), \mathrm{Der}(A))$.

Proof. Comparing the Chevalley coboundary operator (see 3.4)

$$\begin{aligned}
\partial K(X_1, \dots , X_{k+1}) = \tfrac{1}{k!} &\sum_\sigma \operatorname{sign}\sigma\ [X_{\sigma 1}, K(X_{\sigma 2}, \dots , X_{\sigma(k+1)})] \\
&- \tfrac{1}{(k-1)!\, 2!} \sum_\sigma \operatorname{sign}\sigma\ K([X_{\sigma 1}, X_{\sigma 2}], X_{\sigma 3}, \dots , X_{\sigma(k+1)})
\end{aligned}$$

for the adjoint representation of the Lie algebra $\mathrm{Der}(A)$ with the Frölicher-Nijenhuis bracket (1) one sees that

$$[K, L] = [K, L]_\wedge + (-1)^k i(\partial K)L - (-1)^{kl+l} i(\partial L)K, \tag{2}$$

where we have put

(3)
$$\begin{aligned}
&[K, L]_\wedge(X_1, \dots , X_{k+l}) = \\
&\qquad = \tfrac{1}{k!\, l!} \sum_\sigma \operatorname{sign}(\sigma)[K(X_{\sigma 1}, \dots , X_{\sigma k}), L(X_{\sigma(k+1)}, \dots , X_{\sigma(k+l)})]_{\mathrm{Der}(A)}.
\end{aligned}$$

Formula (2) is the same as in [21], p. 100, where it is also stated that from this formula 'one can show (with a good deal of effort) that this bracket defines a graded Lie algebra structure'. Similarly we can write the Lie derivative 5.5 as

$$\mathcal{L}_K = \mathcal{L}_\wedge(K) + (-1)^k i(\partial K), \tag{4}$$

where the action $\mathcal{L}$ of $\operatorname{Der}(A)$ on A is extended to $\mathcal{L}_\wedge : C(\operatorname{Der}(A), \operatorname{Der}(A)) \times C(\operatorname{Der}(A), A) \to C(\operatorname{Der}(A), A)$ by

$$(5)\quad (\mathcal{L}_\wedge(K)\omega)(X_1, \dots, X_{q+k}) = \\ = \tfrac{1}{k!\,l!} \sum_\sigma \operatorname{sign}(\sigma) \mathcal{L}(K(X_{\sigma 1}, \dots, X_{\sigma k}))(\omega(X_{\sigma(k+1)}, \dots, X_{\sigma(k+q)})).$$

Using (4) we see that

$$(6)\qquad \begin{aligned} [\mathcal{L}_K, \mathcal{L}_L] = {} & \mathcal{L}_\wedge(K)\mathcal{L}_\wedge(L) - (-1)^{kl}\mathcal{L}_\wedge(L)\mathcal{L}_\wedge(K) \\ & + (-1)^k i(\partial K)\mathcal{L}_\wedge(L) - (-1)^{kl+k}\mathcal{L}_\wedge(L) i(\partial K) \\ & - (-1)^{kl+l} i(\partial L)\mathcal{L}_\wedge(K) + (-1)^l \mathcal{L}_\wedge(K) i(\partial L) \\ & + (-1)^{k+l} i(\partial K) i(\partial L) - (-1)^{kl+k+l} i(\partial L) i(\partial K), \end{aligned}$$

and from (2) and (4) we get

$$(7)\qquad \begin{aligned} \mathcal{L}_{[K,L]} &= \mathcal{L}_{[K,L]_\wedge} + (-1)^k \mathcal{L}_{i(\partial K)L} - (-1)^{kl+l}\mathcal{L}_{i(\partial L)K} \\ &= \mathcal{L}_\wedge([K,L]_\wedge) + (-1)^{k+l} i(\partial[K,L]_\wedge) \\ &\quad + (-1)^k \mathcal{L}_\wedge(i(\partial K)L) + (-1)^k i(\partial i(\partial K)L) \\ &\quad - (-1)^{kl+l}\mathcal{L}_\wedge(i(\partial L)K) - (-1)^{kl+k} i(\partial i(\partial L)K). \end{aligned}$$

By a straightforward direct computation one checks that

$$(8)\qquad \mathcal{L}_\wedge(K)\mathcal{L}_\wedge(L) - (-1)^{kl}\mathcal{L}_\wedge(L)\mathcal{L}_\wedge(K) = \mathcal{L}_\wedge([K,L]_\wedge).$$

If the derivation i_K of degree k looks at the expression $\mathcal{L}_\wedge(L)\omega$, it sees only the 'wedge' product in 3.4, so we may apply lemma 5.1 (or reprove lemma 5.1 in this situation) and get

$$(9)\qquad i_K \mathcal{L}_\wedge(L)\omega = \mathcal{L}_\wedge(i_K L)\omega + (-1)^{(k-1)l}\mathcal{L}_\wedge(L) i_K \omega.$$

By a straightforward combinatorial computation one can check directly from the definitions that the following formula holds:

$$(10)\qquad \partial(i_K L) = i_{\partial K} L + (-1)^{k-1} i_K \partial L + (-1)^k [K,L]_\wedge.$$

Moreover it is obvious that

$$(11)\qquad \partial[K,L]_\wedge = [\partial K, L]_\wedge + (-1)^k [K, \partial L]_\wedge.$$

We have to show that (6) equals (7). This follows by using (8), twice (9), twice (10), (11), and $\partial\partial = 0$.

That the Frölicher-Nijenhuis bracket defines a graded Lie bracket follows now from the fact that $\mathcal{L} : C(\operatorname{Der}(A), \operatorname{Der}(A)) \to \operatorname{Der}(C(\operatorname{Der}(A), A))$ is injective. □

5.7. Lemma. *For* $K \in C^k(\mathrm{Der}(A), \mathrm{Der}(A))$ *and* $L \in C^l(\mathrm{Der}(A), \mathrm{Der}(A))$ *we have*

$$[\mathcal{L}_K, i_L] = i([K, L]) - (-1)^{k(l-1)}\mathcal{L}(i_L K), \text{ or}$$
$$[i_L, \mathcal{L}_K] = \mathcal{L}(i_L K) + (-1)^k\, i([L, K]).$$

Proof. The two equations are obviously equivalent by graded skew symmetry, and the second one follows by inserting the following formulas, all from 5.6: expand the equation by (4), (2), and use then (10). □

5.8. Remark. As formal consequences of lemma 5.7 we get the following formulae: For $K_i \in C^{k_i}(\mathrm{Der}(A), \mathrm{Der}(A))$ and $L_i \in C^{k_i+1}(\mathrm{Der}(A), \mathrm{Der}(A))$ we have

$$\begin{aligned}[\mathcal{L}_{K_1} + i_{L_1}, \mathcal{L}_{K_2} + i_{L_2}] &= \\ = \mathcal{L}&\left([K_1, K_2] + i_{L_1}K_2 - (-1)^{k_1k_2} i_{L_2}K_1\right) \\ &+ i\left([L_1, L_2]^\wedge + [K_1, L_2] - (-1)^{k_1k_2}[K_2, L_1]\right).\end{aligned} \tag{1}$$

Each summand of this formula looks like a semidirect product of graded Lie algebras, but the mappings

$$\begin{aligned} i &: C(\mathrm{Der}(A), \mathrm{Der}(A)) \to \mathrm{End}_{\mathbb{K}}(C(\mathrm{Der}(A), \mathrm{Der}(A)), [\quad,\quad]) \\ \mathrm{ad} &: C(\mathrm{Der}(A), \mathrm{Der}(A)) \to \mathrm{End}_{\mathbb{K}}(C(\mathrm{Der}(A), \mathrm{Der}(A)), [\quad,\quad]^\wedge)\end{aligned}$$

do not take values in the subspaces of graded derivations. We have instead for $K \in C^k(\mathrm{Der}(A), \mathrm{Der}(A))$ and $L \in C^{l+1}(\mathrm{Der}(A), \mathrm{Der}(A))$ the following relations:

$$\begin{aligned} i_L[K_1, K_2] &= [i_L K_1, K_2] + (-1)^{k_1 l}[K_1, i_L K_2] \\ &- \Big((-1)^{k_1 l} i([K_1, L])K_2 - (-1)^{(k_1+l)k_2} i([K_2, L])K_1\Big)\end{aligned} \tag{2}$$

$$\begin{aligned} [K, [L_1, L_2]^\wedge] &= [[K, L_1], L_2]^\wedge + (-1)^{kl_1}[L_1, [K, L_2]]^\wedge - \\ &- \Big((-1)^{kl_1}[i(L_1)K, L_2] - (-1)^{(k+l_1)l_2}[i(L_2)K, L_1]\Big)\end{aligned} \tag{3}$$

The algebraic meaning of these relations and its consequences in group theory have been investigated in [20]. The corresponding product of groups is well known to algebraists under the name 'Zappa-Szep'-product.

Moreover the Chevalley coboundary operator is a homomorphism from the Frölicher-Nijenhuis bracket to the Nijenhuis-Richardson bracket:

$$\partial[K, L] = [\partial K, \partial L]^\wedge \tag{4}$$

Proof. (1) follows from repeated applications of 5.7. It is easy to show that the images under $\mathcal{L}$ of both sides of (2) coincide, using 5.7, then we just note that $\mathcal{L}$ is injective. For (3) it is again easy to show that the images under i of both sides coincide, again using 5.7; also $i : C(\mathrm{Der}(A), \mathrm{Der}(A)) \to \mathrm{Der}(C(\mathrm{Der}(A), A))$ is injective.

(4) follows from 5.6,(2), (10), and (11), and from 5.3. □

5.9. The graded differential Lie-subalgebra $C_{Z(A)}(\mathrm{Der}(A), \mathrm{Der}(A))$. Clearly a derivation $i_K \in \mathrm{Der}(C(\mathrm{Der}(A), A))$ for $K \in C^k(\mathrm{Der}(A), \mathrm{Der}(A))$ maps the graded differential subalgebra $C_{Z(A)}(\mathrm{Der}(A), A)$ into itself if and only if K is skew $Z(A)$-multilinear, i.e.

$$K \in C_{Z(A)}(\mathrm{Der}(A), \mathrm{Der}(A))$$
$$K(z_1X_1, \dots, z_kX_k) = z_1 \dots z_k K(X_1, \dots, X_k) \text{ for all } z_i \in Z(A) \text{ and } X_i \in \mathrm{Der}(A).$$

Then also the Lie derivation $\mathcal{L}_K$ respects $C_{Z(A)}(\mathrm{Der}(A), A) = \underline{\Omega}_{\mathrm{Der}}(A)$, and so the closed linear subspace $C_{Z(A)}(\mathrm{Der}(A), \mathrm{Der}(A))$ is a graded Lie subalgebra for both brackets $[\quad,\quad]^\wedge$ and $[\quad,\quad]$, and all formulas in this section continue to hold. But note that in the simpler formula 5.6.(2), although $[K, L]$ is $Z(A)$-multilinear, none of the three summands is in $C_{Z(A)}(\mathrm{Der}(A), \mathrm{Der}(A))$. The same applies to 5.6.(4).

5.10. Remark. If $K \in C^1(\mathrm{Der}(A), \mathrm{Der}(A)) = L(\mathrm{Der}(A), \mathrm{Der}(A))$ then formula 5.6.(1) boils down to

$$[K, K](X, Y) = 2([KX, KY] - K[KX, Y] - K[X, KY] + K^2[X, Y]).$$

So if $[K, K] = 0$ then the image of K is a Lie subalgebra of $\mathrm{Der}(A)$. If K is moreover a projection, $K \circ K = K$, then also the kernel is a Lie subalgebra. In this case one could view K as a 'connection' and could say that the kernel and the image of K are 'involutive subbundles' whose 'curvatures vanish', compare with [1], section 5. We shall elaborate on this topic in a later paper.

If $K : \mathrm{Der}(A) \to \mathrm{Der}(A)$ is semisimple with $[K, K] = 0$, then each eigenspace of K is also a Lie subalgebra. For if $KX = \lambda X$ and $KY = \lambda Y$ then we get $(K - \lambda)^2[X, Y] = 0$ and thus $K[X, Y] = \lambda[X, Y]$. In particular the kernel of K is a Lie subalgebra $\lambda = 0$.

If moreover K is $Z(A)$-linear, $K \in C^1_{Z(A)}(\mathrm{Der}(A), \mathrm{Der}(A)) = \mathrm{Der}(A, \underline{\Omega}^1_{\mathrm{Der}}(A))$, then the above Lie subalgebras of $\mathrm{Der}(A)$ are also $Z(A)$-submodules.

6. Description of all derivations in the Chevalley differential graded algebra

6.1. Insertion operators. The space $\mathrm{Hom}^A_A(C^p(\mathrm{Der}(A), A), A)$ is a sort of 'A-dual' of $C^p(\mathrm{Der}(A), A)$. Let us write $\langle \Xi, \varphi \rangle_A \in A$ for the evaluation of the element $\Xi \in \mathrm{Hom}^A_A(C^p(\mathrm{Der}(A), A), A)$ on $\varphi \in C^p(\mathrm{Der}(A), A)$. Note that for $a, b \in A$ we have $\langle \Xi, a.\varphi.b \rangle_A = a.\langle \Xi, \varphi \rangle_A.b$. Then we consider the (closure of the) linear subspace

$$\sum_{0<i<p} C^i(\mathrm{Der}(A), A).C^{p-i}(\mathrm{Der}(A), A) \subset C^p(\mathrm{Der}(A), A)$$

and its annihilator

$$\mathrm{Ann}^p(\mathrm{Der}(A), A) := \left(\sum_{0<i<p} C^i(\mathrm{Der}(A), A).C^{p-i}(\mathrm{Der}(A), A) \right)^{\bigcirc}$$
$$\subseteq \mathrm{Hom}^A_A(C^p(\mathrm{Der}(A), A), A),$$

and set

$$\begin{aligned}\operatorname{Ann}^1(\operatorname{Der}(A),A)&=\operatorname{Hom}^A_A(C^1(\operatorname{Der}(A),A),A),\\ \operatorname{Ann}^0(\operatorname{Der}(A),A)&=\operatorname{Der}(A).\end{aligned}$$

For $\Xi\in\operatorname{Ann}^p(\operatorname{Der}(A),A)$ we consider the 'insertion operator'

$$\begin{gathered}i_\Xi : C^*(\operatorname{Der}(A),A)\to C^{*-p}(\operatorname{Der}(A),A),\\ (i_\Xi\omega)(X_1,\dots,X_l):=\langle\Xi,\omega(\underbrace{\ ,\dots,\ }_{p\text{ times}},X_1,\dots,X_l)\rangle_A,\end{gathered}$$

which is a derivation of degree $-p$ satisfying $i_\Xi|C^q(\operatorname{Der}(A),A)=0$ for $q<p$, since Ξ is an A-bimodule homomorphism and annihilates all 'small' products.

More generally let $K\in C^k(\operatorname{Der}(A),\operatorname{Ann}^p(\operatorname{Der}(A),A))$ be a bounded skew symmetric k-linear mapping $\operatorname{Der}(A)^k\to\operatorname{Ann}^p(\operatorname{Der}(A),A)$. Then we define the bounded linear mapping $i_K : C^l(\operatorname{Der}(A),A)\to C^{l+k-p}(\operatorname{Der}(A),A)$ by $i_K|C^q(\operatorname{Der}(A),A)=0$ for $q<p$, and by

$$\begin{aligned}(1)\quad &(i_K\omega)(X_1,\dots,X_{l+k-p})=\\ &=\tfrac{1}{k!\,(l-1)!}\sum_{\sigma\in\mathcal{S}_{k+l-1}}\operatorname{sign}\sigma.\langle K(X_{\sigma1},\dots,X_{\sigma k}),\omega(\underbrace{\ ,\dots,\ }_{p\text{ times}},X_{\sigma(k+1)},\dots)\rangle_A.\end{aligned}$$

Lemma. *i_K is a derivation of degree $k-p$, $i_K\in\operatorname{Der}_{k-p}(C^*(\operatorname{Der}(A),A))$.*

Proof. Clearly one may write

$$\begin{aligned}(2)\quad &(i_K\omega)(X_1,\dots,X_{l+k-1})=\\ &=\sum_{i_1<\dots<i_k}(-1)^{i_1+\dots+i_k-\frac{k(k+1)}{2}}\langle K(X_{i_1},\dots,X_{i_k}),\\ &\qquad\omega(\underbrace{\ ,\dots,\ }_{p\text{ times}},X_1,\dots,\widehat{X_{i_1}},\dots,\widehat{X_{i_2}},\dots)\rangle_A.\end{aligned}$$

Using expression (2) one may check directly that then we have

$$(3)\qquad [i_X,i_K]=i_{K(X,\dots)}=i_{i_XK}\text{ for any }X\in\operatorname{Der}(A).$$

From (3) it is easy to prove that for $\varphi\in C^m(\operatorname{Der}(A),A)$ and $\psi\in C^n(\operatorname{Der}(A),A)$ we have

$$i_K(\varphi.\psi)=(i_K\varphi).\psi+(-1)^{(k-p)m}\varphi.(i_K\psi),$$

by induction on $k-p+m+n$, for each fixed p. $\square$

6.2. Proposition. *Let $D \in \operatorname{Der}_k(C(\operatorname{Der}(A), A))$ be a graded derivation. Then there are unique $K_0 \in C^k(\operatorname{Der}(A), \operatorname{Der}(A)) = C^k(\operatorname{Der}(A), \operatorname{Ann}^0(\operatorname{Der}(A), A))$ and $K_p \in C^{k+p}(\operatorname{Der}(A), \operatorname{Ann}^p(\operatorname{Der}(A), A))$ for $p = 1, 2, \dots$ such that*

$$D = \mathcal{L}_{K_0} + \sum_{p=1}^{\infty} i_{K_p}.$$

Note that on each fixed component $C^l(\operatorname{Der}(A), A)$ this is a finite sum.

Proof. Let $D \in \operatorname{Der}_k(C(\operatorname{Der}(A), A))$ be a graded derivation. Then the restriction $D|A$ of D to $A = C^1(\operatorname{Der}(A), A)$ is an element of

$$\begin{aligned} K_0 := D|A \in \operatorname{Der}(A, C^k(\operatorname{Der}(A), A)) &= \\ = \operatorname{Der}(A, L(\Lambda^k \operatorname{Der}(A), A)) &= \\ = L(\Lambda^k \operatorname{Der}(A), \operatorname{Der}(A, A)),& \text{ by 2.6} \\ = C^k(\operatorname{Der}(A), \operatorname{Der}(A)).& \end{aligned}$$

By 5.4 we have the Lie derivation $\mathcal{L}_{K_0} \in \operatorname{Der}_k(C(\operatorname{Der}(A), A))$ which coincides with D on $A = C^0(\operatorname{Der}(A), A)$, so that the difference $D - \mathcal{L}_{K_0} \in \operatorname{Der}_k(C(\operatorname{Der}(A), A))$ is a graded derivation with $(D - \mathcal{L}_{K_0})|A = 0$. So

$$\begin{aligned} K_1 := (D - \mathcal{L}_{K_0})|C^1(\operatorname{Der}(A), A) &\in \operatorname{Hom}_A^A(C^1(\operatorname{Der}(A), A), C^{k+1}(\operatorname{Der}(A), A)) \\ &= \operatorname{Hom}_A^A(L(\operatorname{Der}(A), A), L(\Lambda^{k+1} \operatorname{Der}(A), A)) \\ &= L(\Lambda^{k+1} \operatorname{Der}(A), \operatorname{Hom}_A^A(L(\operatorname{Der}(A), A), A)) \text{ by 2.6} \\ &= L(\Lambda^{k+1} \operatorname{Der}(A), \operatorname{Ann}^1(\operatorname{Der}(A), A)). \end{aligned}$$

By 6.1 we get a derivation $i_{K_1} : C(\operatorname{Der}(A), A) \to C(\operatorname{Der}(A), A)$ and the difference $D - \mathcal{L}_{K_0} - i_{K_1} \in \operatorname{Der}_k(C(\operatorname{Der}(A), A))$ now vanishes on $A = C^0(\operatorname{Der}(A), A)$ and $C^1(\operatorname{Der}(A), A)$. Thus the restriction is an A-bimodule homomorphism

$$(D - \mathcal{L}_{K_0} - i_{K_1})|C^2(\operatorname{Der}(A), A) : C^2(\operatorname{Der}(A), A) \to C^{k+2}(\operatorname{Der}(A), A))$$

vanishes on all products of 1-forms. If we consider

$$\begin{aligned} K_2 := (D - \mathcal{L}_{K_0} - i_{K_1})|C^2(\operatorname{Der}(A), A) &\in \operatorname{Hom}_A^A(C^2(\operatorname{Der}(A), A), C^{k+2}(\operatorname{Der}(A), A)) \\ &= \operatorname{Hom}_A^A(C^2(\operatorname{Der}(A), A), L(\Lambda^{k+2} \operatorname{Der}(A), A)) \\ &= L(\Lambda^{k+2} \operatorname{Der}(A), \operatorname{Hom}_A^A(C^2(\operatorname{Der}(A), A), A)) \text{ by 2.6,} \\ &= C^{k+2}(\operatorname{Der}(A), \operatorname{Ann}^2(\operatorname{Der}(A), A)), \end{aligned}$$

then by lemma 6.1 we have a derivation $i_{K_2} \in \operatorname{Der}_k(C(\operatorname{Der}(A), A))$ which vanishes on $C^q(\operatorname{Der}(A), A)$ for $q = 0, 1$ and coincides with $D - \mathcal{L}_{K_0} - i_{K_1}$ on $C^2(\operatorname{Der}(A), A)$, so the derivation $D - \mathcal{L}_{K_0} - i_{K_1} - i_{K_2}$ vanishes on $C^q(\operatorname{Der}(A), A)$ for $q = 0, 1, 2$, and we may repeat the process. $\square$

6.3. Remarks. If we try to expand the graded commutator $[i_K, i_L]$ using 6.2, the resulting formulas are too complicated to be written down easily: we should invent new notation. We refrain from doing this since we also have no use for it.

For $K \in C^k(\mathrm{Der}(A), \mathrm{Ann}^1(\mathrm{Der}(A), A))$ we get $[i_K, d]|A = \mathcal{L}_{\hat{K}}|A$, where $\hat{K} \in C^k(\mathrm{Der}(A), \mathrm{Der}(A))$ is given by $\hat{K}(X_1, \dots, X_k)(a) = K(X_1, \dots, X_k)(da)$. The higher order part in the expansion of $[i_K, d]$ according to 6.2 does not vanish, and it can be written down in principle.

If we want to classify all derivations in $\mathrm{Der}(C_{Z(A)}(\mathrm{Der}(A), A))$ we can just repeat the development in this section, but have to replace $\mathrm{Ann}^k(\mathrm{Der}(A), A)$ everywhere by the annihilator

$$
\begin{aligned}
\mathrm{Ann}^p_{Z(A)}(\mathrm{Der}(A), A) :&= \left(\sum_{0<i<p} C^i_{Z(A)}(\mathrm{Der}(A), A).C^{p-i}_{Z(A)}(\mathrm{Der}(A), A) \right)^{\bigcirc} \\
&\subseteq \mathrm{Hom}^A_A(C^p_{Z(A)}(\mathrm{Der}(A), A), A), \\
\mathrm{Ann}^1_{Z(A)}(\mathrm{Der}(A), A) &= \mathrm{Hom}^A_A(C^1_{Z(A)}(\mathrm{Der}(A), A)), \\
\mathrm{Ann}^0_{Z(A)}(\mathrm{Der}(A), A) &= \mathrm{Ann}^0(\mathrm{Der}(A), A) = \mathrm{Der}(A).
\end{aligned}
$$

7. Diagonal bimodules

7.1. The differential graded algebra $\Omega_{\mathrm{Der}}(A)$. By the universal property of $(\Omega^*(A), d)$ there is a unique bounded homomorphism of graded differential algebras

$$
\begin{array}{ccccccc}
A & \xrightarrow{d} & \Omega^1(A) & \xrightarrow{d} & \Omega^2(A) & \xrightarrow{d} & \cdots \\
\| & & \downarrow{\zeta_1} & & \downarrow{\zeta_2} & & \\
A & \xrightarrow{d} & C^1_{Z(A)}(\mathrm{Der}(A), A) & \xrightarrow{d} & C^2_{Z(A)}(\mathrm{Der}(A), A) & \xrightarrow{d} & \cdots
\end{array}
$$

which is given by

$$(\zeta_k \omega)(X_1, \dots, X_k) = j_{X_k} \dots j_{X_1} \omega \in \Omega^0(A) = A,$$

for $\omega \in \Omega^k(A)$ and $X_i \in \mathrm{Der}(A)$. The kernel of ζ is the space

$$
\begin{aligned}
F^1\Omega^*(A) &= \bigoplus_{k \geq 0} F^1\Omega^k(A) \\
F^1\Omega^k(A) &= \{\omega \in \Omega^k(A) : j_{X_1} \dots j_{X_k} \omega = 0 \text{ for all } X_i \in \mathrm{Der}(A)\},
\end{aligned}
$$

see [6], which is a closed graded differential ideal in $(\Omega(A), d)$ by a short computation. It is part of an obvious filtration which leads to a spectral sequence, see also [6]. The image of the homomorphism ζ is denoted by $(\Omega^*_{\mathrm{Der}}(A), d)$ and it will be equipped with the quotient structure of a convenient vector space, which is a finer structure than that induced from $C_{Z(A)}(\mathrm{Der}(A), A)$.

Note that $\Omega_{\mathrm{Der}}(A)$ is not functorial in A in general; its convenient structure makes it useful as a source for constructions.

7.2. Derivation based bimodules. The space $\Omega^1_{\mathrm{Der}}(A) = \Omega^1(A)/\{\omega \in \Omega^1(A) : j_X\omega = 0 \text{ for all } X \in \mathrm{Der}(A)\}$ has the following remainder of the universal property 3.3.(1):

(1) For any bounded derivation $X : A \to A$ there is a unique bounded A-bimodule homomorphism $i_X : \Omega^1_{\mathrm{Der}}(A) \to A$ such that $X = i_X \circ d$.

We now say that an A-bimodule M is a *derivation-based A-bimodule* if the following property is satisfied:

(2) For any bounded derivation $D : A \to M$ there is a bounded A-bimodule homomorphism $i_D : \Omega^1_{\mathrm{Der}}(A) \to M$ such that $D = i_D \circ d$. In fact, if i_D exists it is unique since the image of d generates $\Omega^1_{\mathrm{Der}}(A)$ as A-bimodule.

By the universal property 3.3.(1) of $\Omega^1(A)$ condition (2) is equivalent to the following:

(3) $\mathrm{Hom}^A_A(\Omega^1(A), M) = \mathrm{Hom}^A_A(\Omega^1_{\mathrm{Der}}(A), M)$.

It is obvious that for an arbitrary index set J the direct product A^J with the product A-bimodule structure has property (2), and also each of it's sub A-bimodules. Let us call a *diagonal bimodules* any A-bimodule which is isomorphic to a submodule of some product A^J. So all diagonal bimodules are derivation based. The concept of diagonal bimodules has been announced in [9].

7.3. Proposition. *An A-bimodule K is diagonal if and only if the 'A-dual' $\mathrm{Hom}^A_A(K, A)$ separates points on K. For each A-bimodule M there exists a universal diagonal quotient $p_M : M \to \mathrm{Diag}(M)$ such that each bimodule homomorphism from M into a diagonal bimodule K factors over $\mathrm{Diag}(M)$:*

$$\begin{array}{ccc} M & \longrightarrow & K \\ {\scriptstyle p_M}\downarrow & & \| \\ \mathrm{Diag}(M) & \longrightarrow & K \end{array}$$

Let Bimod *be the category of all A-bimodules and let* Diag *denote the full subcategory of all diagonal A-bimodules, with ι : Diag $\to$ Bimod the embedding and* Diag : Bimod $:\to$ Diag *the functor from above. Then* Diag *is left adjoint to ι, i.e. we have the following natural correspondence:*

$$\mathrm{Hom}^A_A(\mathrm{Diag}(M), K) \cong \mathrm{Hom}^A_A(M, \iota K)$$

Thus the functor Diag *respects colimits, whereas ι respects limits, and the category* Diag *is complete: products and submodules of modules in* Diag *are again in* Diag.

Proof. The vector space $\mathrm{Hom}^A_A(A^J, A)$ clearly separates points on A^J, so this is also true for each sub bimodule of A^J. For an arbitrary A-bimodule M we consider the following homomorphism of A-bimodules:

$$M \xrightarrow{p_M} A^{\mathrm{Hom}^A_A(M,A)}, \quad M \ni m \mapsto (\varphi(m))_{\varphi \in \mathrm{Hom}^A_A(M,A)} \in A^{\mathrm{Hom}^A_A(M,A)},$$

and we denote by $\mathrm{Diag}(M)$ the image of p_M, a diagonal A-bimodule. If the vector space $\mathrm{Hom}^A_A(M, A)$ separates points on M then p_M is injective and M is diagonal. The kernel of p_M is given by $\ker(p_M) = \{m \in M : \varphi(m) = 0 \text{ for all } \varphi \in$

$\operatorname{Hom}^A_A(M, A)\}$, and obviously any homomorphism $M \to K$ into a diagonal A-bimodule K vanishes on $\ker(p_M)$ and thus factors through p_M.

The remaining statements follow by basic category theory. □

7.4. Examples. We have $\operatorname{Diag}(\Omega^1(A)) = \Omega^1_{\text{Der}}(A)$, since any homomorphism $\Omega^1(A) \to A$ corresponds to a derivation $A \to A$ and thus factors to a homomorphism $\Omega^1_{\text{Der}}(A) \to A$.

Obviously we have $\operatorname{Diag}(A) = A$, but let us consider $\operatorname{Diag}(A\tilde{\otimes}A)$. We have $\operatorname{Hom}^A_A(A\tilde{\otimes}A, A) \cong A$ by $\varphi \mapsto \varphi(1\otimes 1)$, so $\ker(p_{A\tilde{\otimes}A} : A\tilde{\otimes}A \to \operatorname{Diag}(A\tilde{\otimes}A))$ consists of all

(1) $\sum_n a_n\tilde{\otimes}b_n \in A\tilde{\otimes}A$ which satisfy $\sum_n a_n.c.b_n = 0$ for each $c \in A$.

Taking $c = 1$ in (1) we see that $\ker(p_{A\otimes A}) \subset \Omega^1(A)$, and for each element in (1) we have

$$\sum_n a_n\tilde{\otimes}b_n = \sum_n a_n\, db_n,$$
$$i_{\operatorname{ad}(c)} \sum_n a_n\, db_n = \sum_n a_n.(c.b_n - b_n.c) = 0,$$

so that $\ker(p_{A\otimes A})$ is the subspace $\Omega^1(A)_{H-\operatorname{Int}(A)}$ of all elements in $\Omega^1(A)$ which are horizontal with respect to $\operatorname{Int}(A)$, see 3.5.

We also see that $1\otimes a - a\otimes 1 \in \ker(p_{A\tilde{\otimes}A})$ if and only if $a \in Z(A)$, so that $p_{A\tilde{\otimes}A}$ factors as follows:

$$A\tilde{\otimes}A \to A\tilde{\otimes}_{Z(A)}A \to \operatorname{Diag}(A\tilde{\otimes}A). \tag{2}$$

Since Diag is a left adjoint functor, it is right exact, and we get the following diagram with exact rows and exact columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & F^1\Omega^1(A) & \longrightarrow & \Omega^1(A)_{H-\operatorname{Int}(A)} & \longrightarrow & \Omega^1_{\text{Der}}(A)_{H-\operatorname{Int}(A)} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & \Omega^1(A) & \longrightarrow & A\tilde{\otimes}A & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p_{\Omega^1(A)}} & & \downarrow{\scriptstyle p_{A\tilde{\otimes}A}} & & \| & & \\
 & & \Omega^1_{\text{Der}}(A) & \longrightarrow & \operatorname{Diag}(A\tilde{\otimes}A) & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

From this and the factorization (2) it follows, that for commutative A we have $\operatorname{Diag}(A\tilde{\otimes}A) = A$.

7.5. Lemma. *Over the algebra* $\mathrm{Mat}_N(\mathbb{C})$ *of complex* $(N \times N)$*-matrices every bimodule is diagonal.*

Proof. It is well known that every irreducible left Mat_N-module is isomorphic to $\mathbb{C}^N$, and that every left Mat_N-module is semisimple. From this by transfinite induction one can show that every left Mat_N-module L (with its finest locally convex topology) is isomorphic to the direct sum of copies of $\mathbb{C}^N$, $L \cong \mathbb{C}^N \otimes E$ for a vector space E.

Now let M be a Mat_N-bimodule. As a left Mat_N-module we have $M \cong \mathbb{C}^N \otimes E$. Take a minimal projection $p \in \mathrm{Mat}_N$, then $p.M \cong \mathbb{C}.v \otimes E \cong E$ is a right Mat_N-module. By the argument above, as a right Mat_N-module we have $E \cong K \otimes (\mathbb{C}^N)^*$. Thus

$$M \cong C^N \otimes K \otimes (\mathbb{C}^N)^* \cong \mathbb{C}^N \otimes (\mathbb{C}^N)^* \otimes K \cong \mathrm{Mat}_N \otimes K \subset {\mathrm{Mat}_N}^K. \quad \square$$

8. Derivations on the differential graded algebra $\Omega_{\mathrm{Der}}(A)$

8.1. Theorem. *Let A be a convenient algebra. Then we have the following bounded canonical mappings, where on the left hand side all mappings are homomorphisms of graded differential algebras*

$$\begin{array}{ccccc}
\Omega^*(A) & & \mathrm{Der}(A, \Omega^*(A)) & & \\
\big\downarrow \zeta & & \big\downarrow \zeta & & \\
\Omega^*_{\mathrm{Der}}(A) & & \mathrm{Der}(A, \Omega^*_{\mathrm{Der}}(A)) & & \\
\subset \big\downarrow \bar\zeta & & \subset \big\downarrow \bar\zeta & & \\
C_{Z(A)}(\mathrm{Der}(A), A) & & \mathrm{Der}(A, C_{Z(A)}(\mathrm{Der}(A), A)) & = & C_{Z(A)}(\mathrm{Der}(A), \mathrm{Der}(A)) \\
\subset \big\downarrow & & \subset \big\downarrow & & \\
C(\mathrm{Der}(A), A) & & \mathrm{Der}(A, C(\mathrm{Der}(A), A)) & = & C(\mathrm{Der}(A), \mathrm{Der}(A))
\end{array}$$

Then for every element K of degree k in one of the right hand spaces there is a canonical graded derivation i_K (resp. j_K) of degree $k-1$ on the corresponding left hand space which vanishes in degree 0; this corresponds to the graded Lie bracket $[\ \ ,\ \ ]^\wedge$ on the right hand spaces. There is also the corresponding Lie derivation $\mathcal{L}_K$ of degree k on the left hand space, which leads to the Frölicher-Nijenhuis bracket $[\ \ ,\ \ ]$ on the right hand space. The vertical arrows intertwine all these derivations and the right hand ones are homomorphisms for all brackets mentioned.

The proof of this theorem will fill the rest of this section 8.

8.2. Let $D \in \mathrm{Der}_k(\Omega^*_{\mathrm{Der}}(A))$ be a derivation which vanishes on $A = \Omega^0_{\mathrm{Der}}(A)$. Then by setting $K := (D|\Omega^1_{\mathrm{Der}}(A)) \circ d \in \mathrm{Der}(A, \Omega^{k+1}_{\mathrm{Der}}(A))$ we have

$$D(\omega_0 \dots \omega_l) = \sum (-1)^{ik} (\omega_0) \dots D(\omega_i) \dots (\omega_l), \tag{1}$$

so D is uniquely determined by $D|\Omega^1_{\mathrm{Der}}(A)$, thus by K.

Conversely, let $K \in \operatorname{Der}(A, \Omega^{k+1}_{\mathrm{Der}}(A))$ and consider the corresponding homomorphism $\tilde{j}_K \in \operatorname{Hom}^A_A(\Omega^1(A), \Omega^{k+1}_{\mathrm{Der}}(A))$ with $\tilde{j}_K \circ d = K$. We extend it to $\tilde{j}_K : \Omega^*(A) \to \Omega^{*+k}_{\mathrm{Der}}(A)$ by the right hand side of the universal analog of (1), i.e.

$$\tilde{j}_K(\omega_0 \otimes_A \cdots \otimes_A \omega_l) = \sum (-1)^{ik} \zeta(\omega_0) \dots \tilde{j}_K(\omega_i) \dots \zeta(\omega_l) \tag{2}$$

Then $\tilde{j}_K$ is a graded derivation of degree k along ζ. We are going to show that $\tilde{j}_K$ factors to $\Omega^*_{\mathrm{Der}}(A)$, but we need some preparation.

8.3. For $X \in \operatorname{Der}(A) \cong \operatorname{Hom}^A_A(\Omega^1(A), A) \cong \operatorname{Hom}^A_A(\Omega^1_{\mathrm{Der}}(A), A)$ we clearly have the following factorization:

$$\begin{array}{ccc}
\Omega^*(A) & \xrightarrow{j_X} & \Omega^{*-1}(A) \\
\zeta\downarrow & & \zeta\downarrow \\
\Omega^*_{\mathrm{Der}}(A) & \xrightarrow{i_X} & \Omega^{*-1}_{\mathrm{Der}}(A) \\
\downarrow & & \downarrow \\
C^*_{Z(A)}(\operatorname{Der}(A), A) & \xrightarrow{i_X} & C^{*-1}_{Z(A)}(\operatorname{Der}(A), A)
\end{array}$$

For $K \in \operatorname{Der}(A, \Omega^{k+1}_{\mathrm{Der}}(A))$ let us consider the 'graded commutator'

$$i_X\, \tilde{j}_K - (-1)^k \tilde{j}_K\, j_X : \Omega^*(A) \to \Omega^{*+k}_{\mathrm{Der}}(A)$$

It is still a graded derivation along ζ, and by applying 8.2 we see that

$$\begin{gathered}
i_X\, \tilde{j}_K - (-1)^k \tilde{j}_K\, j_X = \tilde{j}_{[X,K]^\Delta} \\
\text{for } [X,K]^\Delta \in \operatorname{Der}(A, \Omega^{k+1}_{\mathrm{Der}}(A)), \\
[X,K]^\Delta(\omega) = i_X K(\omega) - (-1)^k \tilde{j}_K(j_X \omega) = (i_X \circ K)(\omega).
\end{gathered}$$

8.4. Lemma. *For $K \in \operatorname{Der}(A, \Omega^{k+1}_{\mathrm{Der}}(A))$ we have $\tilde{j}_K(F^1(\Omega^l(A))) = 0$ for all l.*

Proof. We do induction on $l + k$. For $l = 0$ we have $F^1(A) = 0$, so the assertion holds. If $k = -1$ then $K = X \in \operatorname{Der}(A)$ and we have $\tilde{j}_X(F^1(\Omega^l(A))) = \zeta(j_X(F^1(\Omega^l(A)))) \subset \zeta(F^1(\Omega^l(A))) = 0$.

Now for the induction step we take $\varphi \in F^1(\Omega^l(A))$, so that $j_{X_1} \dots j_{X_l} \varphi = 0$ for all $X_i \in \operatorname{Der}(A)$. From 8.3 we get

$$i_X\, \tilde{j}_K\, \varphi = (-1)^k \tilde{j}_K(j_X \varphi) + \tilde{j}_{i_X \circ K} \varphi = 0,$$

by induction: the first summand vanishes since $j_X \varphi \in F^1 \Omega^{l-1}(A)$, and the second summand vanishes since $i_X \circ K \in \operatorname{Der}(A, \Omega^k_{\mathrm{Der}}(A))$. □

8.5. Corollary. *For each k the A-bimodule $\Omega^k_{\mathrm{Der}}(A)$ is a derivation based bimodule.*

Proof. By the universal property 3.3.(1) and from 8.4 we get

$$\mathrm{Der}(A,\Omega^k_{\mathrm{Der}}(A)) \cong \mathrm{Hom}^A_A(\Omega^1(A),\Omega^k_{\mathrm{Der}}(A)) \xleftarrow[\cong]{\zeta^*} \mathrm{Hom}^A_A(\Omega^1_{\mathrm{Der}}(A),\Omega^k_{\mathrm{Der}}(A)). \quad \square$$

This result also follows from the fact that $\Omega^k_{\mathrm{Der}}(A)$ is a sub A-bimodule of a direct product A^J, i.e. a diagonal bimodule, see the last remark in 7.2. By the same reason the bimodules $C^k(\mathrm{Der}(A),A)$ and $C^k_{Z(A)}(\mathrm{Der}(A),A)$ are also diagonal bimodules, thus derivation based A-bimodules.

8.6. Insertion operators. For any $K \in \mathrm{Der}(A,\Omega^{k+1}_{\mathrm{Der}}(A))$ we may now define the insertion operator $i_K \in \mathrm{Der}(\Omega^*_{\mathrm{Der}}(A))$ by the following factorization which is due to lemma 8.4:

$$\begin{array}{ccc} \Omega^*(A) & \xrightarrow{\tilde{j}_K} & \Omega^{*+k}_{\mathrm{Der}}(A) \\ {\scriptstyle\zeta}\big\downarrow & & \big\| \\ \Omega^*_{\mathrm{Der}}(A) & \xrightarrow{i_K} & \Omega^{*+k}_{\mathrm{Der}}(A) \end{array}$$

From 8.2 we may now conclude that any derivation $D \in \mathrm{Der}(\Omega^*_{\mathrm{Der}}(A))$ with $D|A = 0$ is of the form i_K for a unique $K \in \mathrm{Der}(A,\Omega^{k+1}_{\mathrm{Der}}(A))$.

For $K, L \in \mathrm{Der}(A,\Omega^*_{\mathrm{Der}}(A))$ of degree $k+1$ and $l+1$, respectively, we have

$$[i_K, i_L] = i_K\, i_L - (-1)^{kl} i_L\, i_K = i_{[K,L]^\wedge}, \text{ where}$$
$$[K,L]^\wedge = i_K \circ L - (-1)^{kl} i_L \circ K : \Omega^1(A) \to \Omega^{k+l+1}_{\mathrm{Der}}(A)$$

8.7. Lie derivations. For $K \in \mathrm{Der}(A,\Omega^k_{\mathrm{Der}}(A))$ we define the *Lie derivation* along K by

$$\mathcal{L}_K = [i_K, d] = i_K\, d - (-1)^{k-1} d\, i_K \in \mathrm{Der}_k(\Omega^*_{\mathrm{Der}}(A)).$$

Similar as in 4.3 one sees that $\mathcal{L}_{Id} = d$.

8.8. Proposition. *For any graded derivation $D \in \mathrm{Der}_k(\Omega^*_{\mathrm{Der}}(A))$ there are unique $K \in \mathrm{Der}(A,\Omega^k_{\mathrm{Der}}(A))$ and $L \in \mathrm{Der}(A,\Omega^{k+1}_{\mathrm{Der}}(A))$ such that*

$$D = \mathcal{L}_K + i_L.$$

We have $L = 0$ if and only if $[D,d] = 0$; and $K = 0$ if and only if $D|A = 0$.

Proof. $D|A : A = \Omega^0_{\mathrm{Der}}(A) \to \Omega^k_{\mathrm{Der}}(A)$ is a derivation with values in the derivation based A-bimodule $\Omega^k_{\mathrm{Der}}(A)$ (7.2), so by the universal property 3.3.(1) and by 8.5 there is a unique $K \in \mathrm{Der}(A,\Omega^k_{\mathrm{Der}}(A))$ with $D|A = K$. But then $(D - \mathcal{L}_K)|A = 0$, so by 8.6 we have $D - \mathcal{L}_K = i_L$ for a unique $L \in \mathrm{Der}(A,\Omega^{k+1}_{\mathrm{Der}}(A))$. $\square$

8.9. The Frölicher-Nijenhuis bracket. For $K \in \operatorname{Der}(A, \Omega^k_{\mathrm{Der}}(A))$ and $L \in \operatorname{Der}(A, \Omega^l_{\mathrm{Der}}(A))$ the graded commutator of the Lie derivations $[\mathcal{L}_K, \mathcal{L}_L]$ commutes with d, so by 8.8 we have $[\mathcal{L}_K, \mathcal{L}_L] = \mathcal{L}_{[K,L]}$ for unique $[K, L] \in \operatorname{Der}(A, \Omega^{k+l}_{\mathrm{Der}}(A))$. We may conclude that this bracket $[\quad,\quad]$ defines a graded Lie algebra structure on $\operatorname{Der}(A, \Omega^*_{\mathrm{Der}}(A))$, because the mapping $\mathcal{L}$ is injective. This bracket is called the *Frölicher-Nijenhuis bracket*.

8.10. For $K \in \operatorname{Der}(A, \Omega^k_{\mathrm{Der}}(A))$ we define $\zeta K \in C^k_{Z(A)}(\operatorname{Der}(A), \operatorname{Der}(A))$ by

$$\zeta K(X_1, \dots, X_k) := i_{X_k} \circ \dots \circ i_{X_1} \circ K : A \to A$$

If we denote for the moment by $\bar\zeta : \Omega^*_{\mathrm{Der}}(A) \to C^*_{Z(A)}(\operatorname{Der}(A), A)$ the embedding of graded differential algebras, then we have

$$\bar\zeta \circ i_K = i_{\zeta K} \circ \bar\zeta : \Omega^*_{\mathrm{Der}}(A) \to C^{*+k-1}_{Z(A)}(\operatorname{Der}(A), A),$$

So the elements $L \in C^k_{Z(A)}(\operatorname{Der}(A), \operatorname{Der}(A))$ for which i_L maps the graded subalgebra $\Omega^*_{\mathrm{Der}}(A)$ into itself are precisely those of the form $L = \zeta K$ for some $K \in \operatorname{Der}(A, \Omega^k_{\mathrm{Der}}(A))$.

8.11. Proposition. *The injective bounded linear mapping*

$$\zeta : \operatorname{Der}(A, \Omega^*_{\mathrm{Der}}(A)) \to C^*_{Z(A)}(\operatorname{Der}(A), \operatorname{Der}(A))$$

is a homomorphism for both brackets $[\quad,\quad]^\wedge$ *and* $[\quad,\quad]$. *If we denote for the moment also by* $\bar\zeta : \Omega^*_{\mathrm{Der}}(A) \to C^*_{Z(A)}(\operatorname{Der}(A), A)$ *the embedding of graded differential algebras, then we have*

$$\bar\zeta \circ i_K = i_{\zeta K} \circ \bar\zeta, \quad \bar\zeta \circ \mathcal{L}_K = \mathcal{L}_{\zeta K} \circ \bar\zeta : \Omega^*_{\mathrm{Der}}(A) \to C^*_{Z(A)}(\operatorname{Der}(A), A),$$

so all the formulas from section 5 continue to hold on $\operatorname{Der}(A, \Omega^*_{\mathrm{Der}}(A))$.

Proof. This is obvious from the considerations above. □

8.12. For $K \in \operatorname{Der}(A, \Omega^k(A))$ we consider $\zeta_k \circ K \in \operatorname{Der}(A, C^k_{Z(A)}(\operatorname{Der}(A), A))$, and the corresponding element

$$\zeta_k(K) \in C_{Z(A)}(\operatorname{Der}(A), \operatorname{Der}(A)) \cong \operatorname{Der}(A, C^k_{Z(A)}(\operatorname{Der}(A), A))$$

then we have:

Lemma. *For* $\omega \in \Omega^q(A)$ *and* $K \in \operatorname{Der}(A, \Omega^k(A))$ *we have*

$$i_{\zeta_k(K)}(\zeta_q \omega) = \zeta_{q+k-1}(j_K \omega). \tag{1}$$

Proof. Both sides,

$$i_{\zeta_k(K)} \circ \zeta, \zeta \circ j_K : \Omega^*(A) \to C^{*+k-1}_{Z(A)}(\operatorname{Der}(A), A),$$

are derivations over $\zeta : \Omega(A) \to C_{Z(A)}(\operatorname{Der}(A), A)$, a homomorphism of graded differential algebras, and both vanish on $A = \Omega^0(A)$, thus it remains to show that they are equal on $\Omega^1(M)$. But for $\omega \in \Omega^1(A)$ we have by 5.1.(1)

$$\begin{aligned}(i_{\zeta_k(K)}(\zeta_1\omega))(X_1, \dots, X_k) &= (\zeta_1\omega)(\zeta_k K(X_1, \dots, X_k)) \\ &= j_{(\zeta_k K(X_1,\dots,X_k))}\omega \\ &= j_{(j_{X_k}\dots j_{X_1}K)}\omega \\ &= j_{X_k}\dots j_{X_1} j_K \omega \\ &= \zeta_k(j_K\omega)(X_1, \dots, X_k) \qquad \square\end{aligned}$$

8.13. Corollary. *For $K, L \in \mathrm{Der}(A, \Omega(A))$ and $\omega \in \Omega(A)$ we have:*

(1) $\zeta([K, L]^\Delta) = [\zeta(K), \zeta(L)]^\wedge$ *for the algebraic brackets.*
(2) $\zeta \mathcal{L}_K \omega = \mathcal{L}_{\zeta(K)} \zeta\omega$ *for the Lie derivations.*
(3) $\zeta([K, L]) = [\zeta(K), \zeta(L)]$ *for the Frölicher-Nijenhuis brackets.*

9. The differential graded algebra $\Omega_{\mathrm{Out}}(A)$

9.1. The differential graded algebra $\Omega_{\mathrm{Out}}(A)$ is the subspace of all forms $\omega \in \Omega_{\mathrm{Der}}(A)$ which are basic with respect to all inner derivations of A. In more detail: For $a \in A$ let $\mathrm{ad}(a) : A \to A$ be given by $\mathrm{ad}(a)b = [a, b] = ab - ba$. Then the space $\mathrm{Int}(A)$ of all these *inner derivations* $\mathrm{ad}(a)$ for $a \in A$ is an ideal in $\mathrm{Der}(A)$, and the quotient $\mathrm{Out}(A) = \mathrm{Der}(A)/\mathrm{Int}(A)$ is called the Lie algebra of outer derivations. Then we define $\Omega^k_{\mathrm{Out}}(A)$ to be the set of all $\omega \in \Omega^k_{\mathrm{Der}}(A)$ which satisfy $i_{\mathrm{ad}(a)}\omega = 0$ and $\mathcal{L}_{\mathrm{ad}(a)}\omega = 0$ for all $a \in A$. It is easily seen to be a differential graded subalgebra.

9.2. Lemma. *The homomorphism $\zeta : \Omega_{\mathrm{Der}}(A) \to C_{Z(A)}(\mathrm{Der}(A), A)$ is part of the following commutative diagram*

$$\begin{array}{ccccc}
\Omega_{\mathrm{Der}}(A) & \xrightarrow{\zeta} & C_{Z(A)}(\mathrm{Der}(A), A) & \longrightarrow & C(\mathrm{Der}(A), A) \\
\uparrow & & \uparrow & & \uparrow \\
\Omega_{\mathrm{Out}}(A) & \xrightarrow{\zeta} & C_{Z(A)}(\mathrm{Out}(A), Z(A)) & \longrightarrow & C(\mathrm{Out}(A), Z(A))
\end{array}$$

where in the lower row we have the graded differential subalgebras of those element which are basic with respect to all inner derivations of A.

Proof. Clearly the homomorphisms in the upper row of the diagram map elements which are basic with respect to all inner derivations to themselves. It just remains to check that in $C_{Z(A)}(\mathrm{Der}(A), A)$ and in $C(\mathrm{Der}(A), A)$ these elements form the sets $C_{Z(A)}(\mathrm{Out}(A), Z(A))$ and $C(\mathrm{Out}(A), Z(A))$, respectively. Let $\omega \in C_{Z(A)}(\mathrm{Der}(A), A)$ or $C(\mathrm{Der}(A), A)$ be basic. Since $i_{\mathrm{ad}(a)}\omega = 0$ for all $a \in A$, the skew symmetric multilinear mapping $\zeta(\omega) : \mathrm{Der}(A)^l \to A$ factors to $\mathrm{Out}(A) \to A$. By formula 5.5 we have

$$\begin{aligned}
0 &= (\mathcal{L}_{\mathrm{ad}(a)}\omega)(X_1, \dots, X_l) \\
&= \mathrm{ad}(a)(\omega(X_1, \dots, X_l) - \sum \omega(X_1, \dots, [\mathrm{ad}(a), X_i], \dots, X_l) \\
&= \mathrm{ad}(a)(\omega(X_1, \dots, X_l) - 0
\end{aligned}$$

for all $a \in A$, since $[\mathrm{ad}(a), X_i] \in \mathrm{Int}(A)$. But this means that ω has values in $Z(A)$. □

9.3. Proposition. *The graded differential algebra $(C_{Z(A)}(\mathrm{Out}(A), Z(A)), d)$ is Morita invariant.*

Proof. $Z(A) = HH^0(A; A)$ and $\mathrm{Out}(A) = HH^1(A; A)$, where $HH^*(A; A)$ is the Hochschild cohomology of A with values in A, and also the action of $\mathrm{Out}(A)$ on $Z(A)$ is described via Hochschild cohomology. Since Hochschild cohomology is Morita invariant, the result follows. □

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