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THE RIEMANNIAN GEOMETRY OF ORBIT SPACES. THE METRIC, GEODESICS, AND INTEGRABLE SYSTEMS

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To Professor L. Tamássy, on the occasion of his 80th anniversary

Abstract. We investigate the rudiments of Riemannian geometry on orbit spaces M/G for isometric proper actions of Lie groups on Riemannian manifolds. Minimal geodesic arcs are length minimising curves in the metric space M/G and they can hit strata which are more singular only at the end points. This is phrased as convexity result. The geodesic spray, viewed as a (strata-preserving) vector field on TM/G, leads to the notion of geodesics in M/G which are projections under $M \to M/G$ of geodesics which are normal to the orbits. It also leads to 'ballistic curves' which are projections of the other geodesics. In examples (Hermitian and symmetric matrices, and more generally polar representations) we compute their equations by singular symplectic reductions and obtain generalizations of Calogero-Moser systems with spin.

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1. Introduction

Differential geometry deals ordinarily with smooth objects on smooth manifolds. However, in many cases this assumption is too restrictive. In various contexts different types of non-smooth manifolds or 'manifolds with singularities' appear naturally, like complex algebraic varieties, orbifolds, V-manifolds [33], [10], limits of Riemannian manifolds with respect to the Gromov-Hausdorff metric [12], etc. In

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the present paper we consider a rather special but interesting and important class of non-smooth manifolds, namely the orbit spaces M/G of Riemannian manifolds M with respect to groups G of isometries. Our goal is to develop the Riemannian geometry of orbit spaces. An orbit space has the structure of a stratified manifold with smooth strata, namely the connected components of the sets $(M/G)_{(H)}$ of orbits of some given orbit type (or isotropy type) (H).

Let us describe the structure of the paper. The basic facts about the orbit stratification of an orbit space M/G is presented in section 2.

An orbit space of a complete Riemannian G-manifold M has a natural structure of a metric space. Some elementary properties of this metric structure are collected in section 3. In particular, we establish some convexity properties of the stratification with respect to distance minimizing curves which we call minimal geodesic arcs. For example, the set $(M/G)_{\text{reg}}$ of regular orbits is a convex open dense submanifold of M/G.

Some basic global differential geometric objects on M/G are defined in section 4. We define the algebra of smooth functions $C^\infty(M/G)$ on M/G as the algebra $C^\infty(M)^G$ of G-invariant smooth functions on M, and the Lie algebra of vector fields $\mathfrak{X}(M/G)$ as the Lie algebra of all derivations of $C^\infty(M/G) = C^\infty(M)^G$ which preserve all ideals of functions vanishing on some stratum. The deep result [35] of G. Schwarz states that the natural homomorphism $\mathfrak{X}(M)^G \to \operatorname{Der}(C^\infty(M/G))$ has $\mathfrak{X}(M/G)$ as image. Each element of $\mathfrak{X}(M/G)$ induces a derivation on the algebra of smooth functions on each stratum and thus defines an ordinary vector field along each stratum. In order to define geodesics which connect different strata we consider the geodesic spray as a vector field on TM/G, the projection of the geodesic spray $\Gamma \in \mathfrak{X}(TM)^G$. The integral curves of the geodesic spray Γ on TM/G stay within the strata, but their projection to M/G connect different strata in M/G. We define geodesics in M/G as projections to M/G of integral curves of Γ on TM/G which are orthogonal to orbits, i.e., in TM their initial vector should be in in $\operatorname{Nor}(M)_x = T_x(G.x)^\perp$ for $x \in M$. More generally, the projection on M/G of an arbitrary integral curve of the geodesic spray Γ on TM/G is called a 'ballistic curve'. Imitating the classical Riemannian case, we establish some properties of the geodesic spray on TM/G and define Jacobi fields along geodesics and along ballistic curves.

Section 5 is devoted to a more systematic study of ballistic curves in M/G for the simple model case when the unitary group $G = SU(n)$ acts on the space $M = H(n)$ of Hermitian matrices by conjugation. Using the singular Hamiltonian reduction by Sjamaar and Lerman [36], we derive the Hamiltonian equation for a ballistic curve in $H(n)/SU(n)$. It turns out that it is a Calogero-Moser system with spin. In the special case when the momentum $Y \in \mathfrak{su}(n)^*$ has maximal isotropy group, i.e. when $Y = \sqrt{-1}(c.1_n + w \otimes w^*)$ where w is a vector with $|w|^2 = -c > 0$ the equation is the classical Calogero Moser system, which reproduces results of Kazhdan, Kostant, and Sternberg, [14].

The last section 6 is devoted to the generalization of the results on ballistic curves to the orbit space V/G where V is an Euclidean space and $G \subset SO(V)$ is a connected subgroup of the orthogonal group whose action on V is polar, i.e. admits a section. We are able to generalize the approach of [14] to this general situation. Related results can be found in [30], and [27], see also [13], and [4], and [5].

The special case of a Riemannian orbit space M/G where G is a discrete proper group of isometries (so that the isotropy groups are all finite) are special cases of Riemannian orbifolds. Work in this direction has been done by [7], [8], and [3]. Also the case of polar representations and thus all the explicit examples at the end of the paper fall into the case of orbifolds.

We thank Nikolai Reshetikhin for helpful discussions and pointing out references [14], [5], and [15]. His paper [32] deals with related aspects.

2. Stratification of orbit spaces

2.1. The setup. Let M be a connected G-manifold, where G is a Lie group. The G-action is called *proper* if $G \times M \to M \times M$, given by $(g, x) \mapsto (g.x, x)$, is a proper mapping. It is well known that proper G-actions admit *slices* ([28], [29], [23]): For each $x \in M$ there exists a submanifold $S_x \subset M$ containing x, an open G-invariant neighborhood W of $G.x$, and a smooth equivariant retraction $r : W \to G.x$ such that $S_x = r^{-1}(x)$. Moreover, for the isotropy group G_x of x we have $G_x.S_x \subseteq S_x$, and $g \in G$ with $g.S_x \cap S_x \neq \emptyset$ must lie in G_x. Moreover, the slice S_x is a manifold and $G_s \subseteq G_x$ for each $s \in S_x$. Finally, the action $G \times S_x \to W$ induces an G-equivariant diffeomorphism $G \times_{G_x} S_x \to W$. This implies that $C^\infty(W)^G = C^\infty(S_x)^{G_x}$ via restriction.

We consider the orbit space M/G and the canonical projection $\pi : M \to M/G$, and we endow the quotient space M/G with the following smooth structure: The quotient topology and the sheaf of smooth real valued functions $U \mapsto C^\infty(U) := C^\infty(\pi^{-1}(U))^G$. A mapping $\varphi : M/G \to M'/G'$ is called *smooth* if it respects these sheafs. For a slice S_x as above on the orbit spaces we have $C^\infty(W/G) = C^\infty(S_x/G_x)$. Therefore the local smooth structure of M/G coincides with the smooth structure of S_x/G_x. A mapping $\varphi : M/G \to M'/G'$ is smooth if and only if $\varphi^* C^\infty(M'/G') \subseteq C^\infty(M/G)$.

2.2. Let H be a closed subgroup of G and let (H) denote the conjugacy class of H. For two closed subgroups H_1 and H_2 we write $(H_1) \leq (H_2)$ if H_1 is conjugated to a subgroup of H_2.

Let $M_{(H)}$ denote the set of points of M whose isotropy groups belong to (H). It is known that $M_{(H)}$ is a smooth submanifold of M for proper actions (see [23], 7.4). Put $(M/G)_{(H)} := \pi(M_{(H)}) = (M_{(H)})/G$, call this the isotropy stratum of type (H), and call any connected component of this an *orbit stratum* of M/G.

Proposition. [35], [23], [29].

1. *The isotropy stratum $(M/G)_{(H)} = M_{(H)}/G$ is a smooth manifold, the inclusion $(M/G)_{(H)} \to M/G$ is smooth, and $\pi : M_{(H)} \to (M/G)_{(H)}$ is a smooth fiber bundle with fiber type G/H.*
2. *We have a smooth fiber bundle $M_{(H)} \to G/N_G(H)$ where $N_G(H)$ is the normalizer of H in G, and where the fiber over $g.N_G(H)$ is the fixed point set $M^{gHg^{-1}} \cap M_{(H)}$. See* [23], *7.3.*
3. *The orbit strata of M/G form a locally finite partition of M/G.*

2.3. Theorem. [23], *6.15ff.*

There exists a unique minimal isotropy type (K) such that

1. *$(M/G)_{(K)}$ is connected, locally connected, open, and dense in M/G.*

(2) *The slice representations at all points of $M_{(K)}$ are trivial, i.e. G_x acts trivially on S_x for all $x \in M_{(K)}$.*
(3) $\dim(M/G)_{(K)} = \dim M - \dim G + \dim K$.

The stratum $(M/G)_{(K)}$ is called the principal isotropy stratum and is denoted by $(M/G)_{\text{reg}}$. Likewise we write $M_{\text{reg}} := M_{(K)}$.

2.4. Let G be a compact group and let $\rho : G \to GL(V)$ be an orthogonal representation of G on a real finite dimensional Euclidean vector space V. Let $\sigma = (\sigma_1, \dots, \sigma_n) : V \to \mathbb{R}^n$, where $\sigma_1, \dots, \sigma_n$ is a system of generators for the algebra $\mathbb{R}[V]^G$ of invariant polynomials on V. The mapping σ is proper and induces a homeomorphism between V/G and the closed subset $\sigma(V) \subset \mathbb{R}^n$, see [34]. Since σ is a polynomial map, $\sigma(V)$ is a semi-algebraic subset, see [31] for an explicit description by polynomial equations and inequalities. By [37] and [18] the semi-algebraic set $\sigma(V)$ has a canonical stratification into smooth algebraic submanifolds, called the *Whitney stratification.* The strata for this stratification are the connected components of the images under σ of the set of points in V where the rank of the system of polynomials $\sigma_1, \dots, \sigma_n$ is constant.

Theorem.

(1) [6] *The mapping $\sigma : V \to \sigma(V)$ induces a bijection $\bar{\sigma} : V/G \to \sigma(V)$ which maps the components of the isotropy strata of V/G diffeomorphically onto the strata of $\sigma(V)$ as a semi-algebraic set.*
(2) [34] *$\sigma^* : C^\infty(\mathbb{R}^n) \to C^\infty(V)^G$ is a surjective homomorphism of algebras. There exists a continuous linear map $\varphi : C^\infty(V)^G \to C^\infty(\mathbb{R}^n)$ with $\sigma^* \circ \varphi = \mathrm{Id}_{C^\infty(\mathbb{R}^n)}$,* [22].

2.5. Remark. Let M be a smooth proper G-manifold. Then the orbit stratification of M/G is locally given as the Whitney stratification of a semi-algebraic subset in a vector space; by [21] this is in turn determined by the algebra of smooth functions on it. Thus the orbit stratification of M/G is determined by $C^\infty(M)^G = C^\infty(M/G)$.

3. The orbit space M/G as a metric space

Let (M, g) be a connected complete Riemannian manifold and let G be a Lie group of isometries which acts properly on M (or equivalently, is closed in the full group of isometries). Then we say that (M, g) is a complete Riemannian G-manifold. Denote by $\pi : M \to \bar{M} = M/G$ the natural projection of M onto the orbit space M/G.

Denote by d the natural metric structure on M/G induced by the Riemannian metric g of M. By definition the distance $d(\bar{p}, \bar{q})$ is the minimum of the lengths of all curves in M which connect the orbits $\bar{p}, \bar{q}$.

Recall that a metric space (X, d) is said to be of *inner type* [2] or a *path metric space* [12] if the distance between any two points p, q is equal to the length of a curve pq connecting these points. Such a curve is called a *minimal geodesic segment.*

3.1. Proposition. *Let (M, g) be a complete Riemannian G-manifold. Then the following holds:*

(1) *The orbit space $(M/G, d)$ with the natural metric is a complete metric space and a path metric space.*
(2) *Any minimal geodesic segment of M/G is the projection of a normal (i.e. orthogonal to orbits) geodesic segment of M which is called a 'horizontal lift'.*
(3) *For every $\bar{p} \in M/G$ there exists $r > 0$ such that each $\bar{q}$ with $d(\bar{p}, \bar{q}) < r$ can be connected to $\bar{p}$ by a unique minimal geodesic segment.*
(4) *Any two horizontal lifts in M of a minimal geodesic segment in M/G differ by the action of an isometry in G. For any normal geodesic c in M the projection $\pi \circ c$ into M/G has the following property: For each t there exists $r > 0$ such that $\pi(c(s))$ for s between t and $t \pm r$ both are minimal geodesic segments.*

Note that even if M is compact, in (3) one cannot choose the same $r > 0$ for all points $\bar{p} \in M/G$ in general, see example 3.3.

Proof. (1) and (2). For $p, q \in M$ there is a point $g.q \in G.q$ such that $d(p, g.q)$ is the distance from p to the (closed) orbit $G.q$. Since M is a complete Riemannian manifold, there is a geodesic c of minimal length from p to $g.q$. This geodesic is orthogonal to the orbit $G.p$: otherwise, by Gauss' lemma, we could find a shorter broken geodesic from p to $g.q$. By a well known lemma ([23], 8.1, or [29]), $c'(t)$ is orthogonal to each orbit which it meets. But then the length of $\pi \circ c$ in M/G equals the length of c in M, which is the distance between the orbits $G.p$ and $G.q$ and thus equals $d(\bar{p}, \bar{q})$.

The metric space $(M/G, d)$ is complete since each bounded closed set is compact: Use that the image of a closed geodesic ball $B_r(x) \subset M$ of radius r with center $x \in M$ is the closed ball $B_r(\pi(x))$ of the same radius in M/G.

(3) and (4). Let $\bar{p}, \bar{q}$ be two points of M/G with sufficiently short distance $d = d(\bar{p}, \bar{q})$ and let γ, δ be two geodesic segments in M of length d connecting the orbits $\bar{p}, \bar{q}$. Transforming one of the geodesics by an appropriate isometry, we may assume that the geodesics start from the same point $p \in \bar{p}$. Since the geodesics are normal to the orbit $\bar{p}$, they belong to the slice $S = \exp_p N$, where N is a neighborhood of the origin in the normal space of the orbit $\bar{p}$ at p. The end points $\gamma(d), \delta(d)$ of these geodesics belong to the orbit $\bar{q}$. Hence, by the main property of a slice, there exist an isometry h in the stabilizer G_p such that $h\gamma(d) = \delta(d)$. Since the geodesics are small, this implies $h\gamma = \delta$ and $\pi(\gamma) = \pi(\delta)$. □

We define the *angle between two minimal geodesic ray segments* in M/G from a point $\bar{p}$ as the minimum of the angles between all their horizontal lifts through $p \in \bar{p}$. The angle is independent of the choice of p.

3.2. Proposition. *Let G be a compact connected Lie group and $\rho : G \to SO(V)$ a polar orthogonal representation into a Euclidean vector space $(V, g = \langle\ ,\ \rangle)$. Then any two points of the orbit space V/G are connected by a unique minimal geodesic segment.*

Proof. By definition of a polar representation there exists a linear section S, i.e., a vector subspace of V which intersects all orbits orthogonally, see [11]. Denote by W the Weyl group of a section S, that is the quotient $W = N_G(S)/Z_G(S)$, where $N_G(S)$ is the subgroup of G which preserves S and $Z_G(S)$ is its normal subgroup which acts trivially on S. It is known that W is a finite group generated by reflections in hyperplanes of S and that the orbit spaces V/G and S/W are isometric, [11]. This reduces the statement to the case when G is a finite group generated by reflections, i.e., the Weyl group of a root system. The orbit space of such group is the closure of a Weyl chamber which is a convex polyhedral cone of Euclidean space with the induced metric. Now the statement is obvious. $\square$

Note that up to now we know only minimal geodesic segments; geodesics as we will treat them in 4.4 below will turn out to be reflected at faces, so there will be many different geodesics connecting two points.

3.3. Example. Statement 3.2 is not true if the representation ρ is not polar. Moreover, there may exist points x, y with arbitrary small distance ε which are connected by several minimal geodesic segments.

Let $G \subset SO(V)$ be a compact connected linear group such that V is the direct sum of two G-invariant subspaces V_1, V_2 of dimension > 1. Choose $x \in V_1, y \in V_2$ such that the stabilizer $H = G_x$ does not act transitively on the orbit $G.y$. For example, we may assume that $H = G_x = \{\mathrm{Id}\}$. Note that here $y \in T_x(G.x)^\perp$, but y is not contained in a slice at x. The maximal radius of a slice at x is just $|x|$.

Then for any $z \in G.y$ the geodesic $\gamma(t) = (1-t)x + tz$ is normal, since $\gamma'(t) = -x + z$ is normal to $X.\gamma(t) = (1-t)X.x + tX.z$ for each $X \in \mathfrak{g} \subset \mathfrak{so}(V)$. It is a minimal geodesic which connects the orbits $G.x$ and $G.y$. Hence it defines a minimal geodesic segment $\bar\gamma$ in the orbit space V/G. If $z, z' \in Gy$ do not belong to one H orbit, then the corresponding geodesics are not G-equivalent and define different minimal geodesic segments of the orbit space connecting $\pi(x)$ and $\pi(y)$.

3.4. Recall that a subset N of a path metric space (M, d) is called *weakly convex* if the induced metric on N is also of path metric type. So any two points in N are connected in M by some minimal geodesic segment which lies in N.

A subset N of a path metric space (M, d) is called *convex* if *any* minimal geodesic segment in M between two points in N lies in N.

Denote by $(M/G)_{\le(H)}$ the set of orbits with orbit type smaller then (H). In particular, $(M/G)_{\le(K)} = (M/G)_{\mathrm{reg}}$ if (K) denotes the minimal orbit type.

Proposition. *$M_{\le(H)}$ is a convex subset of M/G. In particular, $(M/G)_{\mathrm{reg}}$ is a convex open dense submanifold.*

The proof follows from the following lemma.

3.5. Lemma. *Let $\bar p\bar q$ be a minimal geodesic in M/G of length d and let pq be a horizontal lift. Then the stabilizer G_x of any interior point of pq is contained in the stabilizers G_p, G_q of the end points.*

In particular, if $G_p \cap G_q = \{K\}$, the minimal stabilizer group, then all interior points of $\bar p\bar q$ are regular.

Proof. Assume for contradiction that there exist an isometry $h \in G_x \setminus G_p$. Applying h to a minimal geodesic px we obtain a minimal geodesic $(hp)x$ of the same length which connects $Gp = \bar p$ and x. Then the broken geodesic $(hp)xq$ has length d and connects $\bar p$ and $\bar q$. This is impossible.

4. Vector fields, geodesics, and Jacobi fields on orbit spaces

4.1. Smooth vector fields on orbit spaces. Let M be a proper G-manifold and let $\operatorname{Der}(C^\infty(M/G))$ denote the space of all derivations of the real algebra $C^\infty(M/G)$; these are called smooth vector fields on M/G. A vector field $X \in \operatorname{Der}(C^\infty(M/G))$ is called *strata-preserving* if it preserves each ideal in $C^\infty(M/G)$ consisting of functions which vanish on an orbit stratum. We denote by $\mathfrak{X}(M/G)$ the Lie subalgebra of all strata-preserving vector fields.

Theorem. [35] *The canonical mapping*

$$X \mapsto \bar{X}, \quad \mathfrak{X}(M)^G \to \operatorname{Der}(C^\infty(M/G))$$

has image $\mathfrak{X}(M/G)$.

In [35] this is stated for compact G, but the proof works without change also for proper G-actions.

Thus we have the following exact sequence:

$$0 \to \mathfrak{X}(M)^G_{\text{ver}} \to \mathfrak{X}(M)^G \to \mathfrak{X}(M/G) \to 0$$

where $\mathfrak{X}(M)^G_{\text{ver}}$ is the space of all G-invariant vector fields on M which are tangent to the orbits (vertical).

Thus a strata preserving smooth vector field $\bar{X} \in \mathfrak{X}(M/G)$ induces a derivation on the algebra of smooth functions on each stratum and thus a smooth vector field on each stratum which is tangent to this stratum. Moreover, $\bar{X}$ induces a local flow $\operatorname{Fl}^{\bar{X}}$ on each stratum. By using also a lift in $\mathfrak{X}(M)^G$, we get a strata preserving smooth mapping

$$\mathbb{R} \times M/G \supseteq U \xrightarrow{\operatorname{Fl}^{\bar{X}}} M/G$$

which is defined on an open neighborhood U of $0 \times M/G$ in $\mathbb{R} \times M/G$. Clearly for $X \in \mathfrak{X}(M)^G$ the flows Fl^X of X and $\operatorname{Fl}^{\bar{X}}$ of $\bar{X} \in \mathfrak{X}(M/G)$ are related, i.e., $\pi_{M/G} \circ \operatorname{Fl}^X_t = \operatorname{Fl}^{\bar{X}}_t \circ \pi_{M/G}$, since this is true on each orbit stratum.

4.2. If M is a complete Riemannian G-manifold we may also consider the vector space $\mathfrak{X}(M)^G_{\text{hor}}$ of all smooth *horizontal G-invariant* vector fields, which are normal to each orbit which they meet. The space $\mathfrak{X}(M)^G_{\text{hor}}$ is a Lie algebra if and only if the vector subbundle $\operatorname{Nor}(M)|M_{\text{reg}}$ is integrable which is almost equivalent to the fact that M admits a section, since then the horizontal bundle over M_{reg} is integrable; there might be topological difficulties, namely, the leaves of the horizontal bundle might be not closed.

4.3. Differential equations of second order on M/G. Let $\kappa_M : TTM \to TTM$ be the canonical involution. Recall that a vector field Γ on TM is called a differential equation of second order on M if $\kappa_M \circ \Gamma = \Gamma$ or, equivalently, if $T(\pi_M) \circ \Gamma = \operatorname{Id}_{TM}$. It is called a *spray* if it is quadratic in the sense that $\Gamma(m^M_t.X) = m^{TM}_t.T(m^M_t).\Gamma(X)$, where m^M_t is the scalar multiplication by t on TM.

Let M be a proper G-manifold. Then the extension of the G-action to TM is also proper. The projection $\bar{\Gamma}$ of a G-invariant second order differential equation or spray $\Gamma \in \mathfrak{X}(TM)^G$ to $\mathfrak{X}(TM/G)$ is called a second order differential equation or spray on M/G. By 4.1 we know that $\bar\Gamma$ is tangent to all strata of TM/G, thus integral curves of $\bar\Gamma$ make sense and are unique as integral curves of vector fields on manifolds. Clearly, the projection of an integral curve of Γ is an integral curve of $\bar\Gamma$ on TM/G.

4.4. The geodesic spray on M/G. Fix a complete G-invariant Riemannian metric on M and denote by Γ its geodesic spray. The flow lines of Γ are the velocity fields of geodesics on M. The corresponding spray $\bar\Gamma \in \mathfrak{X}(TM/G)$ is called the *geodesic spray* on M/G. Its flow lines are complete and each is contained in one stratum of TM/G.

For $x \in M$ let $\operatorname{Nor}(M)_x = (T_x(G.x))^\perp$, the normal space to the orbit, which we may split as orthogonal direct sum of the subspace $\operatorname{Nor}^{\text{inv}}(M)_x = \operatorname{Nor}(M)_x^{G_x}$ which is invariant under the isotropy group G_x, and its orthogonal complement in $\operatorname{Nor}(M)_x$. We consider $\operatorname{Nor}(M) := \bigcup_{x\in M} \operatorname{Nor}(M)_x \subset TM$, and similarly for $\operatorname{Nor}^{\text{inv}}(M)$. These are G-invariant subsets of TM which can be considered as families of sub vector spaces with jumping dimensions: Over singular strata the dimension may become larger. Note that $\operatorname{Nor}(M)_x = \operatorname{Nor}^{\text{inv}}(M)_x$ if and only if x is a regular point.

$\operatorname{Nor}(M)$ is invariant under the flow of the spray Γ since a geodesic which is orthogonal to one orbit is orthogonal to any orbit it meets. However, $Nor^{inv}(M)$ is not invariant under the flow of the spray Γ, since a geodesic starting at a regular point orthogonally to the orbit may hit later a singular point where its tangent vector is still orthogonal to the orbit but no longer invariant under the (larger) isotropy group. Consequently, $\operatorname{Nor}(M)/G \subseteq TM/G$ is invariant under the flow of the spray $\bar\Gamma$. We may consider $\operatorname{Nor}(M)/G$ as a substitute of the *tangent bundle* of the the orbit space M/G since the normal slices suffice to describe any tangent vector which moves an orbit infinitesimally.

Definition. A *geodesic* on M/G is a curve of the form

$$t \mapsto \left(\pi_{M/G} \circ \operatorname{Fl}_t^{\bar\Gamma}\right)(\bar\xi), \quad \bar\xi \in \operatorname{Nor}(M)/G$$

This definition fits well with concept of minimal geodesic arcs as treated in section 3; geodesics are prolongations of minimal geodesic arcs. Clearly geodesics in M/G are exactly the projections onto M/G of normal geodesics in M.

4.5. Example. Let $G \to O(V)$ be a polar representation of a compact group, with section $\Sigma \subset V$, a linear subspace which meets every orbit orthogonally. Then $V/G = \Sigma/W(\Sigma)$ is represented by a chamber C in Σ. The normal geodesics in V can be chosen to lie in Σ, thus the geodesics in $V/G = \Sigma/W(\Sigma) \cong C$ are straight lines in the interior of C which are reflected by the walls: the incoming angle equals the outgoing angle.

4.6. Questions. (1) Are the geodesics on M/G uniquely determined by the metric space $(M/G, d)$? More precisely: Let M/G and N/H be two orbit spaces of connected complete Riemannian manifolds by proper groups of isometries, and let $\varphi : M/G \to N/H$ be an isometric bijection. Is it true that φ maps geodesics to geodesics? Clearly minimal geodesic arcs are mapped to minimal geodesic arcs.

(2) Is the orbit stratification of M/G determined by the metric space M/G?

(3) Can one lift isometries between orbit spaces to the Riemannian manifolds? Results in this direction can be found in [35] under stronger conditions, and in [19] more generally. See also [16] and [20] for finite groups.

4.7. Ballistic curves. The projection $t \mapsto (\pi_{M/G} \circ \mathrm{Fl}^{\bar\Gamma}_t)(\bar\xi)$ onto M/G of a flow line of the geodesic spray on TM/G with general initial vector $\bar\xi \in TM/G$ (which need *not* be in $\mathrm{Nor}(M)/G$), is called a *ballistic curve.* It depends on external data: $\bar\pi_{M/G} : TM/G \to M/G$ is bigger than the tangent bundle.

4.8. The Jacobian flow. We recall the following result. Here ∇ is a torsion-free covariant derivative on TM which is uniquely determined by its spray Γ. R is the curvature of ∇, and $K : TTM \to TM$ is the connector, i.e., the projection from TTM onto the vertical bundle along the horizontal bundle described by ∇, followed by the the natural projection from the vertical bundle to TM.

Theorem. [24] *Let $\Gamma : TM \to TTM$ be a spray on a manifold M. Then $\kappa_{TM} \circ T\Gamma : TTM \to TTTM$ is a vector field. Consider a flow line*

$$Y(t) = \mathrm{Fl}^{\kappa_{TM} \circ T\Gamma}_t (Y(0))$$

of this field. Then we have:

$c := \pi_M \circ \pi_{TM} \circ Y$ is a geodesic on M.
$\dot c = \pi_{TM} \circ Y$ is the velocity field of c.
$J := T(\pi_M) \circ Y$ is a Jacobi field along c.
$\dot J = \kappa_M \circ Y$ is the velocity field of J.
$\nabla_{\partial_t} J = K \circ \kappa_M \circ Y$ is the covariant derivative of J.
The Jacobi equation is given by:

$$0 = \nabla_{\partial_t}\nabla_{\partial_t} J + R(J, \dot c)\dot c = K \circ TK \circ T\Gamma \circ Y.$$

This implies that in a canonical chart induced from a chart on M the curve $Y(t)$ is given by

$$(c(t), c'(t); J(t), J'(t)).$$

On a complete Riemannian G-manifold M with geodesic spray Γ we may thus consider the G-invariant vector field $\kappa_{TM} \circ T\Gamma : TTM \to TTTM$ and the induced smooth derivation

$$\overline{\kappa_{TM} \circ T\Gamma} \in \mathfrak{X}(TTM/G).$$

We have

$$T(\pi_{TM/G}) \circ \overline{\kappa_{TM} \circ T\Gamma} = \overline{T\pi_{TM} \circ \kappa_{TM} \circ T\Gamma} = \overline{\pi_{TTM} \circ T\Gamma} = \overline{\Gamma \circ \pi_{TM}} = \bar\Gamma \circ \pi_{TM/G}.$$

We consider a flow line

$$\bar Y(t) = \mathrm{Fl}^{\overline{\kappa_{TM} \circ T\Gamma}}_t (\bar Y(0))$$

of this field which respects the orbit stratification. Then we have:

(1) $t \mapsto \bar c(t) := \pi_{M/G} \circ \pi_{TM/G} \circ \bar Y(t) \in M/G$ is a geodesic on M/G if the initial velocity vector $\pi_{TM/G}(\bar Y(0)) \in \mathrm{Nor}(M)/G$ is normal. If not then $\bar c(t)$ is a ballistic curve on M/G.

(2) $t \mapsto \dot{\bar{c}}(t) = \pi_{TM/G} \circ \bar{Y}(t) \in TM/G$ is the velocity field of $\bar{c}$. It respects the orbit stratification of TM/G since it is a flow line of $\bar{\Gamma}$.
(3) $t \mapsto \bar{J}(t) := T(\pi_M)_{/G} \circ \bar{Y}(t) \in TM/G$ can be called a *Jacobi field* along $\bar{c}$. It does not respect the orbit stratification in general.
(4) $\dot{\bar{J}} = \kappa_{M/G} \circ \bar{Y}$ is the velocity field of $\bar{J}$. It respects the orbit stratification.
(5) $\nabla_{\partial_t} \bar{J} = \bar{K} \circ \kappa_{M/G} \circ \bar{Y}$ is the covariant derivative of $\bar{J}$.

5. Example: Hermitian and symmetric matrices

5.1. Simplest Example. Let S^1 act on $\mathbb{R}^2$ by rotations. Then $\mathbb{R}^2/S^1 = [0,\infty)$, and $\overline{\pi_{\mathbb{R}^2}} : T\mathbb{R}^2/S^1 \to \mathbb{R}^2/S^1$ looks as follows: The fiber over 0 is $[0,\infty)$ again, and the fiber over $t > 0$ is $\mathbb{R}^2$. Normal geodesics on $\mathbb{R}^2$ are lines through 0, so geodesics on $\mathbb{R}^2/S^1$ are constant speed curves coming in from infinity on $[0,\infty)$ which are reflected at 0 and go out again at constant speed: $t \mapsto |tv|$. However, ballistic curves seem to carry a charge and behave as being repelled by a field carried by the singular orbit 0, namely

$$t \mapsto \sqrt{(x_1 + tv_1)^2 + (x_2 + tv_2)^2}, \qquad \text{where } x, v \text{ are linearly independent in } \mathbb{R}^2.$$

5.2. Simple Example. Let $SO(2)$ act on the space $S(2)$ of symmetric (2×2)-matrices by conjugation, a polar representation, where the diagonal matrices form a section. A chamber is here given by the halfspace

$$S(2)/SO(2) = C := \left\{ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} : \lambda_1 \geq \lambda_2 \right\}.$$

Let us describe $TS(2)/SO(2) \to S(2)/SO(2) = C$. Over a point $A = \operatorname{diag}(\lambda, \lambda)$ in the wall of C we can use the isotropy group $SO(2)_A = SO(2)$ to put the tangent vector in normal form, so the fiber there is the half space C. The fiber over an interior point is the whole vector space $S(2)$.

As in 4.5 geodesics in $S(2)/SO(2) \cong C$ are straight lines which are reflected by the wall $\{\lambda_1 = \lambda_2\}$. One can compute the ballistic curves. For

$$t \mapsto A + tV = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} + t \begin{pmatrix} v_1 & v_3 \\ v_3 & v_2 \end{pmatrix} \in S(2)$$

the curve of eigenvalues in C is

$$t \mapsto \begin{pmatrix} \lambda_1(t) \\ \lambda_2(t) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} a_1 + a_2 + t(v_1 + v_2) + \sqrt{(a_1 - a_2 + t(v_1 - v_2))^2 + 4t^2 v_3^2} \\ a_1 + a_2 + t(v_1 + v_2) - \sqrt{(a_1 - a_2 + t(v_1 - v_2))^2 + 4t^2 v_3^2} \end{pmatrix}.$$

Here $t \mapsto a_1 + a_2 + t(v_1 + v_2)$ is the component on the wall $\lambda_1 = \lambda_2$ of C which travels with constant speed, whereas $\sqrt{(a_1 - a_2 + t(v_1 - v_2)^2 + 4t^2 v_3^2}$ is the distance from the wall. So again the ballistic curve is being repelled by a field carried by the wall. If it contains one regular orbit and is not a geodesic, then it never hits the wall.

5.3. The space of Hermitian matrices. Let $G = SU(n)$ act on the space $H(n)$ of complex Hermitian $(n \times n)$-matrices by conjugation, where the inner product is given by the (always real) trace $\operatorname{Tr}(AB)$. This is a polar representation, where the diagonal matrices with real entries form a section Σ. A chamber is here given by the quadrant $C \subset \Sigma$ consisting of all real diagonal matrices with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Geodesics in $H(n)/SU(n) \cong C$ are straight lines which are reflected by all walls $\{\lambda_i = \lambda_{i_1} = \cdots = \lambda_{i+k}\}$.

A ballistic curve looks as follows: Let A be a diagonal matrix with eigenvalues $a_1 \geq \cdots \geq a_n$, and let $V = (v_{i,j})$ be a Hermitian matrix. The ballistic curve is then $\lambda(t) = (\lambda_1(t) \geq \cdots \geq \lambda_n(t))$, the curve of eigenvalues of the Hermitian matrix $A + tV$.

5.4. Hamiltonian description. Let us describe ballistic curves as trajectories of a Hamiltonian system on a reduced phase space. Let $T^*H(n) = H(n) \times H(n)$ be the cotangent bundle where we identified $H(n)$ with its dual by the inner product, so the duality is given by $\langle \alpha, A\rangle = \operatorname{Tr}(A\alpha)$. Then the canonical 1-form is given by $\theta(A, \alpha, A', \alpha') = \operatorname{Tr}(\alpha A')$, the symplectic form is $\omega_{(A,\alpha)}((A', \alpha'), (A'', \alpha'')) = \operatorname{Tr}(A'\alpha'' - A''\alpha')$, and the Hamiltonian function for the straight lines $(A+t\alpha, \alpha)$ on $H(n)$ is $h(A, \alpha) = \frac{1}{2} \operatorname{Tr}(\alpha^2)$. The action $SU(n) \ni g \mapsto (A \mapsto gAg^{-1})$ lifts to the action $SU(n) \ni g \mapsto ((A, \alpha) \mapsto (gAg^{-1}, g\alpha g^{-1}))$ on $T^*H(n)$ with fundamental vector fields $\zeta_X(A, \alpha) = (A, \alpha, [X, A], [X, \alpha])$ for $X \in \mathfrak{su}(n)$, and with generating functions $f_X(A, \alpha) = \theta(\zeta_X(A, \alpha)) = \operatorname{Tr}(\alpha[X, A]) = \operatorname{Tr}([A, \alpha]X)$. Thus the momentum mapping $J : T^*H(n) \to \mathfrak{su}(n)^*$ is given by $\langle X, J(A, \alpha)\rangle = f_X(A, \alpha) = \operatorname{Tr}([A, \alpha]X)$. If we identify $\mathfrak{su}(n)$ with its dual via the inner product $\operatorname{Tr}(XY)$, the momentum mapping is $J(A, \alpha) = [A, \alpha]$. Along the line $t \mapsto A + t\alpha$ the momentum mapping is constant: $J(A + t\alpha, \alpha) = [A, \alpha] = Y \in \mathfrak{su}(n)$. Note that for $X \in \mathfrak{su}(n)$ the evaluation on X of $J(A + t\alpha, \alpha) \in \mathfrak{su}(n)^*$ equals the inner product:

$$\langle X, J(A + t\alpha, \alpha)\rangle = \operatorname{Tr}(\tfrac{d}{dt}(A + t\alpha), \zeta_X(A + t\alpha)),$$

which is obviously constant in t; compare with the general result of Riemannian transformation groups, e.g. [23], 8.1.

According to principles of symplectic reduction [1], 4.3.5, [36], [17], [30], we have to consider for a regular value Y (and later for an arbitrary value) of the momentum mapping J the submanifold $J^{-1}(Y) \subset T^*H(n)$. The null distribution of $\omega|J^{-1}(Y)$ is integrable (with jumping dimensions) and its leaves (according to the Stefan-Sussmann theory of integrable distributions) are exactly the orbits in $J^{-1}(Y)$ of the isotropy group $SU(n)_Y$ for the coadjoint action. So we have to consider the orbit space $J^{-1}(Y)/SU(n)_Y$. If Y is not a regular value of J, the inverse image $J^{-1}(Y)$ is a subset which is described by polynomial equations since J is polynomial (in fact quadratic), so $J^{-1}(Y)$ is stratified into submanifolds; symplectic reduction works also for this case, see [36].

5.5. The case of momentum $Y = 0$ gives again geodesics. If $Y = 0$ then $SU(n)_Y = SU(n)$ and $J^{-1}(0) = \{(A, \alpha) : [A, \alpha] = 0\}$, so A and α commute. If A is regular (i.e. all eigenvalues are distinct), using a uniquely determined transformation $g \in SU(n)$ we move the point A into the open chamber $C^o \subset H(n)$, so $A = \operatorname{diag}(a_1 > a_2 > \cdots > a_n)$ and since α commutes with

A it is also in diagonal form. The symplectic form ω restricts to the canonical symplectic form on $C^o \times \Sigma = C^o \times \Sigma^* = T^*(C^o)$. Thus symplectic reduction gives $(J^{-1}(0) \cap (T^*H(n))_{\text{reg}})/SU(n) = T^*(C^o) \subset T^*H(n)$. By [36] we also use symplectic reduction for non-regular A and we get (see in particular [17], 3.4) $J^{-1}(0)/SU(n) = T^*C$, the stratified cotangent cone bundle of the chamber C considered as stratified space. Namely, if one root $\varepsilon_i(A) = a_i - a_{i+1}$ vanishes on the diagonal matrix A then the isotropy group $SU(n)_A$ contains a subgroup $SU(2)$ corresponding to these coordinates. Any matrix α with $[A, \alpha] = 0$ contains an arbitrary hermitian submatrix corresponding to the coordinates i and $i+1$, which may be brought into diagonal form with the help of this $SU(2)$ so that $\varepsilon_i(\alpha) = \alpha_i - \alpha_{i+1} \geq 0$. Thus the tangent vector α with foot point in a wall is either tangent to the wall (if $\alpha_i = \alpha_{i+1}$) or points into the interior of the chamber C. The Hamiltonian h restricts to $C^o \times \Sigma \ni (A, \alpha) \mapsto \frac{1}{2}\sum_i \alpha_i^2$, so the trajectories of the Hamiltonian system here are again straight lines which are reflected at the walls.

5.6. The case of general momentum Y. If $Y \neq 0 \in \mathfrak{su}(n)$ and if $SU(n)_Y$ is the isotropy group of Y for the adjoint representation, then it is well known (see references in 5.4) that we may pass from Y to the coadjoint orbit $\mathcal{O}(Y) = \operatorname{Ad}^*(SU(n))(Y)$ and get

$$J^{-1}(Y)/SU(n)_Y = J^{-1}(\mathcal{O}(Y))/SU(n) = (J^{-1}(Y) \times \mathcal{O}(-Y))/SU(n), \tag{1}$$

where all (stratified) diffeomorphisms are symplectic ones.

5.7. The Calogero Moser system. As the simplest case we assume that $Y' \in \mathfrak{su}(n)$ is not zero but has maximal isotropy group, see [14]: So we assume that Y' has complex rank 1 plus a suitable imaginary multiple of the identity to ensure that $Y' \in \mathfrak{su}(n)$, in more detail: $Y' = \sqrt{-1}(c\mathbb{I}_n + v \otimes v^*)$ for $0 \neq v = (v^i)$ a column vector in $\mathbb{C}^n$. The coadjoint orbit is then $\mathcal{O}(Y') = \{\sqrt{-1}(c\mathbb{I}_n + w \otimes w^*) : w \in \mathbb{C}^n, |w| = |v|\}$, isomorphic to $S^{2n-1}/S^1 = \mathbb{C}P^n$, of real dimension $2n-2$. Consider (A', α') with $J(A', \alpha') = Y'$, choose $g \in SU(n)$ such that $A = gA'g^{-1} = \operatorname{diag}(a_1 \geq a_2 \geq \cdots \geq a_n)$, and let $\alpha = g\alpha' g^{-1}$. Then the entry of the commutator is $[A, \alpha]_{ij} = \alpha_{ij}(a_i - a_j)$. So $[A, \alpha] = gY'g^{-1} =: Y = \sqrt{-1}(c\mathbb{I}_n + gv \otimes (gv)^*) = \sqrt{-1}(c\mathbb{I}_n + w \otimes w^*)$ has zero diagonal entries, thus $0 < w^i \bar{w}^i = -c$ and $w^i = \exp(\sqrt{-1}\theta_i)\sqrt{-c}$ for some θ_i. But then all off-diagonal entries $Y_{ij} = \sqrt{-1} w^i \bar{w}^j = -\sqrt{-1}\, c \exp(\sqrt{-1}(\theta_i - \theta_j)) \neq 0$, and A has to be regular. We may use the remaining gauge freedom in the isotropy group $SU(n)_A = S(U(1)^n)$ to put $w^i = \exp(\sqrt{-1}\theta)\sqrt{-c}$ where $\theta = \sum \theta_i$. Then $Y_{ij} = -c\sqrt{-1}$ for $i \neq j$.

So the reduced space $(T^*H(n))_Y$ is diffeomorphic to the submanifold of $T^*H(n)$ consisting of all $(A, \alpha) \in H(n) \times H(n)$ where $A = \operatorname{diag}(a_1 > a_2 > \cdots > a_n)$, and where α has arbitrary diagonal entries $\alpha_i := \alpha_{ii}$ and off-diagonal entries $\alpha_{ij} = Y_{ij}/(a_i - a_j) = -c\sqrt{-1}/(a_i - a_j)$. We can thus use $a_1, \ldots, a_n, \alpha_1, \ldots, \alpha_n$ as coordinates. The invariant symplectic form pulls back to $\omega_{(A,\alpha)}((A'\alpha'), (A'', \alpha'')) = \operatorname{Tr}(A'\alpha'' - A''\alpha') = \sum(a_i'\alpha_i'' - a_i''\alpha_i')$. The invariant Hamiltonian h restricts to the Hamiltonian

$$h(A, \alpha) = \tfrac{1}{2}\operatorname{Tr}(\alpha^2) = \frac{1}{2}\sum_i \alpha_i^2 + \frac{1}{2}\sum_{i \neq j} \frac{c^2}{(a_i - a_j)^2}.$$

This is the famous Hamiltonian function of the Calogero-Moser completely integrable system, see [25], [26], [14], and [30], 3.1 and 3.3. The corresponding Hamiltonian vector field and the differential equation for the ballistic curve are then

$$H_h = \sum_i \alpha_i \frac{\partial}{\partial a_i} + 2\sum_i \sum_{j\neq i} \frac{c^2}{(a_i-a_j)^3}\frac{\partial}{\partial \alpha_i},$$
$$\ddot a_i = 2\sum_{j\neq i}\frac{c^2}{(a_i-a_j)^3},$$

Note that the ballistic curve avoids the walls of the Weyl chamber C.

5.8. Degenerate cases of non-zero momenta of minimal rank. Let us discuss now the case of non-regular diagonal A. Namely, if one root, say $\varepsilon_{12}(A) = a_1 - a_2$ vanishes on the diagonal matrix A then the isotropy group $SU(n)_A$ contains a subgroup $SU(2)$ corresponding to these coordinates. Consider α with $[A,\alpha] = Y$; then $0 = \alpha_{12}(a_1 - a_2) = Y_{12}$. Thus α contains an arbitrary hermitian submatrix corresponding to the first two coordinates, which may be brought into diagonal form with the help of this $SU(2) \subset SU(n)_A$ so that $\varepsilon_{12}(\alpha) = \alpha_1 - \alpha_2 \geq 0$. Thus the tangent vector α with foot point A in a wall is either tangent to the wall (if $\alpha_1 = \alpha_2$) or points into the interior of the chamber C (if $\alpha_1 > \alpha_2$). Note that then $Y_{11} = Y_{22} = Y_{12} = 0$.

Let us now assume that the momentum Y is of the form $Y = \sqrt{-1}(c\mathbb{I}_{n-2} + v\otimes v^*)$ for some vector $0 \neq v \in \mathbb{C}^{n-2}$. We can repeat the analysis of 5.7 in the subspace $\mathbb{C}^{n-2}$, and get for the Hamiltonian

$$h(A,\alpha) = \tfrac12 \operatorname{Tr}(\alpha^2) = \frac12\sum_{i=1}^n \alpha_i^2 + \frac12 \sum_{3\leq i\neq j}\frac{c^2}{(a_i-a_j)^2},$$
$$H_h = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial a_i} + 2\sum_{3\leq i\neq j}\frac{c^2}{(a_i-a_j)^3}\frac{\partial}{\partial\alpha_i},$$
$$\ddot a_1 = \ddot a_2 = 0, \quad \ddot a_i = 2\sum_{3\leq j\neq i}\frac{c^2}{(a_i-a_j)^3} \text{ for } i>2.$$

So the ballistic curves are just the trajectories of the Calogero-Moser integrable system inside the wall $\{a_1 - a_2 = 0\}$ complemented by a geodesic in the coordinates orthogonal to this wall. Of course we may add other vanishing roots.

5.9. The case of general momentum Y and regular A. Starting again with some regular A' consider (A',α') with $J(A',\alpha') = Y'$, choose $g \in SU(n)$ such that $A = gA'g^{-1} = \operatorname{diag}(a_1 > a_2 > \cdots > a_n)$, and let $\alpha = g\alpha' g^{-1}$ and $Y = gY'g^{-1} = [A,\alpha]$. Then the entry of the commutator is $Y_{ij} = [A,\alpha]_{ij} = \alpha_{ij}(a_i - a_j)$ thus $Y_{ii} = 0$. We may pass to the coordinates a_i and $\alpha_i := \alpha_{ii}$ for $1 \leq i \leq n$ on the one hand, corresponding to $J^{-1}(Y)$ in 5.6.1, and Y_{ij} for $i \neq j$ on the other hand, corresponding to $\mathcal{O}(-Y)$ in 5.6.1, with the linear relation $Y_{ji} = -\overline{Y_{ij}}$ and with $n-1$ non-zero entries $Y_{ij} > 0$ with $i > j$ (chosen in lexicographic order) by applying the remaining isotropy group $SU(n)_A = S(U(1)^n) = \{\operatorname{diag}(e^{\sqrt{-1}\theta_1}, \dots, e^{\sqrt{-1}\theta_n}) : \sum \theta_i \in 2\pi\mathbb{Z}\}$. We may use this canonical form as section

$$(J^{-1}(Y) \times \mathcal{O}(-Y))/SU(n) \to J^{-1}(Y) \times \mathcal{O}(-Y) \subset TH(n) \times \mathfrak{su}(n)$$

to pull back the symplectic or Poisson structures and the Hamiltonian function

$$h(A,\alpha) = \tfrac{1}{2}\operatorname{Tr}(\alpha^2) = \frac{1}{2}\sum_i \alpha_i^2 - \frac{1}{2}\sum_{i\neq j}\frac{Y_{ij}Y_{ji}}{(a_i-a_j)^2},$$

$$\begin{aligned} dh &= \sum_i \alpha_i\, d\alpha_i + \sum_{i\neq j}\frac{Y_{ij}Y_{ji}}{(a_i-a_j)^3}(da_i - da_j) - \frac{1}{2}\sum_{i\neq j}\frac{dY_{ij}.Y_{ji} + Y_{ij}.dY_{ji}}{(a_i-a_j)^2}, \\ &= \sum_i \alpha_i\, d\alpha_i + 2\sum_{i\neq j}\frac{Y_{ij}Y_{ji}}{(a_i-a_j)^3}da_i - \sum_{i\neq j}\frac{Y_{ji}}{(a_i-a_j)^2}dY_{ij}. \qquad (1) \end{aligned}$$

The invariant symplectic form on $TH(n)$ pulls back to $\omega_{(A,\alpha)}((A'\alpha'),(A'',\alpha'')) = \operatorname{Tr}(A'\alpha'' - A''\alpha') = \sum(a_i'\alpha_i'' - a_i''\alpha_i')$ thus to $\sum_i da_i \wedge d\alpha_i$. The Poisson structure on $\mathfrak{su}(n)$ is given by

$$\begin{aligned} \Lambda_Y(U,V) &= \operatorname{Tr}(Y[U,V]) = \sum_{m,n,p}(Y_{mn}U_{np}V_{pm} - Y_{mn}V_{np}U_{pm}) \\ \Lambda_Y &= \sum_{i\neq j,k\neq l}\Lambda_Y(dY_{ij}, dY_{kl})\partial_{Y_{ij}} \otimes \partial_{Y_{kl}} \\ &= \sum_{i\neq j,k\neq l}\sum_{m,n}(Y_{mn}\delta_{ni}\delta_{jk}\delta_{lm} - Y_{mn}\delta_{nk}\delta_{li}\delta_{jm})\partial_{Y_{ij}} \otimes \partial_{Y_{kl}} \\ &= \sum_{i\neq j,k\neq l}(Y_{li}\delta_{jk} - Y_{jk}\delta_{li})\partial_{Y_{ij}} \otimes \partial_{Y_{kl}} \end{aligned}$$

Since this Poisson 2-vector field is tangent to the orbit $\mathcal{O}(-Y)$ and is $SU(n)$-invariant, we can push it down to the orbit space. There it maps dY_{ij} to (remember that $Y_{ii} = 0$)

$$\Lambda_{-Y}(dY_{ij}) = -\sum_{k\neq l}(Y_{li}\delta_{jk} - Y_{jk}\delta_{li})\partial_{Y_{kl}} = -\sum_k (Y_{ki}\partial_{Y_{jk}} - Y_{jk}\partial_{Y_{ki}}).$$

So by (1) the Hamiltonian vector field is

$$\begin{aligned} H_h &= \sum_i \alpha_i\,\partial_{a_i} - 2\sum_{i\neq j}\frac{Y_{ij}Y_{ji}}{(a_i-a_j)^3}\,\partial_{\alpha_i} + \sum_{i\neq j}\frac{Y_{ji}}{(a_i-a_j)^2}\sum_k(Y_{ki}\,\partial_{Y_{jk}} - Y_{jk}\,\partial_{Y_{ki}}) \\ &= \sum_i \alpha_i\,\partial_{a_i} - 2\sum_{i\neq j}\frac{Y_{ij}Y_{ji}}{(a_i-a_j)^3}\,\partial_{\alpha_i} - \sum_{i,j,k}\left(\frac{Y_{ji}Y_{jk}}{(a_i-a_j)^2} - \frac{Y_{ij}Y_{kj}}{(a_j-a_k)^2}\right)\partial_{Y_{ki}} \end{aligned}$$

The differential equation thus becomes (remember that $Y_{jj} = 0$):

$$\begin{aligned} \dot a_i &= \alpha_i \\ \dot\alpha_i &= -2\sum_j\frac{Y_{ij}Y_{ji}}{(a_i-a_j)^3} = 2\sum_j\frac{|Y_{ij}|^2}{(a_i-a_j)^3} \\ \dot Y_{ki} &= -\sum_j\left(\frac{Y_{ji}Y_{jk}}{(a_i-a_j)^2} - \frac{Y_{ij}Y_{kj}}{(a_j-a_k)^2}\right). \end{aligned}$$

Consider the Matrix Z with $Z_{ii} = 0$ and $Z_{ij} = Y_{ij}/(a_i - a_j)^2$. Then the differential equations become:

$$\ddot{a}_i = 2\sum_j \frac{|Y_{ij}|^2}{(a_i - a_j)^3}, \qquad \dot{Y} = [Y^*, Z].$$

This is the Calogero-Moser integrable system with spin, see [5] and [32].

5.10. The case of general momentum Y and singular A. Let us consider the situation of 5.9, when A is not regular. Let us assume again that one root, say $\varepsilon_{12}(A) = a_1 - a_2$ vanishes on the diagonal matrix A. Consider α with $[A, \alpha] = Y$. From $Y_{ij} = [A, \alpha]_{ij} = \alpha_{ij}(a_i - a_j)$ we conclude that $Y_{ii} = 0$ for all i and also $Y_{12} = 0$. The isotropy group $SU(n)_A$ contains a subgroup $SU(2)$ corresponding to the first two coordinates and we may use this to move α into the form that $\alpha_{12} = 0$ and $\varepsilon_{12}(\alpha) \geq 0$. Thus the tangent vector α with foot point A in the wall $\{\varepsilon_{12} = 0\}$ is either tangent to the wall when $\alpha_1 = \alpha_2$ or points into the interior of the chamber C when $\alpha_1 > \alpha_2$. We can then use the same analysis as in 5.9 where we use now that $Y_{12} = 0$.

In the general case, when some roots vanish, we get for the Hamiltonian function, vector field, and differential equation:

$$\begin{aligned}
h(A, \alpha) &= \tfrac{1}{2}\operatorname{Tr}(\alpha^2) = \frac{1}{2}\sum_i \alpha_i^2 + \frac{1}{2}\sum_{\{(i,j):a_i \neq a_j\}} \frac{|Y_{ij}|^2}{(a_i - a_j)^2},\\
H_h &= \sum_i \alpha_i \partial_{a_i} + 2\sum_{a_j \neq a_i} \frac{|Y_{ij}|^2}{(a_i - a_j)^3}\,\partial_{\alpha_i} + \\
&\qquad - \sum_{a_i \neq a_j, k} \frac{Y_{ji}Y_{jk}}{(a_i - a_j)^2}\partial_{Y_{ki}} + \sum_{i, a_j \neq a_k} \frac{Y_{ij}Y_{kj}}{(a_j - a_k)^2}\partial_{Y_{ki}}\\
\ddot{a}_i &= 2\sum_{a_j \neq a_i} \frac{|Y_{ij}|^2}{(a_i - a_j)^3}, \qquad \dot{Y} = [Y^*, Z]
\end{aligned}$$

where we use the same notation as above. It would be very interesting to investigate the reflection behavior of this ballistic curve at the walls.

5.11. Example: symmetric matrices. We finally treat the action of $SO(n) = SO(n, \mathbb{R})$ on the space $S(n)$ of symmetric matrices by conjugation. Following the method of 5.9 and 5.10 we get the following result. Let $t \mapsto A' + t\alpha'$ be a straight line in $S(n)$. Then the ordered set of eigenvalues $a_1(t), \dots, a_n(t)$ of $A' + t\alpha'$ is part of the integral curve of the following vector field:

$$\begin{aligned}
H_h &= \sum_i \alpha_i \partial_{a_i} + 2\sum_{a_j \neq a_i} \frac{Y_{ij}^2}{(a_i - a_j)^3}\,\partial_{\alpha_i} + \\
&\quad + \sum_{a_i \neq a_j, k} \frac{Y_{ij}Y_{jk}}{(a_i - a_j)^2}\partial_{Y_{ki}} - \sum_{i, a_j \neq a_k} \frac{Y_{ij}Y_{jk}}{(a_j - a_k)^2}\partial_{Y_{ki}}\\
\ddot{a}_i &= 2\sum_{a_j \neq a_i} \frac{Y_{ij}^2}{(a_i - a_j)^3}, \qquad \dot{Y} = [Y, Z], \qquad \text{where } Z_{ij} = -\frac{Y_{ij}}{(a_i - a_j)^2},
\end{aligned}$$

where we also note that $Y_{ij} = Z_{ij} = 0$ whenever $a_i = a_j$.

5.12. Remark. Along the same line one can investigate the action of the quaternionic unitary group $Sp(n)$ on the space of all quaternionic hermitian matrices. Since the results are more complicated to write down and since they are a special case of section 6 we do not dwell on them here.

6. Ballistic curves on polar representations

6.1. The setting. Let $\rho : G \to O(V, \langle\ ,\ \rangle)$ be a polar representation of a compact connected semisimple group, see 4.5, with section $\Sigma \subset V$, a linear subspace which meets every orbit orthogonally. Then $V/G = \Sigma/W(\Sigma)$ is represented by a chamber C in Σ. The normal geodesics in V can be chosen to lie in Σ, thus the geodesics in $V/G = \Sigma/W(\Sigma) \cong C$ are straight lines in the interior of C which are reflected by the walls.

By Dadok, [11], proposition 6, which follows from his classification, for any polar representation there exists an isotropy representation of a symmetric space with the same orbits, and it suffices to investigate those latter ones. Thus we can assume that $\mathfrak{l} = \mathfrak{g} \oplus V$ is a reductive decomposition of a compact semisimple Lie algebra $\mathfrak{l}$, where $\mathfrak{g}$ is the compact Lie algebra of G, and where $\langle\ ,\ \rangle$ is an invariant positive definite inner product on $\mathfrak{l}$, the negative of the Killing form, and where $\mathfrak{l} = \mathfrak{g} \oplus V$ is an orthogonal decomposition. Moreover the infinitesimal action ρ' of $\mathfrak{g}$ on V is by the adjoint action, $\rho'(X)A = [X, A]$, so $[\mathfrak{g}, V] \subseteq V$, and $[V, V] \subseteq \mathfrak{g}$. As section Σ we may use any maximal abelian subspace in V. Moreover we shall use the following lemma.

6.2. Lemma. *In the situation above we have:*

(1) *For $A \in V$ let $\mathfrak{g}'$ be the orthogonal complement of the centralizer $Z_{\mathfrak{g}}(A)$ in $\mathfrak{g}$ and let V' be the orthogonal complement to the centralizer $Z_V(A)$ in V such that $\mathfrak{l} = Z_{\mathfrak{g}}(A) \oplus \mathfrak{g}' \oplus Z_V(A) \oplus V'$.*

   Then ad_A induces linear isomorphisms $\mathrm{ad}_A : V' \to [A, V'] = \mathfrak{g}'$ and $\mathrm{ad}_A : \mathfrak{g}' \to [A, \mathfrak{g}'] = V'$.

(2) *An element $A \in V$ is regular for the G-action on V if and only if the centralizer $Z_V(A)$ in V is a maximal commutative subalgebra of V. In this case $Z_V(A)$ is the unique section in V containing A.*

(3) *An element $A \in V$ is regular for the G-action if and only if there exists an element $X \in \mathfrak{g}$ such that $X + A$ is regular in $\mathfrak{l}$, so that $Z_{\mathfrak{l}}(X + A)$ is a Cartan subalgebra of $\mathfrak{l}$.*

(4) *A linear subspace $\Sigma \subset V$ is a section if and only if there exists a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{h} \oplus \Sigma$ is a Cartan subalgebra of $\mathfrak{l}$.*

   In this case, let $R \subset L(\Sigma, \mathbb{R})$ be the system of restricted roots so that we have the orthogonal root space decomposition

$$\mathfrak{l} = \mathfrak{h} \oplus \Sigma \oplus \bigoplus_{\lambda \in R} \mathfrak{l}_\lambda$$

   where each $\mathfrak{l}_\lambda$ has an orthogonal basis E^i_λ, B^i_λ, where $i = 1, \dots, k_\lambda$, and where $E^i_\lambda \in \mathfrak{g}$ and $B^i_\lambda \in V$ are unit vectors, such that $[A, E^i_\lambda] = \lambda(A) B^i_\lambda$ and $[A, B^i_\lambda] = \lambda(A) E^i_\lambda$ for all $A \in \Sigma$.

(5) *Let $\Sigma \subset V$ be a section for the G-action on V and let $A \in \Sigma$. Then for any $\alpha \in V$ with $[A, \alpha] = 0$ there exists some $g \in G_A = Z_G(A)$ with $g.\alpha \in \Sigma$.*

Proof. (1) The result follows from the Fitting decomposition $\mathfrak{l} = Z_{\mathfrak{l}}(A) + \mathrm{ad}_A(\mathfrak{l})$ of the skew-symmetric endomorphism $\mathrm{ad}_A : \mathfrak{l} \to \mathfrak{l}$ which we may write as

$$\mathfrak{l} = (Z_{\mathfrak{g}}(A) \oplus \mathrm{ad}_A(V')) \oplus (Z_V(A) \oplus \mathrm{ad}_A(\mathfrak{g}'))$$

since ad_A interchanges $\mathfrak{g}$ and V.

(2) It is known (see e.g. [11]) that the G-regular elements in a polar G-module V are exactly those $A \in V$ such that $Z_{\mathfrak{g}}(A)$ is of minimal dimension. Moreover, any $A \in V$ is contained in some section Σ, and any section Σ is a maximal commutative subspace of $V \subset \mathfrak{l} = \mathfrak{g} \oplus V$. Since a section Σ generates a compact torus in the Lie group L, there exists an element $A \in \Sigma$ such that $Z_V(A) = \Sigma$. These elements are characterized as those such that $Z_V(A)$ is of minimal dimension, since for any $A \in V$ the centralizer $Z_A(V)$ contains a section Σ. By (1) the element $A \in V$ is G-regular if and only if $\dim(Z_V(A))$ is minimal, i.e. $Z_V(A) = \Sigma$ is a section.

(3) follows from (2).

(4) The first assertion follows from (3), and the rest is well known in the theory of symmetric spaces.

(5) Denote by Σ' a maximal commutative subspace of V containing A and α. Then Σ' is a section. Now the isotropy group $G_A = Z_G(A)$ acts transitively on the set of all sections of V which contain A. So in particular there exists $g \in G_A$ such that $g.\Sigma' = \Sigma$, and hence $g.\alpha \in \Sigma$. $\square$

6.3. Hamiltonian description and symplectic reduction. Generalizing 5.4, under the assumption of 6.1, we consider $TV = V \times V \cong V \times V^* = T^*V$ where we identify V with V^* via the inner product. The canonical 1-form is then given by $\theta(A, \alpha, A', \alpha') = \langle \alpha, A' \rangle$, the symplectic form is $\omega_{(A,\alpha)}((A', \alpha'), (A'', \alpha'')) = \langle A', \alpha'' \rangle - \langle A'', \alpha' \rangle$, the Hamiltonian function for the straight lines $(A + t\alpha, \alpha)$ is $h(A, \alpha) = \frac{1}{2}\langle \alpha, \alpha \rangle$. The action of G on V lifts to the diagonal action on $TV = V \times V$ with fundamental vector field $\zeta_X(A, \alpha) = (A, \alpha, [X, A], [X, \alpha])$ for $X \in \mathfrak{g}$ and with generating function $f_X(A, \alpha) = \theta(\zeta_X(A, \alpha)) = \langle \alpha, [X, A] \rangle = \langle X, [A, \alpha] \rangle$, where we used the invariance of the inner product on $\mathfrak{l}$. Thus the momentum mapping $J : TV \to \mathfrak{g}^* \cong \mathfrak{g}$ for this action is given by $\langle X, J(A, \alpha) \rangle = f_X(A, \alpha)$, so $J(A, \alpha) = [A, \alpha]$. Along each line the momentum is constant, $J(A + t\alpha, \alpha) = [A, \alpha]$.

According to principles of symplectic reduction [1], 4.3.5, [36], [17], [30], we have to consider a regular value $Y \in \mathfrak{g}$ (and later for an arbitrary value) of the momentum mapping J and the submanifold $J^{-1}(Y) \subset TV$. The null distribution of $\omega|J^{-1}(Y)$ is integrable (with jumping dimensions) and its leaves (according to the Stefan-Sussmann theory of integrable distributions) are exactly the orbits in $J^{-1}(Y)$ of the isotropy group G_Y for the coadjoint action. So we have to consider the orbit space $J^{-1}(Y)/G_Y$. If Y is not a regular value of J, the inverse image $J^{-1}(Y)$ is a subset which is described by polynomial equations since J is polynomial (in fact quadratic), so $J^{-1}(Y)$ is stratified into submanifolds; symplectic reduction works also for this case, see [36].

6.4. The case of momentum $Y = 0$ gives again geodesics. If $Y = 0$ then $G_Y = G$ and $J^{-1}(0) = \{(A, \alpha) : [A, \alpha] = 0\}$, so A and α commute. We may use $g \in G$ to move (A, α) such that $A \in C \subset \Sigma \subset V$. If $A \in \Sigma$ is regular for the G-action then by lemma 6.2.2 it is also regular in the sense that $\Sigma = Z_V(A)$, thus

$\alpha \in \Sigma$ also, and the reduced phase space is $TC^o = C^o \times \Sigma$. The Hamiltonian is still $h(A,\alpha) = \frac{1}{2}\langle \alpha, \alpha \rangle$, the line $A + t\alpha$ is in Σ, and thus projects to a geodesic in C which is reflected at walls.

If A is not regular and $[A, \alpha'] = 0$ then there exists g in the isotropy group G_A such that $\alpha = g.\alpha' \in C$, by 6.2.5. If A is in a wall $\{\lambda = 0\}$, and if $\lambda(\alpha) = 0$, then the geodesic is in the same wall. If $\lambda(\alpha) > 0$ then the geodesic is reflected at this wall.

6.5. Symplectic reduction at regular points. If $Y \neq 0 \in \mathfrak{g}$ and if G_Y is the isotropy group of Y for the adjoint representation, then it is well known that we may pass from Y to the adjoint orbit $\mathcal{O}(Y) = \operatorname{Ad}^*(G)(Y)$ and get

$$J^{-1}(Y)/G_Y = J^{-1}(\mathcal{O}(Y))/G = (J^{-1}(Y) \times \mathcal{O}(-Y))/G, \tag{1}$$

where all (stratified) diffeomorphisms are symplectic ones.

We start again with some regular $A \in V$ which we may move into $C^o \subset \Sigma \subset V$ by using some suitable $g \in G$. Given $\alpha \in V$ we consider $[A, \alpha] = Y \in \mathfrak{g}$. According to the restricted root space decomposition 6.2 we can decompose $\alpha = \alpha_\Sigma + \sum_{\lambda \in R, i} \alpha^i_\lambda B^i_\lambda$ where $\alpha_\Sigma \in \Sigma$. Similarly we decompose $Y = Y_\mathfrak{h} + \sum_{\lambda \in R,i} Y^i_\lambda . E^i_\lambda$. Then $Y = [A, \alpha]$ implies $Y_\mathfrak{h} = 0 \in \mathfrak{h}$ and $Y^i_\lambda = \lambda(A)\alpha^i_\lambda$. Since A is regular, $A \in C^o$ and thus $\lambda(A) \neq 0$ for all $\lambda \in R$. Let us use also an orthonormal basis B^i_0 of Σ to expand $\alpha_\Sigma = \sum_i \alpha^i_0 B^i_0$ and $A = \sum_i A^i_0 B^i_0$.

We can thus use as coordinates

$$\begin{aligned}
&(A^i_0, \alpha^i_0) \in C^o \times \Sigma = TC^o \\
&Y^i_\lambda = \lambda(A)\alpha^i_\lambda \in \operatorname{Ad}(G).Y
\end{aligned}$$

The Hamiltonian function in these splitting is given by

$$\begin{aligned}
h(A,\alpha) &= \frac{1}{2}\langle \alpha, \alpha\rangle = \frac{1}{2}\sum_i (\alpha^i_0)^2 + \frac{1}{2}\sum_{\lambda \in R,i} \frac{(Y^i_\lambda)^2}{\lambda(A)^2} \\
dh &= \sum_i \alpha^i_0\, d\alpha^i_0 - \sum_{\lambda \in R,i} \frac{(Y^i_\lambda)^2}{\lambda(A)^3} \sum_k \lambda(B^k_0)\, dA^k_0 + \sum_{\lambda \in R,i} \frac{Y^i_\lambda}{\lambda(A)^2} dY^i_\lambda
\end{aligned}$$

The Poisson 2-field on $\mathfrak{g}$ (which is tangent to each adjoint orbit) is given by

$$\begin{aligned}
\Lambda_Y(dU, dV) &= \langle Y, [U,V]\rangle, \quad Y, U, V \in \mathfrak{g} \\
\Lambda_Y &= \sum_{\lambda,\mu,i,j} \Lambda_Y(dY^i_\lambda, dY^j_\mu)\partial_{Y^i_\lambda} \otimes \partial_{Y^j_\mu} \\
&= \sum_{\lambda,\mu,i,j} \langle Y, [E^i_\lambda, E^j_\mu]\rangle \partial_{Y^i_\lambda} \otimes \partial_{Y^j_\mu} \\
&= \sum_{\lambda,\mu,i,j} \langle Y, \sum_k N^{ijk}_{\lambda\mu} E^k_{\lambda+\mu}\rangle \partial_{Y^i_\lambda} \otimes \partial_{Y^j_\mu} \\
\Lambda_{-Y}(dY^i_\lambda) &= \sum_{\mu,j} \langle -Y, \sum_k N^{ijk}_{\lambda\mu} E^k_{\lambda+\mu}\rangle \partial_{Y^j_\mu}
\end{aligned}$$

where we used the convention $[E^i_\lambda, E^j_\mu] = \sum_k N^{ijk}_{\lambda\mu} E^k_{\lambda+\mu}$. We get the following Hamiltonian vector field

$$H_h = \sum_i \alpha^i_0 \,\partial_{A^i_0} + \sum_{\lambda\in R,i,k} \frac{(Y^i_\lambda)^2}{\lambda(A)^3}\lambda(B^k_0)\,\partial_{\alpha^k_0} - \sum_{\lambda\in R,i} \frac{Y^i_\lambda}{\lambda(A)^2}\sum_{\mu,j}\langle Y, \sum_k N^{ijk}_{\lambda\mu}E^k_{\lambda+\mu}\rangle \partial_{Y^j_\mu}$$

and the differential equation

$$\begin{aligned}\ddot{A}^k_0 &= \sum_{\lambda\in R,i}\frac{(Y^i_\lambda)^2}{\lambda(A)^3}\lambda(B^k_0),\\ \dot{Y}^j_\mu &= -\sum_{\lambda\in R,i}\frac{Y^i_\lambda}{\lambda(A)^2}\langle Y, \sum_k N^{ijk}_{\lambda\mu}E^k_{\lambda+\mu}\rangle = -\sum_{\lambda\in R,i}\frac{Y^i_\lambda}{\lambda(A)^2}\langle Y, [E^i_\lambda, E^j_\mu]\rangle.\end{aligned}$$

Let us now write $Y = \sum_{\lambda\in R,i} Y^i_\lambda.E^i_\lambda =: \sum_{\lambda\in R} Y_\lambda$ where $Y_\lambda \in \mathfrak{l}_\lambda \cap \mathfrak{g}$, and $Z = \sum_{\lambda\in R}\frac{1}{\lambda(A)^2}Y_\lambda \in \mathfrak{g}$. Then the differential equation becomes:

$$\begin{aligned}\ddot{A}^k_0 &= \langle \ddot{A}, B^k_0\rangle = \sum_{\lambda\in R}\frac{\|Y_\lambda\|^2}{\lambda(A)^3}\lambda(B^k_0) \qquad \text{or } \langle \ddot{A}, \quad \rangle = \sum_{\lambda\in R}\frac{\|Y_\lambda\|^2}{\lambda(A)^3}\lambda \ \in \Sigma^*\\ \dot{Y}^j_\mu &= -\Big\langle Y, \sum_{\lambda\in R,i}\frac{Y^i_\lambda}{\lambda(A)^2}[E^i_\lambda, E^j_\mu]\Big\rangle = -\langle Y, [Z, E^j_\mu]\rangle = -\langle [Y,Z], E^j_\mu\rangle,\end{aligned}$$

so that finally we have

$$\langle \ddot{A}, \quad \rangle = \sum_{\lambda\in R}\frac{\|Y_\lambda\|^2}{\lambda(A)^3}\lambda \ \in \Sigma^*, \qquad \dot{Y} = -[Y,Z]. \tag{2}$$

So the ballistic curve in C avoids the walls whenever $Y_\lambda(t) \neq 0$ for all restricted roots $\lambda \in R$, and just one time t.

6.6. Symplectic reduction at singular points. Let us now consider a singular $A \in V$ which we may move into $C \subset \Sigma \subset V$ by using some suitable $g \in G$. Then A is contained in some intersection of walls of C so $\mu(A) = 0$ for $\mu \in R_0 \subset R$. Given $\alpha \in V$ we consider $[A, \alpha] = Y \in \mathfrak{g}$. Using the decompositions $\alpha = \alpha_\Sigma + \sum_{\lambda\in R,i}\alpha^i_\lambda B^i_\lambda$ and $Y = Y_{\mathfrak{h}} + \sum_{\lambda\in R,i} Y^i_\lambda.E^i_\lambda =: Y_{\mathfrak{h}} + \sum_{\lambda\in R} Y_\lambda$ as in 6.5 we see that $Y = [A,\alpha]$ implies $Y_{\mathfrak{h}} = 0 \in \mathfrak{h}$ and $Y^i_\lambda = \lambda(A)\alpha^i_\lambda$ for all $\lambda \in R$. Thus we get $Y_\mu = 0$ for all $\mu \in R_0$. So we can follow the analysis in 6.5 without change if we agree $Y_\mu = 0$ means also $(Y^i_\mu)^2/\mu(A) = 0$ and $Z_\mu = Y_\mu/\mu(A)^2 = 0$. So again the ballistic curve is described by the equations (6.5.2). The second equation $\dot{Y} = [Y, Z]$ shows that $Y_\mu = 0$ along the the whole ballistic curve for $\mu \in R_0$. So the ballistic curve is composed of one just like in (6.5.2) inside the intersection of walls $\{B \in C : \mu(B) = 0 \text{ for all } \mu \in R_0\}$ together with a geodesic (reflected at walls) transversal to this intersection of walls.

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