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NO SLICES ON THE SPACE OF GENERALIZED CONNECTIONS

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Abstract. On a fiber bundle without structure group the action of the gauge group (the group of all fiber respecting diffeomorphisms) on the space of (generalized) connections is shown not to admit slices.

1. Introduction.

In modern mathematics and physics actions of Lie groups on manifolds and the resulting orbit spaces (moduli spaces) are of great interest. For example, the moduli space of principal connections on a principal fiber bundle modulo the group of principal bundle automorphisms is the proper configuration space for Yang–Mills field theory (as e.g. outlined in [3], [18], and [14]). The structure of these orbit spaces usually is quite complicated, but sometimes it can be shown that they are stratified into smooth manifolds. This is usually done by proving a, so called, slice theorem for the group action. Also, very recent research in theoretical physics is connected to moduli spaces: e.g. invariance of Euler numbers of moduli spaces of instantons on 4–manifolds [19], moduli spaces of parabolic Higgs bundles, which are connected to Higgs fields [9], [13]. In algebraic topology moduli spaces play an important role, either, [8], [16], [17], and also the definition of the famous Donaldson polynomials involves moduli spaces ([2]).

The result presented in this paper is connected to a slice theorem for the orbit space of connections on a *principal* fiber bundle modulo the gauge group, proved by [6]. The situation considered in this paper is a generalization of that.

For a general survey on slice theorems and slices see [4], where a slice theorem for the space of solutions of Einstein's equations modulo the diffeomorphism group is proved.

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The non-existence of the slice theorem in the case of (generalized) connections on a fiber bundle modulo the gauge group is connected to the fact that the right action of $\operatorname{Diff}(S^1)$ on $C^\infty(S^1, \mathbb{R})$ by composition admits in general no slices, except when restricted to the space of functions which have finite codimension at all critical points [1]. For more information on (generalized) see [10], [11], or section 9 of [5].

2. Definitions. A *(fiber) bundle* $(p : E \to M, S)$ consists of smooth finite dimensional manifolds E, M, S and a smooth mapping $p : E \to M$. Furthermore, each $x \in M$ has an open neighborhood U such that $E \mid U := p^{-1}(U)$ is diffeomorphic to $U \times S$ via a fiber respecting diffeomorphism:

$$\begin{array}{ccc} E \mid U & \xrightarrow{\ \psi\ } & U \times S \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_1} & \\ & U. & \end{array}$$

In the following we assume that M and S, hence E are compact.

We consider the fiber linear tangent mapping $Tp : TE \to TM$ and its kernel $\ker Tp =: VE$, which is called the *vertical bundle* of E. It is a locally splitting vector subbundle of the tangent bundle TE.

A *connection* on the fiber bundle $(p : E \to M, S)$ is a vector valued 1-form $\Phi \in \Omega^1(E; VE)$ with values in the vertical bundle VE such that $\Phi \circ \Phi = \Phi$ and $\operatorname{im}\Phi = VE$; so Φ is just a projection $TE \to VE$.

The kernel $\ker \Phi$ is a sub vector bundle of TE, it is called the space of *horizontal vectors* or the *horizontal bundle*, and it is denoted by HE. Clearly, $TE = HE \oplus VE$ and $T_uE = H_uE \oplus V_uE$ for $u \in E$.

If $\Phi : TE \to VE$ is a connection on the bundle $(p : E \to M, S)$, then the *curvature* R of Φ is given by the Frölicher-Nijenhuis bracket

$$2R = [\Phi, \Phi] = [\operatorname{Id} - \Phi, \operatorname{Id} - \Phi] \in \Omega^2(E; VE).$$

R is an obstruction against involutivity of the horizontal subbundle in the following sense: If the curvature R vanishes, then horizontal vector fields on E also have a horizontal Lie bracket. Furthermore, we have the *Bianchi identity* $[\Phi, R] = 0$ by the graded Jacobi identity for the Frölicher-Nijenhuis bracket.

3. Local description. Let Φ be a connection on $(p : E \to M, S)$. Let us fix a fiber bundle atlas (U_α) with transition functions $(\psi_{\alpha\beta})$, and let us consider the connection $((\psi_\alpha)^{-1})^*\Phi \in \Omega^1(U_\alpha \times S; U_\alpha \times TS)$, which may be written in the form

$$(((\psi_\alpha)^{-1})^*\Phi)(\xi_x, \eta_y) =: -\Gamma^\alpha(\xi_x, y) + \eta_y \text{ for } \xi_x \in T_xU_\alpha \text{ and } \eta_y \in T_yS,$$

since it reproduces vertical vectors. The Γ^α are given by

$$(0_x, \Gamma^\alpha(\xi_x, y)) := -T(\psi_\alpha).\Phi.T(\psi_\alpha)^{-1}.(\xi_x, 0_y).$$

We consider Γ^α as an element of the space $\Omega^1(U_\alpha; \mathfrak{X}(S))$, a 1-form on U^α with values in the Lie algebra $\mathfrak{X}(S)$ of all vector fields on the standard fiber. The Γ^α are called the *Christoffel forms* of the connection Φ with respect to the bundle atlas (U_α, ψ_α).

The transformation law for the Christoffel forms is

$$T_y(\psi_{\alpha\beta}(x,\quad)).\Gamma^\beta(\xi_x, y) = \Gamma^\alpha(\xi_x, \psi_{\alpha\beta}(x, y)) - T_x(\psi_{\alpha\beta}(\quad, y)).\xi_x.$$

The curvature R of Φ satisfies

$$(\psi_\alpha^{-1})^* R = d\Gamma^\alpha + \tfrac{1}{2}[\Gamma^\alpha, \Gamma^\alpha]^\wedge_{\mathfrak{X}(S)}.$$

Here $d\Gamma^\alpha$ is the exterior derivative of the 1-form $\Gamma^\alpha \in \Omega^1(U_\alpha, \mathfrak{X}(S))$ with values in the convenient vector space $\mathfrak{X}(S)$.

4. The *gauge group* $\operatorname{Gau}(E)$ of the finite dimensional fiber bundle $(p : E \to M, S)$ with compact standard fiber S is, by definition, the group of all fiber respecting diffeomorphisms

$$\begin{array}{ccc} E & \xrightarrow{\ f\ } & E \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ M & \xrightarrow{\ \mathrm{Id}\ } & M. \end{array}$$

The gauge group acts on the space of connections by $\Phi \mapsto f^*\Phi = Tf^{-1}.\Phi.Tf$. By naturality of the Frölicher-Nijenhuis bracket for the curvatures, we have

$$R^{f^*\Phi} = \tfrac{1}{2}[f^*\Phi, f^*\Phi] = \tfrac{1}{2}f^*[\Phi, \Phi] = f^*R^\Phi = Tf^{-1}.R^\Phi.\Lambda^2 Tf.$$

Now it is very easy to describe the infinitesimal action. Let X be a vertical vector field with compact support on E, and consider its global flow Fl^X_t.

Then we have $\frac{d}{dt}|_0(\mathrm{Fl}^X_t)^*\Phi = \mathcal{L}_X\Phi = [X, \Phi]$, the Frölicher Nijenhuis bracket, by [5]. The tangent space of $\operatorname{Conn}(E)$ at Φ is the space $T_\Phi \operatorname{Conn}(E) = \{\Psi \in \Omega^1(E; TE) : \Psi|VE = 0\}$. The "infinitesimal orbit" at Φ in $T_\Phi \operatorname{Conn}(E)$ is $\{[X, \Phi] : X \in C^\infty_c(E \leftarrow VE)\}$.

The isotropy subgroup of a connection Φ is $\{f \in \operatorname{Gau}(E) : f^*\Phi = \Phi\}$. Clearly, this is just the group of all those f which respect the horizontal bundle $HE = \ker \Phi$. It is in general not compact and infinite dimensional. The most interesting object is of course the orbit space $\operatorname{Conn}(E)/\operatorname{Gau}(E)$.

5. Slices. Let $\mathcal{M}$ be a smooth manifold, G a Lie group, $G \times \mathcal{M} \to \mathcal{M}$ a smooth action, $x \in \mathcal{M}$, and let $G_x = \{g \in G : g.x = x\}$ denote the isotropy group at x. A contractible subset $S \subseteq \mathcal{M}$ is called a *slice* at x, if it contains x and satisfies

(1) If $g \in G_x$ then $g.S = S$.
(2) If $g \in G$ with $g.S \cap S \neq \emptyset$ then $g \in G_x$.
(3) There exists a local section $\chi : G/G_x \to G$ defined on a neighborhood V of the identity coset such that the mapping $F : V \times S \to \mathcal{M}$, defined by $F(v, s) := \chi(v).s$ is a homeomorphism onto a neighborhood of x.

We have the following additional properties

(4) For $y \in F(V \times S) \cap S$ we get $G_y \subset G_x$, by (2).
(5) For $y \in F(V \times S)$ the isotropy group G_y is conjugate to a subgroup of G_x, by (3) and (4).

6. Counter-example. [15], *6.7. The action of the gauge group* $\operatorname{Gau}(E)$ *on* $\operatorname{Conn}(E)$ *does not admit slices, for* $\dim M \geq 2$.

We will construct locally a connection, which satisfies that in any neighborhood there exist connections which have a bigger isotropy subgroup. Let $n = \dim S$, and let $h : \mathbb{R}^n \to \mathbb{R}$ be a smooth nonnegative bump function, which satisfies $\operatorname{carr} h = \{s \in \mathbb{R}^n | \ \|s - s_0\| < 1\}$. Put $h_r(s) := rh(s_0 + \frac{1}{r}(s - s_0))$, then $\operatorname{carr} h_r = \{s \in \mathbb{R}^n | \ \|s - s_0\| < r\}$. Then set $h_r^{s_1}(s) := h_r(s - (s_1 - s_0))$ which implies $\operatorname{carr} h_r^{s_1} = \{s \in \mathbb{R}^n | \ \|s - s_1\| < r\}$. Using these functions, we can define new functions f_k for $k \in \mathbb{N}$ as

$$f_k(s) = h^{s_k}_{\|z\|/2^k}(s),$$

where $z := \frac{s_\infty - s_0}{3}$ for some $s_\infty \in \mathbb{R}^n$ and $s_k := s_0 + z(2\sum_{l=0}^k \frac{1}{2^l} - 1 - \frac{1}{2^k})$. Further define

$$f^N(s) := e^{-\frac{1}{\|s - s_\infty\|^2}} \sum_{k=0}^N \tfrac{1}{4^k} f_k(s), \quad f(s) := \lim_{N\to\infty} f^N(s).$$

The functions f^N and f are smooth, respectively, since all the functions f_k are smooth, and on every point s at most one summand is nonzero. $\operatorname{carr} f^N = \bigcup_{k=0}^N \{s \in \mathbb{R}^n | \ \|s - s_k\| < \frac{1}{2^k}\|z\|\}$, $\operatorname{carr} f = \bigcup_{k=0}^\infty \{s \in \mathbb{R}^n | \ \|s - s_k\| < \frac{1}{2^k}\|z\|\}$, f^N and f vanish in all derivatives in all x_k, and f vanishes in all derivatives in s_∞.

Let $\psi : E|U \to U \times S$ be a fiber bundle chart of E with a chart $u : U \xrightarrow{\cong} \mathbb{R}^m$ on M, and let $v : V \xrightarrow{\cong} \mathbb{R}^n$ be a chart on S. Choose $g \in C_c^\infty(M, \mathbb{R})$ with $\emptyset \neq \operatorname{supp}(g) \subset U$ and $dg \wedge du^1 \neq 0$ on an open dense subset of $\operatorname{supp}(g)$. Then we can define a Christoffel form as in 3 by

$$\Gamma := g\, du^1 \otimes f(v)\partial_{v^1} \in \Omega^1(U, \mathfrak{X}(S)).$$

This defines a connection Φ on $E|U$ which can be extended to a connection Φ on E by the following method. Take smooth functions $k_1, k_2 \geq 0$ on M satisfying $k_1 + k_2 = 1$, $k_1 = 1$ on $\operatorname{supp}(g)$, and $\operatorname{supp}(k_1) \subset U$, and take an arbitrary connection Φ' on E, and set $\Phi = k_1\Phi^\Gamma + k_2\Phi'$, where Φ^Γ denotes the connection which is induced locally by Γ. In any neighborhood of Φ there exists a connection Φ^N defined by

$$\Gamma^N := g\, du^1 \otimes f^N(s)\partial_{v_1} \in \Omega^1(U, \mathfrak{X}(S)),$$

and extended like Φ.

Claim: There is no slice at Φ.

Proof: We have to consider the isotropy subgroups of Φ and Φ^N. Since the connections Φ and Φ^N coincide outside of U, we may investigate them locally on $W = \{u : k_1(u) = 1\} \subset U$. The curvature of Φ is given locally on W by 3 as

$$R_U := d\Gamma - \tfrac{1}{2}[\Gamma, \Gamma]^{\mathfrak{X}(S)}_\wedge = dg \wedge du^1 \otimes f(v)\partial_{v^1} - 0. \tag{1}$$

For every element of the gauge group $\operatorname{Gau}(E)$ which is in the isotropy group $\operatorname{Gau}(E)_\Phi$ the local representative over W which looks like $\tilde\gamma : (u, v) \mapsto (u, \gamma(u, v))$ by 3 satisfies

$$T_v(\gamma(u, \quad)).\Gamma(\xi_u, v) = \Gamma(\xi_u, \gamma(u, v)) - T_u(\gamma(\quad , v)).\xi_u, \tag{2}$$

$$g(u)du^1 \otimes f(v) \sum_i \frac{\partial\gamma^1}{\partial v^i}\partial_{v^i} = g(u)du^1 \otimes f(\gamma(u, v))\partial_{v^1} - \sum_{i,j} \frac{\partial \gamma^i}{\partial u^j} du^j \otimes \partial_{v^i}.$$

Comparing the coefficients of $du^j \otimes \partial_{v^i}$ we get the following equations for γ over W.

$$\frac{\partial \gamma^i}{\partial u^j} = 0 \quad \text{for } (i,j) \neq (1,1),$$
$$g(u)f(v)\frac{\partial \gamma^1}{\partial v^1} = g(u)f(\gamma(u,v)) - \frac{\partial \gamma^1}{\partial u^1}. \tag{3}$$

Considering next the transformation $\tilde{\gamma}^* R_U = R_U$ of the curvature 3 we get

$$T_v(\gamma(u, \quad)).R_U(\xi_u, \eta_u, v) = R_U(\xi_u, \eta_u, \gamma(u,v)),$$
$$dg \wedge du^1 \otimes f(v) \sum_i \frac{\partial \gamma^1}{\partial v^i}\partial_{v^i} = dg \wedge du^1 \otimes f(\gamma(u,v))\partial_{v^1}. \tag{4}$$

Another comparison of coefficients yields the equations

$$f(v)\frac{\partial \gamma^1}{\partial v^i} = 0 \quad \text{for } i \neq 1,$$
$$f(v)\frac{\partial \gamma^1}{\partial v^1} = f(\gamma(u,v)), \tag{5}$$

whenever $dg \wedge du^1 \neq 0$, but this is true on an open dense subset of $\operatorname{supp}(g)$. Finally, putting (5) into (3) shows

$$\frac{\partial \gamma^i}{\partial u^j} = 0 \quad \text{for all } i,\ j.$$

Collecting the results on $\operatorname{supp}(g)$, we see that γ has to be constant in all directions of u. Furthermore, wherever f is nonzero, γ^1 is a function of v^1 only and γ has to map zero sets of f to zero sets of f.

Replacing Γ by Γ^N we get the same results with f replaced by f^N. Since $f = f^N$ wherever f^N is nonzero or f vanishes, γ in the isotropy group of Φ obeys all these equations not only for f but also for f^N on $\operatorname{supp} f^N \cup f^{-1}(0)$. On $B := \operatorname{carr} f \setminus \operatorname{carr} f^N$ the gauge transformation γ is a function of v^1 only, hence it cannot leave the zero set of f^N by construction of f and f^N. Therefore, γ obeys all equations for f^N whenever it obeys all equations for f. Thus, every gauge transformation in the isotropy subgroup of Φ is in the isotropy subgroup of Φ^N.

On the other hand, any γ having support in B changing only in v^1 direction not keeping the zero sets of f invariant defines a gauge transformation in the isotropy subgroup of Φ^N which is not in the isotropy subgroup of Φ.

Therefore, there exists in every neighborhood of Φ a connection Φ^N whose isotropy subgroup is bigger than the isotropy subgroup of Φ. Thus, by property 5.5 no slice exists at Φ.

7. Counter-example. [15], *6.8. The action of the gauge group* $\operatorname{Gau}(E)$ *on* $\operatorname{Conn}(E)$ *also admits no slices for* $\dim M = 1$, *i.e. for* $M = S^1$.

The method of 6 is not applicable in this situation, since there is no function g satisfying $dg \wedge du^1 \neq 0$ on an open and dense subset of $\operatorname{supp}(g)$. In this case, however, any connection Φ on E is flat. Hence, the horizontal bundle is integrable, the horizontal foliation induced by Φ exists and determines Φ. Any gauge transformation leaving Φ invariant also has to map leaves of the horizontal foliation to other leaves of the horizontal foliation.

We shall construct connections $\Phi^{\lambda'}$ near Φ^λ such that the isotropy groups in $\operatorname{Gau}(E)$ look radically different near the identity, contradicting 5.5.

Let us assume without loss of generality that E is connected, and then, by replacing S^1 by a finite covering, if necessary, that the fiber is connected. Then there exists a smooth global section $\chi : S^1 \to E$. By [12], p. 95, there exists a tubular neighborhood $\pi : U \subset E \to \operatorname{im}(\chi)$ such that $\pi = \chi \circ p|U$ (i.e. a tubular neighborhood with vertical fibers). This tubular neighborhood then contains an open thickened sphere bundle with fiber $S^1 \times \mathbb{R}^{n-1}$, and since we are only interested in gauge transformations near Id_E, which e.g. keep a smaller thickened sphere bundle inside the larger one, we may replace E by an S^1-bundle. By replacing the Klein bottle by a 2-fold covering, we may finally assume that the bundle is $\operatorname{pr}_1 : S^1 \times S^1 \to S^1$.

Consider now connections where the horizontal foliation is a 1-parameter subgroup with slope λ we see that the isotropy group equals S^1 if λ is irrational, and equals S^1 times the diffeomorphism group of a closed interval if λ is rational.

8. Consequences. *For every compact base manifold M and every compact standard fiber S, which are at least one dimensional, there are connections on the fiber bundle E, where the action of* $\operatorname{Gau}(E)$ *on* $\operatorname{Conn}(E)$ *does not admit slices.*

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