

**THE MULTIGRADED
NIJENHUIS-RICHARDSON ALGEBRA,
ITS UNIVERSAL PROPERTY AND APPLICATIONS**

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ABSTRACT. We define two $(n + 1)$ graded Lie brackets on spaces of multilinear mappings. The first one is able to recognize n -graded associative algebras and their modules and gives immediately the correct differential for Hochschild cohomology. The second one recognizes n -graded Lie algebra structures and their modules and gives rise to the notion of Chevalley cohomology.

1. INTRODUCTION

In this paper we will generalize the construction of Nijenhuis and Richardson which associates to a given vector space V a graded Lie algebra $Alt(V)$ of multilinear alternating mappings $V \times \dots \times V \rightarrow V$ to study Lie algebra structures on V and their deformations, see [9]. Their construction suggests a "principle" which we present here as the starting point for our investigations. The principle is as follows:

Suppose that \mathcal{S} is a type of structures on V , defining for example associative algebras, Lie algebras, modules (over a given Algebra \mathcal{A}) or Lie bialgebras on V . Then there exists a \mathbb{Z} -graded Lie algebra $(\mathcal{E} = \bigoplus_{k \in \mathbb{Z}} \mathcal{E}^k, [\ , \])$ such that $P \in \mathcal{S}$ if and only if $P \in \mathcal{E}^1$ such that $[P, P] = 0$.

In the case where \mathcal{S} is the set of Lie algebra structures on V the space \mathcal{E} can be identified with $Alt(V)$. Moreover if V is equipped with such

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a P , the Chevalley-Eilenberg coboundary operator ∂_P of the adjoint representation of (V, P) is just the adjoint action of P on $Alt(V)$ up to a sign. Another application may be found in [6]. There one uses the graded cohomology of the subalgebras of $Alt(V)$ to classify and to construct formal deformations of (V, P) .

The purpose of this paper is to establish the principle in each of the cited cases. We will do this in more generality which makes the construction even more powerful. Namely, we assume that V is itself graded over \mathbb{Z}^n , ($n = 0, 1, 2, \dots$) and we will define for each \mathcal{S} a graded Lie algebra \mathcal{E} which is now graded over \mathbb{Z}^{n+1} and satisfies the principle. If we don't stress the special choice of n we will speak of *multigraded* algebras. Having defined the multigraded Lie algebra \mathcal{E} , deformation theory and cohomology of \mathcal{S} may be treated at the same time using only the space \mathcal{E} and its properties.

Given a multigraded vector space V we will construct first $M(V)$, a multigraded Lie algebra which is adapted to study the associative structures on V . Using then the multigraded alternator α we define $A(V)$ to be the image of $M(V)$ by α equipped with the unique bracket making α a homomorphism of multigraded Lie algebras. Moreover $A(V)$ satisfies a universal property and describes multigraded Lie algebra structures on V . We call $A(V)$ the *multigraded Nijenhuis-Richardson algebra* of V since it coincides with $Alt(V)$ for $n = 0$. Once having established this multigraded version, the result for module structures follows quite easily.

In this way we rediscover Hochschild and Chevalley-Eilenberg Cohomology for $n \leq 1$, where the differential is given by the adjoint action of P on \mathcal{E} . Their generalizations for $n > 1$ are now obvious and yield a canonical description for multigraded cohomology in both cases.

Moreover one can study now the theory of formal deformations of multigraded algebras L and their modules. Roughly speaking we describe a mapping from the cohomology of the adjoint representation of $A(L)$ into the set of formal deformations of all possible structures on L which may be used to construct and classify these deformations. Such a point of view has also been emphasized by [11], [5], and [4].

2. MULTIGRADED ASSOCIATIVE ALGEBRA STRUCTURES

2.1. Conventions and definitions.. By a *multidegree* we mean an element $x = (x^1, \dots, x^n) \in \mathbb{Z}^n$ for some n . We call it also *n-degree* if we want to stress the special choice of n . We shall need also the inner product of multidegrees $\langle \cdot, \cdot \rangle : \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{Z}$, given by $\langle x, y \rangle = \sum_{i=1}^n x^i y^i$.

An *n-graded vector space* is just a direct sum $V = \bigoplus_{x \in \mathbb{Z}^n} V^x$, where the elements of V^x are said to be homogeneous of multidegree x . To avoid technical problems we assume that vector spaces are defined over a field \mathbb{K} of characteristic 0. In the following X, Y , etc will always denote

homogeneous elements of some multigraded vector space of multidegrees x, y , etc.

By an n -graded algebra $\mathcal{A} = \bigoplus_{x \in \mathbb{Z}^n} \mathcal{A}^x$ we mean an n -graded vector space which is also a \mathbb{K} algebra such that $\mathcal{A}^x \cdot \mathcal{A}^y \subseteq \mathcal{A}^{x+y}$.

- (1) The multigraded algebra (\mathcal{A}, \cdot) is said to be *multigraded commutative* if for homogeneous elements $X, Y \in \mathcal{A}$ of multidegree x, y , respectively, we have $X \cdot Y = (-1)^{\langle x, y \rangle} Y \cdot X$.
- (2) If $X \cdot Y = -(-1)^{\langle x, y \rangle} Y \cdot X$ holds it is called *multigraded anticommutative*.
- (3) An n -graded Lie algebra is a multigraded anticommutative algebra $(\mathcal{E}, [\ , \])$, such that the *multigraded Jacobi identity* holds:

$$[X, [Y, Z]] = [[X, Y], Z] + (-1)^{\langle x, y \rangle} [Y, [X, Z]]$$

Obviously the space $\text{End}(V) = \bigoplus_{\delta \in \mathbb{Z}^n} \text{End}^\delta(V)$ of all endomorphisms of a multigraded vector space V is a multigraded algebra under composition, where $\text{End}^\delta(V)$ is the space of linear endomorphisms D of V of multidegree δ , i.e. $D(V^x) \subseteq V^{x+\delta}$. Clearly $\text{End}(V)$ is a multigraded Lie algebra under the multigraded commutator

$$(4) \quad [D_1, D_2] := D_1 \circ D_2 + (-1)^{\langle \delta_1, \delta_2 \rangle} D_2 \circ D_1.$$

If \mathcal{A} is an n -graded algebra, an endomorphism $D : \mathcal{A} \rightarrow \mathcal{A}$ of multidegree δ is called a *multigraded derivation*, if for $X, Y \in \mathcal{A}$ we have

$$(5) \quad D(X \cdot Y) = D(X) \cdot Y + (-1)^{\langle \delta, x \rangle} X \cdot D(Y).$$

Let us write $\text{Der}^\delta(\mathcal{A})$ for the space of all multigraded derivations of degree δ of the algebra \mathcal{A} , and we put

$$(5) \quad \text{Der}(\mathcal{A}) = \bigoplus_{\delta \in \mathbb{Z}^n} \text{Der}^\delta(\mathcal{A}).$$

The following lemma is standard:

Lemma. *If \mathcal{A} is an n -graded algebra, then the space $\text{Der}(\mathcal{A})$ of multigraded derivations is an n -graded Lie subalgebra under the n -graded commutator.*

It is clear from the definitions that non-graded algebras and \mathbb{Z} -graded algebras are multigraded of multidegree 0 and 1, respectively.

2.2 Associative algebra structures. Let us recall first the construction in the case of non-graded vector spaces which was given in [3], [1]. There a 1-graded Lie algebra $(M(V), [\ , \]^\Delta)$ is described for each vector space V with the property that (V, μ) is an associative algebra if and only if $\mu \in M^1(V)$ and $[\mu, \mu]^\Delta = 0$. This algebra is as follows.

Denote by $M^k(V)$ the space of all $k+1$ -linear mappings $K : V \times \dots \times V \rightarrow V$ and set

$$M(V) := \bigoplus_{k \in \mathbb{Z}} M^k(V).$$

For $K_i \in M^{k_i}(V)$ and $X_j \in V$ we define $j(K_1)K_2 \in M^{k_1+k_2}(M)$ by

$$\begin{aligned} (j(K_1)K_2)(X_0, \dots, X_{k_1+k_2}) &:= \\ &= \sum_{i=0}^{k_2} (-1)^{k_1 i} K_2(X_0, \dots, K_1(X_i, \dots, X_{i+k_1}), \dots, X_{k_1+k_2}). \end{aligned}$$

The graded Lie bracket of $M(V)$ is then given by

$$[K_1, K_2]^\Delta = j(K_1)K_2 - (-1)^{k_1 k_2} j(K_2)K_1.$$

Proposition. ([3], [1])

- (1) $(M(V), [\ , \]^\Delta)$ is a 1-graded Lie algebra.
- (2) If $\mu \in M^1(V)$, so $\mu : V \times V \rightarrow V$ is bilinear, then (V, μ) is an associative algebra if and only if $[\mu, \mu]^\Delta = 0$. \square

Note that $M^0(V) = \text{End}(V)$ is a Lie subalgebra of $M(V)$, and its bracket is the negative of the usual commutator.

The explicit formulas above follow directly from investigating the 1-graded Lie algebra of (1-graded) derivations of certain graded algebras, see [11]. We explain that in the simple case of a finite dimensional V . Then $M(V)$ is canonically isomorphic to the 1-graded Lie algebra $\text{Der}(\bigotimes V^*)$ of derivations of the tensor algebra of V^* , a derivation D of degree k being completely determined by its restriction $V^* \rightarrow \bigotimes^{k+1} V^*$ and hence by a unique $K \in M^k(V)$.

2.3 Multigraded associative algebras. We will give now the multigraded generalization. Of course one can proceed as before by identifying $M(V)$ as the algebra of derivations of some suitable multigraded algebra. But we will generalize 2.2 directly. So let $V = \bigoplus_{x \in \mathbb{Z}^n} V^x$ be an n -graded vector space. We define

$$M(V) := \bigoplus_{(k, \kappa) \in \mathbb{Z} \times \mathbb{Z}^n} M^{(k, \kappa)}(V),$$

where $M^{(k, \kappa)}(V)$ is the space of all $k+1$ -linear mappings $K : V \times \dots \times V \rightarrow V$ such that $K(V^{x_0} \times \dots \times V^{x_k}) \subseteq V^{x_0 + \dots + x_k + \kappa}$. We call k the *form degree* and κ the *weight degree* of K . In 2.2 the mapping K had degree k and X_i had degree -1 in $M(V)$, hence the sign $(-1)^{k_i}$. We define for $K_i \in M^{(k_i, \kappa_i)}(V)$ and $X_j \in V^{x_j}$

$$\begin{aligned} (j(K_1)K_2)(X_0, \dots, X_{k_1+k_2}) &:= \\ &= \sum_{i=0}^{k_2} (-1)^{k_1 i + \langle \kappa_1, \kappa_2 + x_0 + \dots + x_{i-1} \rangle} \\ &\quad \cdot K_2(X_0, \dots, K_1(X_i, \dots, X_{i+k_1}), \dots, X_{k_1+k_2}) \end{aligned}$$

$$[K_1, K_2]^\Delta = j(K_1)K_2 - (-1)^{k_1 k_2 + \langle \kappa_1, \kappa_2 \rangle} j(K_2)K_1.$$

Theorem. *Let V be an n -graded vector space. Then we have:*

- (1) $(M(V), [\ , \]^\Delta)$ is an $(n+1)$ -graded Lie algebra.
- (2) If $\mu \in M^{(1,0,\dots,0)}(V)$, so $\mu : V \times V \rightarrow V$ is bilinear of weight $0 \in \mathbb{Z}^n$, then μ is an associative n -graded multiplication if and only if $[\mu, \mu]^\Delta = 0$.

Proof. The bracket is $(n+1)$ -graded anticommutative. The $(n+1)$ -graded Jacobi identity follows from the formula

$$j([K_1, K_2]^\Delta) = [j(K_1), j(K_2)],$$

the multigraded commutator in $\text{End}(M(V))$. This is a long but elementary calculation. The second assertion follows by writing out the definitions. \square

3. MULTIGRADED LIE ALGEBRA STRUCTURES

3.1. Multigraded signs of permutations. Let $\mathbf{x} = (x_1, \dots, x_k) \in (\mathbb{Z}^n)^k$ be a multi index of n -degrees $x_i = (x_i^1, \dots, x_i^n) \in \mathbb{Z}^n$ and let $\sigma \in \mathcal{S}_k$ be a permutation of k symbols. Then we define the *multigraded sign* $\text{sign}(\sigma, \mathbf{x})$ as follows: For a transposition $\sigma = (i, i+1)$ we put $\text{sign}(\sigma, \mathbf{x}) = -(-1)^{\langle x_i, x_{i+1} \rangle}$; it can be checked by combinatorics that this gives a well defined mapping $\text{sign}(\ , \mathbf{x}) : \mathcal{S}_k \rightarrow \{-1, +1\}$. In fact one may define directly

$$\text{sign}(\sigma, \mathbf{x}) = \text{sign}(\sigma) \text{sign}(\sigma_{|x_1^1|, \dots, |x_k^1|}) \cdots \text{sign}(\sigma_{|x_1^n|, \dots, |x_k^n|}),$$

where $\sigma_{|x_1^j|, \dots, |x_k^j|}$ is that permutation of $|x_1^j| + \cdots + |x_k^j|$ symbols which moves the i -th block of length $|x_i^j|$ to the position σi , and where $\text{sign}(\sigma)$ denotes the ordinary sign of a permutation in \mathcal{S}_k . Let us write $\sigma \mathbf{x} = (x_{\sigma 1}, \dots, x_{\sigma k})$, then we have the following

Lemma. $\text{sign}(\sigma \circ \tau, \mathbf{x}) = \text{sign}(\sigma, \mathbf{x}) \cdot \text{sign}(\tau, \sigma \mathbf{x})$. \square

3.2 Multigraded Nijenhuis-Richardson algebra. We define the *multigraded alternator* $\alpha : M(V) \rightarrow M(V)$ by

$$(1) \quad (\alpha K)(X_0, \dots, X_k) = \frac{1}{(k+1)!} \sum_{\sigma \in \mathcal{S}_{k+1}} \text{sign}(\sigma, \mathbf{x}) K(X_{\sigma 0}, \dots, X_{\sigma k})$$

for $K \in M^{(k,*)}(V)$ and $X_i \in V^{x_i}$. If the ground field is not of characteristic 0 one could omit the combinatorial factor, but one should redo the whole development starting from the point of view of derivations again, see the remark at the end of 2.2. However, the combinatorial factors used here are quite essential, judging from our experience in differential geometry. By lemma 3.1 we have $\alpha^2 = \alpha$ so α is a projection

defined on $M(V)$, homogeneous of multidegree 0, and we set

$$\begin{aligned} A(V) &= \bigoplus_{(k,\kappa) \in \mathbb{Z} \times \mathbb{Z}^n} A^{(k,\kappa)}(V) \\ &:= \bigoplus_{(k,\kappa) \in \mathbb{Z} \times \mathbb{Z}^n} \alpha(M^{(k,\kappa)}(V)). \end{aligned}$$

A long but straightforward computation shows that for $K_i \in M^{(k_i,\kappa_i)}(V)$

$$\alpha(j(\alpha K_1)\alpha K_2) = \alpha(j(K_1)K_2),$$

so the following operator and bracket is well defined:

$$\begin{aligned} i(K_1)K_2 &:= \frac{(k_1 + k_2 + 1)!}{(k_1 + 1)!(k_2 + 1)!} \alpha(j(K_1)K_2) \\ [K_1, K_2]^\wedge &= \frac{(k_1 + k_2 + 1)!}{(k_1 + 1)!(k_2 + 1)!} \alpha([K_1, K_2]^\Delta) \\ &= i(K_1)K_2 - (-1)^{\langle (k_1, \kappa_1), (k_2, \kappa_2) \rangle} i(K_2)K_1 \end{aligned}$$

The combinatorial factor will become clear in 3.4 .

3.3. Theorem. 1. If K_i are as above then

$$\begin{aligned} (i(K_1)K_2)(X_0, \dots, X_{k_1+k_2}) &= \\ &= \frac{1}{(k_1 + 1)!k_2!} \sum_{\sigma \in \mathcal{S}_{k_1+k_2+1}} \text{sign}(\sigma, \mathbf{x}) (-1)^{\langle \kappa_1, \kappa_2 \rangle} \\ &\quad \cdot K_2((K_1(X_{\sigma_0}, \dots, X_{\sigma_{k_1}}), \dots, X_{\sigma_{(k_1+k_2)}})). \end{aligned}$$

2. $(A(V), [\ , \]^\wedge)$ is an $(n+1)$ -graded Lie algebra.

3. If $\mu \in A^{(1,0,\dots,0)}(V)$, so $\mu : V \times V \rightarrow V$ is bilinear n -graded anticommutative mapping of weight $0 \in \mathbb{Z}^n$ then $[\mu, \mu]^\wedge = 0$ if and only if (V, μ) is a n -graded Lie algebra.

Proof. 1. This follows by a straight forward computation.

2. $[\ , \]^\wedge$ is clearly multigraded anticommutative and the multigraded Jacobi identity follows directly from the one of $[\ , \]^\Delta$.

3. Let $\mu \in A^{(1,0,\dots,0)}(V)$, then

$$\begin{aligned} 0 &= [\mu, \mu]^\wedge(X_0, X_1, X_2) \\ &= \frac{3!}{2!3!} \frac{1}{3!} \sum_{\sigma \in \mathcal{S}_3} \text{sign}(\sigma, \mathbf{x}) \cdot [\mu, \mu]^\Delta(X_{\sigma_0}, X_{\sigma_1}, X_{\sigma_2}) \\ &= \sum_{\sigma \in \mathcal{S}_3} \text{sign}(\sigma, \mathbf{x}) \cdot \mu(\mu(X_{\sigma_0}, \mu(X_{\sigma_1}, X_{\sigma_2}))) \end{aligned}$$

which is equivalent to the multigraded Jacobi identity of (V, μ) . \square

We call $(A(V), [\ , \]^\wedge)$ the *multigraded Nijenhuis-Richardson algebra*, since $A(V)$ coincides for $n = 0$ with $Alt(V)$ of [9].

3.4. Universality of the algebra $(A(V), [\ , \]^\wedge)$. Let V be a multigraded vector space and denote by $\mathcal{E}(V)$ the category of multigraded Lie algebras $(E, [\ , \])$ such that

$$\begin{aligned} E^{(k,*)} &= 0 \quad k < -1 \\ E^{(-1,*)} &= V. \end{aligned}$$

If $E, F \in \mathcal{E}(V)$, then a *morphism* $\varphi : E \rightarrow F$ is a homomorphism of multigraded Lie algebras satisfying $\varphi|_{E^{(-1,*)}} = \text{id}_V$. For example $M(V)$ and $A(V)$ are elements of $\mathcal{E}(V)$.

Theorem. $A(V)$ is a final object in $\mathcal{E}(V)$, so for each $E \in \mathcal{E}(V)$ there exists a unique morphism $\varepsilon : E \rightarrow A(V)$. It follows that $A(V)$ is unique up to isomorphism.

Proof. Suppose that $Z \in E^{(k,z)}$ then we define

$$\varepsilon(Z)(X_0, \dots, X_k) = (-1)^{\langle z, x_0 + \dots + x_k \rangle} [X_0, [X_1, \dots, [X_k, Z] \dots]],$$

an element of $E^{(-1,*)} = V$ for $X_i \in V^{x_i}$. Because of the multigraded Jacobi identity $\varepsilon(Z)$ is well defined as an element of $A^{(k,z)}$. So we are left to show that

$$(*) \quad \varepsilon([Z_1, Z_2]) = [\varepsilon(Z_1), \varepsilon(Z_2)]^\wedge$$

We will do this by induction on $k = k_1 + k_2$. For $k < -1$ this is trivially true. Now let $k = -1$, so we may assume that $Z_1 \in V^{z_1}$. Then

$$\begin{aligned} \varepsilon([Z_1, Z_2]) &= [Z_1, Z_2] = (-1)^{\langle z_1, z_2 \rangle} \varepsilon(Z_2)(Z_1) \\ &= i(Z_1)\varepsilon(Z_2) = [Z_1, \varepsilon(Z_2)]^\wedge = [\varepsilon(Z_1), \varepsilon(Z_2)]^\wedge \end{aligned}$$

by Theorem 3.2 and since $\varepsilon|_V = \text{id}_V$. Suppose that $(*)$ is true for $k_1 + k_2 < k$. Then for $k_1 + k_2 = k$ we have

$$\begin{aligned} i(X)\varepsilon([Z_1, Z_2]) &= [X, \varepsilon([Z_1, Z_2])]^\wedge = \varepsilon([X, [Z_1, Z_2]]) \\ &= \varepsilon([[X, Z_1], Z_2] + (-1)^{\langle (-1, x), (k_1, z_1) \rangle} [Z_1, [X, Z_2]]) \\ &= [\varepsilon([X, Z_1]), \varepsilon(Z_2)]^\wedge + (-1)^{\langle (-1, x), (k_1, z_1) \rangle} [\varepsilon(Z_1), \varepsilon([X, Z_2])]^\wedge \\ &= [i(X)\varepsilon(Z_1), \varepsilon(Z_2)]^\wedge + (-1)^{\langle (-1, x), (k_1, z_1) \rangle} [\varepsilon(Z_1), i(X)\varepsilon(Z_2)]^\wedge \\ &= i(X)[\varepsilon(Z_1), Z_2]^\wedge \end{aligned}$$

by induction hypothesis and the fact that $i(X) = [X, \]^\wedge$ is a derivation of degree $(-1, x)$ of $A(V)$. This proves the induction. Remark that for $E = M(V)$ the morphism ε is given by

$$\varepsilon|M^{k,*}(V) = (k+1)! \alpha. \quad \square$$

4. MULTIGRADED MODULES AND COHOMOLOGY

4.1. Multigraded bimodules. Let V and W be multigraded vector spaces and $\mu : V \times V \rightarrow V$ a multigraded algebra structure. A *multigraded bimodule* $\mathcal{M} = (W, \lambda, \rho)$ over $\mathcal{A} = (V, \mu)$ is given by $\lambda, \rho : V \rightarrow \text{End}(W)$ of weight 0 such that

- (1) $[\mu, \mu]^\Delta = 0$ so \mathcal{A} is associative
- (2) $\lambda(\mu(X_1, X_2)) = \lambda(X_1) \circ \lambda(X_2)$
- (3) $\rho(\mu(X_1, X_2)) = (-1)^{\langle x_1, x_2 \rangle} \rho(X_2) \circ \rho(X_1)$
- (4) $\lambda(X_1) \circ \rho(X_2) = (-1)^{\langle x_1, x_2 \rangle} \rho(X_2) \circ \lambda(X_1)$

where $X_i \in V^{x_i}$ and \circ denotes the composition in $\text{End}(W)$.

4.2. Theorem. *Let E be the multigraded vector space defined by*

$$E^{(k,*)} = \begin{cases} V & \text{if } k = 0 \\ W & \text{if } k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then $P \in M^{(1,0,\dots,0)}(E)$ defines a bimodule structure on W if and only if $[P, P]^\Delta = 0$.

Proof. We define

$$\begin{aligned} \mu(X_1, X_2) &:= P(X_1, X_2) \\ \lambda(X)Y &:= P(X, Y) \\ \rho(X)Y &:= (-1)^{\langle x, y \rangle} P(Y, X) \end{aligned}$$

where we suppose the X_i 's $\in V$ and $Y \in W$ to be embedded in E . Then if $Z_i \in E$ be arbitrary we get

$$\begin{aligned} [P, P]^\Delta(Z_0, Z_1, Z_2) &= 2(j(P)P)(Z_0, Z_1, Z_2) \\ &= 2P((Z_0, Z_1), Z_2) - 2P(Z_0, (Z_1, Z_2)). \end{aligned}$$

Now specify $Z_i \in V$ resp. W to get eight independent equations. Four of them vanish identically because of their degree of homogeneity, the others recover the defining equations for the multigraded bimodules. \square

4.3 Corollary. *In the above situation we have the following decomposition of $M(E)$:*

$$M^{(k,q,*)}(E) = \begin{cases} 0 & \text{for } q > 1 \\ L^{(k+1,*)}(V, W) & \text{for } q = 1 \\ M^{(k,*)}(V) \oplus \bigoplus_{k+1} L^{(k,*)}(V, \text{End}(W)) & \text{for } q = 0 \end{cases}$$

where $L^{(k,*)}(V, W)$ denotes the space of k -linear mappings $V \times \dots \times V \rightarrow W$. If P is as above, then $P = \mu + \lambda + \rho$ corresponds exactly to this decomposition. \square

4.4. Hochschild cohomology and multiplicative structures. Let V, W and P be as in Theorem 4.2 and let $\nu : W \times W \rightarrow W$ be a multigraded algebra structure, so $\nu \in M^{(1, -1, 0, \dots, 0)}(E)$. Then for $C_i \in L^{(k_i, c_i)}(V, W)$ we define

$$C_1 \bullet C_2 := [C_1, [C_2, \nu]^\Delta]^\Delta.$$

Since $[C_1, D_2]^\Delta = 0$ it follows that $(L(V, W), \bullet)$ is multigraded commutative. It is the usual extension of the product ν from W to the level of cochains, where the necessary combinatorics is hidden in the brackets.

Theorem. 1. *The mapping $[P, \]^\Delta : M(E) \rightarrow M(E)$ is a differential. We denote its restriction to $L(V, W)$ by δ_P . This generalizes the Hochschild coboundary operator to the multigraded case: If $C \in L^{(k, c)}(V, W)$ then we have for $X_i \in V^{x_i}$*

$$\begin{aligned} (\delta_P C)(X_0, \dots, X_k) &= \lambda(X_0)C(X_1, \dots, X_k) \\ &\quad - \sum_{i=0}^{k-1} (-1)^i C(X_0, \dots, \mu(X_i, X_{i+1}), \dots, X_k) \\ &\quad + (-1)^{k+1+\langle x_0+\dots+x_{k-1}+c, x_k \rangle} \rho(X_k)C(X_0, \dots, X_{k-1}) \end{aligned}$$

The corresponding cohomology will be denoted by $H(\mathcal{A}, \mathcal{M})$, where \mathcal{A} is the multigraded associative algebra (V, μ) , and where \mathcal{M} is the multigraded \mathcal{A} -bimodule (W, λ, ρ)

2. If $[P, \nu]^\Delta = 0$ then δ_P is a derivation of $L(V, W)$ of multidegree $(1, 0, \dots, 0)$. In this case the product \bullet carries over to a multigraded (cup) product on $H(\mathcal{A}, \mathcal{M})$.

Proof. The fact that δ_P is a differential follows directly from the multigraded Jacobi identity since the degree of δ_P is $(1, 0, \dots, 0)$. The formula is easily checked by writing out the definitions. Applying the multigraded Jacobi identity once again one gets immediately that δ_P is a derivation if and only if $[P, \nu]^\Delta = 0$.

By writing out the definitions one shows that $[P, \nu]^\Delta = 0$ is equivalent to the following equations:

$$\begin{aligned} \lambda(X)\nu(Y_1, Y_2) &= \nu(\lambda(X)Y_1, Y_2) \\ \rho(X)\nu(Y_1, Y_2) &= (-1)^{\langle x, y_1 \rangle} \nu(Y_1, \rho(X)Y_2) \\ \nu(\rho(X)Y_1, Y_2) &= (-1)^{\langle x, y_1 \rangle} \nu(Y_1, \lambda(X)Y_2) \end{aligned}$$

in particular we have $(\lambda - \rho) : V \rightarrow \text{Der}(W, \nu)$. \square

4.5 Multigraded Lie modules and Chevalley cohomology. We obtain a corresponding result for Lie modules by applying the multigraded alternator α to $M(E)$, just as we did in section 3 to obtain the Nijenhuis-Richardson bracket.

Theorem. Let $P \in A^{(1,0,\dots,0)}(E)$ then $[P, P]^\wedge = 0$ if and only if

$$(a) \quad [\mu, \mu]^\wedge = 0$$

so $(V, \mu) = \mathfrak{g}$ is a multigraded Lie algebra, and

$$(b) \quad \pi(\mu(X_1, X_2))Y = [\pi(X_1), \pi(X_2)]Y$$

where $\mu(X_1, X_2) = P(X_1, X_2) \in V$ and $\pi(X)Y = P(X, Y) \in W$ for $X, X_i \in V$ and $Y \in W$, and where $[\ , \]$ denotes the multigraded commutator in $\text{End}(W)$. So $[P, P]^\wedge = 0$ is by definition equivalent to the fact that $\mathcal{M} := (W, \pi)$ is a multigraded Lie- \mathfrak{g} module.

If P is as above the mapping $\partial_P := [P, \]^\wedge : A(E) \rightarrow A(E)$ is a differential and its restriction to

$$\bigoplus_{k \in \mathbb{Z}} \Lambda^{(k,*)}(\mathfrak{g}, \mathcal{M}) := \bigoplus_{k \in \mathbb{Z}} A^{(k,1,*)}(E)$$

generalizes the Chevalley-Eilenberg coboundary operator to the multigraded case:

$$\begin{aligned} (\partial_P C)(X_0, \dots, X_k) &= \sum_{i=0}^k (-1)^{\alpha_i(\mathbf{x}) + \langle x_i, c \rangle} \pi(X_i) C(X_0, \dots, \widehat{X}_i, \dots, X_k) \\ &\quad + \sum_{i < j} (-1)^{\alpha_{ij}(\mathbf{x})} C(\mu(X_i, X_j), \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots) \end{aligned}$$

where

$$\begin{cases} \alpha_i(\mathbf{x}) = \langle x_i, x_1 + \dots + x_{i-1} \rangle + i \\ \alpha_{ij}(\mathbf{x}) = \alpha_i(\mathbf{x}) + \alpha_j(\mathbf{x}) + \langle x_i, x_j \rangle \end{cases}$$

We denote the corresponding cohomology space by $H(\mathfrak{g}, \mathcal{M})$.

If $\nu : W \times W \rightarrow W$ is multigraded symmetric (so $\nu \in A^{(1,-1,*)}(E)$) and $[P, \nu]^\wedge = 0$ then ∂_P acts as derivation of multidegree $(1, 0, \dots, 0)$ on the multigraded commutative algebra $(\Lambda(\mathfrak{g}, \mathcal{M}), \bullet)$, where

$$C_1 \bullet C_2 := [C_1, [C_2, \nu]^\wedge]^\wedge \quad C_i \in \Lambda^{(k_i, c_i)}(\mathfrak{g}, \mathcal{M}).$$

In this situation the product \bullet carries over to a multigraded symmetric (cup) product on $H(\mathfrak{g}, \mathcal{M})$.

Proof. Apply the multigraded alternator α to the results of 4.1, 4.2, 4.3, and 4.4. \square

The formulas we obtained here are not that surprising since they are standard in the non-graded case. The new feature of our approach lies in the fact that we can formulate deformation equations and cohomology at once inside a multigraded Lie algebra (which we denoted $M(E)$, $A(V)$ respectively). Then all the "different" results we obtained are consequences of only "one" fact, namely the multigraded Jacobi identity. In the line of [11] it seems to us that this procedure should be somehow extended to other structures defined on a (multigraded) vector space, for example coalgebras, comodules and then of course to bialgebras such as Hopf algebras and Lie bialgebras. The latter one was discussed in [7].

5. STRUCTURES AND THEIR FORMAL DEFORMATIONS

5.1. Structures. We fix once for all a \mathbb{Z}^n graded Lie algebra $(\mathcal{E}, [\cdot, \cdot])$ and a multidegree $\theta \in \mathbb{Z}^n$ such that $\langle \theta, \theta \rangle + 1 \equiv 0 \pmod{2}$ (i.e. θ has one odd number of odd components).

By definition, a *structure (of degree θ) of \mathcal{E}* is an element $P \in \mathcal{E}^\theta$ such that $[P, P] = 0$. We denote by $\mathcal{S}^\theta(\mathcal{E})$ the set of structures of degree θ of \mathcal{E} .

If $P \in \mathcal{S}^\theta(\mathcal{E})$, then the adjoint action $\partial_P := \text{ad } P$ of P on \mathcal{E} is a differential homogeneous of degree θ , since $\langle \theta, \theta \rangle + 1 \equiv 0 \pmod{2}$. We denote by

$$H(\mathcal{E}, \partial_P) = \bigoplus_{x \in \mathbb{Z}^n} H^x(\mathcal{E}, \partial_P)$$

its cohomology, where

$$H^x(\mathcal{E}, \partial_P) = \mathcal{E}^x \cap \ker \partial_P / \partial_P \mathcal{E}^{x-\theta}.$$

As ∂_P is a derivation of \mathcal{E} , $H(\mathcal{E}, \partial_P)$ has a unique \mathbb{Z}^n -graded Lie algebra structure making the natural map $\ker \partial_P \rightarrow H(\mathcal{E}, \partial_P)$ a surjective homomorphism of graded Lie algebras.

Observe that $[\cdot, \cdot]$ is a structure of degree $e_1 = (1, 0, \dots, 0)$ of $A(\mathcal{E})$:

$$[\cdot, \cdot] \in \mathcal{S}^{e_1}(A(\mathcal{E})).$$

To avoid confusion as well as to make the notations lighter, we denote in the sequel by $\mathbb{H}(\mathcal{E})$ the space $H(A(\mathcal{E}), \partial_{[\cdot, \cdot]})$ and by \mathbb{D} the differential $\partial_{[\cdot, \cdot]}$.

As mentioned above, many useful algebraic structures on a vector space are particular instances of the abstract notion of structure introduced here (associative algebras, Lie algebras, graded or not, Lie bialgebras for instance). This leads to a unified way to study these various algebraic structures, what we shall now illustrate for their formal deformations.

5.2. Formal deformations, Equivalences. We denote by \mathcal{E}_λ^x the space of formal power series in the parameter λ with coefficients in \mathcal{E}^x ($x \in \mathbb{Z}^n$). The space $\mathcal{E}_\lambda = \bigoplus_{x \in \mathbb{Z}^n} \mathcal{E}_\lambda^x$ has a canonical multigraded Lie algebra structure extending that of \mathcal{E} :

$$\left[\sum_k \lambda^k X_k, \sum_l \lambda^l Y_l \right] = \sum_k \lambda^k \sum_{i+j=k} [X_i, Y_j].$$

By definition, a *formal deformation* of a structure $P \in \mathcal{S}^\theta(\mathcal{E})$ is an element $P_\lambda \in \mathcal{S}^\theta(\mathcal{E}_\lambda)$ such that $P_0 = P$. Two such deformations P_λ and P'_λ are said to be *equivalent* if $P'_\lambda = \varphi_\lambda(P_\lambda)$ for some automorphism $\varphi_\lambda = \sum_k \lambda^k \varphi_k$ ($\varphi_k \in A^0(\mathcal{E})$, $k \in \mathbb{N}$) of \mathcal{E}_λ such that $\varphi_0 = \text{id}_\mathcal{E}$, the identity on \mathcal{E} . In the sequel, such a φ_λ will be called an *equivalence*.

Lemma. (i) A mapping $\varphi_\lambda : \mathcal{E}_\lambda \rightarrow \mathcal{E}_\lambda$ is an equivalence if and only if it is solution of a formal differential equation

$$\frac{d}{d\lambda} \varphi_\lambda = \varphi_\lambda \circ T_\lambda, \quad \varphi_0 = id_{\mathcal{E}},$$

where $T_\lambda \in A^0(\mathcal{E})_\lambda$ is a \mathbb{D} -cocycle.

(ii) Let φ_λ be an equivalence and let $C_\lambda \in A(\mathcal{E})_\lambda$ be a \mathbb{D} -cocycle. If $\mathbb{H}^0(\mathcal{E}) = 0$, then $\varphi_\lambda^* C_\lambda$ is a \mathbb{D} -cocycle cohomologous to C_λ .

Here, φ_λ^* denotes the natural action of φ_λ on $A(\mathcal{E})_\lambda$:

$$(\varphi_\lambda^* C_\lambda)(X_0, \dots, X_k) = \varphi_\lambda(C_\lambda(\varphi_\lambda^{-1}(X_0), \dots, \varphi_\lambda^{-1}(X_k))).$$

Proof. (i) Applying $\varphi_\lambda^{-1} \frac{d}{d\lambda}$ to the members of the equation

$$\varphi_\lambda([X, Y]) = [\varphi_\lambda(X), \varphi_\lambda(Y)] \quad (X, Y \in \mathcal{E})$$

shows that $T_\lambda = \varphi_\lambda^{-1} \frac{d\varphi_\lambda}{d\lambda}$ is a \mathbb{D} -cocycle. Conversely, the unique solution of

$$\frac{d}{d\lambda} \varphi_\lambda = \varphi_\lambda \circ T_\lambda, \quad \varphi_0 = id_{\mathcal{E}},$$

which is given stepwise by

$$(k+1)\varphi_{k+1} = \sum_{i+j=k} \varphi_i \circ T_j, \quad \varphi_0 = id_{\mathcal{E}},$$

is an equivalence if $\mathbb{D}T_\lambda = 0$. Indeed, as $\varphi_0 = id_{\mathcal{E}}$, it is a bijective mapping. Moreover,

$$\frac{d}{d\lambda} (\varphi_\lambda([\varphi_\lambda^{-1}(X), \varphi_\lambda^{-1}(Y)])) = \varphi_\lambda((\mathbb{D}T_\lambda)(\varphi_\lambda^{-1}(X), \varphi_\lambda^{-1}(Y))) = 0$$

for all $X, Y \in \mathcal{E}$. Thus

$$\varphi_\lambda([\varphi_\lambda^{-1}(X), \varphi_\lambda^{-1}(Y)]) = \varphi_\lambda([\varphi_\lambda^{-1}(X), \varphi_\lambda^{-1}(Y)])|_{\lambda=0} = [X, Y].$$

(ii) Assume that $\mathbb{D}C = 0$, where $C \in A(\mathcal{E})$. As easily seen, one has

$$\frac{d}{d\lambda} (\varphi_\lambda^* C) = \varphi_\lambda^* [C, \varphi_\lambda^{-1} \frac{d}{d\lambda} \varphi_\lambda]^\wedge.$$

Since $\mathbb{H}^0(\mathcal{E}) = 0$, $\varphi_\lambda^{-1} \frac{d}{d\lambda} \varphi_\lambda$ is a coboundary. It thus reads $ad T_\lambda$ for some $T_\lambda \in \mathcal{E}_\lambda^0 = A^{(-1,0)}(\mathcal{E})$. Noticing that $ad T_\lambda = i(T_\lambda)[\ , \]$, it follows immediately from the Jacobi identity in $A(\mathcal{E})$ that

$$[ad T_\lambda, C]^\wedge = i(T_\lambda)\mathbb{D}C + \mathbb{D}(i(T_\lambda)C).$$

Thus

$$\frac{d}{d\lambda} (\varphi_\lambda^* C) = \varphi_\lambda^* (\mathbb{D}i(T_\lambda)C) = \mathbb{D}(\varphi_\lambda^* (i(T_\lambda)C))$$

because $\mathbb{D}C = 0$ and, obviously, $\varphi_\lambda^* \circ \mathbb{D} = \mathbb{D} \circ \varphi_\lambda^*$. Therefore,

$$\varphi_\lambda^* C = C + \mathbb{D} \int_0^\lambda \varphi_\mu^* (i(T_\mu)C) d\mu.$$

Now, if each component C_k of C_λ is a \mathbb{D} -cocycle, then

$$\varphi_\lambda^* C_\lambda = C_\lambda + \mathbb{D} \left(\sum_k \lambda^k \int_0^\lambda \varphi_\mu^* (i(T_\mu)C_k) d\mu \right)$$

is cohomologous to C_λ . \square

5.3. Proposition. *Let \mathcal{E} be a Lie subalgebra of $A(V)$ for some $(n-1)$ -graded vector space V . Assume that $\mathcal{E}^{(-1,*)} = V^*$. Then*

(i) *If $C \in A^0(\mathcal{E})$ is a \mathbb{D} -cocycle, then $C = \mathbb{D}T$ for some $T \in A^0(V)$ such that $[T, \mathcal{E}] \subset \mathcal{E}$.*

(ii) *The equivalences φ_λ of \mathcal{E} are the mappings of the form S_λ^* where $S_\lambda \in A^0(V)_\lambda$ and $S_0 = id_V$.*

Proof. (i) Set $T = C|V$. Then $T \in A^0(V)$ and $C = \mathbb{D}T$. Indeed, let $Y \in A^{(k,y)}(V)$. For $k = -1$, $C(Y) = (\mathbb{D}T)(Y)$ by definition of T . Now, by induction on k , if $X \in A^{(-1,x)}(V) = V^x$, then

$$\begin{aligned} (-1)^{\langle x,y \rangle} i(X)C(Y) &= (\mathbb{D}C)(X, Y) + C([X, Y]) + (-1)^{k+\langle x,y \rangle} [Y, C(X)] \\ &= [[X, Y], T] + (-1)^{k+\langle x,y \rangle} [Y, [X, T]] \\ &= [X, (\mathbb{D}T)(Y)] = (-1)^{\langle x,y \rangle} i(X)((\mathbb{D}T)(Y)). \end{aligned}$$

Thus $C(Y) = (\mathbb{D}T)(Y)$ for all Y .

(ii) It is clear that S_λ^* is an equivalence. Conversely, if φ_λ is an equivalence, we know that $\varphi_\lambda^{-1} \frac{d}{d\lambda} \varphi_\lambda$ is a \mathbb{D} -cocycle. It is thus of the form $\mathbb{D}T_\lambda$ for some $T_\lambda \in A^0(V)_\lambda$. The equation

$$\frac{d}{d\lambda} S_\lambda = S_\lambda \circ T_\lambda, \quad S_0 = id_V,$$

has, obviously, a unique solution. But then, for an arbitrary $X \in \mathcal{E}$, one has

$$\frac{d}{d\lambda} \varphi_\lambda(X) = \varphi_\lambda(\mathbb{D}T_\lambda)(X), \quad \frac{d}{d\lambda} S_\lambda^* X = S_\lambda^*(\mathbb{D}T_\lambda)(X)$$

and thus φ_λ and S_λ^* coincide on \mathcal{E} since $\varphi_0 = S_0^*$. \square

5.4. We now turn to generalize to arbitrary structures $P \in \mathcal{S}^\theta(\mathcal{E})$ the results obtained for the Lie algebras in [1], [6]. We only indicate the non obvious adaptations of the proofs, referring otherwise the reader to the appropriate papers. As before, $(\mathcal{E}, [\ , \])$ denotes a \mathbb{Z}^n -graded Lie algebra and $\theta \in \mathbb{Z}^n$ is assumed to be such that $\langle \theta, \theta \rangle + 1 \equiv 0 \pmod{2}$. We also denote by $Pol(\mathcal{E})$ the space of polynomials on \mathcal{E} . Let η be the map $A(\mathcal{E}) \rightarrow Pol(\mathcal{E})$ given by

$$\eta(C) : X \rightarrow \frac{1}{(k+1)!} C(X, \dots, X)$$

for $C \in A^{(k,c)}(\mathcal{E})$ and set

$$\eta_P(C) = \eta(C)(P)$$

for $P \in \mathcal{S}^\theta(\mathcal{E})$.

5.5. Lemma. *Let $C \in A^{(k,c)}(\mathcal{E})$, $P \in \mathcal{S}^\theta(\mathcal{E})$ and $X \in \mathcal{E}^x$ be given such that $k > 0$ and $\langle x, x \rangle + 1 \equiv 0 \pmod{2}$. Then we have*

$$\eta(\mathbb{D}C)(X) = (-1)^{\langle x, c \rangle} [X, \eta(C)(X)] - \frac{1}{2} \eta(i[X, X])C(X)$$

$$\partial_P \eta_P(i(X)C) + \eta_P(i(\partial_P X)C) = (-1)^{\langle \theta, x+c \rangle} [X, \eta_P(C)] \text{ if } \mathbb{D}C = 0.$$

Proof. This we get by straightforward computations applying theorem 4.4 to expand $\mathbb{D}C(X, \dots, X)$ and $\mathbb{D}C(X, P, \dots, P)$ respectively. \square

Now comes our main result about formal deformation of structures of \mathcal{E} .

5.6. Theorem. *(i) Let $C_\lambda \in (\bigoplus_{k\theta+c=0} A^{(k,c)}(\mathcal{E}))_\lambda$ be such that $\mathbb{D}C_\lambda = 0$. For each $P \in \mathcal{S}^\theta(\mathcal{E})$, the unique solution of the equation*

$$\frac{d}{d\lambda} P_\lambda + \eta(C_\lambda)(P_\lambda) = 0, \quad P_0 = P,$$

is a formal deformation of P , which will be said to be associated to the cocycle C_λ .

(ii) Formal deformations which are associated to cohomologous cocycles are equivalent.

(iii) If $\mathbb{H}^0(\mathcal{E}) = 0$ and P_λ is associated to C_λ , then each deformation equivalent to P_λ is associated to a cocycle cohomologous to C_λ .

(iv) For a given $P \in \mathcal{S}^0(\mathcal{E})$, the image of

$$\eta_{P\#} : \mathbb{H}(\mathcal{E}) \rightarrow \mathcal{E}$$

lies in the center of $H(E, \partial_P)$. If $H^\theta(\mathcal{E}, \partial_P) \subset \text{im } \eta_{P\#}$, then each deformation of P is associated to some cocycle.

Proof. (i) The proof goes as in ([1], Proposition 15.2), without major change : simply, substitute the first equation of lemma 5.5 to Proposition 15.1 in [1].

(ii) Assume that $C'_\lambda = C_\lambda + \mathbb{D}A_\lambda$ and denote by P_λ and P'_λ the deformations of the same $P \in \mathcal{S}^\theta(\mathcal{E})$ associated to C_λ and C'_λ respectively. Set

$$B_\lambda = \sum_k \lambda^k \int_0^\lambda (\varphi_\mu^*(i(T_\mu)C_k)) d\mu$$

(see the end of the proof of the Lemma in 5.2). Then, the equations

$$\frac{d}{d\lambda} \varphi_\lambda = \varphi_\lambda \circ \text{ad } T_\lambda, \quad \varphi_0 = \text{id}_\mathcal{E},$$

and

$$T_\lambda = \varphi_\lambda^{-1}(\eta(A_\lambda - B_\lambda)(P'_\lambda))$$

have a unique solution $\varphi_\lambda, T_\lambda$, where φ_λ is an equivalence and $T_\lambda \in A^{(-1,0)}(\mathcal{E})_\lambda$. Indeed, it follows from the first that φ_k is uniquely expressed in terms of T_0, \dots, T_{k-1} . The same holds true for the k -th component of φ_λ^{-1} , a polynomial in $\varphi_0, \dots, \varphi_k$. Thus, the k -th component of the right member of the second equation only depends on T_0, \dots, T_{k-1} . It follows that the two equations may be uniquely solved by induction. Taking account of the fact that

$$\varphi_\lambda^* C_\lambda = C'_\lambda + \mathbb{D}(B_\lambda - A_\lambda),$$

one easily sees that P'_λ and $\varphi_\lambda(P_\lambda)$ both are associated to C'_λ . As $P'_0 = \varphi_0(P_0) = P$, one has thus $P'_\lambda = \varphi_\lambda(P_\lambda)$.

(iii) Let φ_λ be an equivalence. As $\mathbb{H}^0(\mathcal{E}) = 0$, one has $\frac{d\varphi_\lambda}{d\lambda} = \varphi_\lambda \circ \text{ad } T_\lambda$ for some $T_\lambda \in \mathcal{E}_\lambda^0$, and by lemma in 5.2, $\varphi_\lambda^* C_\lambda = C_\lambda + \mathbb{D}B_\lambda$. Computing $\frac{d}{d\lambda} \varphi_\lambda(P_\lambda)$ easily shows that $\varphi_\lambda(P_\lambda)$ is associated to $C_\lambda + \mathbb{D}(B_\lambda + \varphi_\lambda(T_\lambda))$.

(iv) The fact that $\text{im } \eta_{P\#}$ lies in the center of $H(\mathcal{E}, \partial_P)$ follows immediately from the second equation in lemma 5.5. The proof of the second part of (iv) goes as in ([6], Prop.4.4). \square

REFERENCES

1. M. De Wilde, P. B. A. Lecomte, *Formal deformations of the Poisson Lie algebra of a symplectic manifold and star-products. Existence, equivalence, derivations*, Deformation theory of algebras and structures and applications, M. Hazewinkel, M. Gerstenhaber, Eds., Kluwer Academic Publishers, Dordrecht, 1988, pp. 897–960.
2. A. Frölicher, A. Nijenhuis, *Theory of vector valued differential forms. Part I*, Indagationes Math **18** (1956), 338–359.
3. M. Gerstenhaber, *On the deformation of rings and algebras*, Ann. of Math. **(2) 57** (1953), 591–603.
4. M. Gerstenhaber, S. D. Schack, *Algebraic cohomology and deformation theory*, Deformation theory of algebras and structures and applications, M. Hazewinkel, M. Gerstenhaber, Eds, Kluwer Academic Publishers, Dordrecht, 1988, pp. 11–264.
5. W. M. Goldman, J. J. Millson, *The deformation theory of representations of fundamental groups of compact Kähler manifolds*, Publ. Math. IHES **67** (1988), 43–96.
6. P. B. A. Lecomte, *Applications of the cohomology of graded Lie algebras to formal deformations of Lie algebras*, Letters in Math. Physics **13** (1987), 157–166.
7. P. Lecomte, C. Roger, *Modules et cohomologies des bigèbres de Lie*, Preprint.
8. P. W. Michor, *Remarks on the Frölicher-Nijenhuis bracket*, Proceedings of the Conference on Differential Geometry and its Applications, Brno 1986, D. Reidel, 1987.
9. A. Nijenhuis, R. Richardson, *Deformation of Lie algebra structures*, J. Math. Mech. **17** (1967), 89–105.
10. H. Schicketanz, *Graded cohomology and derivations of the Frölicher-Nijenhuis algebra*, Ph.D thesis, Univ. of Vienna 1989.
11. M. Schlessinger, J. Stasheff, *The Lie algebra structure of tangent cohomology and deformation theory*, J. Pure Appl. Algebra **38** (1985), 313–322.

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