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TOWARDS THE CHERN-WEIL HOMOMORPHISM IN NON COMMUTATIVE DIFFERENTIAL GEOMETRY

Andreas Cap
Peter W. Michor

Institut für Mathematik, Universität Wien,
Strudlhofgasse 4, A-1090 Wien, Austria.

1. Introduction

In this short review article we sketch some developments which should ultimately lead to the analogy of the Chern-Weil homomorphism for principle bundles in the realm of non commutative differential geometry. Principal bundles there should have Hopf algebras as structure 'cogroups'. Since the usual machinery of Lie algebras, connection forms, etc., just is not available in this setting, we base our approach on the Frölicher–Nijenhuis bracket. See [9] for an account of the classical theory using this approach.

In this paper we give an outline of the construction of a non commutative analogy of the Frölicher–Nijenhuis bracket as well as some simple applications. For simplicity we work in a purely algebraic setting but the whole theory can also be developed for topological algebras as well as for the so called convenient algebras (see [5]) which are best suited for differentiation and take care of completed tensor products. For a detailed exposition in the latter setting see [1] and [2].

2. Universal differential forms

2.1. Let A be a unital associative algebra over a commutative field K of characteristic zero. Then the graded algebra $\Omega_*(A)$ of universal differential forms over A is constructed as follows (see [6] and [7]): The tensor product $A\otimes A$ is an A–bimodule and the multiplication map $\mu : A \otimes A \to A$ is a bimodule homomorphism by associativity. Hence the kernel of μ is an A–bimodule which we denote by $\Omega_1(A)$. We define $d : A \to \Omega_1(A)$ by $d(a) = 1 \otimes a - a \otimes 1$.

The map d can be characterized by a universal property as follows: Let M be an A–bimodule. A linear map $D : A \to M$ is called a derivation if and only if for any $a, b \in A$ we have $D(ab) = D(a) \cdot b + a \cdot D(b)$. Obviously d is a derivation and thus for any bimodule homomorphism $\varphi : \Omega_1(A) \to M$ the map $\varphi \circ d : A \to M$ is a derivation. It can be proved that any derivation is of this form:

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2.2. Proposition. *For any A-bimodule M the canonical linear mapping d^* : $\operatorname{Hom}_A^A(\Omega_1(A), M) \to \operatorname{Der}(A; M)$, given by $\varphi \mapsto \varphi \circ d$ is an isomorphism.*

Clearly the module $\Omega_1(A)$ is determined by this universal property up to canonical isomorphism.

2.3. Now we define the spaces of differential forms of higher degree by $\Omega_k(A) := \Omega_1(A) \otimes_A \cdots \otimes_A \Omega_1(A)$ (k factors). Moreover we put $\Omega_0(A) = A$ and $\Omega(A) = \oplus_{k=0}^{\infty} \Omega_k(A)$. Then $\Omega(A)$ is a graded algebra with the tensor product as multiplication. Next put $\bar{A} := A/K$, where the ground field K is identified with the multiples of the unit of A. Then one proves that the map $a \otimes \bar{b} \mapsto ad(b)$ is an isomorphism between $A \otimes \bar{A}$ and $\Omega_1(A)$. Consequently the map $a_0 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_k \mapsto a_0 d(a_1) \ldots d(a_k)$ defines an isomorphism $A \otimes \overbrace{\bar{A} \otimes \cdots \otimes \bar{A}}^{k\text{-times}} \to \Omega_k(A)$, and it turns out that $a_0 d(a_1) \ldots d(a_k) \mapsto d(a_0) d(a_1) \ldots d(a_k)$ gives a well defined map $d : \Omega_k(A) \to \Omega_{k+1}(A)$ for any k. Then it can be shown that $(\Omega(A), d)$ is a graded differential algebra, i.e. that $d(\omega_p \omega_q) = d\omega_p \omega_q + (-1)^p \omega_p d\omega_q$ for all $\omega_p \in \Omega_p(A)$ and $\omega_q \in \Omega_q(A)$. Again this algebra is characterized by a universal property:

2.4. Proposition. *Let $(B = \oplus_{k=0}^{\infty} B_k, \delta)$ be an arbitrary unital graded differential algebra, $\varphi_0 : A \to B_0$ a homomorphism of unital algebras. Then there is a unique homomorphism $\varphi : \Omega(A) \to B$ of graded differential algebras which restricts to φ_0 in degree zero.*

In particular this result shows that the construction of the algebra of universal differential forms is functorial.

3. Construction of the Frölicher–Nijenhuis bracket

The construction is based on the classification of all graded derivations of the graded algebra $\Omega(A)$.

3.1 Definition. The space $\operatorname{Der}_k \Omega(A)$ consists of all *(graded) derivations* of degree k, i.e. all bounded linear mappings $D : \Omega(A) \to \Omega(A)$ with $D(\Omega_\ell(A)) \subset \Omega_{k+\ell}(A)$ and $D(\varphi\psi) = D(\varphi)\psi + (-1)^{k\ell} \varphi D(\psi)$ for $\varphi \in \Omega_\ell(A)$.

Lemma. *The space $\operatorname{Der} \Omega(A) = \bigoplus_k \operatorname{Der}_k \Omega(A)$ is a graded Lie algebra with the graded commutator $[D_1, D_2] := D_1 \circ D_2 - (-1)^{k_1 k_2} D_2 \circ D_1$ as bracket. So the bracket is graded anticommutative, $[D_1, D_2] = -(-1)^{k_1 k_2}[D_2, D_1]$, and satisfies the graded Jacobi identity $[D_1, [D_2, D_3]] = [[D_1, D_2], D_3] + (-1)^{k_1 k_2}[D_2, [D_1, D_3]]$.*

3.2. A derivation $D \in \operatorname{Der}_k \Omega(A)$ is called *algebraic* if $D \mid \Omega_0(A) = 0$. Then $D(a\omega) = aD(\omega)$ and $D(\omega a) = D(\omega)a$ for $a \in A$, so D restricts to a bounded bimodule homomorphism, an element of $\operatorname{Hom}_A^A(\Omega_l(A), \Omega_{l+k}(A))$. Since we have $\Omega_l(A) = \Omega_1(A) \otimes_A \cdots \otimes_A \Omega_1(A)$ and since for a product of one forms we have $D(\omega_1 \otimes_A \cdots \otimes_A \omega_l) = \sum_{i=1}^{l} (-1)^{ik} \omega_1 \otimes_A \cdots \otimes_A D(\omega_i) \otimes_A \cdots \otimes_A \omega_l$, the derivation D is uniquely determined by its restriction $K := D|\Omega_1(A) \in \operatorname{Hom}_A^A(\Omega_1(A), \Omega_{k+1}(A))$; we write $D = j(K) = j_K$ to express this dependence. Note the defining equation $j_K(\omega) = K(\omega)$ for $\omega \in \Omega_1(A)$. Since it will be very important in the sequel we will use the notation $\Omega_k^1 = \Omega_k^1(A) := \operatorname{Hom}_A^A(\Omega_1(A), \Omega_k(A))$ and $\Omega_*^1 = \Omega_*^1(A) = \bigoplus_{k=0}^{\infty} \Omega_k^1(A)$.

It can be shown that for any $K \in \Omega^1_k(A)$ the formula

$$j_K(\omega_0 \otimes_A \cdots \otimes_A \omega_\ell) = \sum_{i=0}^{\ell} (-1)^{ik} \omega_0 \otimes_A \cdots \otimes_A K(\omega_i) \otimes_A \cdots \otimes_A \omega_k$$

for $\omega_i \in \Omega_1(A)$ defines an algebraic graded derivation $j_K \in \operatorname{Der}_k \Omega(A)$ and any algebraic derivation is of this form. Thus $K \mapsto j_K$ is an isomorphism from $\Omega^1_*(A)$ to the space of algebraic graded derivations of $\Omega(A)$ Since the graded commutator of two algebraic derivations is clearly again algebraic we can define a graded Lie bracket $[\quad,\quad]^\Delta$ on the space $\Omega^1_*(A)$ by $j([K,L]^\Delta) := [j_K, j_L]$. This bracket is called the *algebraic bracket*; it is an analogy of the one used in [3].

3.3. The differential d is a graded derivation of $\Omega(A)$ of degree one which is not algebraic. In analogy to the well known formula for the Lie derivative along vector fields we now define the *Lie derivative* along a field valued form $K \in \Omega^1_k(A)$ by $\mathcal{L}_K := [j_K, d] \in \operatorname{Der}_k \Omega(A)$. Then one proves that for any derivation $D \in \operatorname{Der}_k \Omega(A)$ there are unique elements $K \in \Omega^1_k$ and $L \in \Omega^1_{k+1}$ such that $D = \mathcal{L}_K + j_L$. Moreover $L = 0$ if and only if $[D, d] = 0$ and D is algebraic if and only if $K = 0$.

For elements $K \in \Omega^1_k$ and $L \in \Omega^1_\ell$ one immediately verifies that $[[\mathcal{L}_K, \mathcal{L}_L], d] = 0$, so we have $[\mathcal{L}(K), \mathcal{L}(L)] = \mathcal{L}([K,L])$ for a uniquely defined $[K,L] \in \Omega^1_{k+\ell}$. This vector valued form $[K,L]$ is called the *abstract Frölicher-Nijenhuis bracket* of K and L. Clearly this bracket defines a graded Lie algebra structure on the space $\Omega^1_*(A)$.

4. Distributions and Integrability

4.1. Distributions. By a *distribution* in an algebra A we mean a sub-A-bimodule $\mathcal{D}$ of $\Omega_1(A)$.

The distribution $\mathcal{D}$ is called *globally integrable* if there exists a sub algebra B of A such that $\mathcal{D}$ is the subspace generated by $A(d(B))$ and $d(B)A$.

The distribution $\mathcal{D}$ is called *splitting* if it is a direct summand in $\Omega_1(A)$ or equivalently if there is a projection $P \in \Omega^1_1(A) = \operatorname{Hom}^A_A(\Omega_1(A), \Omega_1(A))$ onto $\mathcal{D}$, i.e. $P \circ P = P$ and $\mathcal{D} = P(\Omega_1(A))$. Then there is a complementary sub module $\ker P \subset \Omega_1(A)$.

The distribution $\mathcal{D}$ is called *involutive* if the ideal $(\mathcal{D})_{\Omega_*(A)}$ generated by $\mathcal{D}$ in the graded algebra $\Omega_*(A)$ is stable under d, i.e. if $d(\mathcal{D}) \subset (\mathcal{D})_{\Omega_*(A)}$.

4.2. Comments. One should think of this as follows: In ordinary differential geometry $\mathcal{D}$ should be the A-bimodule of those 1-forms which annihilate the sub bundle of TM. Global integrability then means that it is integrable and that the space of functions which are constant along the leaves of the foliation generates those forms. This is a strong condition: There are foliations where this space of functions consists only of the constants, and this can be embedded into any manifold. So in $C^\infty(M)$ there are always involutive distributions which are not globally integrable. To prove some Frobenius theorem a notion of local integrability would be necessary.

4.3 Curvature and cocurvature. Let $P \in \Omega^1_1(A) = \operatorname{Hom}^A_A(\Omega_1(A), \Omega_1(A))$ be a projection, then the image $P(\Omega_1(A))$ is a splitting distribution, called the *vertical distribution* of P and the complement $\ker P$ is also a splitting distribution, called the *horizontal* one. $\bar{P} := Id_{\Omega_1(A)} - P$ is a projection onto the horizontal distribution.

We consider now the Frölicher-Nijenhuis bracket $[P,P]$ of P and define

$$\begin{aligned} R = R_P &= [P,P]\circ P && \text{the } \textit{curvature.} \\ \bar R = \bar R_P &= [P,P]\circ \bar P && \text{the } \textit{cocurvature,} \end{aligned}$$

The curvature and the cocurvature are elements of $\Omega^1_2(A) = \operatorname{Hom}^A_A(\Omega_1(A),\Omega_2(A))$. The curvature kills elements of the horizontal distribution, so it is *vertical.* The cocurvature kills elements of the vertical distribution.

Then one proves:

Proposition. *With P, R and $\bar R$ as above we have:*

1. (Bianchi Identity)

$$\begin{aligned} &[P, R+\bar R] = 0 \\ &2[R,P] = j_R \bar R + j_{\bar R} R, \end{aligned}$$

where the insertion operators are extended to $\Omega^1_*(A)$ in the obvious way.

2. The curvature R is zero if and only if the horizontal distribution is involutive. The cocurvature $\bar R$ is zero if and only if the vertical distribution $P(\Omega_1(A))$ is involutive.

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Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A-1090 Wien, Austria.

E-mail address: cap@awirap.bitnet, cap@pap.univie.ac.at, michor@awirap.bitnet, michor@pap.univie.ac.at